Math 176/Econ 135 Mathematics of Finance Lecture 1

Professor Jeffrey Ludwig

- Jeff’s background
  - MIT for undergraduate in aeronautics and astronautics
  - MIT for Masters and PhD in computer science
  - Career on Wall Street in quantitative trading
    - Credit Suisse, Proprietary Index Arbitrage Trader
    - PIMCO Portfolio Manager, Head of Equity Derivatives
    - Millennium Partners Hedge Fund, Portfolio Manager
  - Most recently Director of Jump Labs
    - R&D for Jump Trading, a high frequency trading (HFT) firm
    - Focus on long term collaborations with academia

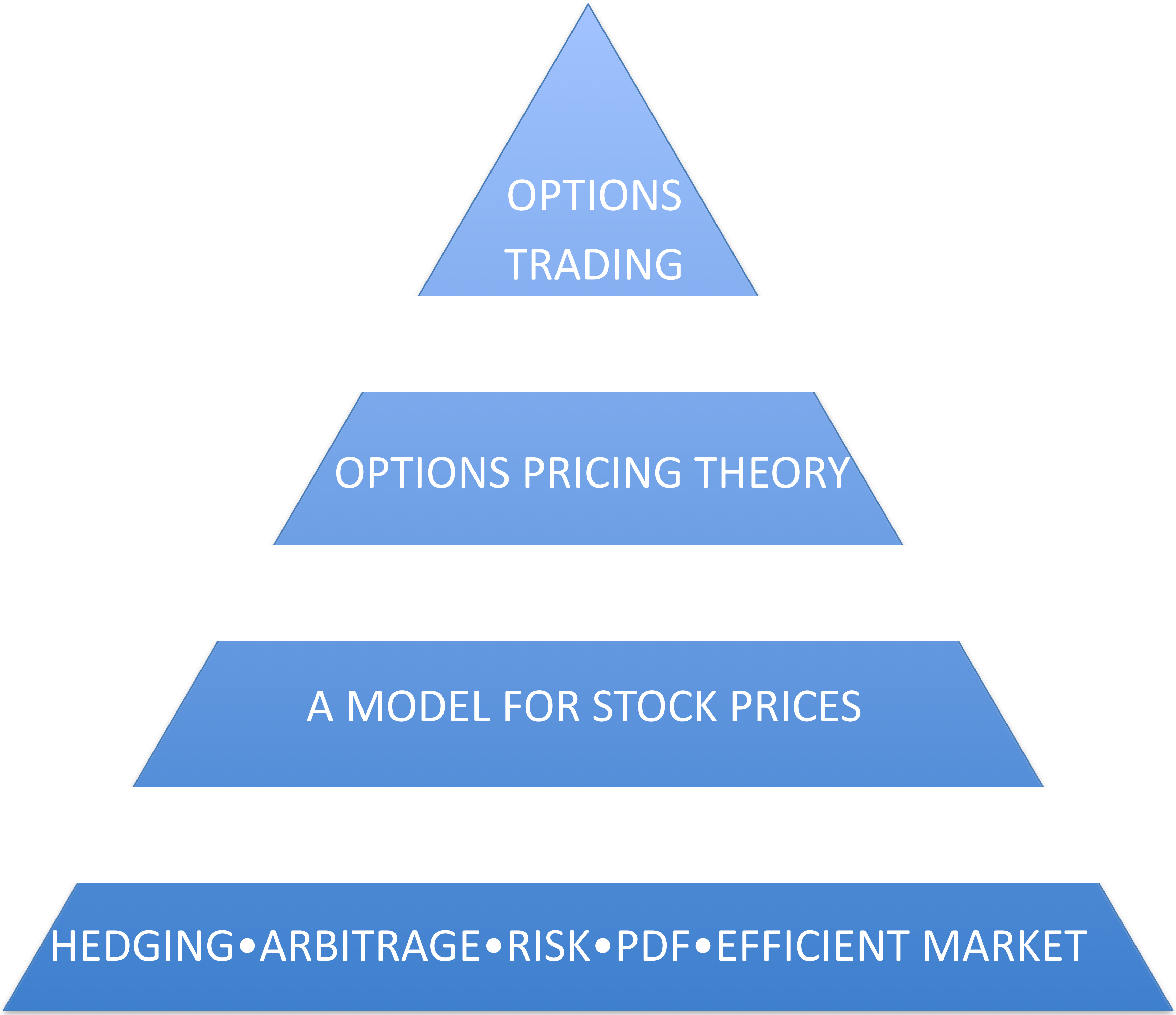
Sailing

Rogue Waves

Rogue waves are an open water phenomenon, in which winds, currents, non-linear phenomena such as solitons, and other circumstances cause a wave to briefly form that is far larger than the "average" large occurring wave

