Homotopy Theory

Contents

1 Covering Spaces

1.1 Locally Trivial Maps

Let $p : E \to B$ be continuous and $U \subset B$ be open.

Definition 1.1. A **trivialization** of p *over* U is a homeomorphism

$$\phi : p^{-1}(U) \to U \times F$$

over U, i.e., it satisfies $\mathrm{pr}_1 \circ \phi = p$.

Remark 1. If F is **discrete**, then we have a covering space. If F is a ***vector space***, we have a vector bundle.

Definition 1.2. The map p is **locally trivial** is there exists a open cover $\{U_\alpha\}_{\alpha \in I}$ of B such that p has a trivilization over each $U_\alpha \in \{U_\alpha\}_{\alpha \in I}$

Remark 2. A locally trivial map is also called a **bundle** or **fibre bundle** and a local trivialization a **bundle chart**.

Remark 3. If p is locally trivial, the set of all $b \in B$ for which $p^{-1}(b)$ is homeomorphic to F, is open and closed in B

Definition 1.3. If the fibers are homeomorphic to F, we call F the **typical fiber**.

Definition 1.4. A **covering space** of B is a locally trivial map

$$p : E \to B$$

with dicrete fibers.

Example 1.1. If we take $p : \mathbb{R} \to S^1$ to be the map

$$p(t) = e^{2\pi i t}$$

Then p is a covering for S^1. Indeed, if we let $U \subset S^1$ be a open set that is represented in the solid silver point in the white circle below, then we have the trivialization

$$\phi : p^{-1}(U) \to U \times \mathbb{Z}$$

So $\mathbb{Z}$ is the typical fiber.

Definition 1.5. Let $(G, *)$ be a group and E be a set. A **group action** is the map $\psi : G \times E \to E$, where

$$\psi(g, x) = g \cdot x$$

such that

$$g_1 \cdot (g_2 * x) = (g_1 \cdot g_2) * x$$
$$e \cdot x = x$$

In other words, it is associative and the identity acting on any element in E give the same element back.

Definition 1.6. A group action $G \times E \to E$ of a discrete group G on E is called **properly discontinuous** if for any $e \in E$ there exists a open neighborhood U of e such that

$$U \cap gU = \emptyset$$

for $g = \not{e}$

Example 1.2. Let $(\mathbb{Z}, +)$ be a group and $\psi : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be a group action defined as

$$\psi(k, x) = k + x$$

Consider $k = 2$ and $x = \pi$ and the open neighborhood $U = (\pi - \epsilon, \pi + \epsilon)$. Then we have

$$U \cap kU = (\pi - \epsilon, \pi + \epsilon) \cap (\pi + 2 - \epsilon, \pi + 2 + \epsilon) = \emptyset$$

so ψ is properly discontinuous.

Definition 1.7. A group action $\psi : G \times E \to E$ is **transitive** if for any $(x, y) \in G \times E$ there exists a $g \in G$ such that

$$x = g \cdot y$$

Definition 1.8. A **left G-principle covering** is the following data:

1. $p : E \to B$ a covering
2. A properly discontinuous group action $\psi : G \times E \to E$ such that

   $$p(gx) = p(x)$$

   for all $(g, x) \in G \times E$.
3. For any $x \in B$ and $a, b \in p^{-1}(\{x\})$, there exists a $g \in G$ such that

   $$a = g \cdot b$$

   In other words the induced action on each fiber is *transitive.*

This means that G acts on E nicley, i.e., G preserves fibers. G acts on fibers so there is an action

$$\Phi : G \times p^{-1}(x) \to p^{-1}(x)$$

where the action is also transitive.

Remark 4. If G is an abelian group, then we have a local system.

Example 1.3. Let $p : E \to B$ be a left G-principle covering. We show that this covering induces a homeomorphism of the orbit space E/G with B , i.e.,

$$E/G \cong B$$

Define the map $\Psi : E/G \to B$ as

$$\Psi(\bar{a_k}) = p(a_k)$$

Clearly this is surjective. Assume that

$$\Psi(\bar{a_k}) = \Psi(\bar{a_j})$$

then $p(a_k) = p(a_j) = b$ and $a_k, a_j \in p^{-1}(b)$. By transitivity there exists an $h \in G$ such that $a_j = ha_k$ so

$$\bar{a_j} = \bar{a_k}$$

Definition 1.9. Let $p : E \to B$ be a covering. A **deck transformation** of p is a homeomorphism $\alpha : E \to E$ such that

$$p \circ \alpha = p$$

that is α lifts p.

Remark 5. In some sense deck transformation are the symmetries of covering spaces.

Example 1.4. Consder the covering space $p : \mathbb{R} \to S^1$ given by

$$p(t) = e^{2\pi i t}$$

then the deck transformations is the homeomorphims $T_n : \mathbb{R} \to \mathbb{R}$ given by

$$T_n(t) = t + n$$

for $n \in \mathbb{Z}$

Example 1.5. Let $p : E \to B$ be a left G-principle covering, E connected. Then for each $g \in G$ define the left translation $l_g : E \to E$ as

$$l_g(t) = g \cdot t$$

Then

$$p \circ l_g(t) = p(g \cdot t) = p(t)$$

so this is a deck transformation. Then we can define a the following homomorphism $l : G \to \text{Aut}(p)$ defined as follows

$$l(g) = l_g$$

This is injective since, if $l(g) = l(h)$, then we have that for $t \in E$, $g \cdot t = h \cdot t$, or

$$t = g^{-1}h \cdot t$$

Let U be a open neighborhood of t, then

$$U \cap g^{-1}hU = \{t\}$$

and since the group action is properly discontinuous $g^{-1}h = e$, so $h = g$.
Furthermore l is surjective since if $x \in E$ and since a automorphims α is determined by a value at singe point x, $\alpha(x) \in p^{-1}(p(x))$ and the group action is transitive we have that for any $\beta \in Aut(p)$ there exists a $g \in G$ such that

$$\alpha(x) = g \cdot \beta(x) = l_g(\beta(x))$$

so its is surjective. So the connected principle coverings are the connected coverings with the largest automorphism group.

