

# Search-Based Software Engineering

Genetic Programming  
Part I

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Lehrstuhl für Software Engineering II



John Koza: Genetic Programming: On the Programming of Computers by Means of Natural Selection, 1992

# Genetic Programming

- 
- 1: Randomly create an *initial population* of programs from the available primitives (more on this in Section 2.2).
  - 2: **repeat**
  - 3:     *Execute* each program and ascertain its fitness.
  - 4:     *Select* one or two program(s) from the population with a probability based on fitness to participate in genetic operations (Section 2.3).
  - 5:     Create new individual program(s) by applying *genetic operations* with specified probabilities (Section 2.4).
  - 6: **until** an acceptable solution is found or some other stopping condition is met (e.g., a maximum number of generations is reached).
  - 7: **return** the best-so-far individual.

**Algorithm 1.1:** Genetic Programming

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# GA vs. GP

- GA operates with a population of potential solutions which are iteratively improved using the mechanisms of selection, crossover, mutation, and replacement
- GP operates with a population of potential solutions which are iteratively improved using the mechanisms of selection, crossover, mutation, and replacement
- There are two main distinctions between GP and GA:
  1. The form of representation used:
    1. GA operates on genotypes; GP operates on phenotypes
    2. GP Representation = trees
  2. The more open-ended nature of the evolutionary process in GP:
    1. Size is variable not fixed length
    2. Number of elements used in the final solution as well as their interconnections must be open to evolution

# Programs, not numbers

- Programs are highly structured.
- GP mostly (but not exclusively) uses syntax tree to represent solutions.
  - $\text{max}(x + x, x + 3 * y)$   
 $= (\text{max } (+ x x) (+ x (* 3 y)))$
- Obviously, it is easier to implement GP with some languages: high-level ADT, garbage collection, etc

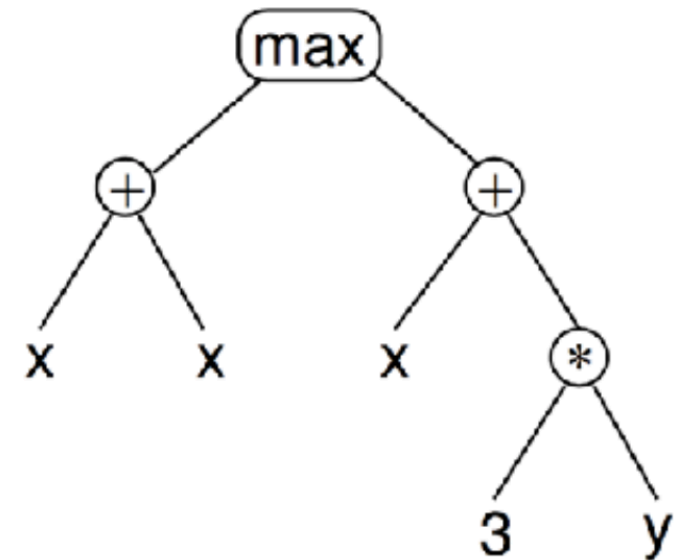
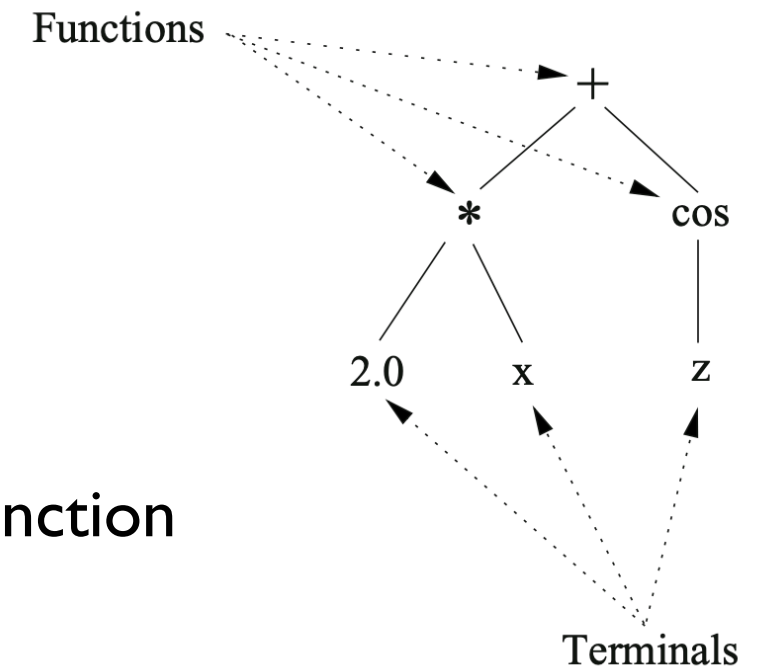


Figure 2.1: GP syntax tree representing  $\text{max}(x+x, x+3*y)$ .

# Classical GP

- Programs generated using two sets:
  - Terminal set: Items of arity 0
  - Function set: Items of arity  $> 0$
- The input to a GP function can be the result of another function
- Required property: **Type Closure**
  - Each function should be able to handle gracefully all values it might ever receive as input
  - All terminals must be allowable inputs for all functions
  - The output from any function must be a permitted input to any other function
  - Closure ensures generated programs are syntactically correct



# Sufficiency

- Is the given set of terminals and non-terminals sufficient to express the solution to the problem?
- Unless there is a theoretical guarantee that comes WITH the problem, this is hard to answer.
- We can always approximate.

