Second Order Approximation of Supersonic Flow

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The purpose of calculating the second order approximation of inner region first to improve the accuracy of the pressure solution and second to find more information about the outer region. The second order approximation shows that the inner solution loses its validity at $O(\epsilon^{-1})$ which implied that outer solution must be present and consequently it would define the characteristic scale of the outer region. Knowing this characteristic scale is important in order to construct the asymptotic solution in the outer region.

- Question 1. Let $\omega = 1 - u$ and assuming $1 - u$ is small. We seek the solution u and v immediately behind the shock. We could re-write the Shock polar as

$$1 - u + \frac{v^2}{1-u} = \frac{1-\gamma}{1+\gamma} - \frac{2}{(\gamma+1)M_\infty^2}\frac{(1-u)^2 + v^2}{(1-u)^2} + 1,$$

and re-write it in terms of ω

$$\omega + \frac{v^2}{\omega} = \frac{2}{1+\gamma} - \frac{2}{(\gamma+1)M_\infty^2}\frac{\omega^2 + v^2}{\omega^2},$$

let v be expressed as

$$v = a\omega + b\omega^2 + \dots\ ,$$

substituting v into the Shock polar gives

$$\omega + a^2\omega + O(\omega^2) = \frac{2}{1+\gamma} - \frac{2}{(\gamma+1)M_\infty^2}\big(1 + a^2 + 2ab\omega\big) + O(\omega^2),$$

considering terms for $O(1)$ leads to $a = \sqrt{M_\infty^2 - 1} = \beta$. The $O(\omega)$ terms gives

$$b = -\frac{M_\infty^4}{4\beta}(\gamma+1),$$

- Question 2. Consider

$$u = 1 + \epsilon\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots\ , \quad v = \epsilon\left(-\beta\frac{\partial\tilde{\varphi}}{\partial\xi}\right) + \dots\ , \quad a^2 = \frac{1}{M_\infty^2} - \epsilon(\gamma-1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots\ .$$

thus

$$uv = -\epsilon\beta\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots,$$

$$a\sqrt{u^2+v^2-a^2} = \left[\frac{1}{M_\infty^2} - \epsilon(\gamma-1)\frac{\partial\tilde{\varphi}}{\partial\xi}\right]^{1/2}\left[1 + \frac{\epsilon}{2}\frac{\partial\tilde{\varphi}}{\partial\xi} - \frac{1}{M_\infty^2} - \epsilon(\gamma-1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots\right]^{1/2},$$
$$a\sqrt{u^2+v^2-a^2} = \left[\frac{\beta^2}{M_\infty^4} + O(\epsilon)\right]^{1/2} + \dots,$$

$$u^2 - a^2 = 1 + \frac{\epsilon}{2}\frac{\partial\tilde{\varphi}}{\partial\xi} - \frac{1}{M_\infty^2} + \epsilon(\gamma-1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots,$$
$$u^2 - a^2 = \frac{\beta^2}{M_\infty^2} + \epsilon(\gamma+1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots,$$

plug these into

$$\begin{aligned}
\frac{dy}{dx} &= \frac{uv + a\sqrt{u^2+v^2-a^2}}{u^2-a^2},\\
&= \frac{uv + a\sqrt{u^2+v^2-a^2}}{u^2-a^2},\\
&= \frac{\frac{\beta}{M_\infty^2} + \dots}{\frac{\beta^2}{M_\infty^2} + \epsilon(\gamma+1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots},\\
&= \frac{1}{\beta}\left[1 + \epsilon\frac{M_\infty^2}{\beta^2}(\gamma+1)\frac{\partial\tilde{\varphi}}{\partial\xi} + \dots\right]^{-1}.
\end{aligned}$$

Note that there is a discrepancy term $\frac{M_\infty^2}{2}$ in the equation above and solution equation (3.4.47). Now we need to find $\frac{\partial\tilde{\varphi}}{\partial\xi}$ where

$$\tilde{\varphi} = -\frac{Y_+(\xi)}{\beta} + \frac{\tilde{y}\chi\big[Y_+'(\xi)\big]^2}{\beta^2}, \quad \xi = s - 2\frac{\chi}{\beta}\tilde{y}Y_+'(\xi), \quad \chi = \frac{M_\infty^4}{4\beta}(\gamma+1).$$

thus

$$\begin{aligned}
\frac{\partial\tilde{\varphi}}{\partial\xi} &= \frac{\partial\tilde{\varphi}}{\partial s}\frac{\partial s}{\partial\xi},\\
\frac{\partial\tilde{\varphi}}{\partial\xi} &= \big(-\frac{1}{\beta}\big)Y_+'(\xi)\frac{\big[1 - 2\frac{\chi}{\beta}\tilde{y}Y_+''(\xi)\big]}{\big[1 - 2\frac{\chi}{\beta}\tilde{y}Y_+''(\xi)\big]},\\
\frac{\partial\tilde{\varphi}}{\partial\xi} &= -\frac{1}{\beta}Y_+'(\xi).
\end{aligned}$$

- Question 3. Let $\tilde{\varphi} = f(\zeta) + \dots$ and $\zeta = \frac{\xi-\xi_0}{\tilde{y}^{1/2}}$

$$\frac{\partial\tilde{\varphi}}{\partial\xi} = \frac{\partial\tilde{\varphi}}{\partial\zeta}\frac{\partial\zeta}{\partial\xi} = f'\tilde{y}^{-1/2}, \quad \frac{\partial\tilde{\varphi}}{\partial\tilde{y}} = -\frac{\xi-\xi_0}{2\tilde{y}^{3/2}}f'.$$

We can rewrite equation (3.4.47) as

$$\chi\big(f'\big)^2 - \frac{\zeta}{2}f' = 0,$$

solving for f'

$$f'\big(\chi f' - \frac{\zeta}{2}\big) = 0,$$

$f' = 0$ leads to a trivial solution so we consider the second solution

$$\chi f' - \frac{\zeta}{2} = 0, \quad f(\zeta) = \frac{\zeta^2}{4\chi} + c.$$

Comparing $f(\zeta)$ with equation (3.4.38)

$$\frac{\zeta^2}{4\chi} + c = -\frac{Y_+(x_M)}{\beta} + \frac{(\xi - x_M)^2}{4\chi\tilde{y}} + \dots\ ,$$

leads to $\xi_0 = x_M$ and the constant $c = -\frac{Y_+(\xi_0)}{\beta}$.

Note that entropy stays constant along each streamline but it increases when a streamline crosses a shock wave which is why shock waves could be considered as vorticity generators. For weak shock waves we could consider vorticity to be zero.

Any quantity that remains unperturbed must be set to its counterpart in the free stream flow. For example, the velocity upstream of the front shock, the flow is unperturbed and therefore $\hat{V}_1$ coincides with V_∞.

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.