1 Self-Grading Declaration

Based on the guidelines provided, **I self-assign a grade of 2/2 for this homework**. I have made a legitimate effort to complete 100% of the problems.

2 Assignment 1

Assignment 1

Consider a tensor with components

$$[\boldsymbol{S}] = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

in a basis $\{\boldsymbol{e_i}\}$. Suppose one considers a second basis $\{\boldsymbol{e_i^*}\}$ that is generated by rotating the first basis by 45° counter-clockwise about $\boldsymbol{e_1}$. What will S^*_{12} be?
See A&G 1.17, 1.18, and 1.19 [1].

The new vectors can be described:

$$\begin{aligned} \boldsymbol{e_1^*} &= \boldsymbol{e_1} \\ \boldsymbol{e_2^*} &= cos(\pi/4)\boldsymbol{e_2} + sin(\pi/4)\boldsymbol{e_3} \\ \boldsymbol{e_3^*} &= -sin(\pi/4)\boldsymbol{e_2} + cos(\pi/4)\boldsymbol{e_3} \end{aligned}$$

Thus, the component S^*_{12}

$$S^*_{12} = \boldsymbol{e_1^*} \cdot \boldsymbol{T}\boldsymbol{e_2^*} = \boldsymbol{e_1^*} \cdot (T_ij\boldsymbol{e_i} \otimes \boldsymbol{e_j})\boldsymbol{e_2^*} = 4.95\,(usingPython[2])$$

A supporting Python script was used to verify manual results and avoid performing cumbersome calculations by hand. The latest version can be found in the GitHub repository.

Python 3.13 Code

```python
import numpy as np

division_lines_count = 30

def run_assignment_1():
    S = np.array([
        [1, 3, 4],
        [3, 2, 5],
        [4, 5, 3]
    ])

    e1_star = np.array([1, 0, 0])
    e2_star = np.array([0, np.cos(np.pi/4), np.sin(np.pi/4)])
```

```python
    S12_star = e1_star @ S @ e2_star

    print("Assignment 1 Results\n" + "-" * division_lines_count)
    print(f"Transformed tensor component S*       = {S12_star:.4f}")

if __name__ == "__main__":
    run_assignment_1()
```

Assignment 2

Consider a scalar valued function $g(\boldsymbol{x}) = \exp(\boldsymbol{x} \cdot \boldsymbol{x})$.

(a) What is the directional derivative of g at $\boldsymbol{x} = 2\boldsymbol{e_1} + 3\boldsymbol{e_2}$ in the direction $\boldsymbol{h} = (\boldsymbol{e_1} + \boldsymbol{e_3})/\sqrt{2}$?

(b) What is $\boldsymbol{grad}(g)$?

See, for example, A&G 2.3. [1]

(**a**)

$$
\begin{aligned}
g &= \frac{d}{d\alpha} e^{(\boldsymbol{x}+\alpha\boldsymbol{h})\cdot(\boldsymbol{x}+\alpha\boldsymbol{h})}\bigg|_{\alpha=0} \\
&= e^{\boldsymbol{x}\cdot\boldsymbol{x}} 2\boldsymbol{x}\cdot\boldsymbol{h} \\
&= e^{x_i \cdot x_i} \cdot 2x_j h_j.
\end{aligned}
$$

Now plugging $\boldsymbol{x} = 2\boldsymbol{e_1} + 3\boldsymbol{e_2}$ and $\boldsymbol{h} = (\boldsymbol{e_1} + \boldsymbol{e_3})/\sqrt{2}$ into the scalar products:

$$
\begin{aligned}
x_i x_i &= \sqrt{2} \\
x_j h_j &= 13
\end{aligned}
$$

Thus:

$$
g = 26e^{\sqrt{2}}
$$

(**b**) From the definition of gradient [3]:

$$
grad(g) = \nabla g = e^{x_i \cdot x_i} \cdot 2x_j h_j
$$

Assignment 3

Using the divergence theorem, re-express

$$\int_{\partial B} 5\cos(2x_i) n_i \, da$$

as a volume integral over B. See A&G 2.18. [1]