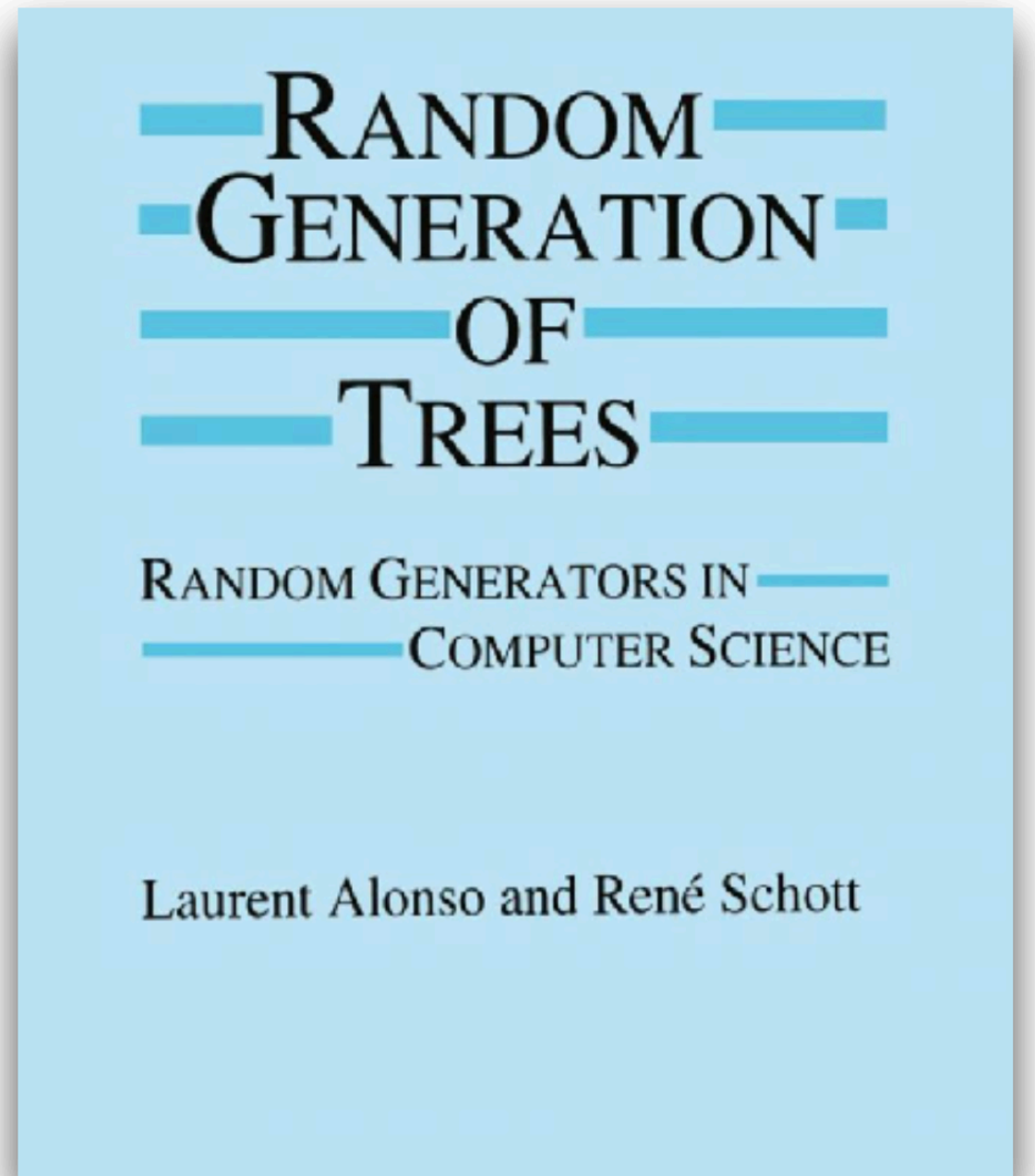
# Ephemeral Random Constants

- It is impossible to include all real numbers in the terminal set
- Ephemeral random constants (ERCs):
  - Every time this terminal is chosen in the construction of an initial tree (or a new subtree to use in an operation like mutation), a different random value is generated which is then used for that particular terminal, and which will remain fixed for the rest of the run.
  - After the initial generation, new constants are created through the recombination of existing ERCs through arithmetic expressions
  - The value of the constant is constant for a tree, but may differ from one tree to another



# Initialisation

- What is a random tree?
- We need to limit the size of the tree (i.e. depth): we do not want arbitrarily large trees as solutions.
- Many initialisation methods: full, grow, ramped half-and-half, and others.



# Full Initialisation

- Grow full trees
- Add non-terminal nodes only until the depth limit is reached.
- Then only add terminals as leaves.
- All trees are fully grown.

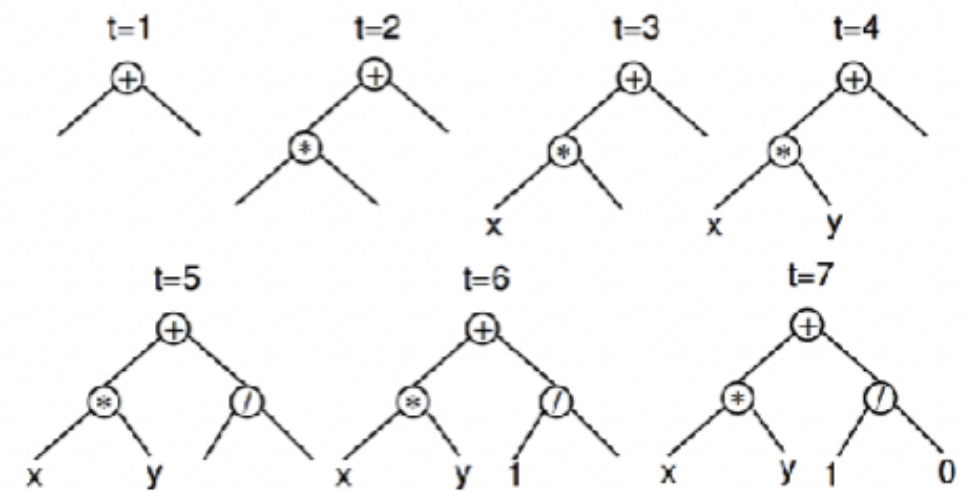
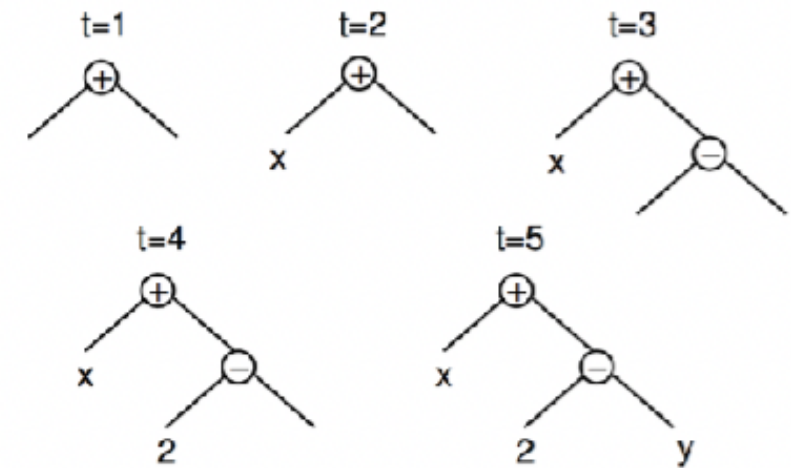


Figure 2.3: Creation of a full tree having maximum depth 2 using the full initialisation method ( $t = \text{time}$ ).

