

Arm Final Round Interviews

Take-home exercise on Heat Equation

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1 Presentation of the heat equation

Let u_0 be a given function for x belonging to $[0,1]$, we seek a function u depending on time and space solution of :

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = 0, & x \in]0, 1[, t > 0, \\ u(t, 0) = u(t, 1), & t > 0, \\ u(0, x) = \sin(\pi x), & x \in [0, 1]. \end{cases}$$

$u_0(x)$ is the initial condition and the boundary conditions are $u_t(0)$ and $u_t(1)$. The diffusion coefficient $\alpha = 1$.

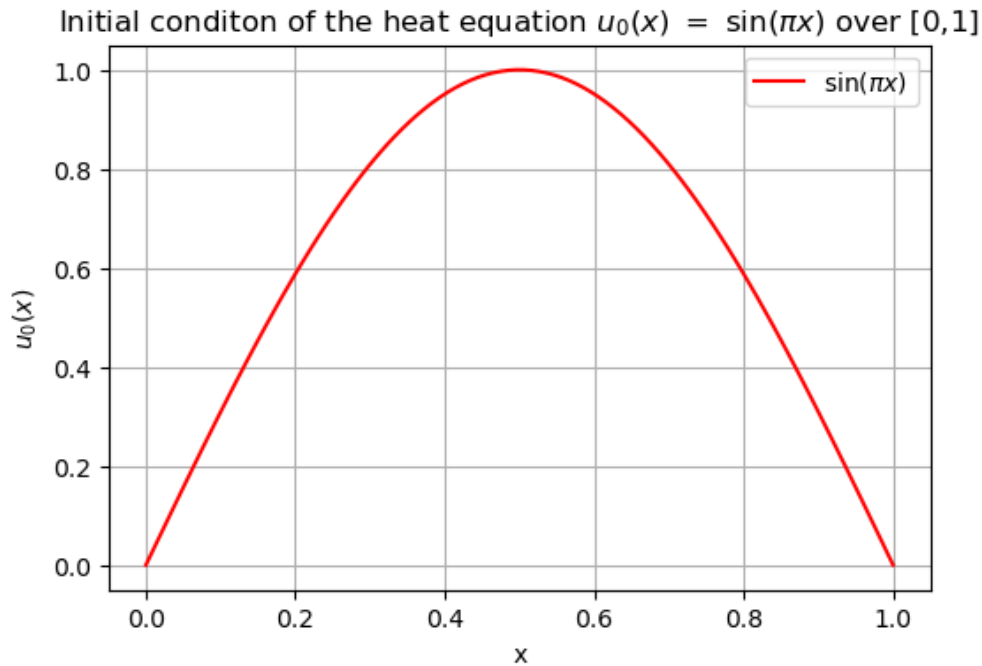


Figure 1: u_0 is the initial temperature $u_0(x_i)$, with a sequence of points $(x_i)_{0 \leq i \leq N+1}$.

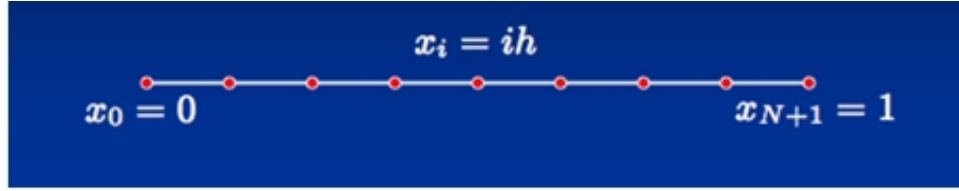


Figure 2: **Discretization of the interval** $[0, 1]$, $h = 1/(N + 1)$, with a sequence of points $(x_i)_{0 \leq i \leq N+1}$.

2 Finite difference approximations

- We will discretize $[0, T] \times [0, 1]$, (T is known).
- We define a uniform mesh of steps h , $h = \frac{1}{N+1}$, of intervals $[0, 1]$ as the set of points $(x_i = ih)_{1 \leq i \leq N}$.
- We define a uniform mesh of steps Δt , $\Delta t = \frac{T}{M}$, of interval $[0, T]$ as the set of points $(t_n = n\Delta t)_{0 \leq n \leq M}$.

Approximate solution to our problem by calculating the values u_i^n , for $n = 1, \dots, M$ and $i = 1, \dots, N$, which are supposed to approximate the values $u(t_n, x_i)$, for $n = 1, \dots, M$ and $i = 1, \dots, N$.

2.0.1 Centered-finite-differences and First order explicit time-stepping

The second-order spatial derivative $\frac{\partial^2 u}{\partial x^2}(t_n, x_i)$ can be approximated using the centered finite difference formula:

$$\frac{\partial^2 u}{\partial x^2}(t_n, x_i) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

where h is the spatial discretization step.

