Separated Flow Past Thin Aerofoil

Exercise 9 - Page 129

- Question 1. Consider

$$f(z) = i\frac{dY_+}{dx} + (C_r + iC_i)(z - x_s)^\alpha + \dots,$$

Let $z - x_s = re^{i\vartheta}$ which yields to

$$\begin{aligned} f(z) &= iY'_+ + r^\alpha\big[C_r\cos(\alpha\vartheta) - C_i\sin(\alpha\vartheta)\big] + ir^\alpha\big[C_r\sin(\alpha\vartheta) + C_i\cos(\alpha\vartheta)\big], \\ f(z) &= p_1 + iv_1. \end{aligned}$$

The pressure is zero at separation point $p_1\Big|_{\vartheta=0} = 0$

$$\Re\{f(z)\}\Big|_{\vartheta=0} = p_1\Big|_{\vartheta=0} = r^\alpha\big[C_r\cos(\alpha\vartheta) - C_i\sin(\alpha\vartheta)\big] = 0,$$

the term $C_r\cos(\alpha\vartheta) - C_i\sin(\alpha\vartheta)$ can be zero only if $C_r = 0$.

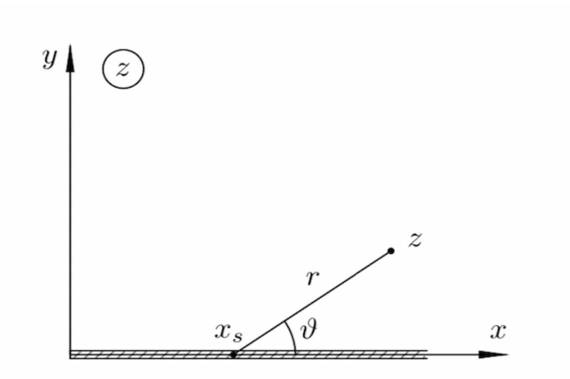


Figure 1: Complex plane near the separation point as illustrated in Ruban (2015)

Now turn our attention to condition $v_1 = Y'_+(x)$

$$\Im\{f(z)\} = v_1 = r^\alpha\big[C_r\cos(\alpha\vartheta) - C_i\sin(\alpha\vartheta)\big] + Y'_+ = Y'_+,$$

thus $C_i\cos(\alpha\pi) = 0$ since C_i cannot be zero we let $\cos^{-1}(0) = \alpha\pi = \frac{\pi}{2}$ which implies $\alpha = 1/2$. As a result we find that

$$p_1 = -r^{1/2}C_i\sin\frac{\vartheta}{2}.$$

- Question 2. Consider a flat aerofoil and set the coordinates (x, y) at the origin such that it coincides with the leading edge of the flat aerofoil

$$Q(x) = \frac{x}{\sqrt{(x+1)(x-\sqrt{x_s})}} \quad \text{at} \quad y = 0,\ x \in (\sqrt{x_s}, \infty)$$

Having $p_1 + iv_1 = \frac{i\alpha_*}{Q(z)}$ implies $v_1 = \frac{\alpha_*}{Q(x)} = Y_s'$.

$$\begin{aligned} Y_s' &= \frac{\alpha_*}{x}\sqrt{(x+1)(x-\sqrt{x_s})} \\ &= \frac{i\alpha_*}{x}\sqrt{x}\big(1+1/x\big)^{1/2}\sqrt{x}\big(1-\sqrt{x_s}/x\big)^{1/2}, \\ &= \alpha_*\big(1+\frac{1}{2x}+\dots\big)\big(1-\frac{\sqrt{x_s}}{2x}+\dots\big), \\ &= \alpha_*\left(1+\frac{1}{2x}(1-\sqrt{x_s})\right)+\dots. \end{aligned}$$

Integrating Y_s' w.r.t x gives us

$$Y_s = \alpha_* x + \alpha_*\big(1-\sqrt{x_s}\big)x + \dots \quad \text{as} \quad x \to \infty.$$

Similarly we find that

$$Y_T = \alpha_* x - \alpha_*\big(1-\sqrt{x_s}\big)x + \dots \quad \text{as} \quad x \to \infty.$$

- Question 3. Consider $\alpha = \epsilon\alpha$

$$y = \pm a x^{-1/2},$$

Focusing on Y_s where

$$a = \alpha_*\left[Lx + \sqrt{Lx}\big(1-\sqrt{x_s}\big)\right],$$

thus

$$\epsilon Y_s = a\sqrt{x},$$

$$a\sqrt{x} = \epsilon\alpha_*\left[Lx + \sqrt{Lx}\big(1-\sqrt{x_s}\big) + \dots\right],$$

let $\epsilon\alpha_* = \alpha$ which yields to

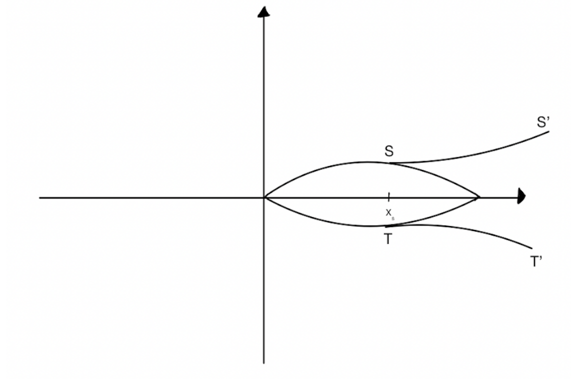
$$a = \alpha\big[\sqrt{L}\big(1-\sqrt{x_s}\big) + \dots\big].$$

Given that the drag force is defined as

$$D = \frac{1}{4}\rho V_\infty^2 a^2 \pi,$$

replaying a in the drag force, it becomes

$$D = \frac{1}{4}\rho V_\infty^2 L\pi\big[\alpha\big(1-\sqrt{x_s}\big)\big]^2,$$



- Question 4.Choose a branch cut of the function $Q(z) = \left(z - x_s\right)^{-1/2}$.

