Computer Science Department

CS504

Digital Logic & Computer Organization

Lecture 2

Lecture Outline (Chapter 2)

- ★ **Boolean Algebra (Sections 2.2, 2.3)**
- ★ **Some Properties of Boolean Algebra (Section 2.4)**
  - **Dual of a Boolean Expression**
  - **Boolean Operator Precedence**
  - **Useful Theorems**
  - **Complementing Functions**
- ★ **Expression Simplification (Section 2.5)**
- ★ **Canonical Forms (Section 2.6)**
  - **Minterms & Maxterms**
  - **Purpose Of The Index**
  - **Standard Order**
  - **Sum-Of-Minterms (SOM) Canonical Form**
  - **Product-Of-Maxterms (POM) Canonical Form**

Boolean Algebra

- Invented by George Boole in 1854.
- An algebraic structure defined by a set $B = \{0, 1\}$, together with two binary operators (+ and ·) and a unary operator ($\overline{}$).

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	Identity element
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	Idempotence
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	Complement
9.	$\overline{\overline{X}} = X$			Involution
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Some Properties of Boolean Algebra

- **Boolean Algebra is defined in general by a set *B* that can have more than two values.**
- **A two-valued Boolean algebra is also know as <u>Switching Algebra</u>, where the Boolean set *B* is restricted to 0 and 1.**
- **Binary logic circuits are represented by switching algebra.**
- **The <u>dual</u> of an algebraic expression is obtained by interchanging + and · and interchanging 0's and 1's.**
- **The identities appear in <u>dual</u> pairs. When there is only one identity on a line the identity is <u>self-dual</u>, i. e., the dual expression = the original expression.**
- **Sometimes, the dot symbol '•' (AND operator) is not written when the meaning is clear.**

Dual of a Boolean Expression

- **Example: $F = (A + \overline{C}) \cdot B + 0$**

  dual $F = (A \cdot \overline{C} + B) \cdot 1 = A \cdot \overline{C} + B$
- **Example: $G = X \cdot Y + \overline{(W + Z)}$**

  dual $G = (X+Y) \cdot \overline{(W \cdot Z)} = (X+Y) \cdot (\overline{W}+\overline{Z})$
- **Example: $H = A \cdot B + A \cdot C + B \cdot C$**

  dual $H = (A+B) \cdot (A+C) \cdot (B+C)$
- **Unless it happens to be self-dual, the dual of an expression does not equal the expression itself**
- **Are any of these functions self-dual? H is self-dual**

  $(A+B)(A+C)(B+C)=(A+BC)(B+C)=AB+AC+BC$

Boolean Operator Precedence

- **The order of evaluation is:**
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example:** $F = A(B + C)(C + \overline{D})$

Boolean Algebraic Proof – Example 1

- $A + A \cdot B = A$ (Absorption Theorem)

Proof Steps	Justification
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	Identity element: $A \cdot 1 = A$
$= A \cdot (1 + B)$	Distributive
$= A \cdot 1$	$1 + B = 1$
$= A$	Identity element

- **Our primary reason for doing proofs is to learn:**
  - Careful and efficient use of the postulates and theorems of switching algebra, and
  - How to choose the appropriate postulate or theorem to apply to make forward progress, irrespective of the application.

Boolean Algebraic Proof – Example 2

- $AB + \overline{A}C + BC = AB + \overline{A}C$ (Consensus Theorem)

<u>Proof Steps</u>	<u>Justification</u>
$AB + \overline{A}C + BC$	
$= AB + \overline{A}C + 1 \cdot BC$	Identity element
$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$	Complement
$= AB + \overline{A}C + ABC + \overline{A}BC$	Distributive
$= AB + ABC + \overline{A}C + \overline{A}CB$	Commutative
$= AB \cdot 1 + ABC + \overline{A}C \cdot 1 + \overline{A}CB$	Identity element
$= AB\,(1+C) + \overline{A}C\,(1 + B)$	Distributive
$= AB\,.\,1 + \overline{A}C\,.\,1$	$1+X = 1$
$= AB + \overline{A}C$	Identity element

Useful Theorems

- Minimization
  $X\,Y + \overline{X}\,Y = Y$
- Minimization (dual)
  $(X+Y)(\overline{X}+Y) = Y$
- Absorption
  $X + X\,Y = X$
- Absorption (dual)
  $X \cdot (X + Y) = X$
- Simplification
  $X + \overline{X}\,Y = X + Y$
- Simplification (dual)
  $X \cdot (\overline{X} + Y) = X \cdot Y$
- DeMorgan's
- $\overline{X + Y} = \overline{X} \cdot \overline{Y}$
- DeMorgan's (dual)
- $\overline{X \cdot Y} = \overline{X} + \overline{Y}$

Truth Table to Verify DeMorgan's

$$\overline{X + Y} = \overline{X} \cdot \overline{Y} \qquad \overline{X \cdot Y} = \overline{X} + \overline{Y}$$

