Jerrold Marsden and Alan Weinstein

CALCULUS I

Springer

Geometry Formulas

Area of rectangle	$A = lw$
Area of circle	$A = \pi r^2$
Area of triangle	$A = \frac{1}{2}bh$
Surface Area of sphere	$A = 4\pi r^2$
Lateral Surface Area of cylinder	$A = 2\pi rh$
Volume of box	$V = lwh$
Volume of sphere	$V = \frac{4}{3}\pi r^3$
Volume of cylinder	$V = \pi r^2 h$
Volume of cone	$V = \frac{1}{3}$ (area of base) $\times$ (height)

Trigonometric Identities

Pythagorean

$$\cos^2\theta + \sin^2\theta = 1,\ 1 + \tan^2\theta = \sec^2\theta,\ \cot^2\theta + 1 = \csc^2\theta$$

Parity

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,\ \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta,\ \sec(-\theta) = \sec\theta,\ \cot(-\theta) = -\cot\theta$$

Co-relations

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right),\ \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right),\ \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Addition formulas

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{(\tan\theta + \tan\phi)}{(1 - \tan\theta\tan\phi)}$$

$$\tan(\theta - \phi) = \frac{(\tan\theta - \tan\phi)}{(1 + \tan\theta\tan\phi)}$$

Double-angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{(1 - \tan^2\theta)}$$

Half-angle formulas

$$\sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2} \quad \text{or} \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2} \quad \text{or} \quad \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta} \quad \text{or} \quad \tan\theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$$

Product formulas

$$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$$

$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$$

$$\sin\theta\cos\phi = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)]$$

Derivatives

1. $\dfrac{d(au)}{dx} = a\dfrac{du}{dx}$

2. $\dfrac{d(u+v-w)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} - \dfrac{dw}{dx}$

3. $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

4. $\dfrac{d(u/v)}{dx} = \dfrac{v(du/dx) - u(dv/dx)}{v^2}$

5. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$

6. $\dfrac{d(u^v)}{dx} = vu^{v-1}\dfrac{du}{dx} + u^v(\ln u)\dfrac{dv}{dx}$

7. $\dfrac{d(e^u)}{dx} = e^u\dfrac{du}{dx}$

8. $\dfrac{d(e^{au})}{dx} = ae^{au}\dfrac{du}{dx}$

9. $\dfrac{da^u}{dx} = a^u(\ln a)\dfrac{du}{dx}$

10. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\,\dfrac{du}{dx}$

11. $\dfrac{d(\log_a u)}{dx} = \dfrac{1}{u(\ln a)}\,\dfrac{du}{dx}$

12. $\dfrac{d\sin u}{dx} = \cos u\dfrac{du}{dx}$

13. $\dfrac{d\cos u}{dx} = -\sin u\dfrac{du}{dx}$

14. $\dfrac{d\tan u}{dx} = \sec^2 u\dfrac{du}{dx}$

15. $\dfrac{d\cot u}{dx} = -\csc^2 u\dfrac{du}{dx}$

16. $\dfrac{d\sec u}{dx} = \tan u\sec u\dfrac{du}{dx}$

17. $\dfrac{d\csc u}{dx} = -(\cot u)(\csc u)\dfrac{du}{dx}$

18. $\dfrac{d\sin^{-1}u}{dx} = \dfrac{1}{\sqrt{1-u^2}}\,\dfrac{du}{dx}$

19. $\dfrac{d\cos^{-1}u}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\,\dfrac{du}{dx}$

20. $\dfrac{d\tan^{-1}u}{dx} = \dfrac{1}{1+u^2}\,\dfrac{du}{dx}$

21. $\dfrac{d\cot^{-1}u}{dx} = \dfrac{-1}{1+u^2}\,\dfrac{du}{dx}$

22. $\dfrac{d\sec^{-1}u}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\,\dfrac{du}{dx}$

23. $\dfrac{d\csc^{-1}u}{dx} = \dfrac{-1}{u\sqrt{u^2-1}}\,\dfrac{du}{dx}$

24. $\dfrac{d\sinh u}{dx} = \cosh u\dfrac{du}{dx}$

25. $\dfrac{d\cosh u}{dx} = \sinh u\dfrac{du}{dx}$

26. $\dfrac{d\tanh u}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$

27. $\dfrac{d\coth u}{dx} = -(\operatorname{csch}^2 u)\dfrac{du}{dx}$

28. $\dfrac{d\operatorname{sech} u}{dx} = -(\operatorname{sech} u)(\tanh u)\dfrac{du}{dx}$

29. $\dfrac{d\operatorname{csch} u}{dx} = -(\operatorname{csch} u)(\coth u)\dfrac{du}{dx}$

30. $\dfrac{d\sinh^{-1}u}{dx} = \dfrac{1}{\sqrt{1+u^2}}\,\dfrac{du}{dx}$

31. $\dfrac{d\cosh^{-1}u}{dx} = \dfrac{1}{\sqrt{u^2-1}}\,\dfrac{du}{dx}$

32. $\dfrac{d\tanh^{-1}u}{dx} = \dfrac{1}{1-u^2}\,\dfrac{du}{dx}$

33. $\dfrac{d\coth^{-1}u}{dx} = \dfrac{1}{1-u^2}\,\dfrac{du}{dx}$

34. $\dfrac{d\operatorname{sech}^{-1}u}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\,\dfrac{du}{dx}$

35. $\dfrac{d\operatorname{csch}^{-1}u}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\,\dfrac{du}{dx}$

Continued on overleaf.

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Anglin/Lambek: The Heritage of Thales.
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CALCULUS I

Second Edition

With 528 Figures

Springer

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To Nancy and Margo

Preface

The goal of this text is to help students learn to use calculus intelligently for solving a wide variety of mathematical and physical problems.