Based on the divergence theorem, we can re-express the expression by computing the derivative:

$$\begin{aligned} f(\boldsymbol{x}) &= 5\cos(2x_i) \\ \frac{df}{dx_i} &= f_{,i} = -5 \cdot \sin(2x_i) \cdot 2 \cdot \delta_{ii} \\ &= -30\sin(2x_i) \end{aligned}$$

Therefore, by the divergence theorem:

$$\int_{\partial B} 5\cos(2x_i) n_i \, da = \int_B \frac{\partial}{\partial x_i} \left(5\cos(2x_i)\right) \, dv = \int_B -30\sin(2x_i) \, dv$$

Hence, the surface integral is re-expressed as the volume integral:

$$\int_{\partial B} 5\cos(2x_i) n_i \, da = \int_B -30\sin(2x_i) \, dv$$

Assignment 4

Consider the motion of a cube $B = [0,1] \times [0,1] \times [0,1]$ given by

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) = a\big(4X_1\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3\big),$$

where $a > 0$ is a given real number.

(a) Describe in words and images the motion.

(b) Consider the point $\boldsymbol{X} = d\boldsymbol{e}_1$, where $0 < d < 1$ is given.

i. What is the deformation gradient at $\boldsymbol{X}$?

ii. What is the displacement gradient at $\boldsymbol{X}$?

iii. What is the stretch at $\boldsymbol{X}$ in the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ directions?

(**a**) The vector $\boldsymbol{x}$ which represents the motion of the body can be re-expressed as:

$$\boldsymbol{x} = \begin{bmatrix} 4aX_1 \\ aX_2 \\ aX_3 \end{bmatrix} = \begin{bmatrix} 4a \\ a \\ a \end{bmatrix} \boldsymbol{X}$$

Which geometrically represents a volumetric change proportional to the real parameter $a > 0$, with the deformation being four times greater in the $\boldsymbol{e_1}$ direction compared to the $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ directions. The volume of the deformed configuration V_f relative to the original volume V_0 is given by

$$V_f = (4a\boldsymbol{X_1}) \cdot \big((a\boldsymbol{X_2}) \times (a\boldsymbol{X_3})\big) = 4a^3 \overbrace{\big(\boldsymbol{X_1} \cdot (\boldsymbol{X_2} \times \boldsymbol{X_3})\big)}^{V_0}. \tag{1}$$

For example, if $a = 1$, the body expands by a factor of 4 in the $\boldsymbol{e_1}$ direction, while remaining unchanged in the $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ directions, with a total volume change of $V_f = 4V_0$. This example is represented in **Fig. 1**

i. We start of by computing the deformation grandient $\boldsymbol{F} = F_{ij}$:

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial \chi_1}{\partial X_1} & \dfrac{\partial \chi_1}{\partial X_2} & \dfrac{\partial \chi_1}{\partial X_3} \\ \dfrac{\partial \chi_2}{\partial X_1} & \dfrac{\partial \chi_2}{\partial X_2} & \dfrac{\partial \chi_2}{\partial X_3} \\ \dfrac{\partial \chi_3}{\partial X_1} & \dfrac{\partial \chi_3}{\partial X_2} & \dfrac{\partial \chi_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 4a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Which is the same $\forall \boldsymbol{X}$ (regardless of position).

ii. $\boldsymbol{H}$ can be simply computed as:

$$\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{1} = \begin{bmatrix} 4a-1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{bmatrix}$$

Which also is independent of the chosen point $\boldsymbol{X}$.