Definition 1.10. The category G-SET has the following data:

1. Objects are left G-sets, i.e., A set X with a left G action.
2. Morphisms are G-equivariant maps,(G-maps for short), i.e. $f : X \to Y$, X, Y G-sets, such that
$$f(ax) = af(x)$$
$a \in G$, $x \in X$.
3. Composition is composition of homomorphisms

Definition 1.11. The category COV_B has the following data:

1. Objects are covering spaces $p : E \to B$ over B.
2. Morphisms are maps of covering spaces, hence continuous functions $f : E_1 \to E_2$ such that the following diagram commutes:
$$\begin{tikzcd} & B & \\ E_1 \arrow[ur, "p_1"] \arrow[rr, "f"] & & E_2 \arrow[ul, "p_2"'] \end{tikzcd}$$

We now construct the **associated coverings** functor:

Lemma 1.1. *Let $p : E \to B$ be a right G-principle covering and F be a set with a left G-action. There is a covering $E \times F \to B$.*

Let $p : E \to B$ be a right G-principle covering and F be a set with a left G-action. Define $E \times_G F$ to be
$$E \times_G F = E \times F / \sim$$
where $\sim$ is a equivalence relation defined as: $(x, f) \sim (xg^{-1}, gf)$ for $x \in E, f \in F, g \in G$. Then we can define a continuous map $p_F : E \times_G F \to B$ as
$$p_F(\overline{(x, f)}) = p(x)$$
Note that this is a covering since p is a right G-principle covering, so there exists a homeomorphism
$$\phi : p^{-1}(U) \to U \times F$$

Remark 6. $E \times_G F$ is a kind of "tensor product", in that we take the product and allow elements of G to pass back and forth "across" $\times$.

Lemma 1.2. *A G-equivariant map $\Psi : F_1 \to F_2$ induces a map of coverings.*

Given a map between two G-sets, $\Psi : F_1 \to F_2$, we can make a map of coverings $\mathrm{id}\times_G\Psi : E \times_G F_1 \to E \times_G F_2$ defined as
$$\mathrm{id} \times_G \Psi(\overline{(x, f)}) = (x, \bar{\Psi}(f))$$

It's easy to show the the map satisfies that a triangular diagram commutes.
The above two lemmas assemble into a functor

$$A(p) : G\text{-}\mathbf{SET} \to \mathbf{COV}_B,$$

which sends a G-set F to $E \times_G F$. The functor is *associated* to p, which gives us a well defined target.

Remark 7. If $A(p)$ is an equivalence of categories, then the G-principle covering $p : E \to B$ over a path connected space B is called **universal**.

1.2 Fiber Transport

Definition 1.12. A map $p : E \to B$ has the **homotopy lifting property (HLP)** if for the space X the following holds: For any homotopy $h : X \times I \to B$ and each map $a : X \to E$ such that

$$p \circ a = h \circ i$$

where $i(x) = (x, 0)$, there exists a homotopy $H : X \times I \to E$ such that

$$p \circ H = h$$

$$H \circ i = a$$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dotted, "H"] & B \end{tikzcd}$$

Example 1.6. Consider the projection map $p : B \times F \to B$, i.e.,

$$p((b, f)) = b$$

and let $a(x) = (a_1(x), a_2(x))$. Now we construct H. From the condition $p \circ a = h \circ i$, we must have that $a_1(x) = h(x, 0)$. So define

$$H(x, t) = (h(x, t), a_2(x))$$

then we have

$$(p \circ H)(x, t) = p(h(x, t), a_2(x)) = h(x, t)$$

and

$$(H \circ i)(x) = H((x, 0)) = (h(x, 0), a_2(x)) = a$$

Remark 8. If a map $p : E \to B$ has the HLP for all spaces, it is called a *fibration.*

To proceed, we need to recall a definition.

Definition 1.13. Let $X, Y \in \mathbf{Spaces}$. Consider $\Pi(X, Y)$. Let $\alpha : U \to X$ and $\beta : Y \to V$. Composition with α and β yield functors

$$\beta_{\#} = \Pi_{\#}(\beta) : \Pi(X, Y) \to \Pi(X, V).$$

and

$$\alpha^{\#} = \Pi^{\#}(\alpha) : \Pi(X, Y) \to \Pi(U, Y)$$

where $f \mapsto \beta f$ and $[K] \mapsto [\beta K]$ in the first place, and $f \mapsto f\alpha$ and $[K] \mapsto [K(\alpha \times \mathrm{id})]$ in the second case.

Let $P : E \to B$ be a covering with the HLP for I and a point. We define the **transport** functor

$$T_p : \Pi(B) \to \mathrm{SET}$$

where $b \mapsto \pi_0(F_b)$ and $[v] \mapsto v_{\#}$. We associate to each path $v : I \to B$ from b to c the map $v_{\#}$ in the following way: Let $x \in F_b$. We pick a lifting $V : I \to E$ of v with $V(0) = x$. Set $v_{\#}[x] = [V(1)]$. This functor provides us with a right group action

$$\pi_0(F_b) \times \pi_1(B, b) \to \pi_0(F_b),$$

defined as $(x, [v]) \mapsto v_{\#}(x)$.