The first-order time derivative $\frac{\partial u}{\partial t}(t_n, x_i)$ can be approximated using the explicit Euler method:

$$\frac{\partial u}{\partial t}(t_n, x_i) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

where Δt gives the frequency with which the outputs are produced.

2.0.2 2D Heat map

$$R_i^n = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t} - \frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1}))}{h^2},$$

Figure 3

$$\max_{1 \leq i \leq N} |R_i^n| \leq C(\Delta t + h^2).$$

Figure 4

Figure 5: Explicit Euler scheme consistency error.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = 0, \quad i = 1, \dots, N, \quad n = 0, \dots, M-1,$$

Figure 6: Discretization of the heat equation.

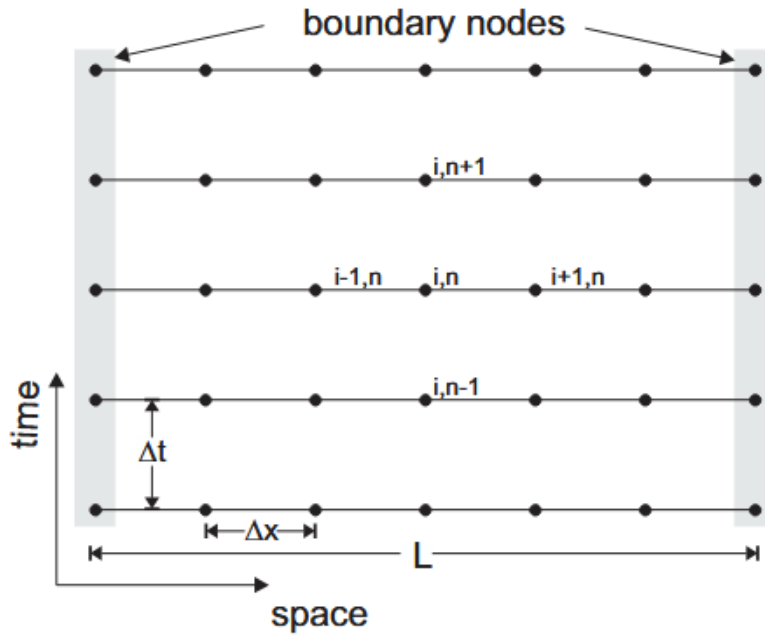


Figure 7: Finite difference discretization of the 1D heat equation.

The stability condition for this problem is for $\alpha = 1$: $\frac{\Delta t}{h^2} \leq \frac{1}{2}$.

3 Conclusion

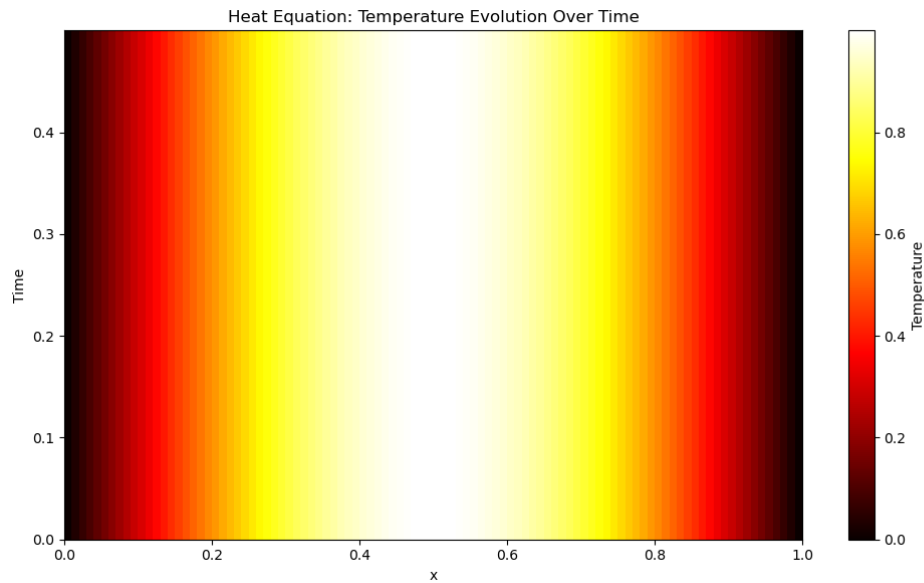


Figure 8: 2D heatmap for the 1D heat equation.

In Figure 8, $u(t, 0) = u(t, 1)$ ensures no abrupt temperature discontinuities at the edges .

Heat Diffusion Effect The heat equation works to spread the sinusoidal temperature profile over time. Higher-temperature regions diffuse outward, creating the smooth transition we observe.

Essentially, starting with $u(0, x) = \sin(\pi x)$ aligns perfectly with the periodic boundary condition, making the temperature distribution symmetric and wavelike as it evolves.