Math 176/Econ 135
Mathematics of Finance
Lecture 14

Professor Jeffrey Ludwig

OPTIONS TRADING

FORMULA FOR $C_E(S,t)$

MODEL FOR STOCHASTIC PROCESS $S(t)$

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Outline for Today

- Solve the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- For the value of a European call option

$$c(t) = S(t)N(d_1) - e^{-r(T-t)}\mathrm{X}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

Review

- Geometric Brownian Motion as a SDE

$$dS = \mu S dt + \sigma S dw$$

- Ito's Lemma

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \left(\frac{1}{2}\right) \frac{\partial^2 V}{\partial S^2} (dS)^2 + \frac{\partial^2 V}{\partial S \partial t} dS dt$$

- Apply V = ln(S) in a risk-neutral world where μ = r

$$S(T) = S(t) \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) (T - t) + \sigma v \right\}$$

$$C_X(t) = e^{-r(T-t)} E\{\max\left[0, S(T) - X\right] | S(t)\}$$

Ito's Lemma with GBM

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \left(\frac{1}{2}\right)\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial t}dSdt$$

- With $dS = \mu S dt + \sigma S dw$ and $(dS)^2 = \sigma^2 S^2 dt$

$$dV = \frac{\partial V}{\partial S}(\mu S dt + \sigma S dw) + \frac{\partial V}{\partial t}dt + \left(\frac{1}{2}\right)\frac{\partial^2 V}{\partial S^2}(\sigma^2 S^2 dt) + \frac{\partial^2 V}{\partial S \partial t}(\mu S dt + \sigma S dw)dt$$

$$dV = \sigma S \frac{\partial V}{\partial S}dw + \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \left(\frac{1}{2}\right)\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt$$

Now Build a Portfolio Π

- Build a portfolio Π where you buy one option V and sell Δ units of the stock

$$\Pi = V - S \cdot \Delta$$
$$d\Pi = dV - dS \cdot \Delta$$

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$$dV = \sigma S \frac{\partial V}{\partial S} dw + \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \left(\frac{1}{2} \right) \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$
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- Plug in the SDEs for dV and dS

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$$dS = \mu S dt + \sigma S dw$$

- Collecting terms we have

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dw + \left(\mu S \frac{\partial V}{\partial S} - \mu S \Delta + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

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- Choosing $\Delta = \frac{\partial V}{\partial S}$ eliminates the stochastic component, and the changes to the portfolio value must now be deterministic:

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$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

How to solve the Black-Scholes PDE?

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$V(S,t) = Kv(x,\tau)$$

$$\tau = \frac{\sigma^2}{2} \cdot (T - t)$$

$$x = \log\left(\frac{S}{K}\right)$$

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$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial \tau}\cdot\frac{d\tau}{dt} = K\frac{\partial v}{\partial \tau}\cdot\frac{-\sigma^2}{2}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x}\cdot\frac{dx}{dS} = K\frac{\partial v}{\partial x}\cdot\frac{1}{S}.$$

$$V(S,t) = Kv(x,\tau) \qquad \tau = \frac{\sigma^2}{2}\cdot(T-t) \qquad x = \log\left(\frac{S}{K}\right)$$

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) \\
&= \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x}\frac{1}{S}\right) \\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{1}{S} \\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{dx}{dS}\cdot\frac{1}{S} \\
&= K\frac{\partial v}{\partial x}\cdot\frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2}\cdot\frac{1}{S^2}.
\end{aligned}$$

How to solve the Black-Scholes PDE?

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$K\frac{\partial v}{\partial \tau} \cdot \frac{-\sigma^2}{2} + \frac{\sigma^2}{2} S^2 \left(K\frac{\partial v}{\partial x} \cdot \frac{-1}{S^2} + K\frac{\partial^2 v}{\partial x^2} \cdot \frac{1}{S^2} \right) + rS\left(K\frac{\partial v}{\partial x} \cdot \frac{1}{S} \right) - rKv = 0$$

$$k = \frac{r}{\sigma^2/2}$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

$$v = e^{\alpha x + \beta \tau} u(x, \tau)$$

$$v_\tau = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_\tau$$

$$v_x = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} u_x$$

$$v_{xx} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} u_x + e^{\alpha x + \beta \tau} u_{xx}$$

$$\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - ku$$

$$u_\tau = u_{xx} + [2\alpha + (k-1)]u_x + [\alpha^2 + (k-1)\alpha - k - \beta]u$$