There is a left action of $\pi_b = \Pi(B)(b, b)$ on F_b given by $(a, x) \mapsto a_{\#}(x)$ which commutes with the right action of G. We say F_b is a (π_b, G)-set. Fix $x \in F_b$. For each $a \in \pi_b$, there exists a unique $\gamma_x(a) \in G$ such that $a \cdot x = x \cdot \gamma_x(a)$ since the action of G is free and transitive. The assignment $a \mapsto \gamma_x(a)$ is a homomorphism $\gamma_x : \pi_b \to G$. Since $\pi_1(B, b)$ is the opposite group to π_b, we set $\delta_x(a) = \gamma_x(a)^{-1}$. Then $\delta_x : \pi_1(B, b) \to G$ is a group homomorphism.

We now reach the whole point of this:

Theorem 1.1. *Let $p : E \to B$ be a right principle–G covering with a path connected total space. Then the following sequence is exact:*

$$1 \to \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, p(x)) \xrightarrow{\delta_x} G \to 1.$$

Corollary 1.1. *E is simply connected iff δ_x is an isomorphism. I.e., if E is simply connected, G is isomorphic to the fundamental group of B.*

We now come to the *point* of this chapter.

1.3 Classification of Coverings

1.4 Coverings of Simplicial Sets

2 Elementary Homotopy Theory

2.1 Mapping Cylinder

We first recall the definition of homotopy.

Definition 2.1 (Homotopy)**.** Let $X, Y \in \mathbf{Top}$ and $f, g : X \to Y$ be continuous maps. A **homotopy** from f to g is a continuous map

$$H : X \times I \to Y,$$

$(x,t) \to H(x,t) = H_t(x)$ such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for $x \in X$. I.e., $H_0 = f, H_1 = g$.

We denote a homotopy from f to g as $H : f \simeq g$. A homotopy $H_t : X \to Y$ is said to be relative to A if $H_t|_A$ does not depend on t. I.e., H_t is constant on A. A space X is **contractible** if it is homotopy equivalent to a point. A map $f : X \to Y$ is **null-homotopic** if it is homotopic to a constant map.

Definition 2.2 (Topological Sum)**.** Let $X, Y \in \mathbf{Top}$. We define the *topological sum* $X + Y$ as the disjoint union $X \sqcup Y$ with the topology defined as the topology generated by X and Y.

Let $f : X \to Y$ be a continuous function. We now construct the **mapping cylinder** $Z(f)$ of f as the pushout:

$$\begin{array}{ccc} X + X & \xrightarrow{\mathrm{id}+f} & X + Y \\ {\scriptstyle\langle i_0, i_1\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\langle j, \mathbf{J}\rangle} \\ X \times I & \xrightarrow{\quad a \quad} & Z(f) \end{array},$$

where $Z(f) = X \times I + Y/(f(x) \sim (x,1))$, $J(y) = y$, $j(x) = (x,0)$, and $i_t(x) = (x,t)$. From the construction, we have the projection $q : Z(f) \to Y$, where $(x,t) \to f(x)$ and $y \mapsto y$. We thus have the relations $qj = f$ and $qJ = \mathrm{id}$. The map Jq is homotopic to the identity relative to Y, where the homotopy is given by the identity on Y and contracts I to 1 relative to 1. We thus have a decomposition of f into a closed embedding J and a homotopy equivalence q.

Via the universal properties of pushouts, we have that continuous maps $\beta : Z(f) \to B$ correspond bijectively to pairs $h : X \times I \to B$ and $\alpha : Y \to B$ such that $h(x,1) = \alpha f(x)$.

We now consider *homotopy* commutative diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{\quad f' \quad} & Y' \end{array}$$

together with homotopies $\Phi : f'\alpha \simeq \beta f$. When the digram is strictly commutative, it depicts a morphism in the category of arrows in $\mathbf{Top}$. We can thus consider the data (α, β, Φ) as a kind of "generalized" morphism.

These data induce a map $\chi = Z(\alpha, \beta, \Phi) : Z(f) \to Z(f')$ defined by $\chi(y) = \beta(y), y \in Y$ and

$$\chi(x,s) = \begin{cases} (\alpha(x), 2s), & x \in X, s \leq 1/2 \\ \Phi_{2s-1}(x), & x \in X, s \geq 1/2. \end{cases}$$

We thus have the following diagram commutes:

$$\begin{tikzcd} X + Y \arrow[r] \arrow[d, "\alpha+\beta"] & Z(f) \arrow[d, "Z(\alpha,\beta,\Phi)"] \\ X' + Y' \arrow[r] & Z(f') \end{tikzcd}$$

The composition of two such morphisms is homotopic to a morphism of the same type.

2.2 Suspension

We will start of with some basic definitions. Let $X \in \mathbf{Top}$ and $x_0 \in X$.

Definition 2.3. We call the pair (X, x_0) a **pointed space** with base point $x_0 \in X$. A **pointed map** $f : (X, x_0) \to (Y, y_0)$ is a continous map $f : X \to Y$ that sends the base point of one space to the other ($x_0 \mapsto y_0$).

Definition 2.4. A homotopy $H : X \times I \to Y$ is pointed if H_t is pointed for each $t \in I$.

