Arm Final Round Interviews

Take-home exercise on Heat Equation

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Contents

1 Presentation of the heat equation

Let u_0 be a given function for x belonging to $[01]$, we seek a function u depending on time and space solution of :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) = 0, & x \in]0,1[,\ t > 0, \\ u(t,0) = u(t,1), & t > 0, \\ u(0,x) = sin(\pi x), & x \in [0,1]. \end{cases}$$

$u_0(x)$ is the initial condition and the boundary conditions are $u_t(0)$ and $u_t(1)$. The diffusion coefficient $\alpha = 1$.

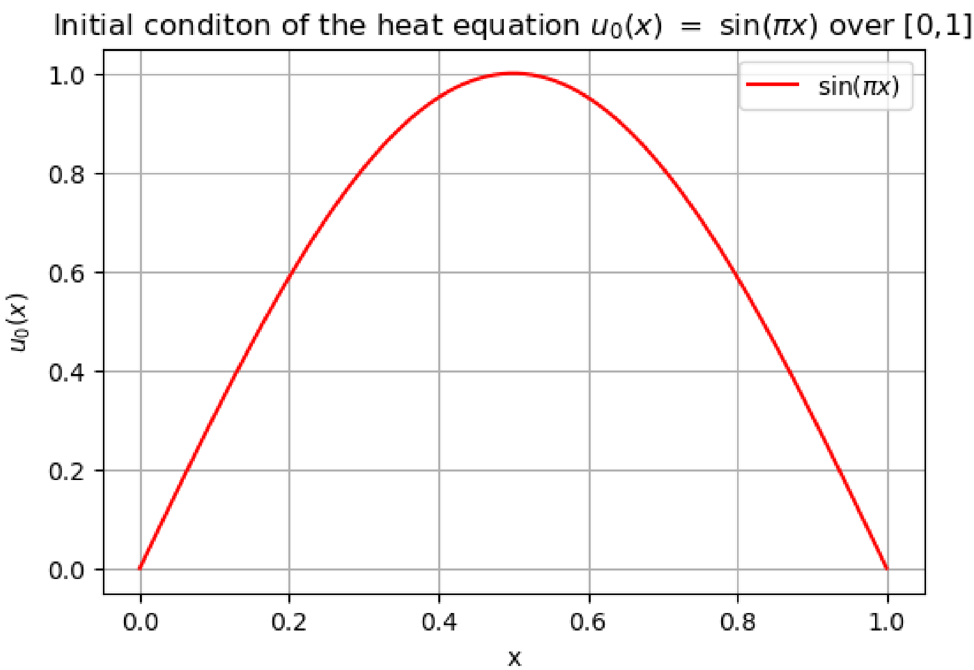

Figure 1: u_0 **is the initial temperature** $u_0(x_i)$, with a sequence of points $(x_i)_{0 \leq i \leq N+1}$.

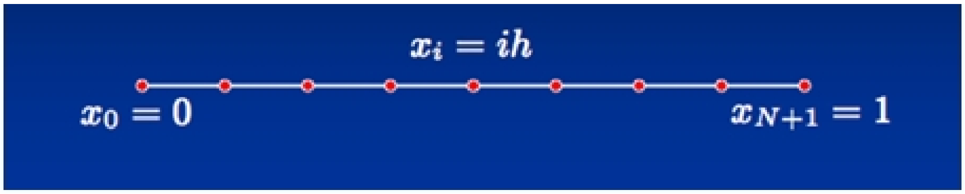

Figure 2: **Discretization of the interval** $[0, 1]$**,** $h = 1/(N + 1)$, with a sequence of points $(x_i)_{0 \le i \le N+1}$.

2 Finite difference approximations

- We will discretize $[0, T] \times [0, 1]$, (T is known).
- We define a uniform mesh of steps h, $h = \frac{1}{N+1}$, of intervals $[0, 1]$ as the set of points $(x_i = ih)_{1 \le i \le N}$.
- We define a uniform mesh of steps Δt, $\Delta t = \frac{T}{M}$, of interval $[0, T]$ as the set of points $(t_n = n\Delta t)_{0 \le n \le M}$.

Approximate solution to our problem by calculating the values u_i^n, for $n = 1, \ldots, M$ and $i = 1, \ldots, N$, which are supposed to approximate the values $u(t_n, x_i)$, for $n = 1, \ldots, M$ and $i = 1, \ldots, N$.

2.0.1 Centered-finite-differences and First order explicit time-stepping

The second-order spatial derivative $\frac{\partial^2 u}{\partial x^2}(t_n, x_i)$ can be approximated using the centered finite difference formula:

$$\frac{\partial^2 u}{\partial x^2}(t_n, x_i) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

where h is the spatial discretization step.