# Grow Initialisation

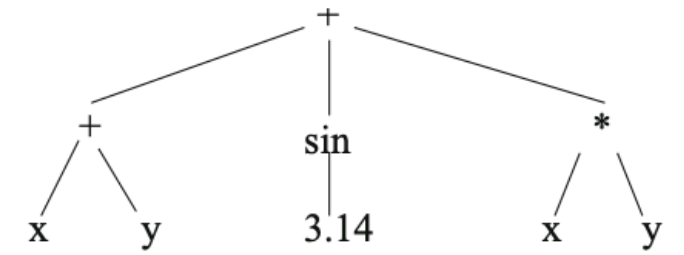
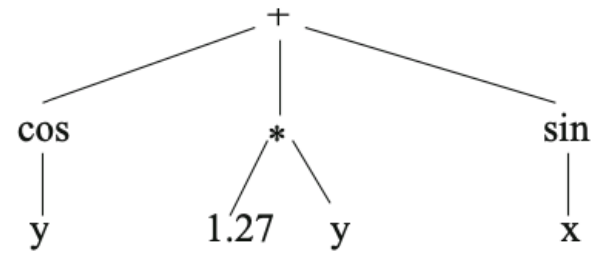
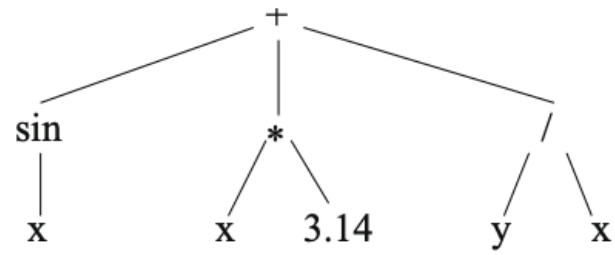
- Grow various trees
- Add any node while there are empty slots and the depth limit is not reached.
- Results in trees of various sizes, but the ratio between terminals and non-terminals will bias the average size.



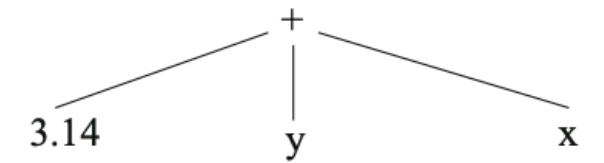
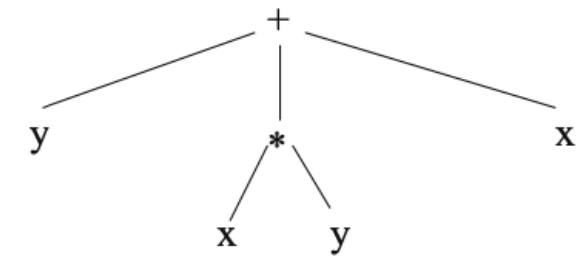
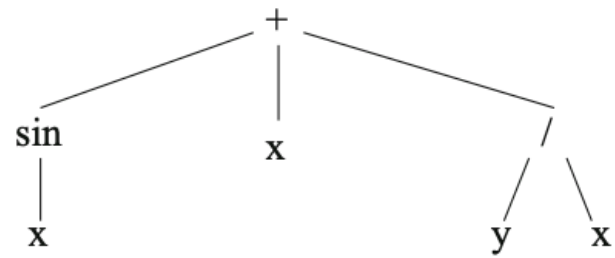
**Figure 2.4:** Creation of a five node tree using the **grow** initialisation method with a maximum depth of 2 ( $t = \text{time}$ ). A terminal is chosen at  $t = 2$ , causing the left branch of the root to be closed at that point even though the maximum depth had not been reached.

# Full vs. Grow

Full



Grow



# Ramped Half and Half

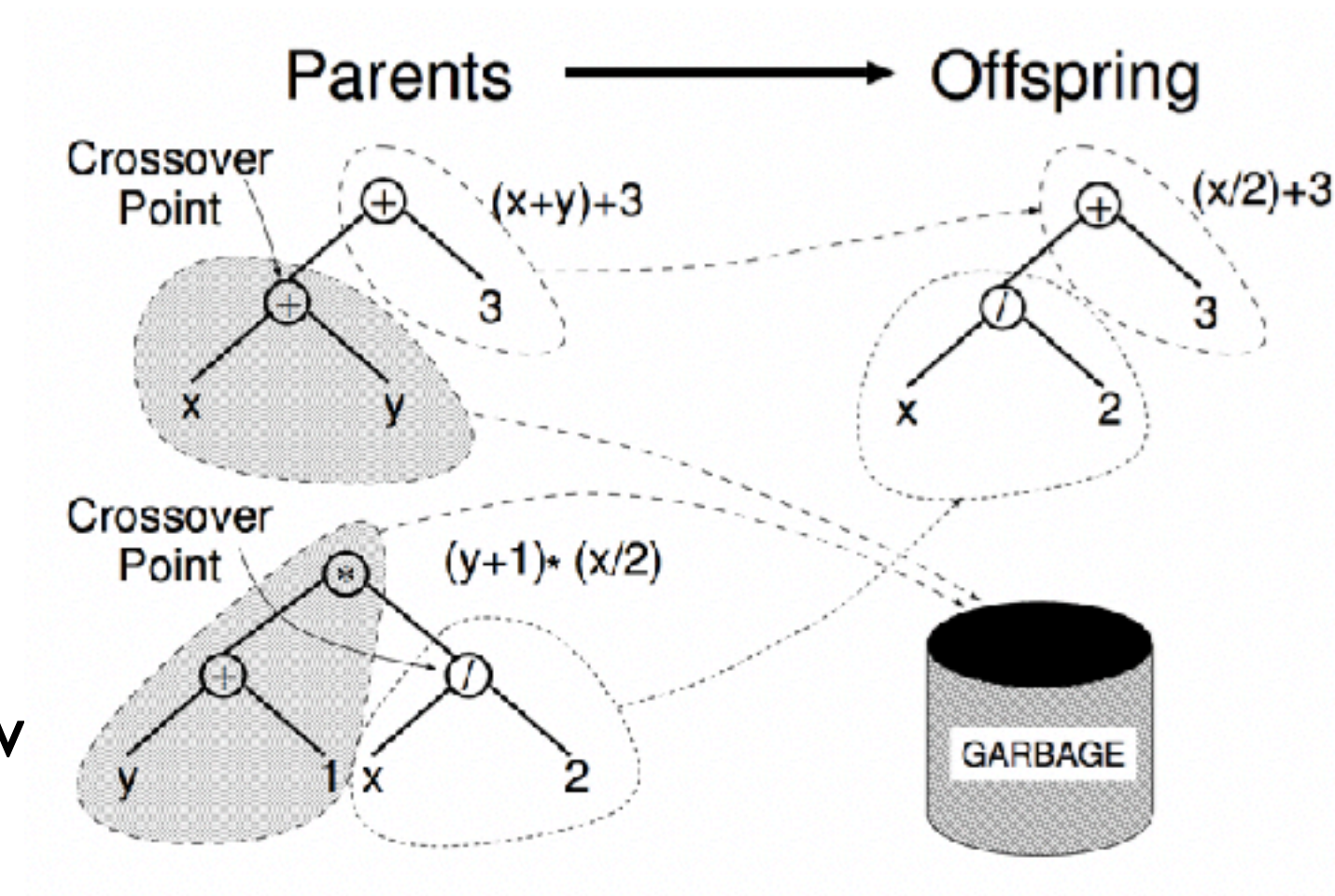
- Half of population is initialised with Full method
- Half of population is initialised with Grow method
- A better diversity in terms of shapes and size
- Ramped method tends to generate *bushy* trees. Some programs have highly asymmetric shape, which is hard to achieve with ramped method.
- Various methods have been developed to sample trees with sizes that are more uniformly distributed (highly sophisticated combinatorics).

