Arm Final Round Interview

Take home exercie on Heat Equation

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Contents

1 Presentation of the heat equation

Let u_0 be a given function for x belonging to $[01]$, we seek a function u depending on time and space solution of :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) = 0, & x \in]0,1[,\ t > 0, \\ u(t,0) = u(t,1), & t > 0, \\ u(0,x) = sin(\pi x), & x \in [0,1]. \end{cases}$$

$u_0(x)$ is the initial condition and the boundary conditions are $u_t(0)$ and $u_t(1)$. The diffusion coefficient $\alpha = 1$.

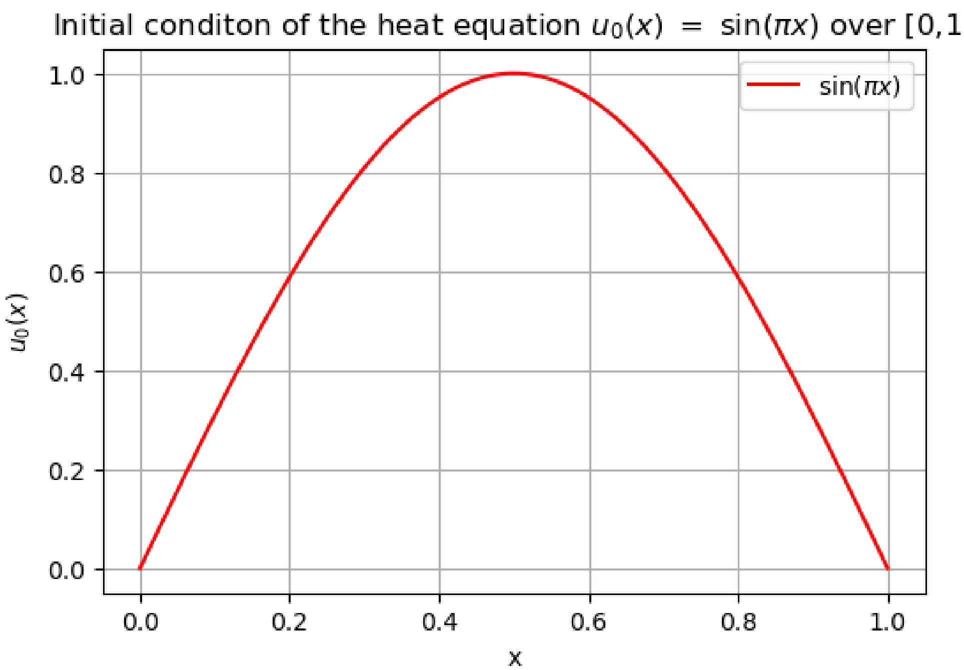

Figure 1: u_0 **is the initial temperature** $u_0(x_i)$, with a sequence of points $(x_i)_{0 \le i \le N+1}$.

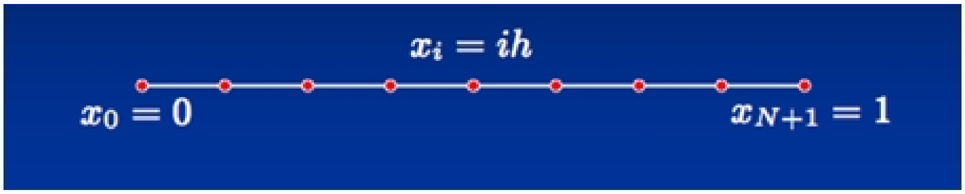

Figure 2: **Discretization of the interval** $[0, 1]$**,** $h = 1/(N + 1)$, with a sequence of points $(x_i)_{0\leq i\leq N+1}$

2 Finite difference approximations

- We will discretize $[0, T] \times [0, 1]$, (T is known).
- We define a uniform mesh of steps h, $h = \frac{1}{N+1}$, of intervalle $[0, 1]$ as the set of points $(x_i = ih)_{1\leq i\leq N}$.
- We define a uniform mesh of steps Δt, $\Delta t = \frac{T}{M}$, of intervalle $[0, T]$ as the set of point $(t_n = n\Delta t)_{0\leq n\leq M}$.

Approximate solution to our problem by calculating values u_i^n, for $n = 1, \ldots, M$ and $i = 1, \ldots, N$, which are supposed to approximate the values $u(t_n, x_i)$, for $n = 1, \ldots, M$ and $i = 1, \ldots, N$.

2.0.1 Centered-finite-differences and First order explicit time-stepping

The second-order spatial derivative $\frac{\partial^2 u}{\partial x^2}(t_n, x_i)$ can be approximated using the centered finite difference formula:

$$\frac{\partial^2 u}{\partial x^2}(t_n, x_i) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

where h is the spatial discretization step.