This book is an outgrowth of our teaching of calculus at Berkeley, and the present edition incorporates many improvements based on our use of the first edition. We list below some of the key features of the book.

Examples and Exercises

The exercise sets have been carefully constructed to be of maximum use to the students. With few exceptions we adhere to the following policies.

- The section *exercises are graded* into three consecutive groups:

(a) The first exercises are routine, modelled almost exactly on the examples; these are intended to give students confidence.
(b) Next come exercises that are still based directly on the examples and text but which may have variations of wording or which combine different ideas; these are intended to train students to think for themselves.
(c) The last exercises in each set are difficult. These are marked with a star (★) and some will challenge even the best students. Difficult does not necessarily mean theoretical; often a starred problem is an interesting application that requires insight into what calculus is really about.

- The *exercises come in groups* of two and often four similar ones.
- *Answers* to odd-numbered exercises are available in the back of the book, and every other odd exercise (that is, Exercise 1, 5, 9, 13, . . .) has a complete solution in the student guide. Answers to even-numbered exercises are not available to the student.

Placement of Topics

Teachers of calculus have their own pet arrangement of topics and teaching devices. After trying various permutations, we have arrived at the present arrangement. Some highlights are the following.

- *Integration* occurs early in Chapter 4; *antidifferentiation* and the $\int$ notation with motivation already appear in Chapter 2.

- *Trigonometric functions* appear in the first semester in Chapter 5.
- The *chain rule* occurs early in Chapter 2. We have chosen to use rate-of-change problems, square roots, and algebraic functions in conjunction with the chain rule. Some instructors prefer to introduce $\sin x$ and $\cos x$ early to use with the chain rule, but this has the penalty of fragmenting the study of the trigonometric functions. We find the present arrangement to be smoother and easier for the students.
- *Limits* are presented in Chapter 1 along with the derivative. However, while we do not try to hide the difficulties, technicalities involving epsilonics are deferred until Chapter 11. (Better or curious students can read this concurrently with Chapter 2.) Our view is that it is very important to teach students to differentiate, integrate, and solve calculus problems as quickly as possible, without getting delayed by the intricacies of limits. After some calculus is learned, the details about limits are best appreciated in the context of l'Hôpital's rule and infinite series.
- *Differential equations* are presented in Chapter 8 and again in Sections 12.7, 12.8, and 18.3. Blending differential equations with calculus allows for more interesting applications early and meets the needs of physics and engineering.

Prerequisites and Preliminaries

A historical introduction to calculus is designed to orient students before the technical material begins.

Prerequisite material from algebra, trigonometry, and analytic geometry appears in Chapters R, 5, and 14. These topics are treated completely: in fact, analytic geometry and trigonometry are treated in enough detail to serve as a first introduction to the subjects. However, high school algebra is only lightly reviewed, and knowledge of some plane geometry, such as the study of similar triangles, is assumed.

Several *orientation quizzes* with answers and a *review section* (Chapter R) contribute to bridging the gap between previous training and this book. Students are advised to assess themselves and to take a pre-calculus course if they lack the necessary background.

Chapter and Section Structure

The book is intended for a three-semester sequence with six chapters covered per semester. (Four semesters are required if pre-calculus material is included.)

The length of chapter sections is guided by the following typical course plan: If six chapters are covered per semester (this typically means four or five student contact hours per week) then approximately two sections must be covered each week. Of course this schedule must be adjusted to students' background and individual course requirements, but it gives an idea of the pace of the text.

Proofs and Rigor

Proofs are given for the most important theorems, with the customary omission of proofs of the intermediate value theorem and other consequences of the completeness axiom. Our treatment of integration enables us to give particularly simple proofs of some of the main results in that area, such as the fundamental theorem of calculus. We de-emphasize the theory of limits, leaving a detailed study to Chapter 11, after students have mastered the fundamentals of calculus—differentiation and integration. Our book *Calculus Unlimited* (Benjamin/Cummings) contains all the proofs omitted in this text and additional ideas suitable for supplementary topics for good students. Other references for the theory are Spivak's *Calculus* (Benjamin/Cummings & Publish or Perish), Ross' *Elementary Analysis*: *The Theory of Calculus* (Springer) and Marsden's *Elementary Classical Analysis* (Freeman).

Calculators

Calculator applications are used for motivation (such as for functions and composition on pages 40 and 112) and to illustrate the numerical content of calculus (see, for instance, p. 142). Special calculator discussions tell how to use a calculator and recognize its advantages and shortcomings.

Applications

Calculus students should not be treated as if they are already the engineers, physicists, biologists, mathematicians, physicians, or business executives they may be preparing to become. Nevertheless calculus is a subject intimately tied to the physical world, and we feel that it is misleading to teach it any other way. Simple examples related to distance and velocity are used throughout the text. Somewhat more special applications occur in examples and exercises, some of which may be skipped at the instructor's discretion. Additional connections between calculus and applications occur in various section supplements throughout the text. For example, the use of calculus in the determination of the length of a day occurs at the end of Chapters 5, 9, and 14.

Visualization

The ability to visualize basic graphs and to interpret them mentally is very important in calculus and in subsequent mathematics courses. We have tried to help students gain facility in forming and using visual images by including plenty of carefully chosen artwork. This facility should also be encouraged in the solving of exercises.

Computer-Generated Graphics

Computer-generated graphics are becoming increasingly important as a tool for the study of calculus. High-resolution plotters were used to plot the graphs of curves and surfaces which arose in the study of Taylor polynomial approximation, maxima and minima for several variables, and three-dimensional surface geometry. Many of the computer drawn figures were kindly supplied by Jerry Kazdan.