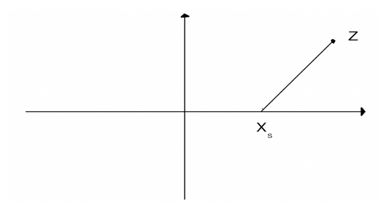
  To the right side of $x = x_s$ we have $\rho = x - x_s$ and $\vartheta = 0$

$$Q(x) = \left(\rho e^{i\vartheta}\big|_{\vartheta=0}\right)^{-1/2} = (x - x_s)^{-1/2}.$$

  To the left side of $x = x_s$ at $\vartheta = \pi$

$$Q(x) = i(-x + x_s)^{-1/2}.$$

  Now let's consider $\Phi(z) = F(z)Q(z)$. We choose the contour of integration as shown in the Figure 1. It is composed of an interval $[0, R]$ of the real-axis and a quarter-circle C_R whose radius R is large enough to ensure that point z lies inside the contour C. Also we introduce a small semi-circle C_r to deal with possible singularity at separation point $z = x_s$.



$$\Phi(z) = \frac{1}{2\pi} \oint_C \frac{\Phi(\xi)}{\xi - z} d\xi. \tag{1}$$

  Given the far field condition $f(z) \to 0$ as $z \to \infty$ we find that

$$\Phi(z) \to \quad \text{as} \quad z \to .$$

  Consequently the integral alone C_R is calculated as

$$\lim_{R\to\infty} \int_{C_R} \frac{\Phi(\xi)}{\xi - z} d\xi = 0,$$

  when calculating the integral along $[0, R]$ of the real-axis we need to examine possible singularity of the function $\Phi(\xi)$ and at the separation point s

$$Q(z) = O(z) \quad \text{as} \quad z \to 0,$$

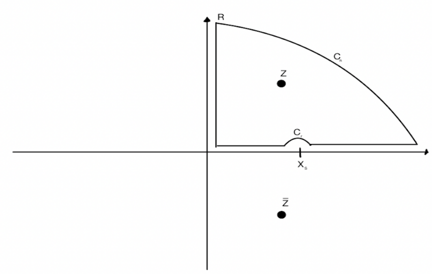
  thus $Q(z)$ remains finite which means the integration through this point does not require any especial treatment.

$$\Phi(z) = F(z)Q(z) = O(z - x_s)^{-1/2} \quad \text{as} \quad z \to x_s,$$

the integral c_r is estimated to be

$$\int_{c_r} \frac{\Phi(\xi)}{\xi - z} d\xi = O(\rho^{1/2}),$$

these integral calculation render (1) in the form



$$\Phi(z) = \frac{1}{2\pi i} \int_0^{\infty} \frac{\Phi(\xi)}{\xi - z} d\xi,$$

Now we need to consider the point $\bar{z}$ which is the complex conjugate of z. By Cauchy Theorem

$$\lim_{R\to\infty} \int_{C_R} \frac{\Phi(\xi)}{\xi - \bar{z}} d\xi = 0,$$

this is equivalent to

$$\Phi(z) = \frac{1}{2\pi i} \int_0^{\infty} \frac{\overline{\Phi(\xi)}}{\xi - z} d\xi = 0,$$

Now let's consider $\xi \in (-\infty, 0)$ and this interval we know $\Im\{F(z)\} = 0$.

$$\begin{aligned}
\Phi(\xi) + \overline{\Phi(\xi)} &= F(\xi)Q(\xi) + \overline{F(\xi)Q(\xi)}, \\
&= \frac{-i}{\sqrt{x_s - x}} \left[F(\xi) + \overline{F(\xi)}\right] \\
&= \frac{-i2}{\sqrt{x_s - x}} \Im\{F(\xi)\} = 0.
\end{aligned}$$

Now let's consider $\xi \in (0, x_s)$ and this interval we know $\Im\{F(z)\} = Y_+'(x)$.

$$\Phi(\xi) + \overline{\Phi(\xi)} = \frac{-i2}{\sqrt{x_s - x}} \Im\{F(\xi)\} = Y_+'(\xi).$$

Now let's consider $\xi \in (x_s, \infty)$ and this interval we know $\Re\{F(z)\} = 0$.

$$\Phi(\xi) + \overline{\Phi(\xi)} = 2(\sqrt{x_s - x})^{-1/2} \Re\{F(\xi)\} = 0.$$

Finally using the expression $F(z) = \frac{\Phi(z)}{Q(z)}$, we are able to find $F(z)$ as

$$F(z) = -\frac{\sqrt{z - x_s}}{\pi} \int_0^{x_s} \frac{Y_+'(\zeta)}{(x_s - \zeta)^{1/2}()\zeta - z} d\zeta.$$

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.