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$$\alpha = -\tfrac{k-1}{2}$$

$$\beta = \alpha^2 + (k-1)\alpha - k = -\tfrac{(k+1)^2}{4}$$

$$u_\tau = u_{xx}$$

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{\frac{-(x-s)^2}{4\tau}} \, \mathrm{d}s$$

$$u(x,0) = u_0(x)$$

$$V(S,T) = \max(S-K,0) = \max(Ke^x - K, 0)$$

but $V(S,T) = Kv(x,0)$ so $v(x,0) = \max(e^x - 1, 0)$

$$v = e^{\alpha x + \beta \tau} u(x,\tau)$$

$$\begin{aligned} u(x,0) &= e^{-(-\frac{(k-1)}{2})x - (-\frac{(k+1)^2}{4})\cdot 0} v(x,0) \\ &= e^{(\frac{(k-1)}{2})x} \max(e^x - 1, 0) \\ &= \max\left(e^{\left(\frac{(k+1)}{2}\right)x} - e^{\left(\frac{(k-1)}{2}\right)x}, 0\right) \end{aligned}$$

$$u(x,0) = u_0(x)$$

$u_0(x) > 0$ when $x > 0$ $\qquad$ $u_0(x) = 0$ for $x \leq 0$

$$u_\tau = u_{xx}$$

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$$u(x,0) = u_0(x) = \max\left(e^{\left(\frac{(k+1)}{2}\right)x} - e^{\left(\frac{(k-1)}{2}\right)x}, 0 \right)$$

$$z = \frac{(s-x)}{\sqrt{2\tau}} \qquad \mathrm{d}z = \left(-\frac{1}{\sqrt{2\tau}}\right) \mathrm{d}x$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0\left(z\sqrt{2\tau} + x \right) e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0 \left(z\sqrt{2\tau} + x \right) e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

$$u_0 \; > \; 0 \text{ when } \; z \; > \; -\frac{x}{\sqrt{2\tau}}$$

$$u_0 \; = \max \left(e^{\left(\frac{(k+1)}{2}\right)x} - e^{\left(\frac{(k-1)}{2}\right)x}, 0 \right)$$

$$u(x,\tau) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k+1}{2}\left(x+z\sqrt{2\tau}\right)} e^{-\frac{z^2}{2}} \, \mathrm{d}z - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k-1}{2}\left(x+z\sqrt{2\tau}\right)} e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

$$u(x,\tau) = \; I_1 - I_2$$

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k+1}{2}\left(x+z\sqrt{2\tau}\right)} e^{-\frac{z^2}{2}} \, \mathrm{d}z$$

$$\begin{aligned}
&\frac{k+1}{2}\left(x+z\sqrt{2\tau}\right) - \frac{z^2}{2} \\
&\quad = \left(\frac{-1}{2}\right)\left(z^2 - \sqrt{2\tau}\,(k+1)\,z\right) + \left(\frac{k+1}{2}\right)x \\
&\quad = \left(\frac{-1}{2}\right)\left(z^2 - \sqrt{2\tau}\,(k+1)\,z + \tau\frac{(k+1)^2}{2}\right) + \left(\frac{(k+1)}{2}\right)x + \tau\frac{(k+1)^2}{4} \\
&\quad = \left(\frac{-1}{2}\right)\left(z - \sqrt{\tau/2}\,(k+1)\right)^2 + \frac{(k+1)\,x}{2} + \frac{\tau\,(k+1)^2}{4}.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k+1}{2}\left(x+z\sqrt{2\tau}\right)} e^{-\frac{z^2}{2}} \, \mathrm{d}z \\
&\qquad = \frac{e^{\frac{(k+1)x}{2}+\tau\frac{(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(z-\sqrt{\tau/2}(k+1)\right)^2} \, \mathrm{d}z
\end{aligned}$$

$$I_1 = \frac{e^{\frac{(k+1)x}{2}+\tau\frac{(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(z-\sqrt{\tau/2}(k+1)\right)^2} \, \mathrm{d}z$$

$$y = z - \sqrt{\tau/2}\,(k+1) \qquad \mathrm{d}y = \mathrm{d}z.$$

$$I_1 = \frac{e^{\frac{(k+1)x}{2}+\tau\frac{(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}-\sqrt{\tau/2}(k+1)}^{\infty} e^{\left(-\frac{y^2}{2}\right)} \, \mathrm{d}y$$

$$I_1 = e^{\frac{(k+1)x}{2}+\tau\frac{(k+1)^2}{4}} N(d_1) \qquad d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}\,(k+1)$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