# Selection

- Nothing different really, except:
  - GP evolves programs;
  - The fitness of the program is usually measured by executing the candidate program;
  - This can be time consuming, despite the evaluation essentially being inherently parallel.

# Crossover

- Initial idea
- Randomly choose two crossover points in parent trees;
- Cut and swap subtrees below the crossover points.



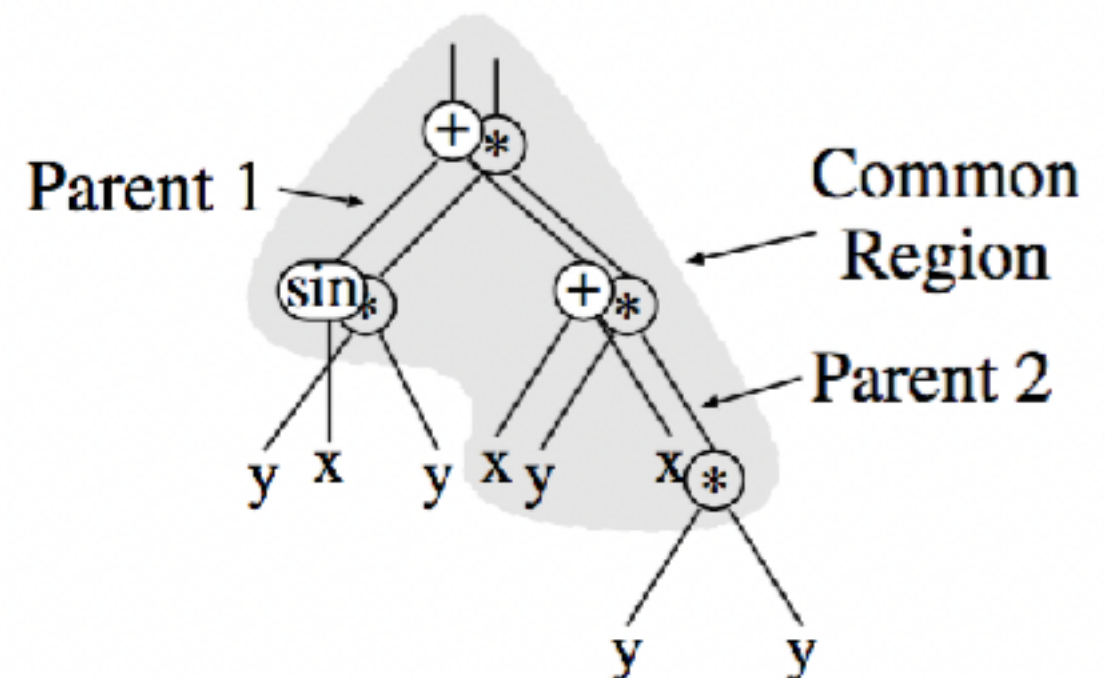
# Crossover

- Often crossover points are NOT sampled with random distribution:
- Average branching factor is 2 or more, which means the majority of the nodes are leaves, which means the majority of branches will simply cut a single leaf.
- Type-aware crossover (Koza 1992):
  - 90% chance of choosing a non-terminal node
  - 10% chance of choosing a terminal node



# Uniform Crossover

- Find the common region between two parents.
- For each node in the common region, flip a coin to decide whose node to take; when taking a non-terminal node, take its subtree.
- Mixes code nearer to the root more often, compared to other crossover operators.