Let $\mathbf{Top}^0$ denote the category of pointed topological spaces.

Definition 2.5. Let $(X, x) \in \mathbf{Top}^0$. The **suspension** of (X, x) is the space

$$\Sigma X := X \times I/(X \times \partial I \cup \{x\} \times I).$$

The basepoint of ΣX is the space we identified to a point.

Remark 9. The definition we use here is often referred to as **reduced suspension**. Reduced suspension can be used to construct a homomorphism of homotopy groups, to which a very important theorem (*Freudenthal Suspension Theorem*) can be applied. This gives you a shot at determining the higher homotopy groups of spaces, including the all-important spheres.

Example 2.1. Consider $X = S^1$ and $I = (0, 1)$. Then the space $X \times \partial I \cup \{x_0\} \times I$ becomes the green and orange parts of the beautifully created godly figure below: Further we can see that the suspension ΣX becomes a sphere.

No we prove a useful proposition:

Theorem 2.1. *$K : (X, x_0) \times I \to (Y, y_0)$ is a pointed homotopy from k_{y_0} to itself if and only if it maps $X \times \partial I \cup \{x\} \times I$ to the base point y_0.*

Proof. $\Rightarrow$: Suppose that $K : (X, x_0) \times I \to (Y, y_0)$ is a pointed homotopy from k_{y_0} to itself, i.e., $K(x, 0) = k_{y_0}(x) = y_{y_0}$, $K(x, 1) = k_{y_0}(x) = y_{y_0}$ and $K_t(x_0) = y_0$ for all $t \in I$. $\square$

Suppose we have a map of pointed spaces $f : (I, \partial I) \to (Y, y_0)$. Suppose there were a homotopy from f to the constant map k_{y_0}. This would then be given by a map $H : (I, \partial I) \times I$ such that $H(-, 0) = f, H(-, 1) = k_{y_0}$ and $H|_{\partial I \times I} = k_{y_0}$. We thus have that a map $K : (X, x_0) \times I \to (Y, y_0)$ is a pointed homotopy from k_{y_0} to itself if and only if $(X \times \partial I \cup \{x\} \times I)$ is mapped to y_0.

Definition 2.6. Let $f, g : X \to Y$ be continuous maps and K is a subset of X, then we say that f and g are **homotopic relative to** K if there exists a homotopy $H : X \times I \to Y$ between f and g such that $H(k, t) = f(k) = h(k)$ for $k \in K$ and $t \in I$.

Now we can construct a group structure as follows: For a homotopy map $K : X \times I \to Y$ there is a pointed map $\bar{K} : (\Sigma X, X \times \partial I \cup \{x\} \times I) \to (Y, y_0)$, and homtopies relative to $X \times \partial I$ corresponds to pointed homptopies $\Sigma X \to Y$. Then homotopy set $[\Sigma X, Y]^0 := \{[f] | f : \Sigma X \to Y \text{ pointed maps}\}$ has a group structure with the binary operation (written additively) being $[f] + [g] = [f + g]$ where

$$(f+g)(x,t) = \begin{cases} f(x, 2t) & t \le \frac{1}{2} \\ g(x, 2t-1) & \frac{1}{2} \le t \end{cases}$$

The identity is the homotopy class of the constant map to x_0, the inverse of $[f(x,t)]$ is the homotopy class containing the function $-f(x,t) = f(x, 1-t)$. The associativity of this operation is given by the straight line homotopy H described in the picture below:

If $f : X \to Y$ is a pointed map, then $f \times \text{id}(I)$ induces a map

$$\Sigma f : \Sigma X \to \Sigma Y$$

by mapping $(x, t) \mapsto (f(x), t)$. With this we can define the funtor $\Sigma : \mathbf{Top}^0 \to \mathbf{Top}^0$ which induces a functor for pointed homotopies.

Definition 2.7. The **smash product** between two pointed spaces (A, a) and (B, b) is defined as

$$A \wedge B := A \times B / A \times b \cup a \times B = A \times B / A \vee B$$

Definition 2.8. Let $(X, x) \in \mathbf{Top}^0$. The k**-fold suspension** of (X, x) is the space

$$\Sigma^k X := X \wedge (I^k / \partial I^k).$$

Remark 10. This definition of the k-fold suspension is somehow canonically homeomorphic to $X \times I^n / (X \times \partial I^n \cup \{x\} \times \partial I^n)$

So just like before we can construct k group structures as follows: We define k binary operation $(+_i)$ on the homotopy set $[\Sigma^k X, Y]^0$, which depends on the I-coordinates, as follows:

$$(f +_i g)(x,t) = \begin{cases} f(x, t_1, ..., t_{i-1}, 2t_i, t_{i+1}, ..) & t_i \le \frac{1}{2} \\ g(x, t_1, ..., t_{i-1}, 2t_1 - 1, t_{i+1}, ...) & \frac{1}{2} \le t_i \end{cases}$$

We now show that all the group structures coincide and are abelian for $n \geq 2$. First we prove the commutation rule for $n = 2$, which can be done by unraveling the defintions

$$(a +_1 b) +_2 (c +_1 d) = (a +_2 c) +_1 (b +_2 d)$$

Theorem 2.2. *Suppose the set M carries two composition laws $+_1$ and $+_2$ with neutral elements e_i. Suppose further that the commutation rule holds. Then $+_1 = +_2 = +$, $e_1 = e_2 = e$, and the composition is associative and commutative.*

The suspension induces a map $\Sigma_* : [A, Y]^0 \to [\Sigma A, \Sigma Y]^0$ mapping $[f] \mapsto [\Sigma f]$, also called the suspension. If $A = \Sigma X$, then Σ_* is a homomorphism, because addition in $[\Sigma X, Y]^0$ is transformed by Σ_* into $+_1$.
Now suppose that $X = S^0 = \{\pm e_1\}$ with base point e_1. The we have the canonical homeomorphism.

$$I^n/\partial I^n \cong \Sigma^n S^0$$

Definition 2.9. Let (X, x_0) be pointed topological space, then the n-th homotopy group is:

$$\pi_n(X, x_0) = [I^n/\partial I^n, X]^0 = [(I^n, \partial I^n), (X, x_0)].$$

Furthermore these groups are abelian for $n \geq 2$, where we can use we can use each n coordinates to define a group structure.