Calculation for I_2 identical, except that $(k+1)$ is replaced by $(k-1)$

$$u(x,\tau) = e^{\frac{(k+1)x}{2} + \frac{\tau(k+1)^2}{4}} N(d_1) \; - e^{\frac{(k-1)x}{2} + \frac{\tau(k-1)^2}{4}} N(d_2)$$

$$d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\tfrac{\tau}{2}}\,(k+1) \qquad d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\tfrac{\tau}{2}}\,(k-1)$$

$$v(x,\tau) \;=\; e^{-\frac{(k-1)x}{2} - \frac{(k+1)^2\tau}{4}} u(x,\tau)$$

$$x \;=\; \log\left(S/K\right)$$

$$\tau = \left(\tfrac{1}{2}\right)\sigma^2(T-t)$$

$$V(S,t) = Kv(x,\tau)$$

$$k = \frac{r}{\sigma^2/2}$$

Black-Scholes Formula for European call option

- Strike price: K
- Risk-free rate: r

- Volatility σ

- Expiration time: T
- Stock Price: S(t)

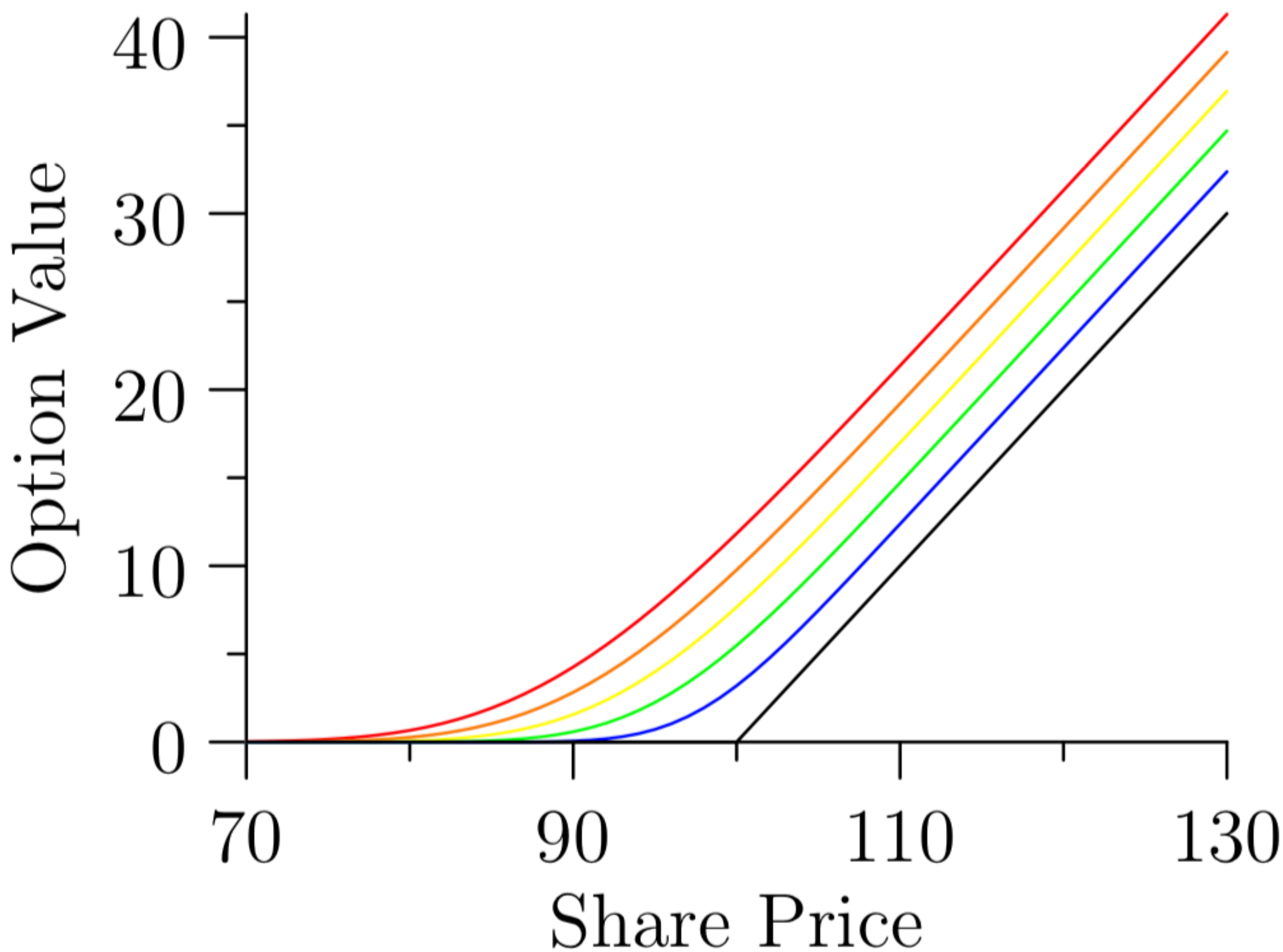
$$c(t) = S(t)N(d_1) - e^{-r(T-t)}\mathrm{K}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

Black-Scholes Call Option Value



Summary

- The Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Solved for the value of a European call option

$$c(t) = S(t)N(d_1) - e^{-r(T-t)}\mathrm{K}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

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