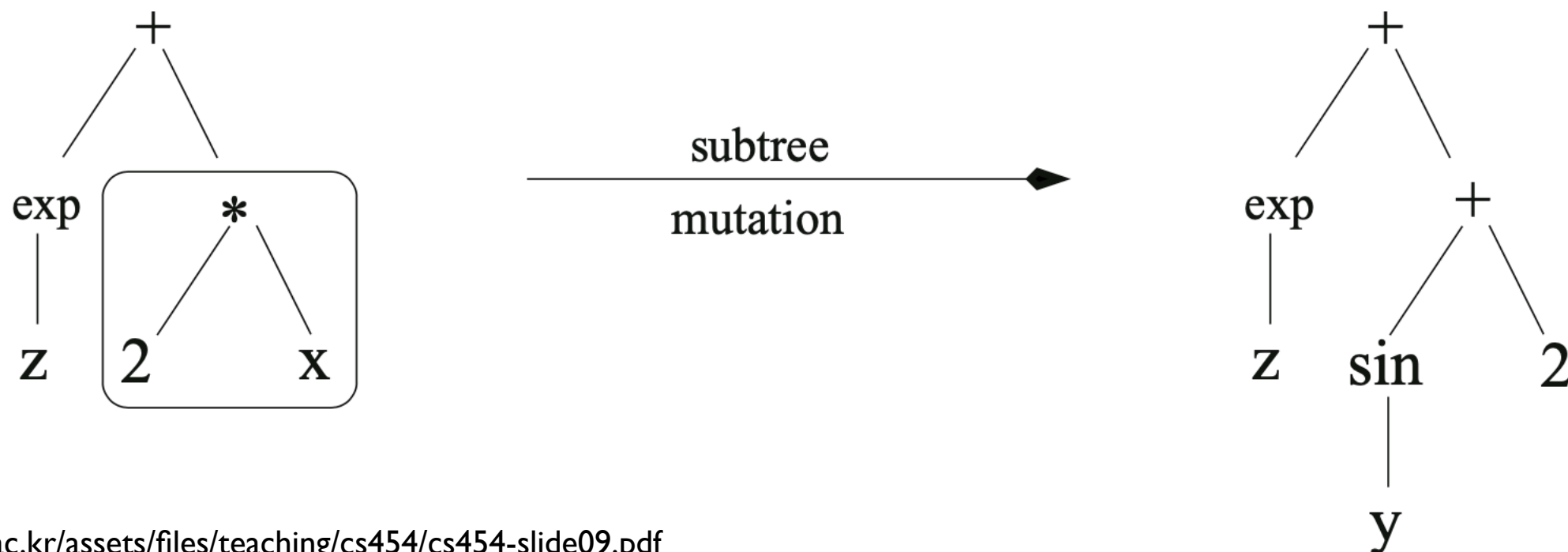


# Size-fair Crossover

- First crossover point in one parent chosen randomly;
- Measure the subtree size;
- Constrain the size of the second subtree to be chosen from the other parent

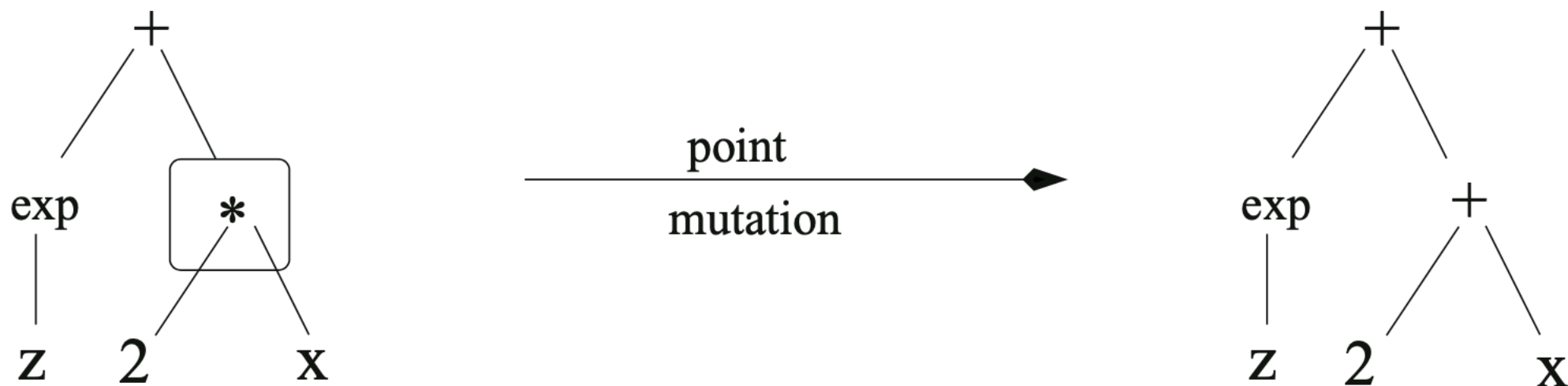
# Subtree Mutation

- Subtree mutation (a.k.a. headless chicken mutation):
  - Choose a subtree
  - Replace it with a randomly generated subtree



# Point Mutation

- For each node:
  - With a certain probability, replace the node with another node of same arity.
- Independently consider all nodes; may mutate more than one.



# ... and many more

- Hoist mutation: create a new individual, which is a randomly chosen subtree of the parent.
- Shrink mutation: replace a randomly chosen subtree with a randomly chosen terminal node.
- Permutation mutation: change the order of function arguments in trees.
- Systematic constant mutation: use external optimisation to tune the constants in the expression tree.

# Bloat

- Average size of trees in the population remains relatively static for certain number of generations, then:
- It increases rapidly and significantly. This growth in size is not accompanied by improvement in fitness.
- Many attempt to explain why this happens; no unified theory yet.

# Three Theoretical Attempts

- **Replication accuracy theory** (McPhee and Miller 1995): success of a GP individual depends on its ability to have offsprings that are functionally similar to itself, hence the tendency to repeat itself.
- **Removal bias theory** (Soule and Foster 1998): inactive (dead) code usually lies lower in the tree, and are smaller than average. When replaced (and removed), larger subtrees take their place, increasing the tree size.
- **Program Search Space theory** (Langdon and Poli 1997): above certain size, there is no correlation between size and fitness, but there are more longer programs, so they are just sampled more often.