2.3 Loop Space

Definition 2.10. The **Loop Space** ΩY of Y is the subset of the path space Y^I (with compact open topology) consisting of the loops in Y with basepoint y. I.e.,

$$\Omega Y = \{w \in Y^I : w(0) = w(1) = y\}.$$

The constant loop k is the basepoint. We thus have that a pointed map $f : X \to Y$ induces a pointed map $\Omega f : \Omega X \to \Omega Y$, where $w \mapsto f \circ w$. In other words, we have a functor $\Omega : \mathbf{Top}^0 \to \mathbf{Top}^0$. This functor is compatible with homotopies, in that a pointed homotopy H_t yields a pointed homotopy ΩH_t. We can also define the loop space as the space of pointed maps $F^0(S^1, Y)$.

Before we proceed, we take a brief detour into mapping spaces and their arithmetic. Let $X, Y \in \mathbf{Top}$. For $K \subset X$ and $U \subset Y$, we set $W(K, U) = \{f \in Y^X : f(K) \subset U\}$. The *compact-open* topology (CO-topology) on Y^X is the topology which has a subbasis sets of the form $W(K, U)$ where K is compact and U is open. Note that a continous map $f : X \to Y$ induces continous maps $f^Z : X^Z \to Y^Z, g \mapsto f \circ g$, and $Z^f : Z^Y \to Z^X, g \mapsto g \circ f$.

Proposition 2.1. *The evaluation map $e_{X,Y} = e : Y^X \times X \to Y, (f, x) \mapsto f(x)$ is continuous.*

Proof. Let U be an open neighborhood of $f(x)$. Since f is continuous and locally compact, there exists a compact neighborhood K of x such that $f(K) \subset U$. We thus have $W(K,U) \times K$ maps into U. □

Definition 2.11. Let $f : X \times Y \to Z$ be continuous. The **adjoint map** $f^\wedge : X \to Z^Y$ is the continuous map defined as $f^\wedge(x)(y) = f(x,y)$.

We have thus obtained a set map $\alpha : Z^{X\times Y} \to (Z^Y)^X, f \mapsto f^\wedge$. Let $e^{Y,Z}$ be continous. Then a continuous map $\phi : X \to Z^Y$ induces a continous map

$$\phi^\wedge = e_{Y,Z} \circ (\phi \times id_Y) : X \times Y \to Z^Y \times Y \to Z.$$

We thus have another set map $\beta : (Z^Y)^X \to Z^{X\times Y}, \phi \mapsto \phi^\wedge$.

Proposition 2.2. *Let $e_{Y,Z}$ be continuous. Then α, β are inverse bijections, so that ϕ, f is continous iff $\phi^\wedge, f^\wedge$ is, respectively.*

Corollary 2.1. *If $H : X \times Y \times I \to Z$ is a homotopy, then $H^\wedge : X \times I \to Z^Y$ is a homotopy. Hence $[X \times Y, Z] \to [X, Z^Y], [f] \mapsto [f^\wedge]$ is well defined. If, moreover, $e_{Y,Z}$ is continuous (e.g. Y locally compact), this map is bijective.*

We can now define a dual notion of homotopy:

Definition 2.12 (Homotopy, 2nd)**.** We have a continuous evaluation map $e_t : Y^I \to X, w \mapsto w(t)$. Let $f_0, f_1 : X \to Y$. Then, a homotopy from f_0 to f_1 is a continuous map $h : X \to Y^I$ such that $e_i \circ h = f_i, i = 1,2$. Since I is locally compact, there is a bijection between continous functions $X \times I \to Y$ and continous functions $X \to Y^I$.

Now, let $(X, x_0), (Y, y_0)$ be pointed spaces. Denote by $F^0(X,Y)$ the space of pointed maps with CO$-$*topology* as a subspace of $F(X,Y)$. In $F^0(X,Y)$ we use the constant map as the basepoint. If $f : X \times Y \to Z$ is a continuous map, recall we constructed its adjoint as $f^\wedge(x)(y) = f(x,y)$. In order for this map to be *pointed*, we need the basepoint of X to be sent to the constant map. I.e., for all $y \in Y$, we need $f^\wedge(x_0)(y) = f(x_0,y) = z_0$. Thus $x_0 \times Y$ needs to be sent to the basepoint of Z. However, any morphism in Z^Y must *also* be pointed, so that we need for all $x \in X$, $f^\wedge(x)(y_0) = f(x,y_0) = z_0$. Thus $X \times y_0$ must be sent to the basepoint of Z as well. So, we have that $f^\wedge$ is a morphism of pointed spaces if and only if $f(X \times y_0 \cup x_0 \times Y) = z_0$. Does this subspace look familiar?