The first-order time derivative $\frac{\partial u}{\partial t}(t_n, x_i)$ can be approximated using the explicit Forward Euler method:

$$\frac{\partial u}{\partial t}(t_n, x_i) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

where Δt is the time step size.

2.0.2 2D Heat map

$$R_i^n = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t} - \frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1})}{h^2},$$

Figure 3

$$\max_{1 \le i \le N} |R_i^n| \le C(\Delta t + h^2).$$

Figure 4

Figure 5: Overall Caption for Both Images

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} = 0,\ i = 1, \cdots, N,\ n = 1, \cdots, M-1,$$

Figure 6: Discretization of the heat equation.

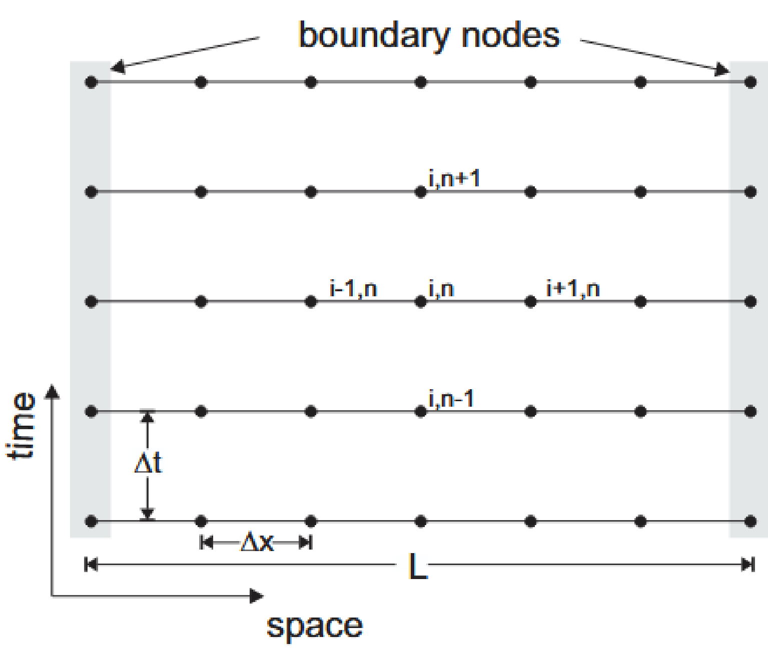

Figure 7: Finite difference discretization of the 1D heat equation.

The stability condition for this problem is for $\alpha = 1 : \frac{\Delta_t}{h^2} \le \frac{1}{2}$.

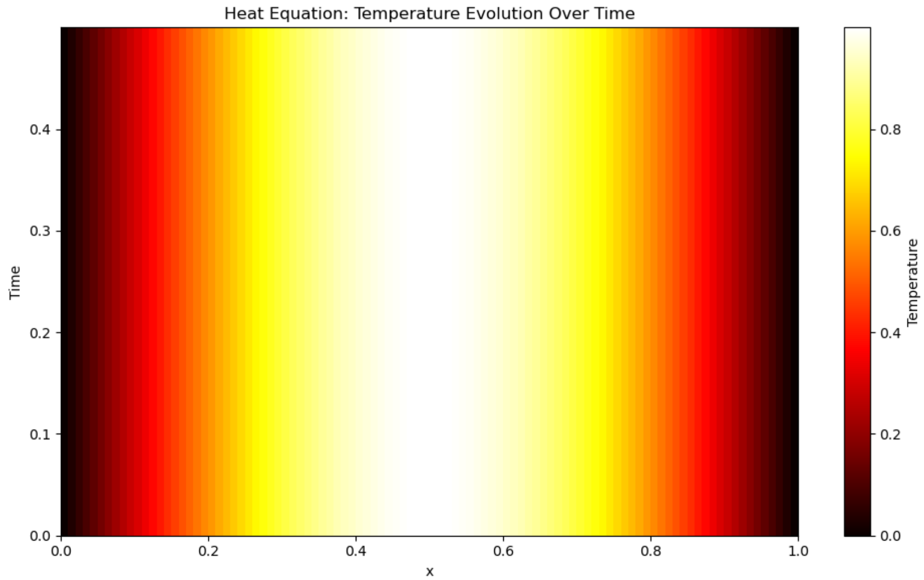

Figure 8: 2D heatmap for the 1D heat equation.

2.0.3 Conclusion

$u(t, 0) = u(t, 1)$ ensures no abrupt temperature discontinuities at the edges.

Heat Diffusion Effect The heat equation works to spread the sinusoidal temperature profile over time. Higher temperature regions diffuse outward, creating the smooth transition you observe.

Essentially, starting with: $u(0, x) = \sin(\pi x)$ aligns perfectly with the periodic boundary condition, making the temperature distribution symmetric and wave-like as it evolves.