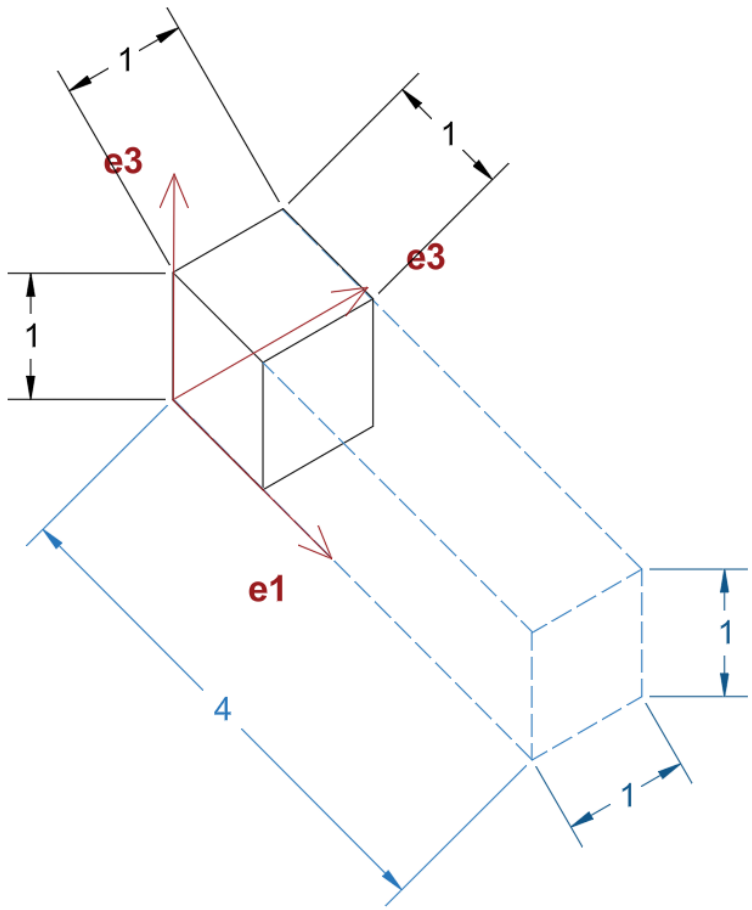

Figure 1: Representation of motion for $a = 1$.

iii. The first step consists in computing $\boldsymbol{C}$, the Green-Cauchy deformation tensor:

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{F}^2 = \begin{bmatrix} 16a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$$

The strain at direction θ can be defined as:

$$\lambda_\theta = \sqrt{\boldsymbol{NCN}}$$

Where $\boldsymbol{N}$ is the vector that represents the observed direction. Note that if $\boldsymbol{N}$ is any of the principal directions $\boldsymbol{e_i}$, then $\lambda_i = \sqrt{C_{ii}}$. Thus,

$$\lambda_1 = \sqrt{C_{11}} = F_{11} = 4a$$

$$\lambda_2 = \sqrt{C_{22}} = F_{22} = a$$

$$\lambda_3 = \sqrt{C_{33}} = F_{33} = a$$

Assignment 5

Consider a unit-cube body $B = [0,1] \times [0,1] \times [0,1]$ subjected to a simple shear motion

$$\boldsymbol{x} = \chi(\boldsymbol{X}) = (X_1 + \gamma X_2)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3,$$

where γ is a given parameter. What is the volume of the cube after it is deformed?

The first step is to compute the deformation gradient:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_{ii}* = 1$$

* Since $\boldsymbol{F}$ is an upper triangular matrix teh determinant can easily be calculated as F_{ii}

Here J is the *volumetric Jacobian*, which represents the ratio of the deformed volume dv to the reference volume dv_R:

$$J = \frac{dv}{dv_R} = det\boldsymbol{F} = 1.$$

Therefore, the volume of the cube after deformation is:

$$V = JV_0 = 1 \cdot V_0,$$

which means the volume remains unchanged.

Conclusion: under this simple shear deformation, the unit cube undergoes a shape change (distortion) but its volume is preserved.

3 References

[1] L. Anand and S. Govindjee, *Example Problems for Continuum Mechanics of Solids.* Independently published, 2020. Publication date: July 25, 2020.

[2] C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant, K. Sheppard, T. Reddy, W. Weckesser, H. Abbasi, C. Gohlke, and T. E. Oliphant, "Array programming with NumPy," *Nature*, vol. 585, pp. 357–362, Sept. 2020.

[3] L. Anand and S. Govindjee, *Continuum Mechanics of Solids.* Oxford University Press, 07 2020.