Supplements

Student Guide Contains

- Goals and guides for the student
- Solutions to every other odd-numbered exercise
- Sample exams

Instructor's Guide Contains

- Suggestions for the instructor, section by section
- Sample exams
- Supplementary answers

Misprints

Misprints are a plague to authors (and readers) of mathematical textbooks. We have made a special effort to weed them out, and we will be grateful to the readers who help us eliminate any that remain.

Acknowledgments

We thank our students, readers, numerous reviewers and assistants for their help with the first and current edition. For this edition we are especially grateful to Ray Sachs for his aid in matching the text to student needs, to Fred Soon and Fred Daniels for their unfailing support, and to Connie Calica for her accurate typing. Several people who helped us with the first edition deserve our continued thanks. These include Roger Apodaca, Grant Gustafson, Mike Hoffman, Dana Kwong, Teresa Ling, Tudor Ratiu, and Tony Tromba.

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Berkeley, California Alan Weinstein

How to Use this Book: A Note to the Student

Begin by orienting yourself. Get a rough feel for what we are trying to accomplish in calculus by rapidly reading the Introduction and the Preface and by looking at some of the chapter headings.

Next, make a preliminary assessment of your own preparation for calculus by taking the quizzes on pages 13 and 14. If you need to, study Chapter R in detail and begin reviewing trigonometry (Section 5.1) as soon as possible.

You can learn a little bit about calculus by reading this book, but you can learn to use calculus only by practicing it yourself. You should do many more exercises than are assigned to you as homework. The answers at the back of the book and solutions in the student guide will help you monitor your own progress. There are a lot of examples with complete solutions to help you with the exercises. The end of each example is marked with the symbol ▲.

Remember that even an experienced mathematician often cannot "see" the entire solution to a problem at once; in many cases it helps to begin systematically, and then the solution will fall into place.

Instructors vary in their expectations of students as far as the degree to which answers should be simplified and the extent to which the theory should be mastered. In the book we have arranged the theory so that only the proofs of the most important theorems are given in the text; the ends of proofs are marked with the symbol ■. Often, technical points are treated in the starred exercises.

In order to prepare for examinations, try reworking the examples in the text and the sample examinations in the Student Guide without looking at the solutions. Be sure that you can do all of the assigned homework problems.

When writing solutions to homework or exam problems, you should use the English language liberally and correctly. A page of disconnected formulas with no explanatory words is incomprehensible.

We have written the book with your needs in mind. Please inform us of shortcomings you have found so we can correct them for future students. We wish you luck in the course and hope that you find the study of calculus stimulating, enjoyable, and useful.

Jerry Marsden
Alan Weinstein

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Introduction

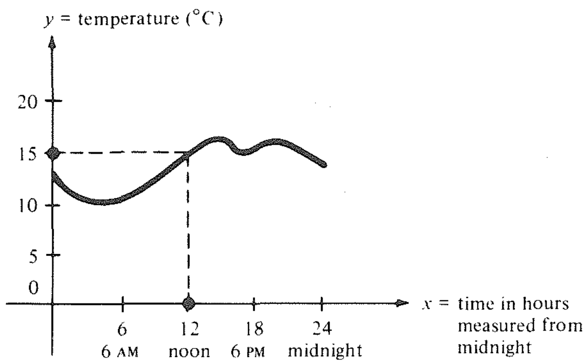
Calculus has earned a reputation for being an essential tool in the sciences. Our aim in this introduction is to give the reader an idea of what calculus is all about and why it is useful.

Calculus has two main divisions, called differential calculus and integral calculus. We shall give a sample application of each of these divisions, followed by a discussion of the history and theory of calculus.

Differential Calculus

The graph in Fig. I.1 shows the variation of the temperature y (in degrees Centigrade) with the time x (in hours from midnight) on an October day in New Orleans.

Figure I.1. Temperature in °C as a function of time.



Each point on the graph indicates the temperature at a particular time. For example, at $x = 12$ (noon), the temperature was 15°C. The fact that there is exactly one y for each x means that y is a *function of* x.

The graph as a whole can reveal information more readily than a table. For example, we can see at a glance that, from about 5 A.M to 2 P.M., the temperature was rising, and that at the end of this period the maximum temperature for the day was reached. At 2 P.M. the air cooled (perhaps due to a brief shower), although the temperature rose again later in the afternoon. We also see that the lowest temperature occurred at about 5 A.M.

We know that the sun is highest at noon, but the highest temperature did not occur until 2 hours later. How, then, is the high position of the sun at noon reflected in the shape of the graph? The answer lies in the concept of *rate of change*, which is the central idea of differential calculus.

At any given moment of time, we can consider the rate at which temperature is changing with respect to time. What is this rate? If the graph of temperature against time were a segment of a straight line, as it is in Fig. I.2, the answer would be easy. If we compare the temperature measurement at

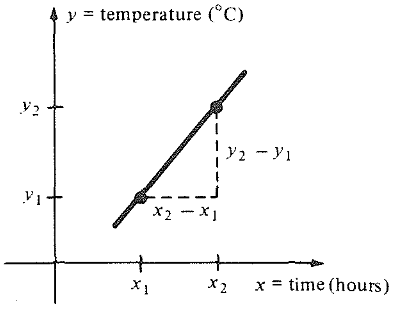


Figure I.2. The ratio $(y_2 - y_1)/(x_2 - x_1)$ is the ratio of change of temperature with respect to time.

times x_1 and x_2, the ratio $(y_2 - y_1)/(x_2 - x_1)$ of change in temperature to change in time, measured in degrees per hour, is the rate of change. It is a basic property of straight lines that this ratio, called the *slope* of the line, does not depend upon which two points are used to form the ratio.