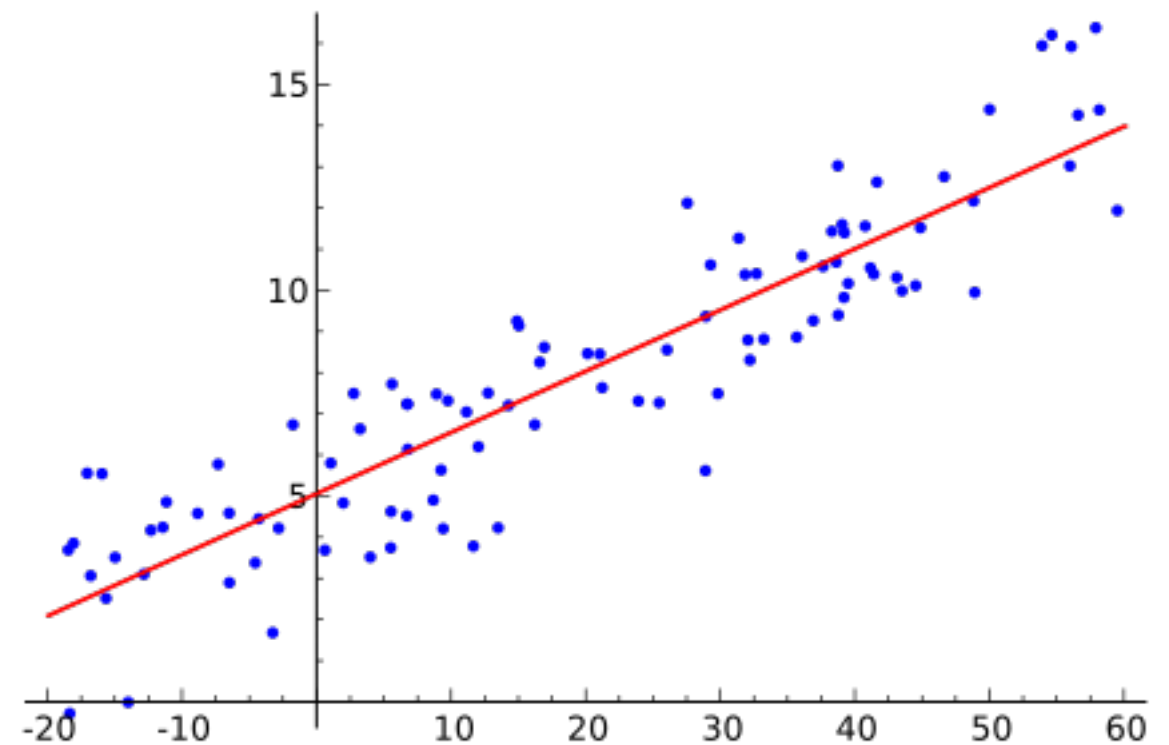
# Bloat Control

- Size and depth limit: do not accept too large individuals into population
- Bloat-aware genetic operators: do not generate too large individuals
- Bloat aware selection: consider program size as part of selection pressure



# Symbolic Regression

- Regression analysis estimates the relationship between variables: for example, linear regression is to find the set of  $(a, b, c)$  such that  $y = ax + b$  fits the given data points with minimum error
- Symbolic regression searches for the model itself in the space of all possible equations
- Fitness: The usual choice for fitness is to minimise MSE (Mean Square Error): for each data point, measure the squared error, and get their average.



# Example: Evolving Fault Localisation Formulas

- Shin Yoo. "Evolving human competitive spectra-based fault localisation techniques." International Symposium on Search Based Software Engineering. Springer, Berlin, Heidelberg, 2012.

# Fault Localisation

```
int mid(int x, int y, int z) {  
    int m;  
    m = z;  
    if (y < z) {  
        if (x < y)  
            m = y;  
        else if (x < z)  
            m = y;  
    } else {  
        if (x > y)  
            m = y;  
        else if (x > z)  
            m = x;  
    }  
    return m;  
}
```

# Fault Localisation

mid(3,3,5) == 3 Pass

mid(1,2,3) == 2 Pass

mid(3,2,1) == 2 Pass

mid(5,5,5) == 5 Pass

mid(5,3,4) == 4 Pass

mid(2,1,3) == 2 | Fail

# Fault Localisation

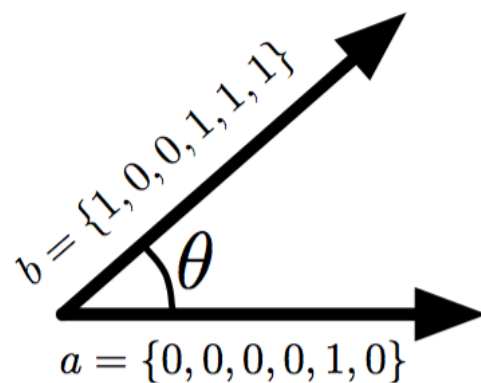
Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3
int m;	✓	✓	✓	✓	✓	✓
m = z;	✓	✓	✓	✓	✓	✓
if(y < z) {	✓	✓	✓	✓	✓	✓
if(x < y)	✓	✓			✓	✓
m = y;		✓				
else if(x < z)	✓				✓	✓
m = y;	✓					✓
} else {			✓	✓		
if(x > y)			✓	✓		
m = y;			✓			
else if(x > z)				✓		
m = x;						
return m;	✓	✓	✓	✓	✓	✓
	Pass	Pass	Pass	Pass	Pass	Fail

Error Vector



# Suspiciousness score

- Insight: Entities in a program that are primarily executed by failed test cases are more likely to be faulty than those that are primarily executed by passed test cases
- Each component (row) is **ranked** according to their **similarity** to the **error vector**
  - Many similarity coefficients exist.



$$Ochiai(a, b) = \cos(\theta)$$

- **Ochiai** similarity is equivalent to the cosine of the angle between two vectors in an n-dimensional space

# Fault Localisation

<b>Tarantula</b>	$S(s) = \frac{failed(s)/totalfailed}{failed(s)/totalfailed + passed(s)/totalpassed}$
<b>Ochiai</b>	$S(s) = \frac{failed(s)}{\sqrt{totalfailed \cdot (failed(s) + passed(s))}}$
<b>Op2</b>	$S(s) = failed(s) - \frac{passed(s)}{totalpassed + 1}$
<b>Barinel</b>	$S(s) = 1 - \frac{passed(s)}{passed(s) + failed(s)}$
<b>DStar</b>	$S(s) = \frac{failed(s)^*}{passed(s) + (totalfailed - failed(s))}$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	
m = z;	✓	✓	✓	✓	✓	✓	
if(y < z) {	✓	✓	✓	✓	✓	✓	
if(x < y)	✓	✓			✓	✓	
m = y;		✓					
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$



# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	
if(y < z) {	✓	✓	✓	✓	✓	✓	
if(x < y)	✓	✓			✓	✓	
m = y;		✓					
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	
if(x < y)	✓	✓			✓	✓	
m = y;		✓					
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	
m = y;		✓					
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					0
else if(x < z)	✓				✓	✓	
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					0
else if(x < z)	✓				✓	✓	0.71
m = y;	✓					✓	
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					0
else if(x < z)	✓				✓	✓	0.71
m = y;	✓					✓	0.83
} else {			✓	✓			
if(x > y)			✓	✓			
m = y;			✓				
else if(x > z)				✓			
m = x;							
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					0
else if(x < z)	✓				✓	✓	0.71
m = y;	✓					✓	0.83
} else {			✓	✓			0
if(x > y)			✓	✓			0
m = y;			✓				0
else if(x > z)				✓			0
m = x;							0
return m;	✓	✓	✓	✓	✓	✓	
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$



# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
if(x < y)	✓	✓			✓	✓	0.63
m = y;		✓					0
else if(x < z)	✓				✓	✓	0.71
m = y;	✓					✓	0.83
} else {			✓	✓			0
if(x > y)			✓	✓			0
m = y;			✓				0
else if(x > z)				✓			0
m = x;							0
return m;	✓	✓	✓	✓	✓	✓	0.5
	Pass	Pass	Pass	Pass	Pass	Fail	

$$\text{Suspicious}(s) = \frac{\text{fail}(s) / \text{totalfail}}{\text{fail}(s) / \text{totalfail} + \text{pass}(s) / \text{total pass}}$$

