

**Arm Final Round Interview**

**Take home exercise on Heat Equation**

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## 1 Presentation of the heat equation

Let  $u_0$  be a given function for  $x$  belonging to  $[0,1]$ , we seek a function  $u$  depending on time and space solution of :

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = 0, & x \in ]0, 1[, t > 0, \\ u(t, 0) = u(t, 1), & t > 0, \\ u(0, x) = \sin(\pi x), & x \in [0, 1]. \end{cases}$$

$u_0(x)$  is the initial condition and the boundary conditions are  $u_t(0)$  and  $u_t(1)$ . The diffusion coefficient  $\alpha = 1$ .

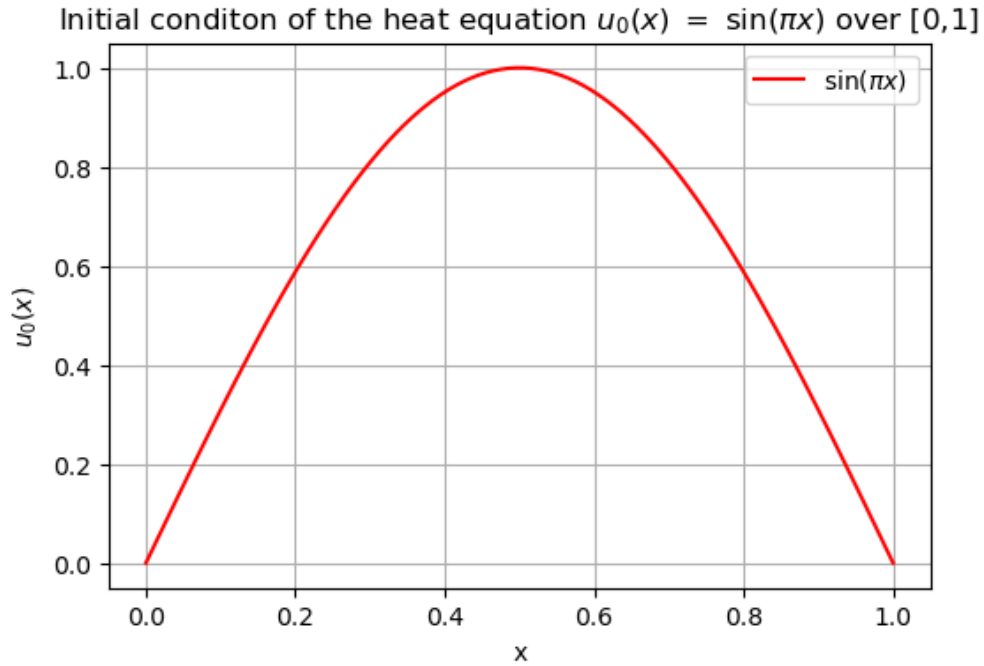


Figure 1:  $u_0$  is the initial temperature  $u_0(x_i)$ , with a sequence of points  $(x_i)_{0 \leq i \leq N+1}$ .

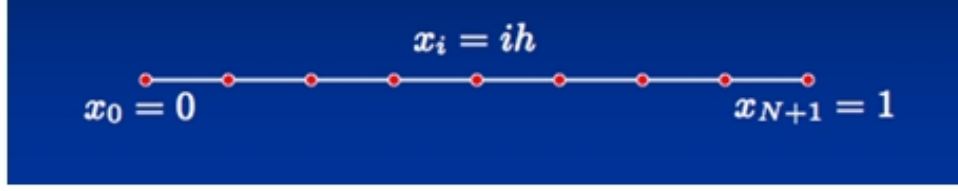


Figure 2: **Discretization of the interval**  $[0, 1]$ ,  $h = 1/(N + 1)$ , with a sequence of points  $(x_i)_{0 \leq i \leq N+1}$

## 2 Finite difference approximations

- We will discretize  $[0, T] \times [0, 1]$ , ( $T$  is known).
- We define a uniform mesh of steps  $h$ ,  $h = \frac{1}{N+1}$ , of intervals  $[0, 1]$  as the set of points  $(x_i = ih)_{1 \leq i \leq N}$ .
- We define a uniform mesh of steps  $\Delta t$ ,  $\Delta t = \frac{T}{M}$ , of interval  $[0, T]$  as the set of points  $(t_n = n\Delta t)_{0 \leq n \leq M}$ .

Approximate solution to our problem by calculating the values  $u_i^n$ , for  $n = 1, \dots, M$  and  $i = 1, \dots, N$ , which are supposed to approximate the values  $u(t_n, x_i)$ , for  $n = 1, \dots, M$  and  $i = 1, \dots, N$ .

### 2.0.1 Centered-finite-differences and First order explicit time-stepping

The second-order spatial derivative  $\frac{\partial^2 u}{\partial x^2}(t_n, x_i)$  can be approximated using the centered finite difference formula:

$$\frac{\partial^2 u}{\partial x^2}(t_n, x_i) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

where  $h$  is the spatial discretization step.