Let $p : X \times Y \to X \wedge Y$ be the quotient map. If $g : X \wedge Y \to Z$ is given, we can form the composition of the map $g \circ p : X \times Y \to Z$, which is nothing more than a map from $X \times Y \to Z$ which sends $X \times y_0 \cup x_0 \times Y$ to z_0. Let $\alpha^0(g)$ denote the adjoint of $g \circ p$, which is by construction an element of $F^0(X, F^0(Y,Z))$. In this manner we obtain a set map

$$\alpha^0 : F^0(X \wedge Y, Z) \to F^0(X, F^0(Y,Z)).$$

Note that the evaluation map $F^0(X,Y) \times X \to Y$ must factor over the quotient space $F^0(X,Y) \wedge X$, and so induces $e^0_{X,Y} : F^0(X,Y) \wedge X \to Y$.

Now, let $e^0_{X,Y}$ be continous. Given a pointed map $\phi : X \to F^0(Y, Z)$, we can form $\phi^\wedge = \beta^0(\phi) = e^0_{X,Y} \circ (\phi \wedge \mathrm{id}) : X \wedge Y \to Z$, and hence a set map

$$\beta^0 : F^0(X, F^0(Y, Z)) \to F^0(X \wedge Y, Z).$$

We have a familiar proposition:

Proposition 2.3. *Let $e^0_{X,Y}$ be continuous. Then α^0 and β^0 are inverse bijections.*

Corollary 2.2.

$$[X \wedge Y, Z]^0 \to [X, F^0(Y, Z)]^0, [f] \mapsto [\alpha^0(f)]$$

is well defined. Moreover, if $e^0_{X,Y}$ is continuous, then this map is bijective.

Theorem 2.3 (Exponential Law)**.** *Let X and Y be locally compact. Then the pointed adjunction map*

$$\alpha^0 : F^0(X \wedge Y, Z) \to F^0(X, F^0(Y, Z))$$

is a homeomorphism. A similar unpointed version holds.

Now, back to loop spaces. Recall that ΩY is pointed with basepoint the constant map k.

Proposition 2.4. *The product of loops defines a multiplication*

$$m : \Omega Y \times \Omega Y \to \Omega Y, (u, v) \mapsto u * v$$

with the following properties:

1. *m is continuous*
2. *the maps $u \mapsto k * u$ and $u \mapsto u * k$ are pointed homotopic to the identity.*
3. *$m(m \times id)$ and $m(id \times m)$ are pointed homotopic.*
4. *the maps $u \mapsto u * \overline{u}$ and $u \mapsto \overline{u} * u$ are pointed homotopic to the constant map.*

3 Homotopy Limits and Colimits

We first describe the necessity of deriving limits and colimits. Let I be a small category, and consider two diagrams $D, D' : I \to$ **Top**. If one has a natural transformation $f : D \to D'$, then there is an induced map $\mathrm{colim} D \to \mathrm{colim} D'$. If f is a natural weak equivalence ($D(i) \to D'(i)$ is a weak equivalence for all i), it does not in general follow that $\mathrm{colim} D \to \mathrm{colim} D'$ is a weak equivalence.

4 Fibrations and Cofibrations

4.1 Compactly Generated Spaces

Given a map $f : X \times Y \to Z$, we would like to topologize the set of continuous functions $C(Y, Z)$ in such a way that f is continuous if and only if the adjoint

$$\tilde{f} : X \to C(Y, Z), \tilde{f}(x)(y) = f(x, y)$$

is continuous. For instance:

1. We would like an action of a topological group $G \times Z \to Z$ to correspond to a continuous function $G \to \mathrm{Hom}_{iso}(Z, Z)$, where Hom_{iso} is given the subspace topology.
2. We would like a homotopy $f : I \times Y \to Z$ to correspond to a path $\tilde{f} : I \to C(Y, Z)$ of functions.
3. The evaluation map
$$C(Y, Z) \times Y \to Z, (f, y) \mapsto f(y)$$
should be continuous.

Such a topology does not exist for **Top**. However, we can restrict to a full subcategory of spaces called "compactly generated" spaces.

Definition 4.1. A topological space X is said to be **compactly generated** if X is Hausdorff and a subset $A \subset X$ is closed if and only if $A \cap C$ is closed for every compact $C \subset X$.

Example 4.1. Examples of compactly generated spaces include:

1. locally compact Hausdorff spaces (e.g. manifolds)
2. metric spaces
3. CW complexes

CG will denote the full subcategory of compactly generated topological spaces.

4.2 Fibrations

The standard example of fibrations are usually fiber bundles. I.e., fiber bundles over paracompact spaces are always fibrations.

5 Homology with Local Coefficients

6 Bundle Theory

6.1 Fiber Bundles with Structure Group

Supposed $p : E \to B$ is a fiber bundle with typical fiber F. For each $b \in B$, $p^{-1}(b)$ is homeomorphic to the fiber, but the homeomorphism is chart dependent. Hence two charts give rise to a homeomorphism $F \cong p^{-1}(b) \cong F$. This is element of the homeomorphism group Homeo(F). More precisely, given two charts $\phi : U \times F \to p^{-1}(U)$ and $\phi' : U' \times F \to p^{-1}(U')$, there is a function $\theta_{\phi,\phi'} : U \cap U' \to \text{Homeo}(F)$ so that

$$\phi'(b, f) = \phi(b, \theta(b)(f))$$

for all $b \in U \cap U'$.