Course Overview

- Course content
  - Financial Markets, Risk, Hedging and Arbitrage
  - Review of Probability Theory
  - Basic Options: Puts, Calls, and Put/Call Parity
  - Black–Scholes Option Pricing Theory
- Exams
  - Two exams during class
  - Final exam
- Course Grade:
  - 10% homework
  - 25% each exam
  - 40% final exam
- My office hours


OPTIONS
TRADING
OPTIONS PRICING THEORY
A MODEL FOR STOCK PRICES
HEDGING•ARBITRAGE•RISK•PDF•EFFICIENT MARKET

Outline for Today

- Hedging
  – Managing bets or investments to eradicate risk
- Arbitrage
  – Capturing profit without risk
- Expected Value for discrete random variables
  – When should I play the Powerball lottery?
- Fair Bet
  – Expected value is zero

Simple Sports Betting

- You’re betting $100 on the UCLA vs. USC football game
- UCLA fan offers you 3:1 odds on UCLA winning
- USC fan offers you 3:2 odds on USC winning
- Possible profits with a single bet, in dollars

	UCLA Wins	USC Wins
UCLA fan (3:1 odds)	–100	+300
USC fan (3:2 odds)	+150	–100

Is there a way to bet with both fans such that we are guaranteed to win?

- Total bet is \$100
  - Place \$x bet with UCLA fan
  - Place \$(100 – x) with USC fan
- Summary of possible profits, in dollars

	UCLA Wins	USC Wins
UCLA fan (3:1 odds)	$-x$	$+3 \bullet x$
USC fan (3:2 odds)	$+3/2 \bullet (100 - x)$	$-(100 - x)$

Example of "Hedging"

- Hedging: managing bets or investments to eradicate risk

If we don't want to lose money…

	UCLA Wins	USC Wins
UCLA fan (3:1 odds)	$-x$	$+3 \bullet x$
USC fan (3:2 odds)	$+3/2 \bullet (100 - x)$	$-(100 - x)$

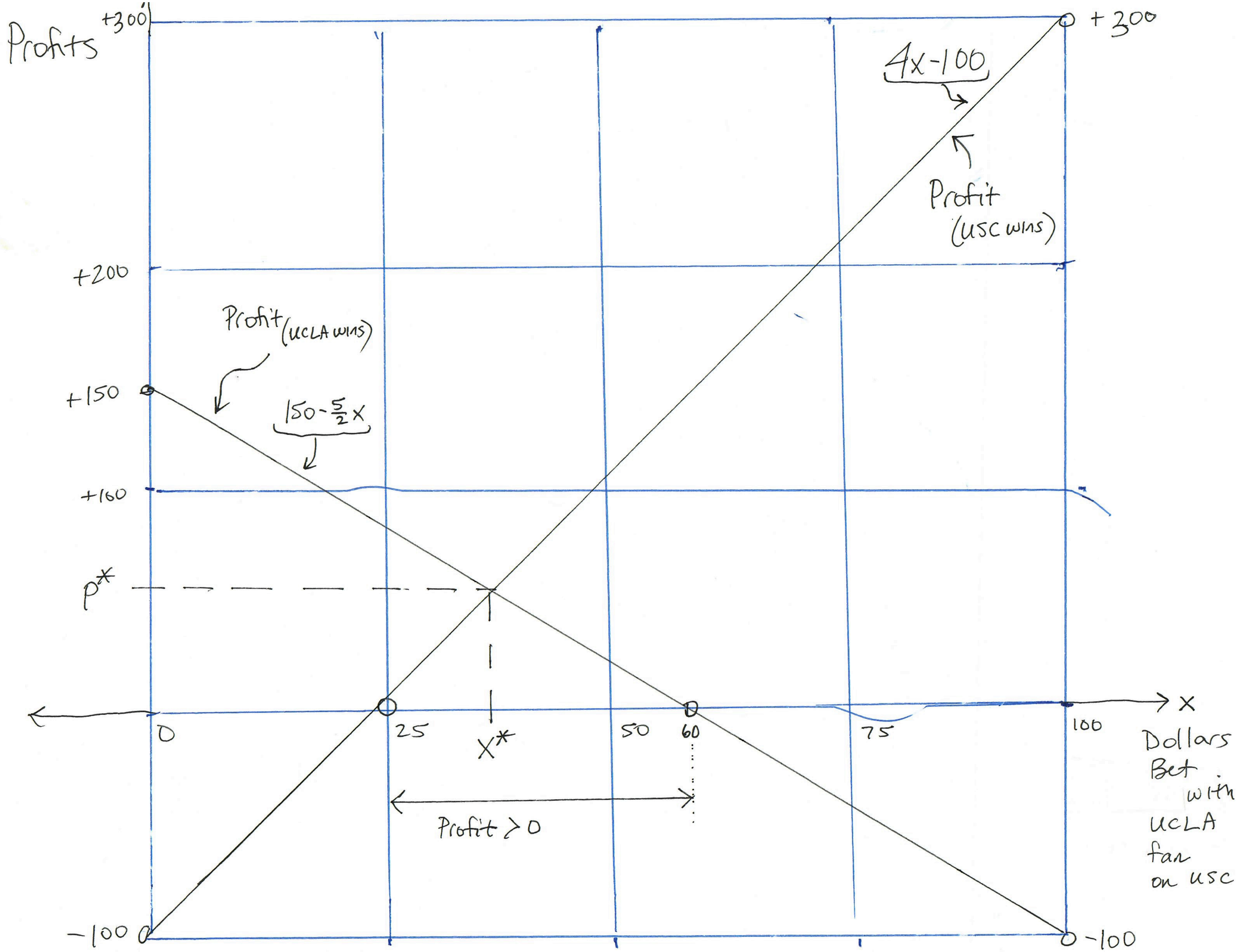
- Set $\text{Profit}_{(\text{UCLA wins})} > 0$