X	Y	$X \cdot Y$	$X+Y$	$\overline{X}$	$\overline{Y}$	$\overline{X+Y}$	$\overline{X} \cdot \overline{Y}$	$\overline{X \cdot Y}$	$\overline{X}+\overline{Y}$
0	0	0	0	1	1	1	1	1	1
0	1	0	1	1	0	0	0	1	1
1	0	0	1	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0	0

- Generalized DeMorgan's Theorem:

$$\overline{X_1 + X_2 + \ldots + X_n} = \overline{X_1} \cdot \overline{X_2} \cdot \ldots \cdot \overline{X_n}$$

$$\overline{X_1 \cdot X_2 \cdot \ldots \cdot X_n} = \overline{X_1} + \overline{X_2} + \ldots + \overline{X_n}$$

Complementing Functions

- Use DeMorgan's Theorem:
  1. Interchange AND and OR operators
  2. Complement each constant and literal
- Example: Complement $F = \overline{x}y\overline{z} + x\overline{y}\overline{z}$

$$\overline{F} = (x + \overline{y} + z)(\overline{x} + y + z)$$

- Example: Complement $G = (\overline{a} + bc)\overline{d} + e$

$$\overline{G} = (a\,(\overline{b} + \overline{c}) + d)\,\overline{e}$$

Expression Simplification

- **An application of switching Boolean algebra**
- **Simplify to contain the smallest number of <u>literals</u> (variables that may or may not be complemented)**

$$AB + \overline{A}CD + \overline{A}BD + \overline{A}C\overline{D} + ABCD$$

$$= AB + ABCD + \overline{A}CD + \overline{A}C\overline{D} + \overline{A}BD$$

$$= AB + AB(CD) + \overline{A}C(D + \overline{D}) + \overline{A}BD$$

$$= AB + \overline{A}C + \overline{A}BD = B(A + \overline{A}D) + \overline{A}C$$

$$= B(A + D) + \overline{A}C \quad \text{(has only 5 literals)}$$

Next … Canonical Forms

- Minterms & Maxterms
- Purpose Of The Index
- Standard Order
- Sum-Of-Minterms (SOM) Canonical Form
- Product-Of-Maxterms (POM) Canonical Form
- Conversions Between Canonical Forms
- Representation Of Function Complements

Minterms

- Minterms are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for n variables.
- Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:

  XY (both normal)

  $X\overline{Y}$ (X normal, Y complemented)

  $\overline{X}Y$ (X complemented, Y normal)

  $\overline{X}\,\overline{Y}$ (both complemented)
- Thus there are four minterms of two variables.

Maxterms

- **Maxterms** are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for *n* variables.
- **Example:** Two variables (X and Y) produce 2 x 2 = 4 combinations:

$X + Y$ (both normal)

$X + \overline{Y}$ (X normal, Y complemented)

$\overline{X} + Y$ (X complemented, Y normal)

$\overline{X} + \overline{Y}$ (both complemented)

Minterms & Maxterms for 2 variables

- Two variable minterms and maxterms.

x	y	Index	Minterm	Maxterm
0	0	0	$m_0 = \overline{x}\,\overline{y}$	$M_0 = x + y$
0	1	1	$m_1 = \overline{x}\,y$	$M_1 = x + \overline{y}$
1	0	2	$m_2 = x\,\overline{y}$	$M_2 = \overline{x} + y$
1	1	3	$m_3 = x\,y$	$M_3 = \overline{x} + \overline{y}$

- The minterm m_i should evaluate to 1 for each combination of x and y.
- The maxterm is the complement of the minterm

Minterms & Maxterms for 3 variables

x	y	z	Index	Minterm	Maxterm
0	0	0	0	$m_0 = \bar{x}\,\bar{y}\,\bar{z}$	$M_0 = x + y + z$
0	0	1	1	$m_1 = \bar{x}\,\bar{y}\,z$	$M_1 = x + y + \bar{z}$
0	1	0	2	$m_2 = \bar{x}\,y\,\bar{z}$	$M_2 = x + \bar{y} + z$
0	1	1	3	$m_3 = \bar{x}\,y\,z$	$M_3 = x + \bar{y} + \bar{z}$
1	0	0	4	$m_4 = x\,\bar{y}\,\bar{z}$	$M_4 = \bar{x} + y + z$
1	0	1	5	$m_5 = x\,\bar{y}\,z$	$M_5 = \bar{x} + y + \bar{z}$
1	1	0	6	$m_6 = x\,y\,\bar{z}$	$M_6 = \bar{x} + \bar{y} + z$
1	1	1	7	$m_7 = x\,y\,z$	$M_7 = \bar{x} + \bar{y} + \bar{z}$

Maxterm M_i is the complement of minterm m_i

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Purpose Of The Index

- Minterms and Maxterms are designated with an index
- The index number corresponds to a binary pattern
- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - ‘1’ means the variable is “Not Complemented” and
  - ‘0’ means the variable is “Complemented”.
- For Maxterms:
  - ‘0’ means the variable is “Not Complemented” and
  - ‘1’ means the variable is “Complemented”.