7 Simplicial Homotopy Theory

7.1 Definitions

In this section we recover classical homotopy theory in the context of simplicial sets. First, recall that a simplicial set is a functor $\Delta^{op} \to \mathbf{Set}$. The category Δ fully embeds in the category of simplicial sets via $[n] \mapsto \text{Hom}([n],)$. Let $X \in \mathbf{sSet}$. Then, there are simplicial structure maps associated with any morphism $[n] \xrightarrow{f} [m]$, i.e.,

$$X[m] \xrightarrow{X[f]} X[n]$$

which map the m-simplices to the n-simplices via composition with f. We can think of $X[m]$ as assigning $[m]$ to the set of m-simplicies of some object constructed from simplices.

7.2 Homology

7.3 Cohomology

7.4 Homotopy Theory of Simplicial Sets

Without further ado, we have the following definition:

Definition 7.1. Let $f, g : X \to Y$ be two simplicial maps. A **homotopy** η from f to g, denoted $\eta : f \implies g$ is a morphism $\eta : X \times \Delta^1 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \Delta^0 \arrow[d, "id\times d^1"'] \arrow[dr, "f"] & \\
X \times \Delta^1 \arrow[r, "\eta"] & Y \\
X \times \Delta^0 \arrow[u, "id\times d^0"] \arrow[ur, "g"'] &
\end{tikzcd}.$$

Note that the commutativity of this diagram implies that simplex-wise, $\eta(x, 0) = f(x)$ and $\eta(x, 1) = g(x)$ for all simplices.

Definition 7.2 (Kan Complex)**.** A **Kan complex** is a simplicial set X such that, for any map $\Lambda^n_k \to X$, we have a lift $\Delta^n \to X$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "\iota"] & X \\ \Delta^n \arrow[ur] & \end{tikzcd}.$$

Definition 7.3. Let $f : X \to Y$ be a simplicial map. f is a **Kan Fibration** if for a diagram of the following form, we have a diagonal lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "s"] \arrow[d, "\iota"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "y"] \arrow[ur, dotted] & Y \end{tikzcd}.$$

Note that a Kan complex is a simplicial set such that the map to the singleton set is a Kan fibration.

To define a right adjoint for $sd : sSet \to sSet$, we begin investigating what properties one must have. The defining property of adjoints is $\hom(sd(X), Y) \cong \hom(X, RY)$. For $X = \Delta^n$, we have $\hom(sd(\Delta^n), Y) \cong \hom(\Delta^n, RY) \cong (RY)_n$. We thus define $Ex(Y)$ as the simplicial set given by $(ExY)_n := \hom(sd(\Delta^n), Y)$.

We have a natural transformation $id_{sSet} \to Ex(X)$:

$$\begin{tikzcd} X_m \arrow[r] \arrow[d] & Ex(X)_m \arrow[d] \\ X_n \arrow[r] & Ex(X)_m \end{tikzcd}$$

where the horizontal components are induced by the adjoint maps

$$\begin{tikzcd} sd(X_m) \arrow[r, "LV"] \arrow[d] & X_m \arrow[d] \\ sd(X_n) \arrow[r, "LV"] & X_m \end{tikzcd}$$

where LV is the last vertex map, mapping $sd(\Delta^n) = C(sd(\partial\Delta^n)) \to \Delta^n$. Concretely, we have that an n-simplex of $Ex(X)$ is a simplicial map $sd(\Delta^n) \to X$. Then, the inclusion $X \hookrightarrow Ex(X)$ sends an n-simplex of X, i.e. a simplicial map $\Delta^n \xrightarrow{\sigma} X$, to the composition

$$sd(\Delta^n) \to \Delta^n \xrightarrow{\sigma} X,$$

an n-simplex of $Ex(X)$.

Definition 7.4. Consider a diagram of the form

$$X \to Ex(X) \to Ex(Ex(X)) \to \cdots \to Ex^k(X) \to \cdots .$$

Then,

$$Ex^\infty(X) := colim_{k\in\mathbb{N}}[X \to Ex(X) \to Ex(Ex(X)) \to \cdots \to Ex^k(X) \to \cdots].$$

Theorem 7.1. *For all $X \in sSet$, we have $Ex^\infty(X)$ is Kan.*

Proof. To be added at a later date. □

Definition 7.5. A simplicial map $f : X \to Y$ is a **weak equivalence** if $Ex^\infty(f) : Ex^\infty(X) \to Ex^\infty(Y)$ is a simplicial homotopy equivalence.

In order to prove that $X \to Ex^\infty(X)$ is a weak equivalence, we need the simplicial whitehead theorem:

Theorem 7.2 (Simplicial Whitehead Theorem)**.** *Let $X, Y \in Kan$. A simplicial map $f : X \to Y$ is a simplicial homotopy equivalence if and only for all α and β making the following square commute, there exists a diagonal making the upper triangle commute strictly and the lower triangle commute up to homotopy:*

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\forall\alpha"] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\forall\beta"'] \arrow[ur, dotted, "\exists d"] & Y \end{tikzcd}.$$

The homotopy $h : \Delta^1 \times \Delta^n \to Y$ for the bottom triangle must be constant on $\Delta^1 \times \partial\Delta^n$.

This means we have a homotopy $\Delta^1 \times \Delta^n \to Y$ from $f \circ d$ to the bottom map whose restriction to $\Delta^1 \times \partial\Delta^n$ is a constant homotopy, meaning it factors as
$\Delta^1 \times \partial\Delta^n \to \Delta^n \to Y$.