$$-x + 3/2 \bullet (100 - x) > 0$$
$$-x + 150 - 3/2x > 0$$
$$150 - (5/2) \bullet x > 0$$
$$x < 60$$

- Set $\text{Profit}_{(\text{USC wins})} > 0$

$$3x - (100 - x) > 0$$
$$3x - 100 + x > 0$$
$$4x - 100 > 0$$
$$x > 25$$

- Choose **$25 < x < 60$** and profit is positive no matter what happens! (recall x is the amount we bet with UCLA fan)



Profits
+300
+300
4x-100
Profit (USC wins)
+200
Profit (UCLA wins)
+150
150 - 5/2 x
+100
p*
X*
0
25
50
60
75
100
x
Dollars Bet with UCLA fan on USC
Profit > 0
-100
-100

"Arbitrage": profit without risk

- Let's define x* as the number of dollars bet with the UCLA fan to guarantee a profit p* *no matter who wins the game*

$$\text{Profit}_{(\text{UCLA wins})} = \text{Profit}_{(\text{USC wins})} = p^*$$

$$150 – (5/2)x^* = 4x^* – 100$$

$$250 = (13/2)x^*$$

$$x^* = 500/13$$

$$x^* = \$38.46$$

$$p^* = 4x^* – 100$$

$$p^* = 4(500/13) – 100$$

$$p^* = \$53.85$$

Derivation of General Case

- Team A fan takes Team A and gives you O_A: 1 odds: you place \$x bet with Team A fan
- Team B fan takes Team B and gives you O_B: 1 odds: you place \$(n–x) bet with Team B fan
- Summary of possible profits, in dollars:

	Team A wins	Team B wins
Team A fan (O_A:1 odds)	$-x$	$+O_A x$
Team B fan (O_B:1 odds)	$+O_B(n-x)$	$-(n-x)$

- Profit_A = profit given Team A wins
- Profit_B = profit given Team B wins
- Choose x such that $\text{Profit}_A > 0$ and $\text{Profit}_B > 0$

	Team A wins	Team B wins
Team A fan (O_A:1 odds)	$-x$	$+O_A x$
Team B fan (O_B:1 odds)	$+O_B(n-x)$	$-(n-x)$

$\text{Profit}_A > 0$: $-x + O_B(n-x) > 0$

$-(1+O_B)x + O_B n > 0$

$x < O_B n/(1+O_B)$

$\text{Profit}_B > 0$: $O_A x - (n-x) > 0$

$(1 + O_A)x - n > 0$

$x > n / (1 + O_A)$

$$n/(1 + O_A) < x < O_B n/(1+O_B)$$

Back to previous example for UCLA vs. USC game:

$n = 100$ $\quad O_A = 3$ $\quad O_B = 3/2$

$$100/(1+3) < x < 3/2(100)/(1+3/2)$$

$$25 < x < 60$$

"Arbitrage": profit without risk

- A bet of x^* with fan A guarantees a profit p^* *no matter who wins the game*

Set $\text{Profit}_A = \text{Profit}_B$ and solve for x^*:

$$\text{Profit}_A = \text{Profit}_B$$

$$-x^* + O_B(n - x^*) = O_A x^* - (n - x^*)$$

$$x^* = n(1 + O_B) / (2 + O_A + O_B)$$

$$p^* = n(O_A O_B - 1) / (2 + O_A + O_B)$$

Expected Value

- Suppose that X_d is a discrete random variable that takes on values $\{x_1, x_2, x_3, ..., x_n\}$ with probabilities $\{p_1, p_2, p_3, ..., p_n\}$

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$$\sum_{i=1}^{n} p_i = 1$$

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- For example, if X_d takes values of 1,2, and 4 with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively, then we have

$$E(X_d) = (1 \cdot \frac{1}{4}) + (2 \cdot \frac{1}{2}) + (4 \cdot \frac{1}{4}) = 2\frac{1}{4}$$

Powerball (23 August 2017)

$758.7 million jackpot

Five white balls are drawn without replacement from a drum that holds 69 balls, each bearing a number between 1 and 69, where order does not matter. Then, a red Powerball is drawn from a drum holding 26 balls, each bearing a number between 1 and 26.