Returning to Fig. I.1, we may ask for the rate of change of temperature with respect to time at noon. We cannot just use a ratio $(y_2 - y_1)/(x_2 - x_1)$; since the graph is no longer a straight line, the answer would depend on which points on the graph we chose. One solution to our problem is to draw the line l which best fits the graph at the point $(x, y) = (12, 15)$, and to take the slope of this line (see Fig. I.3). The line l is called the *tangent line* to the temperature

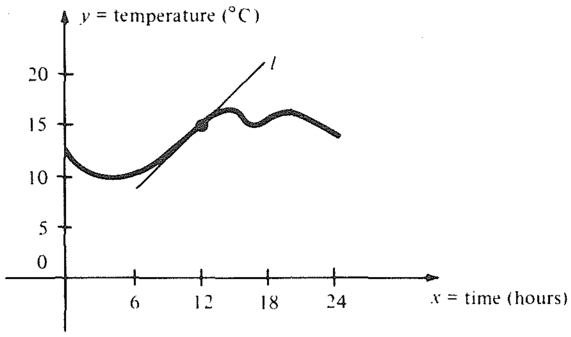


Figure I.3. The rate of change of temperature with respect to time when $x = 12$ is the slope of the line l.

curve at $(12, 15)$; its slope can be measured with a ruler to be about 1°C per hour. By drawing tangent lines to the curve at other points, the reader will find that for no other point is the slope of the tangent line as great as 1°C per hour. Thus, the high position of the sun at noon is reflected by the fact that the rate of change of temperature with respect to time was greatest then.

The example just given shows the importance of rates of change and tangent lines, but it leaves open the question of just what the tangent line *is*. Our definition of the tangent line as the one which "best fits" the curve leaves much to be desired, since it appears to depend on personal judgment. Giving a mathematically precise definition of the tangent line to the graph of a function in the xy plane is the first step in the development of *differential calculus*. The slope of the tangent line, which represents the rate of change of y with respect to x, is called the *derivative* of the function. The process of determining the derivative is called *differentiation*.

The principal tool of differential calculus is a series of rules which lead to a formula for the rate of change of y with respect to x, given a formula for y in terms of x. (For instance, if $y = x^2 + 3x$, the derivative at x turns out to be $2x + 3$.) These rules were discovered by Isaac Newton (1642–1727) in England and, independently, by Gottfried Leibniz (1642–1716), a German working in France. Newton and Leibniz had many precursors. The ancient Greeks, notably Archimedes of Syracuse (287–212 B.C.), knew how to construct the tangent lines to parabolas, hyperbolas, and certain spirals. They were, in effect, computing derivatives. After a long period with little progress, development of Archimedes' ideas revived around 1600. By the middle of the seventeenth century, mathematicians could differentiate powers (i.e., the functions $y = x, x^2, x^3$, and so on) and some other functions, but a *general* method, which could be used by anyone with a little training, was first developed by Newton and Leibniz in the 1670's. Thanks to their work, it is no longer difficult or time-consuming to differentiate functions.

Integral Calculus and the Fundamental Theorem

The second fundamental operation of calculus is called *integration*. To illustrate this operation, we consider another question about Fig. I.1: What was the *average* temperature on this day?

We know that the average of a list of numbers is found by adding the entries in the list and then dividing by the number of entries. In the problem at hand, though, we do not have a finite list of numbers, but rather a continuous graph.

As we did with rates of change, let us look at a simpler example. Suppose that the temperature changed by jumps every two hours, as in Fig. I.4. Then we could simply add the 12 temperature readings and divide by 12 to get the average.

We can interpret this averaging process graphically in the following way. Let $y_1, \dots, y_{12}$ be the 12 temperature readings, so that their average is $y_{\text{ave}} = \frac{1}{12}(y_1 + \cdots + y_{12})$. The region under the graph, shaded in Fig. I.5, is composed of 12 rectangles. The area of the ith rectangle is (base) $\times$ (height) $= 2y_i$, so the total area is $A = 2y_1 + 2y_2 + \cdots + 2y_{12} = 2(y_1 + \cdots + y_{12})$. Comparing this with the formula for the average, we find that $y_{\text{ave}} = A/24$. In other words, the average temperature is equal to the area under the graph,

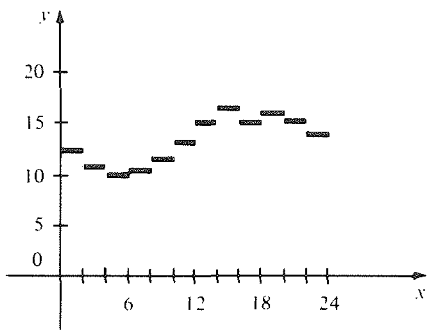


Figure I.4. If the temperature changes by jumps, the average is easy to find.

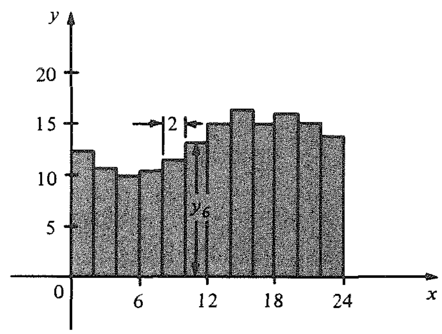


Figure I.5. The area of the ith rectangle is $2y_i$.

divided by the length of the time interval. Now we can guess how to define the average temperature for Fig. I.1. It is simply the area of the region under the graph (shaded in Fig. I.6) divided by 24.