Standard Order

- **All variables should be present in a minterm or maxterm and should be listed in the <u>same order</u> (usually alphabetically)**
- **Example: For variables a, b, c:**
  - **Maxterms $(a + b + \bar{c})$, $(\bar{a} + b + \bar{c})$ are in standard order**
  - **However, $(b + \bar{a} + c)$ is NOT in standard order**

    $(\bar{a} + c)$ does NOT contain all variables
  - **Minterms $(a\ b\ \bar{c})$ and $(\bar{a}\ b\ \bar{c})$ are in standard order**
  - **However, $(b\ a\ \bar{c})$ is not in standard order**

    $(\bar{a}\ c)$ does not contain all variables

Sum-Of-Minterms (SOM)

- Sum-Of-Minterms (SOM) canonical form:
  Sum of minterms of entries that evaluate to '**1**'

x	y	z	F	Minterm
0	0	0	0	
0	0	1	**1**	$m_1 = \bar{x}\,\bar{y}\,z$
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	**1**	$m_6 = x\,y\,\bar{z}$
1	1	1	**1**	$m_7 = x\,y\,z$

Focus on the '**1**' entries

$$F = m_1 + m_6 + m_7 = \sum(1, 6, 7) = \bar{x}\,\bar{y}\,z + x\,y\,\bar{z} + x\,y\,z$$

Sum-Of-Minterms Examples

- $F(a, b, c, d) = \sum(2, 3, 6, 10, 11)$
- $F(a, b, c, d) = m_2 + m_3 + m_6 + m_{10} + m_{11}$

$$\bar{a}\,\bar{b}\,c\,\bar{d} + \bar{a}\,\bar{b}\,c\,d + \bar{a}\,b\,c\,\bar{d} + a\,\bar{b}\,c\,\bar{d} + a\,\bar{b}\,c\,d$$

- $G(a, b, c, d) = \sum(0, 1, 12, 15)$
- $G(a, b, c, d) = m_0 + m_1 + m_{12} + m_{15}$

$$\bar{a}\,\bar{b}\,\bar{c}\,\bar{d} + \bar{a}\,\bar{b}\,\bar{c}\,d + a\,b\,\bar{c}\,\bar{d} + a\,b\,c\,d$$

Product-Of-Maxterms (POM)

- Product-Of-Maxterms (POM) canonical form:
  Product of maxterms of entries that evaluate to '**0**'

x	y	z	F	Maxterm
0	0	0	1	
0	0	1	1	
0	1	0	**0**	$M_2 = (x + \overline{y} + z)$
0	1	1	1	
1	0	0	**0**	$M_4 = (\overline{x} + y + z)$
1	0	1	1	
1	1	0	**0**	$M_6 = (\overline{x} + \overline{y} + z)$
1	1	1	1	

Focus on the '**0**' entries

$$F = M_2 \cdot M_4 \cdot M_6 = \prod (2, 4, 6) = (x+\overline{y}+z)\ (\overline{x}+y+z)\ (\overline{x}+\overline{y}+z)$$

Product-Of-Maxterms Examples

- $F(a, b, c, d) = \prod(1, 3, 6, 11)$
- $F(a, b, c, d) = M_1 \cdot M_3 \cdot M_6 \cdot M_{11}$

$$(a+b+c+\overline{d})\ (a+b+\overline{c}+\overline{d})\ (a+\overline{b}+\overline{c}+d)\ (\overline{a}+b+\overline{c}+\overline{d})$$

- $G(a, b, c, d) = \prod(0, 4, 12, 15)$
- $G(a, b, c, d) = M_0 \cdot M_4 \cdot M_{12} \cdot M_{15}$

$$(a+b+c+d)\ (a+\overline{b}+c+d)\ (\overline{a}+\overline{b}+c+d)\ (\overline{a}+\overline{b}+\overline{c}+\overline{d})$$

Observations

- We can implement any function by "ORing" the minterms corresponding to the '**1**' entries in the function table. A minterm evaluates to '**1**' for its corresponding entry.
- We can implement any function by "ANDing" the maxterms corresponding to '**0**' entries in the function table. A maxterm evaluates to '**0**' for its corresponding entry.
- The same Boolean function can be expressed in two canonical ways: Sum-of-Minterms (SOM) and Product-of-Maxterms (POM).
- If a Boolean function has fewer '**1**' entries then the SOM canonical form will contain fewer literals than POM. However, if it has fewer '**0**' entries then the POM form will have fewer literals than SOM.

The End

Questions?