The first-order time derivative  $\frac{\partial u}{\partial t}(t_n, x_i)$  can be approximated using the explicit Euler method:

$$\frac{\partial u}{\partial t}(t_n, x_i) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

where  $\Delta t$  gives the frequency with which the outputs are produced.

### 2.0.2 2D Heat map

$$R_i^n = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\Delta t} - \frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1}))}{h^2},$$

Figure 3

$$\max_{1 \leq i \leq N} |R_i^n| \leq C(\Delta t + h^2).$$

Figure 4

Figure 5: Overall Caption for Both Images

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} = 0, \quad i = 1, \dots, N, \quad n = 1, \dots, M-1,$$

Figure 6: Discretization of the heat equation.

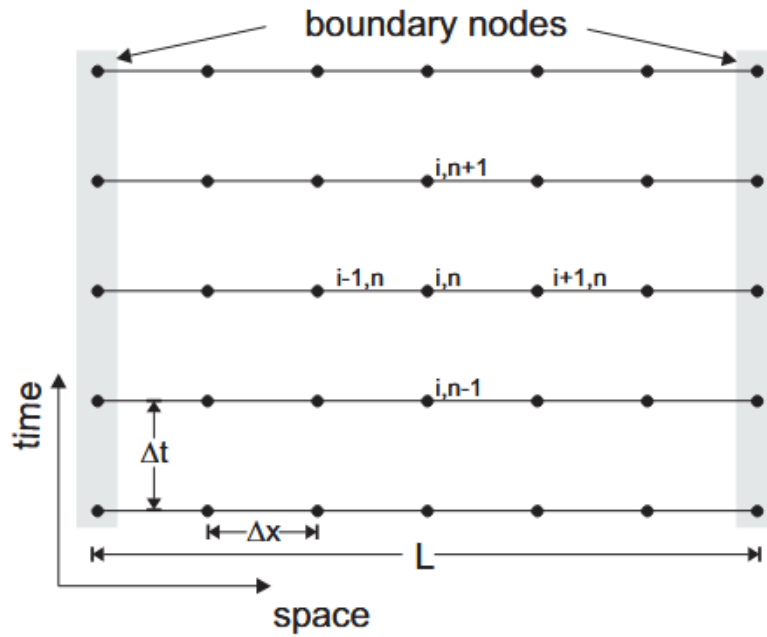


Figure 7: Finite difference discretization of the 1D heat equation.

The stability condition for this problem is for  $\alpha = 1$  :  $\frac{\Delta t}{h^2} \leq \frac{1}{2}$ .

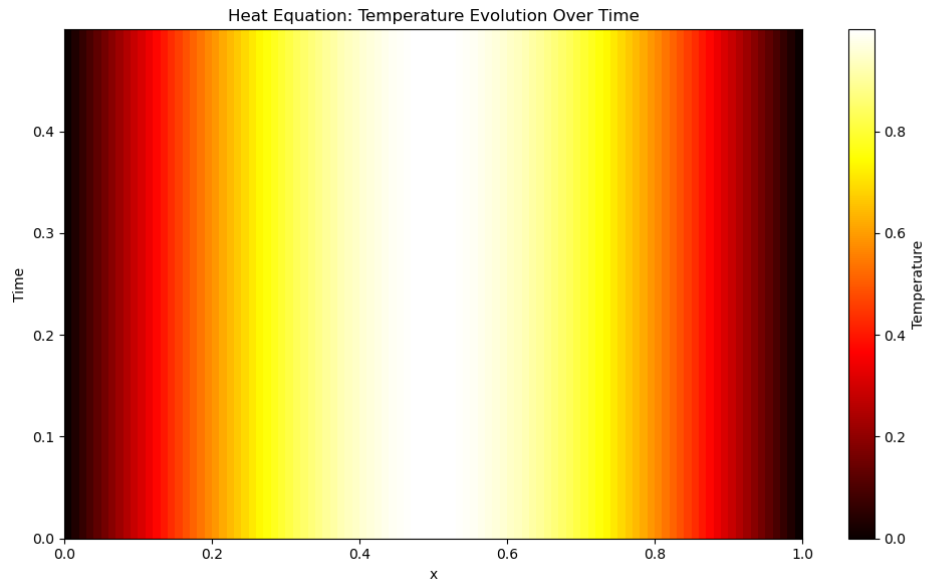


Figure 8: 2D heatmap for the 1D heat equation.

### 2.0.3 Conclusion

$u(t, 0) = u(t, 1)$  ensures no abrupt temperature discontinuities at the edges.

**Heat Diffusion Effect** The heat equation works to spread the sinusoidal temperature profile over time. Higher-temperature regions diffuse outward, creating the smooth transition we observe.

Essentially, starting with  $u(0, x) = \sin(\pi x)$  aligns perfectly with the periodic boundary condition, making the temperature distribution symmetric and wavelike as it evolves.