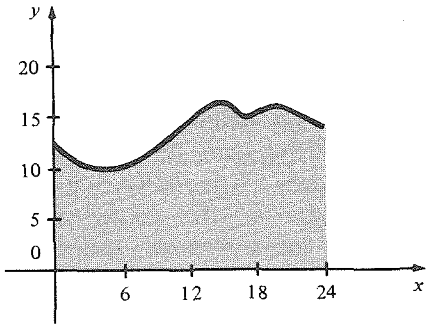


Figure I.6. The average temperature is 1/24 times the shaded area.

The area under the graph of a function on an interval is called the *integral* of the function over the interval. Finding integrals, or *integrating*, is the subject of the *integral calculus*.

Progress in integration was parallel to that in differentiation, and eventually the two problems became linked. The ancient Greeks knew the area of simple geometric figures bounded by lines, circles and parabolas. By the middle of the seventeenth century, areas under the graph of x, x^2, x^3, and other functions could be calculated. Mathematicians at that time realized that the slope and area problems were related. Newton and Leibniz formulated this relationship precisely in the form of the *fundamental theorem of calculus*, which states that integration and differentiation are inverse operations. To suggest the idea behind this theorem, we observe that if a list of numbers $b_1, b_2, \ldots, b_n$ is given, and the differences $d_1 = b_2 - b_1$, $d_2 = b_3 - b_2, \ldots, d_{n-1} = b_n - b_{n-1}$ are taken (this corresponds to differentiation), then we can recover the original list from the d_i's and the initial entry b_1 by adding (this corresponds to integration): $b_2 = b_1 + d_1$, $b_3 = b_1 + d_1 + d_2, \ldots,$ and finally $b_n = b_1 + d_1 + d_2 + \cdots + d_{n-1}$.

The fundamental theorem of calculus, together with the rules of differentiation, brings the solution of many integration problems within reach of anyone who has learned the differential calculus.

The importance and applicability of calculus lies in the fact that a wide

Figure I.7. Quantities related by the operations of calculus. (The independent variable is in brackets.)

Distance traveled along a road	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Velocity	[Time]
Velocity	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Acceleration	[Time]
Cost of living	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Inflation rate	[Time]
Total cost of some goods	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Marginal cost	[Quantity]
Height above sea level on a trail	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Steepness	[Distance]
Mass of a rod	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Linear density	[Length]
Height of a tree	$\underset{\text{Integration}}{\overset{\text{Differentiation}}{\rightleftarrows}}$	Growth rate	[Time]

variety of quantities are related by the operations of differentiation and integration. Some examples are listed in Fig. I.7.

The primary aim of this book is to help you learn *how* to carry out the operations of differentiation and integration and *when* to use them in the solution of many types of problems.

The Theory of Calculus

We shall describe three approaches to the theory of calculus. It will be simpler, as well as more faithful to history, if we begin with integration.

The simplest function to integrate is a constant $y = k$. Its integral over the interval $[a, b]$ is simply the area $k(b - a)$ of the rectangle under its graph (see Fig. I.8). Next in simplicity are the functions whose graphs are composed of several horizontal straight lines, as in Fig. I.9. Such functions are called step functions. The integral of such a function is the sum of the areas of the rectangles under its graph, which is easy to compute.

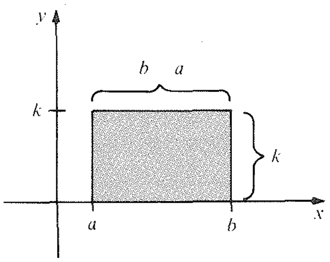


Figure I.8. The integral of the constant function $y = k$ over the interval $[a, b]$, is just the area $k(b - a)$ of this rectangle.

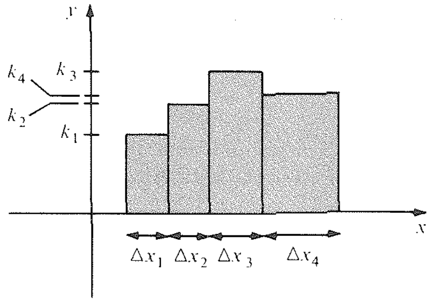


Figure I.9. The integral over $[a, b]$ of this step function is $k_1\Delta x_1 + k_2\Delta x_2 + k_3\Delta x_3 + k_4\Delta x_4$, where k_i is the value of y on the ith interval, and Δx_i is the length of that interval.

There are three ways to go from the simple problem of integrating step functions to the interesting problem of integrating more general functions, like $y = x^2$ or the function in Fig. I.1. These three ways are the following.

1. **The method of exhaustion.** This method was invented by Eudoxus of Cnidus (408–355 B.C.) and was exploited by Archimedes of Syracuse (287–212 B.C.) to calculate the areas of circles, parabolic segments, and other figures. In terms of functions, the basic idea is to *compare* the function to be integrated with step functions. In Fig. I.10, we show the graph of $y = x^2$ on $[0, 1]$, and step functions whose graphs lie below and above it. Since a figure inside another figure has a smaller area, we may conclude that the integral of $y = x^2$

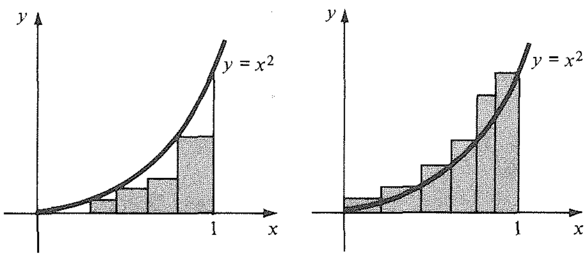
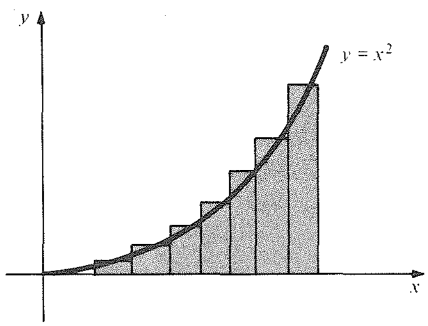


Figure I.10. The integral of $y = x^2$ on the interval [0, 1] lies between the integrals of the two functions.

on [0, 1] lies between the integrals of these two step functions. In this way, we can get lower and upper estimates for the integral. By choosing step functions with shorter and shorter "steps," it is reasonable to expect that we can *exhaust* the area between the rectangles and the curve and, thereby, calculate the area to any accuracy desired. By reasoning with *arbitrarily* small steps, we can in some cases determine the exact area—that is just what Archimedes did.