Theorem 7.3. *For any simplicial set X, we have $X \to Ex^\infty(X)$ is a weak equivalence.*

Definition 7.6. The derived internal hom functor $RHom : sSet^{op} \times sSet \to sSet$ is defined as

$$RHom(X, Y) = Hom(Ex^\infty(X), Ex^\infty(Y)).$$

7.5 How doth one goest about deriving a functor?

We have now encountered our first derived functor. Is there a general process for deriving functors? Indeed there is, and we investigate this in the current section.

First, what is a derived functor used for? In general, functors need not preserve weak equivalences. Indeed, we have that regular internal hom $Hom(X, Y)$ does not preserve weak equivalences, meaning that if $f : X \to Y$ is a weak equivalence, then

$Hom(Z, f) : Hom(Z, X) \to Hom(Z, Y)$ need not be a weak equivalence. In order to solve this problem of deriving arbitrary functors, we need the language of *relative categories*.

Definition 7.7. A **relative category** is a category C together with a subcategory $W \subset C$ with the same objects as C. Morphisms in W are called weak equivalences. A **relative functor** $(C, W) \to (C', W')$ is a functor $F : C \to C'$ that maps W to W'. Small relative categories and relative functors form a category **RelCat**.

Example 7.1. The relative category of simplicial sets is formed by $(sSet, s.w.e.'s)$.

Definition 7.8. Let $f : C \to D$ be a map of chain complexes. f is called a **quasi-isomorphism** if $H(f) : H(C) \to H(D)$ is an isomorphism of graded abelian groups.

Example 7.2. If $f : X \to Y$ is a simplicial weak equivalence, then $C(f) : C(X) \to C(Y)$ is a quasi-isomorphism. Chain complexes and quasi-isomorphisms form a relative category.

Example 7.3. The derived mapping space functor $RHom(X, Y)$ for simplicial sets preserves weak equivalences.

Definition 7.9. In a relative category (C, W), $A, B \in C$ are said to be **weakly equivalent** if there is a finite zig-zag of weak equivalences connecting A and B:

$$A = X_0 \leftarrow X_1 \to X_2 \leftarrow \cdots \to X_n = B.$$

Definition 7.10. Suppose (C, W_C) and (D, W_D) are relative categories and $F : C \to D$ is a functor that need not preserve weak equivalences. We say that F is **right derivable** if there is a full subcategory $C' \subset C$, with the inclusion functor denoted ι, and a resolution functor $R : C \to C'$ that preserves weak equivalences together with a natural weak equivalence $r : id_C \to \iota \circ R$ such that the restriction of F to C' preserves weak equivalences.

If we have a relative category, we have the *potential* to do homotopy theory. By that I mean, we have the notion of weak equivalence, the correct form of equivalence for doing homotopy theory. We will now build on the relative category structure with an eye towards doing homotopy theory.

Definition 7.11. The **homotopy category** of a relative category (C, W) is defined as an ordinary category D equipped with a relative functor $F : (C, W) \to (D, \mathbf{Iso}_D)$ such that for any other pair (D', F'), the category of functors $D \to D'$ making the diagram

$$\begin{tikzcd} & C \arrow[dl, "F"'] \arrow[dr, "F'"] & \\ D \arrow[rr] & & D' \end{tikzcd}$$

commute is equivalent to the the terminal category with one morphism.

So far, here is what we have done:

- define a simplicial weak equivalence;
- derived Hom via Ex^{∞};
- want to see what is necessary to derive generic functors (to do so we need more tools);
- define relative categories (appropriate categories for doing homotopy theory) and relative functors;
- describe the requirements for deriving a functor between relative categories;
- initial description of the homotopy category of a relative category;

7.6 Homotopy Limits and Colimits

Let (C, W) be a relative category, and let I be a small category. We can put a relative category structure on the functor category $(C, W)^I$ as follows: a morphism in $(C, W)^I$ is a weak equivalence if it is a natural weak equivalence, meaning if $t : F \to G$ is a natural transformation, then t is a weak equivalence in $(C, W)^I$ for all $i \in I$, $t(i) : F(i) \to G(i)$ is in W.

8 Sheaf Cohomology

Definition 8.1. Let X be a topological space. A presheaf $\mathcal{F}$ of sets on X consists of the following data:

- For every open subset $U \subset X$, a set $\mathcal{F}(U)$. We assume $\mathcal{F}(\emptyset)$ is a set with one element.
- For every inclusion $V \subset U$, a mapping of sets $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the restriction map.
- We require that for a triple inclusion $W \subset V \subset U$, $\rho_{WU} = \rho_{WV}\rho_{VU}$.

Definition 8.2. Let $\mathcal{F}$ be a presheaf over X. One says $\mathcal{F}$ is a sheaf if for every open set V of X, and every open covering $\{U_i\}_{i \in I}$ of V and for every family $\{s_i\}$, where $s_i \in \mathcal{F}(U_i)$, such that $\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j)$, there exists a unique $s \in \mathcal{F}(V)$ such that $\rho_{U_i, V}(s) = s_i$.

For a sheaf $\mathcal{F}$, we will use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Definition 8.3. Let $\mathcal{F}$ be a presheaf of sets on a space X, and let $x \in X$. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is the quotient of the set $\amalg_{U : x \in U} \mathcal{F}(U)$ by the following equivalence relation: $s \in \mathcal{F}(U) \sim s' \mathcal{F}(V)$ iff there exists an open set $x \in W \subset U \cap V$ such that s and s' have the same restriction to W.

Elements of $\mathcal{F}_x$ are called germs at x of sections of $\mathcal{F}$.