# Fault Localisation

Statement	3,3,5	1,2,3	3,2,1	5,5,5	5,3,4	2,1,3	Susp
m = y;	✓					✓	0.83
else if(x < z)	✓				✓	✓	0.71
if(x < y)	✓	✓			✓	✓	0.63
int m;	✓	✓	✓	✓	✓	✓	0.5
m = z;	✓	✓	✓	✓	✓	✓	0.5
if(y < z) {	✓	✓	✓	✓	✓	✓	0.5
return m;	✓	✓	✓	✓	✓	✓	0.5
m = y;		✓					0
} else {			✓	✓			0
if(x > y)			✓	✓			0
m = y;			✓				0
else if(x > z)				✓			0
m = x;							0
	Pass	Pass	Pass	Pass	Pass	Fail	

# Fault Localisation

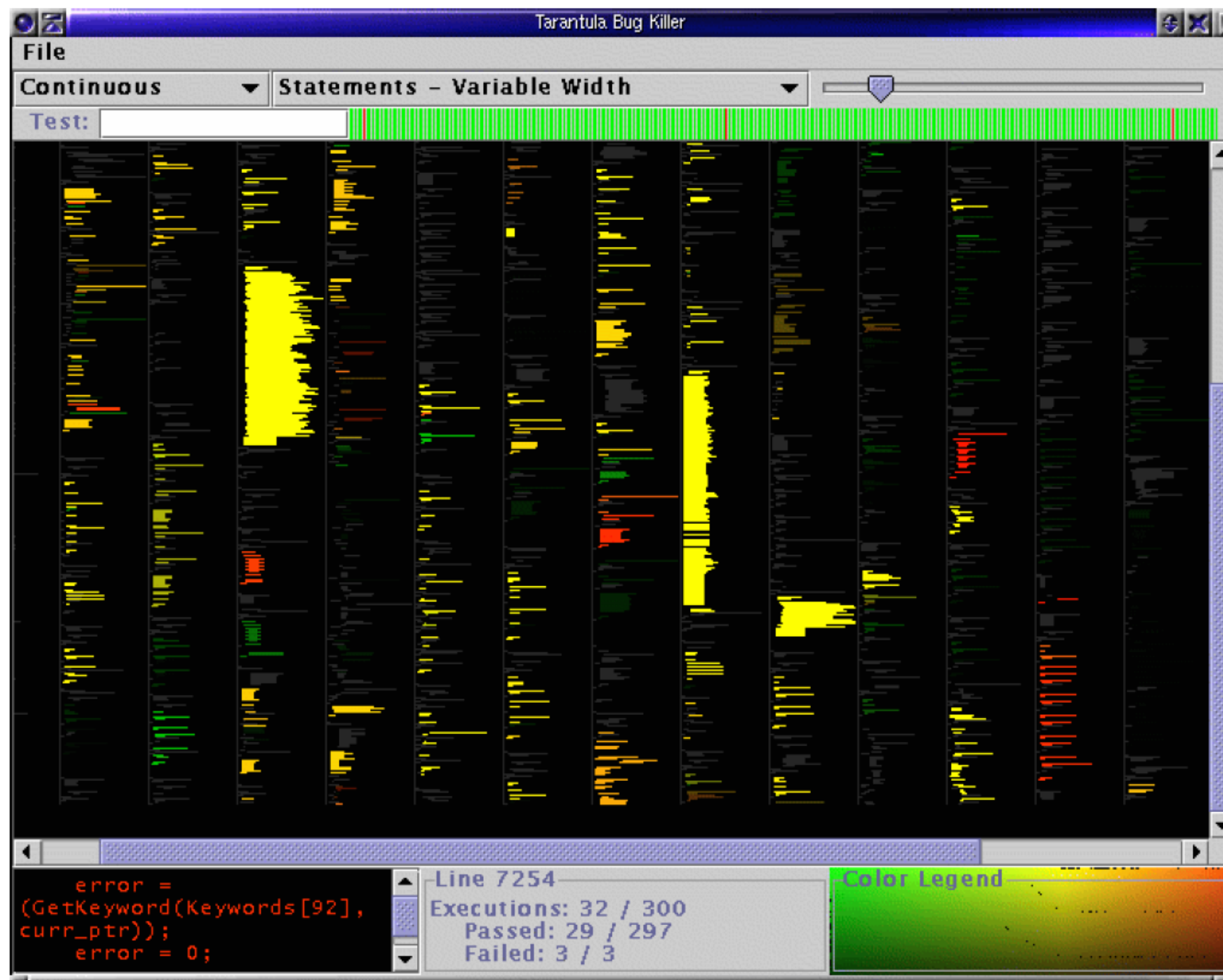
```
int mid(int x, int y, int z) {  
    int m;  
    m = z;  
    if (y < z) {  
        if (x < y)  
            m = y;  
        else if (x < z)  
            m = y;  
    } else {  
        if (x > y)  
            m = y;  
        else if (x > z)  
            m = x;  
    }  
    return m;  
}
```

# Fault Localisation

```
int mid(int x, int y, int z) {  
    int m;  
    m = z;  
    if (y < z) {  
        if (x < y)  
            m = y;  
        else if (x < z)  
            m = y; // should be m = x;  
    } else {  
        if (x > y)  
            m = y;  
        else if (x > z)  
            m = x;  
    }  
    return m;  
}
```

Hue = red + %pass/(%pass+%fail) \* range

# Tarantula



# GZoltar

Resource - commons-math/src/org/apache/commons/math3/complex/Complex.java - Eclipse Platform

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Project Explorer

commons-math

Complex.java

```
294     return createComplex(real / divisor,  
295                          imaginary / divisor);  
296 }  
297  
298 /** {@inheritDoc} */  
299 public Complex reciprocal() {  
300     if (isNaN)  
301         return NaN;  
302  
303     if (real == 0.0 && imaginary == 0.0)  
304         return NaN;  
305  
306     if (isInfinite)  
307         return ZERO;  
308  
309     if (FastMath.abs(real) < FastMath.abs(imaginary)) {  
310         double q = real / imaginary;  
311         double scale = 1. / (real * q + imaginary);  
312         return createComplex(scale * q, -scale);  
313     } else {  
314         double q = imaginary / real;  
315         double scale = 1. / (imaginary * q + real);  
316         return createComplex(scale, -scale * q);  
317     }  
318 }  
319  
320 /**  
321  * Test for the equality of two Complex objects.  
322  * If both the real and imaginary parts of two complex numbers  
323  * are exactly the same, and neither is {@code Double.NaN}, the two  
324  * Complex objects are considered to be equal.  
325  * All {@code NaN} values are considered to be equal - i.e. if either  
326  * (or both) real and imaginary parts of the complex number are equal  
327  * to {@code Double.NaN}, the complex number is equal to  
328  * {@code NaN}.  
329  */
```