2. **The method of limits.** This method was fundamental in the seventeenth-century development of calculus and is the one which is most important today. Instead of comparing the function to be integrated with step functions, we *approximate* it by step functions, as in Fig. I.11. If, as we allow the steps to get shorter and shorter, the approximation gets better and better, we say that the integral of the given function is the *limit* of these approximations.

Figure I.11. The integral of this step function is an approximation for the integral of x^2.



3. **The method of infinitesimals.** This method, too, was invented by Archimedes, but he kept it for his personal use since it did not meet the standards of rigor demanded at that time. (Archimedes' use of infinitesimals was not discovered until 1906. It was found as a *palimpsest*, a parchment which had been washed and reused for some religious writing.)[1] The infinitesimal method was also used in the seventeenth century, especially by Leibniz. The idea behind this method is to consider any function as being a step function whose graph has infinitely many steps, each of them infinitely small, or *infinitesimal*, in length. It is impossible to represent this idea faithfully by a drawing, but Fig. I.12 suggests what is going on.

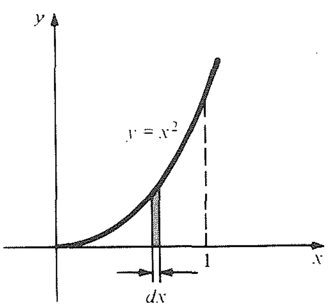


Figure I.12. The integral of x^2 on [0, 1] may be thought of as the sum of the areas of infinitely many rectangles of infinitesimal width dx.

Each of these three methods—exhaustion, limits, and infinitesimals—has its advantages and disadvantages. The method of exhaustion is the easiest to comprehend and to make rigorous, but it is usually cumbersome in applications. Limits are much more efficient for calculation, but their theory is considerably harder to understand; indeed, it was not until the middle of the nineteenth century with the work of Augustin-Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897), among others, that limits were given a firm mathematical foundation. Infinitesimals lead most quickly to answers to many problems, but the idea of an "infinitely small" quantity is hard to comprehend fully,[2] and the method can lead to wrong answers if it is not used carefully. The mathematical foundations of the method of infinitesimals were not

[1] See S. H. Gould, *The method of Archimedes*, American Mathematical Monthly **62**(1955), 473–476.

[2] An early critic of infinitesimals was Bishop George Berkeley, who referred to them as "ghosts of departed quantities" in his anticalculus book, *The Analyst* (1734). The city in which this calculus book has been written is named after him.

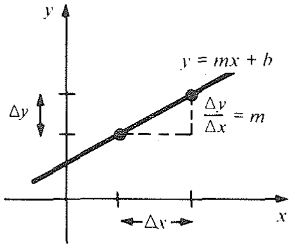


Figure I.13. If $y = mx + b$, the rate of change of y with respect to x is constant and equal to m.

established until the twentieth century with the work of the logician, Abraham Robinson (1918–1974).[3]

The three methods used to define the integral can be applied to differentiation as well. In this case, we replace the piecewise constant functions by the linear functions $y = mx + b$. For a function of this form, a change of Δx in x produces a change $\Delta y = m\Delta x$ in y, so the rate of change, given by the ratio $\Delta y/\Delta x$, is equal to m, independent of x and of Δx (see Fig. I.13).

1. **The method of exhaustion.** To find the rate of change of a general function, we may compare the function with linear functions by seeing how straight lines with various slopes cross the graph at a given point. In Fig. I.14, we show the graph of $y = x^2$, together with lines which are more and less steep at the point $x = 1, y = 1$. By bringing our comparison lines closer and closer together, we can calculate the rate of change to any accuracy desired; if the algebra is simple enough, we can even calculate the rate of change exactly.

The historical origin of this method can be found in the following definition of tangency used by the ancient Greeks: "the tangent line touches the curve, and in the space between the line and curve, no other straight line can be interposed."[4]

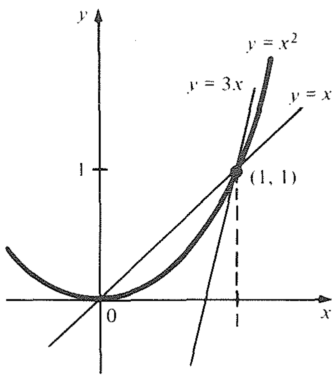
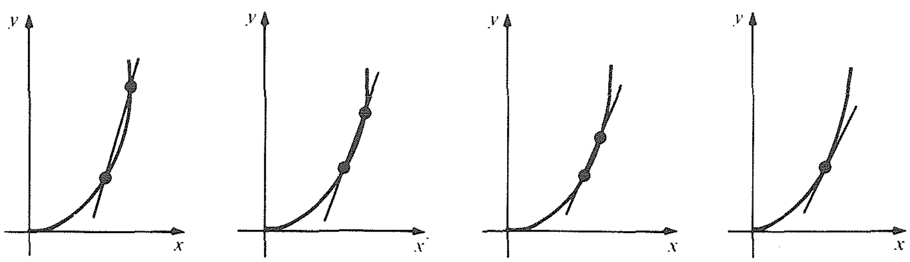


Figure I.14. The rate of change of $y = x^2$ at $x = 1$ lies between 1 and 3.