The first-order time derivative $\frac{\partial u}{\partial t}(t_n, x_i)$ can be approximated using the explicit Euler method:

$$\frac{\partial u}{\partial t}(t_n, x_i) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

where Δt gives the frequency with which the outputs are produced.

2.0.2 2D Heat map

$$R_i^n = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t} - \frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1})}{h^2},$$

Figure 3

$$\max_{1 \leq i \leq N} |R_i^n| \leq C(\Delta t + h^2).$$

Figure 4

Figure 5: Explicit Euler scheme consistency error.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} = 0,\ i = 1, \cdots, N,\ n = 1, \cdots, M-1,$$

Figure 6: Discretization of the heat equation.

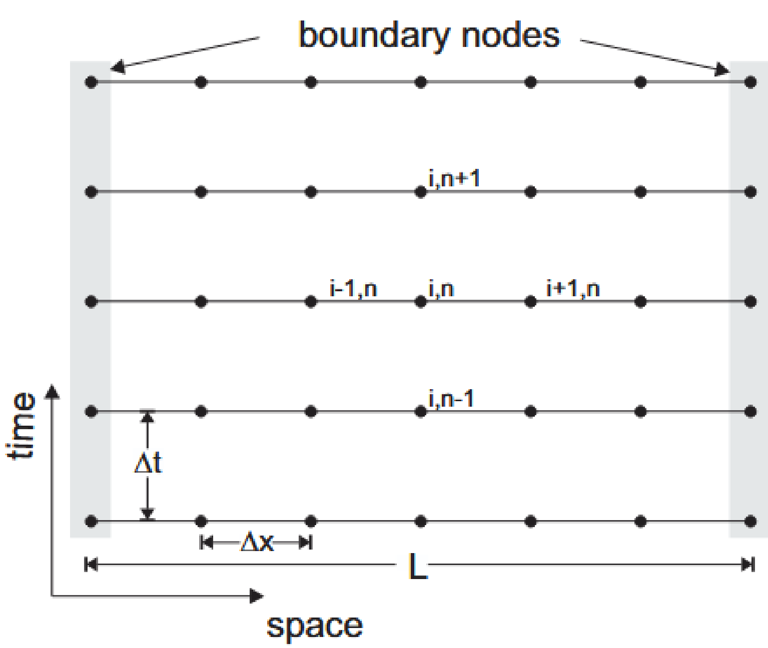

Figure 7: Finite difference discretization of the 1D heat equation.

The stability condition for this problem is for $\alpha = 1 : \frac{\Delta_t}{h^2} \leq \frac{1}{2}$.

3 Conclusion

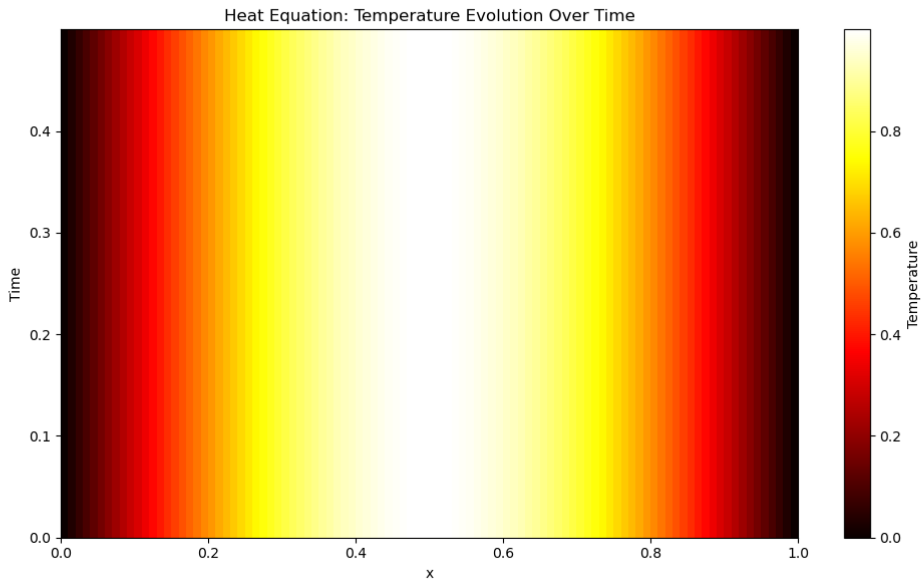

Figure 8: 2D heatmap for the 1D heat equation.

In Figure 8, $u(t, 0) = u(t, 1)$ ensures no abrupt temperature discontinuities at the edges .

Heat Diffusion Effect The heat equation works to spread the sinusoidal temperature profile over time. Higher-temperature regions diffuse outward, creating the smooth transition we observe.

Essentially, starting with $u(0, x) = \sin(\pi x)$ aligns perfectly with the periodic boundary condition, making the temperature distribution symmetric and wavelike as it evolves.