Powerball (23 August 2017)

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What is the probability of winning the jackpot?

Number of possible outcomes is:

$$\binom{69}{5}\binom{26}{1} = \frac{69 \times 68 \times 67 \times 66 \times 65}{5 \times 4 \times 3 \times 2 \times 1} \times \frac{26}{1} = 292201338$$

So odds of winning is:

$$\frac{1}{292201338}$$

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Powerball (23 August 2017)

MATCHING COMBINATION	PRIZES	CURRENT ODDS (1 IN ...)	PREVIOUS ODDS (1 IN ...)
5 white balls and the Powerball	**The grand prize**	292,201,338	175,223,510
5 white balls	**$1,000,000**	11,688,054	5,153,633
4 white balls and the Powerball	**$50,000 (formerly $10,000)**	913,129	648,976
4 white balls	**$100**	36,525	19,088
3 white balls and the Powerball	**$100**	14,494	12,245
3 white balls	**$7**	580	360
2 white balls and the Powerball	**$7**	701	706
1 white balls and the Powerball	**$4**	92	111
The Powerball	**$4**	38	55

$758.7 million jackpot

[A. Horton, "How Powerball manipulated the odds to make another massive jackpot," *Washington Post*, 22 Aug. 2017. https://www.washingtonpost.com/news/wonk/wp/2017/08/22/how-powerball-manipulated-the-odds-to-make-another-massive-jackpot]

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What is the expected payout for buying a $2 ticket, with a $758.7 million jackpot?

$$\frac{\$758.7M}{292201338}+\frac{\$1M}{11688054}+\frac{\$50000}{913129}+\frac{\$100}{36525}+\frac{\$100}{14494}+\frac{\$7}{580}+\frac{\$7}{701}+\frac{\$4}{92}+\frac{\$4}{38}$$

$$\$2.597+\$0.086+\$0.055+\$0.003+\$0.007+\$0.012+\$0.010+\$0.044+\$0.105$$

$2.92

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(At $500 million jackpot, expected payout is $2.03)

How would things look under the old rules?

Fair Bet

- A bet is *fair* when the expected value of the winnings is zero
- Let's revisit the UCLA / USC football game:
  - UCLA fan gives us 3:1 odds
  - What is the implied probability of UCLA winning, based on the UCLA fan's view, assuming a fair bet?
  - How does this compare to the USC fan's view?

Fair Bet

- $P_{UCLA}(UCLA)$ is the probability that UCLA will win implicitly believed by the UCLA fan when he gives us odds. Our winnings W in a single bet with the UCLA fan satisfy:

$$W = \begin{cases} -100, & \text{with probability} = P_{UCLA}(UCLA) \\ +300, & \text{with probability} = (1 - P_{UCLA}(UCLA)) \end{cases}$$

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- Assuming a fair bet, the expected value of our winnings $E(W)$ must be zero:

$$E(W) = (-100) \cdot P_{UCLA}(UCLA) + 300 \cdot (1 - P_{UCLA}(UCLA)) = 0$$

$$P_{UCLA}(UCLA) = \frac{3}{4} = 75\%$$

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$$P_A(A) = \frac{O_A}{1 + O_A}$$

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- In the general case of Fan A offering a fair bet with odds of $O_A : 1$ we have

$$P_A(A) = \frac{O_A}{1 + O_A}$$

- What does the USC fan implicitly believe to be the probability that USC will win?

$$P_{USC}(USC) = \frac{\frac{3}{2}}{1 + \frac{3}{2}} = \frac{3}{5} = 60\%$$

Summary

- Hedging
  - Managing bets or investments to eradicate risk
  - Choose x such that $n/(1 + O_A) < x < O_B n/(1+O_B)$ to hedge
- Arbitrage
  - Capturing profit without risk
  - Choose x = x* and earn an arbitrage profit p*
- Expected Value for discrete random variables
  - When should I play the Powerball lottery?
  - Play when Expected Value of winning exceeds cost of ticket
- Fair Bet
  - The person offering the bet should be willing to take either side of the bet: the expected value of our winnings is zero