2. **The method of limits.** To approximate the tangent line to a curve we draw the *secant* line through two nearby points. As the two points become closer and closer, the slope of the secant approaches a limiting value which is the rate of change of the function (see Fig. I.15).

Figure I.15. The rate of change of a function is the limit of the slopes of secant lines drawn through two points on the graph.

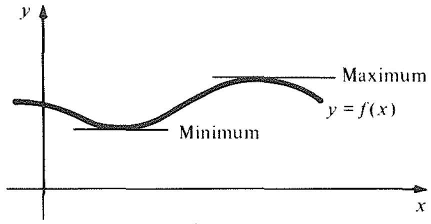


[3] A calculus textbook based upon this work is H. J. Keisler, *Elementary Calculus*, Prindle, Weber, and Schmidt, Boston (1976).

[4] See C. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover, New York, p. 57. The method of exhaustion is not normally used in calculus courses for differentiation, and this book is no exception. However, it could be used and it is intellectually satisfying to do so; see *Calculus Unlimited*, Benjamin/Cummings (1980) by J. Marsden and A. Weinstein.

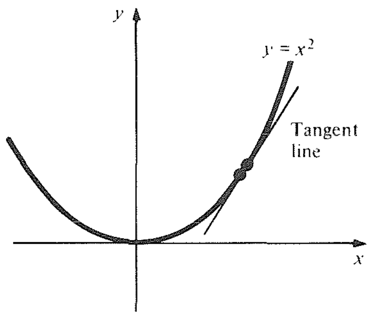
This approach to rates of change derives from the work of Pierre de Fermat[5] (1601–1665), whose interest in tangents arose from the idea, due originally to Kepler, that the slope of the tangent line should be zero at a maximum or minimum point (Fig. I.16).

Figure I.16. The slope of the tangent line is zero at a maximum or minimum point.



3. **The method of infinitesimals.** In this method, we simply think of the tangent line to a curve as a secant line drawn through two infinitesimally close points on the curve, as suggested by Fig. I.17. This idea seems to go back to Galileo[6] (1564–1642) and his student Cavalieri (1598–1647), who defined instantaneous velocity as the ratio of an infinitely small distance to an infinitely short time.

Figure I.17. The tangent line may be thought of as the secant line through a pair of infinitesimally near points.



As with integration, infinitesimals lead most quickly to answers (but not always the right ones), and the method of exhaustion is conceptually simplest. Because of is computational power, the method of limits has become the most widely used approach to differential calculus. It is this method which we shall use in this book.

The Power of Calculus (The Calculus of Power)

To end this introduction, we shall give an example of a practical problem which calculus can help us to solve.

The sun, which is the ultimate source of nearly all of the earth's energy, has always been an object of fascination. The relation between the sun's position and the seasons was predicted by early agricultural societies, some of which developed quite sophisticated astronomical techniques. Today, as the earth's resources of fossil fuels dwindle, the sun has new importance as a *direct* source of energy. To use this energy efficiently, it is useful to know just how

[5] Fermat is also famous for his work in number theory. Fermat's last theorem: "If n is an integer greater than 2, there are no positive integers x, y, and z such that $x^n + y^n = z^n$," remains unproven today. Fermat claimed to have proved it, but his proof has not been found, and most mathematicians now doubt that it could have been correct.

[6] Newton's acknowledgment, "If I have seen further than others, it is because I have stood on the shoulders of giants," probably refers chiefly to Galileo, who died the year Newton was born. (A similar quotation from Lucan (39–65 A.D.) was cited by Robert Burton in the early 1600's—"Pygmies see further than the giants on whose shoulders they stand.")

much solar radiation is available at various locations at different times of the year.

From basic astronomy we know that the earth revolves about the sun while rotating about an axis inclined at 23.5° to the plane of its orbit (see Fig. I.18). Even assuming idealized conditions, such as a perfectly spherical earth

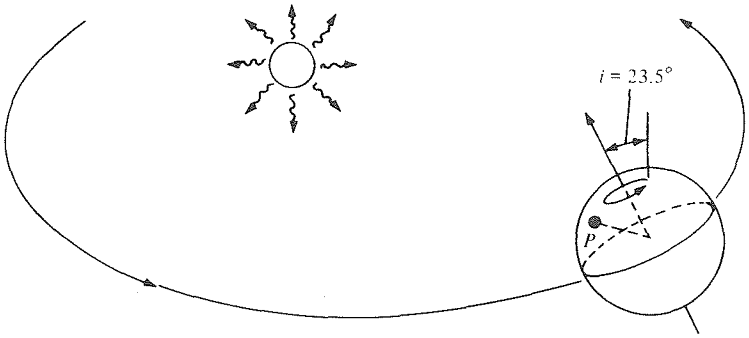


Figure I.18. The earth revolving about the sun.

revolving in a circle about the sun, it is not a simple matter to predict the length of the day or the exact time of sunset at a given latitude on the earth on a given day of the year.

In 1857, an American scientist named L. W. Meech published in the *Smithsonian Contributions to Knowledge* (Volume 9, Article II) a paper entitled "On the relative intensity of the heat and light of the sun upon different latitudes of the earth." Meech was interested in determining the extent to which the variation of temperature on the surface of the earth could be correlated with the variations of the amount of sunlight impinging on different latitudes at different times. One of Meech's ultimate goals was to predict whether or not there was an open sea near the north pole—a region then unexplored. He used the integral calculus to sum the total amount of sunlight arriving at a given latitude on a given day of the year, and then he summed this quantity over the entire year. Meech found that the amount of sunlight reaching the atmosphere above polar regions was surprisingly large during the summer due to the long days (see Fig. I.19). The differential calculus is used to predict the shape of graphs like those in Fig. I.19 by calculating the slopes of their tangent lines.