Warnings

Diagnostic Report

org.apache.commons.math3.complex

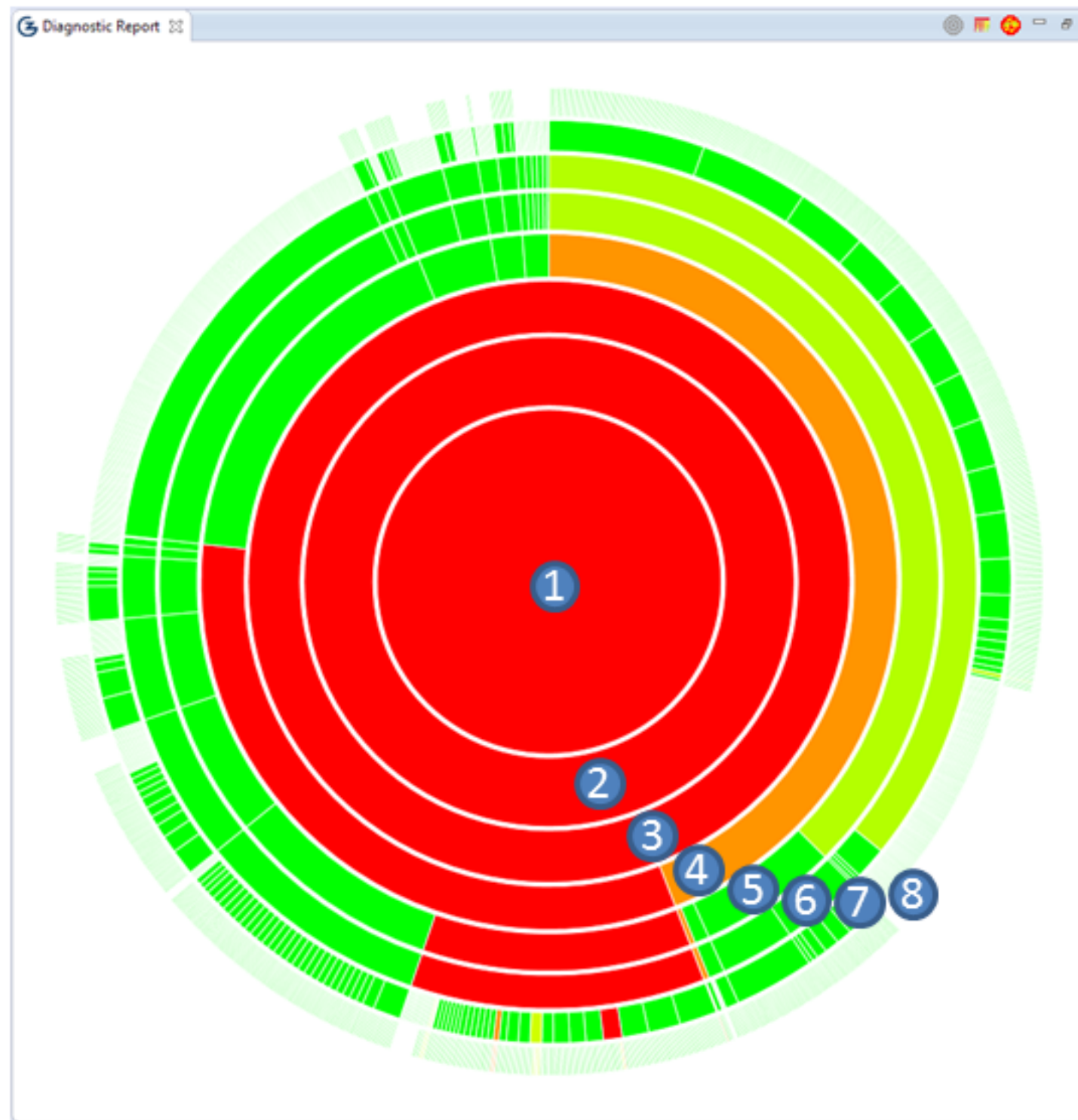
- Complex
- 1: Complex
- NaN: Complex
- INF: Complex
- ONE: Complex
- ZERO: Complex
- serialVersionUID: long
- imaginary: double
- real: double
- isNaN: boolean
- isInfinite: boolean
- Complex(double)
- Complex(double, double)
- abs(): double
- add(Complex): Complex
- add(double): Complex
- conjugate(): Complex
- divide(Complex): Complex
- divide(double): Complex
- reciprocal(): Complex
- equals(Object): boolean
- hashCode(): int
- getImaginary(): double
- getReal(): double
- isNaN(): boolean
- isInfinite(): boolean
- multiply(Complex): Complex
- multiply(int): Complex
- multiply(double): Complex
- neatOf(): Complex

Tasks Regression-Zoltar JUnit Console

Set	Set Cardinality	Runtime (ms)	Failure Trace
Set 1	12	337	
org.apache.commons.math3.RetryRunnerTest		1	
org.apache.commons.math3.complex.ComplexTest6		0	
org.apache.commons.math3.complex.ComplexTest2		0	
org.apache.commons.math3.complex.QuaternionTest		90	
org.apache.commons.math3.complex.ComplexUtilsTest		108	
org.apache.commons.math3.complex.ComplexFieldTest		2	
org.apache.commons.math3.complex.ComplexTest3		0	
org.apache.commons.math3.complex.RootsOfUnityTest		76	
org.apache.commons.math3.complex.ComplexTest		31	
org.apache.commons.math3.complex.ComplexTest_BUG		4	
org.apache.commons.math3.complex.FrenchComplexFormatTest		25	
org.apache.commons.math3.complex.ComplexTest5		0	
All Tests	14	346	

Failure Trace

/commons-math/src/  
org.apache.commons.math3.complex/Complex.java/  
Complex/reciprocal | Likelihood: 1.000



1  
workspace.project.packageroot.package.file.class.method.line  
2  
3  
4  
5  
6  
7  
8

# GP Implementation

- Functions: add, sub, mul, div, sqrt
- Terminals: ep, ef, np, nf, l
- Fitness function:

$$E(\tau, p, b) = \frac{\text{Ranking of } b \text{ according to } \tau}{\text{Number of statements in } p} * 100$$

$$\text{fitness}(\tau, B, P) = \frac{1}{n} \sum_{i=1}^n E(\tau, p_i, b_i) \text{ (to be minimised)}$$