Meech realized that, since the sunlight reaching the polar regions arrives at such a low angle, much of it is absorbed by the atmosphere, so one cannot conclude the existence of "a brief tropical summer with teeming forms of vegetable and animal life in the centre of the frozen zone." Thus, Meech's calculations fell short of permitting a firm conclusion as to the existence or not of an open sea at the North Pole, but his work has recently taken on new importance. Graphs like Fig. I.19 on the next page have appeared in books devoted to meteorology, geology, ecology (with regard to the biological energy balance), and solar energy engineering.

Even if one takes into account the absorption of energy by the atmosphere, on a summer day the middle latitudes still receive more energy at the earth's surface than does the equator. In fact, the hottest places on earth are not at the equator but in bands north and south of the equator. (This is enhanced by climate: the low-middle latitudes are much freer of clouds than the equatorial zone.)[7]

[7] According to the *Guinness Book of World Records*, the world's highest temperatures (near 136°F) have occurred at Ouargla, Algeria (latitude 32°N), Death Valley, California (latitude 36°N), and Al'Aziziyah, Libya (latitude 32°N). Locations in Chile, Southern Africa, and Australia approach these records.

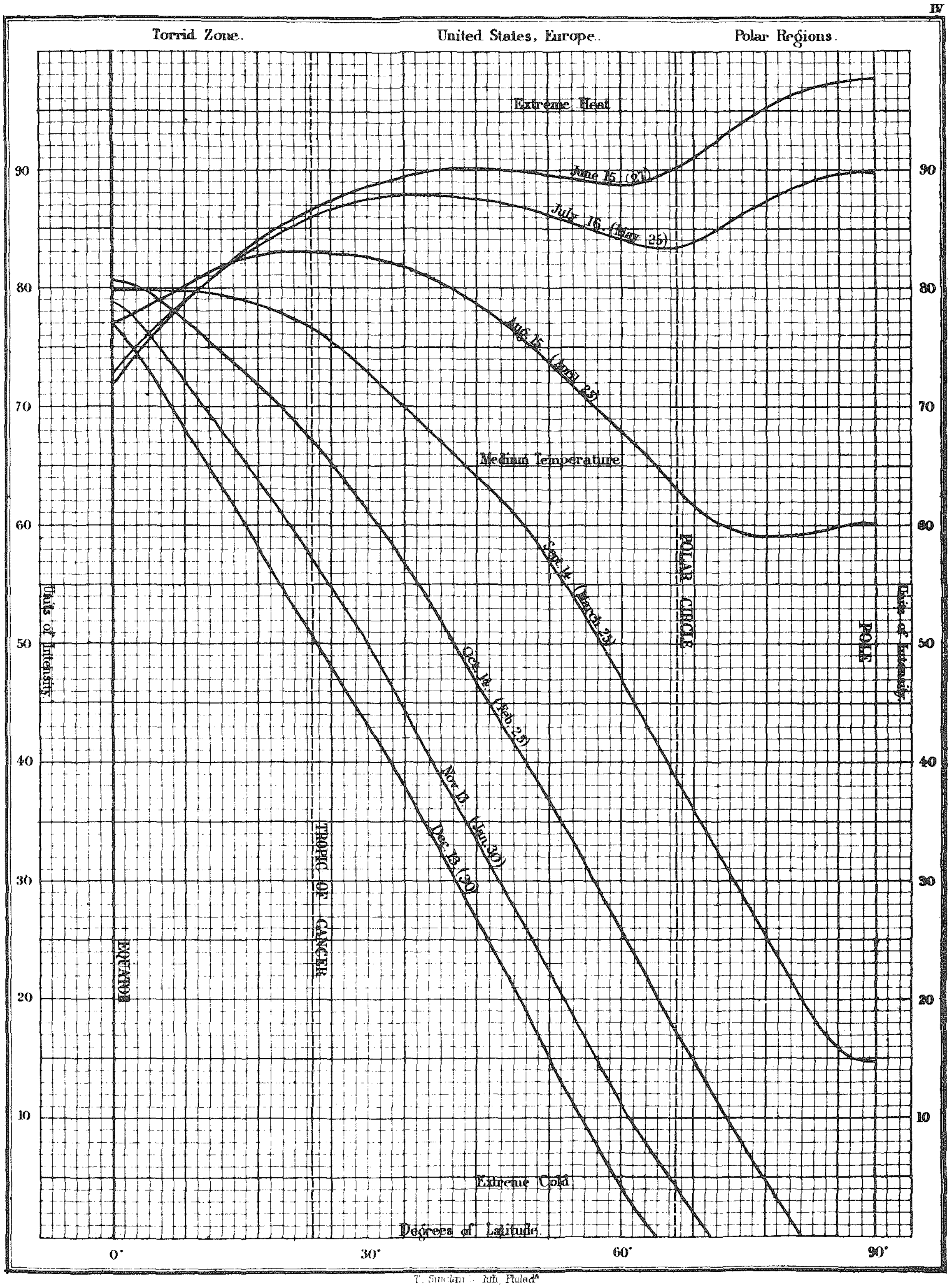


Figure I.19. The sun's diurnal intensity along the meridian, at intervals of 30 days.

As we carry out our study of calculus in this book, we will from time to time in supplementary sections reproduce parts of Meech's calculations (slightly simplified) to show how the material being learned may be applied to a substantial problem. By the time you have finished this book, you should be able to read Meech's article yourself.