Fluid dynamics part II

Exercise 1, page 14.

Q1 (a) show $\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$

$f(x) = \tan x = \frac{\sin x}{\cos x}$

$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$, $f'' = 2\sec^2 x \cdot \tan x$

$f''' = 2\sec^4 x + 4\sec x^2 \tan^2 x.$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

$$f''' = 4\sec^2 x \tan^2 x + 2\sec^4 x = 2\sec^2 x\,(2 + \sec x)$$

$$f(x) = \tan(0) + \sec^2(0)\,x + 2\sec^2(0)\tan(0)\frac{x^2}{2!} + \left[4\sec^2(0)\tan^2(0) + 2\sec^4(0)\right]\frac{x^3}{6}$$

$$f(x) = 0 + x + 0 + [0 + 2]\frac{x^3}{6} + \cdots$$

$$f(x) = \tan x = x + \frac{x^3}{3} + \cdots$$

(b) $\lim_{x\to 0} \frac{\sin x - \tan x}{1 - \sqrt{1+x^3}}$ = inserting taylor Expansion:

$$\lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \cdots - x - \frac{x^3}{3} + \cdots}{1 - (1+x^3)^{1/2}} = \frac{-\frac{x^3}{2}}{1 - \left[1 + \frac{1}{2}x^3 + \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right)x^6 \cdots\right]}$$

$$\lim_{x\to 0} \frac{-x^3/2 + \cdots}{1 - 1 - \frac{x^3}{2} + \cdots} = \frac{-\frac{x^3}{2}}{-\frac{x^3}{2}} \approx 1$$

(ii) $$\lim_{x\to\infty} x^2\left[\left(8+\frac{1}{x^2}\right)^{1/3}-2\right] = \lim_{x\to\infty} x^2\left[8^{1/3}\left(1+\frac{1}{8x^2}\right)^{1/3}-2\right]$$

$$= \lim_{x\to\infty} 2x^2\left[\left(1+(8x^2)^{-1}\right)^{1/3}-1\right] =$$

$$= \lim_{x\to\infty} 2x^2\left[1+\frac{1}{3}\frac{1}{8x^2}+\frac{1}{3}\frac{1}{2}\left(-\frac{2}{3}\right)\frac{1}{(8x^2)^2}+\cdots-1\right]$$

$$= \lim_{x\to\infty} 2x^2\left[\frac{1}{24x^2}-\frac{1}{72x^4}+\cdots\right]$$

$$= \lim_{x\to\infty} \frac{1}{12}-\frac{1}{36x^2}+\cdots = \frac{1}{12}$$

(iii) $$\lim_{x\to 0} \frac{1-\cos(x^2)}{x\sin(x^3)} = \lim_{x\to 0} \frac{1-\left[1-\frac{x^4}{2}+\frac{x^8}{4!}+\cdots\right]}{x\left[x^3-\frac{x^9}{3!}+\cdots\right]}$$

$$\lim_{x\to 0} \frac{1-1+\frac{x^4}{2}+\frac{x^8}{4!}}{x^4} = \frac{1}{2}+\frac{x^4}{4!}+\cdots$$

$$= 1/2.$$

(iv) $$\lim_{x\to 0} \frac{\ln(1+\sqrt{x})-\sqrt{x}}{x} = \lim_{x\to 0} \frac{\sqrt{x}-\frac{x}{2}+\frac{x^{3/2}}{3}+\cdots-\sqrt{x}}{x}$$

$$= \lim_{x\to 0} -\frac{1}{2}+\frac{x^{1/2}}{3}+\cdots = -1/2.$$

Q2. $\lim_{x \to 0} \sin x$, $f(x) = \sin x = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2} + \frac{f'''(x_0)(x-x_0)^3}{3!} - \dots$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f'''' = \sin x$

$$\sin(x) = \underset{①}{0} + \underset{②}{x} + \underset{③}{0} - \underset{④}{\frac{x^3}{3!}} + 0$$

(1.22) $a_0 = \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)}$

$$f(x) = \cos x = \sum_{n=0}^{N} a_n \phi_n(x) \quad \text{where } \{\phi_n(x)\} = 1, x, x^2, \dots, x^n, \dots$$

$\phi_n(x) =$ asymptotic sequence

$$f(x) = a_0(1) + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad \text{as} \quad x \to 0^+$$

$$a_0 = \lim_{x \to 0} \frac{f(x)}{\phi_0(x)} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

$$a_1 = \lim_{x \to 0} \frac{\cos(x)}{x} = \frac{1}{0} = \infty$$

$$a_2 = \lim_{x \to 0} \frac{\cos(x)}{x^2} = \frac{1}{0} \quad \text{singularity at } x=0$$

Note that $\lim_{n \to \infty} \cos(2n\pi) = 1$

$\lim_{n \to \infty} \cos\left(\frac{\pi}{2} + n\pi\right) = 0$

Thus $\cos(\infty)$ doesn't exist

$\lim_{x \to \infty} \cos(x)$? ?.

Q3. (a) $$Ei(x) = \int_{-\infty}^{x} \frac{e^{\xi}}{\xi} d\xi \quad \text{as} \quad x \to \infty$$

$$u = \frac{1}{\xi} \qquad u' = -\frac{1}{\xi^2}$$

$$v = e^{\xi} \qquad v' = e^{\xi} d\xi$$

$$Ei(x) = \frac{e^{\xi}}{\xi}\Big|_{-\infty}^{x} + \int \frac{e^{\xi}}{\xi^2} d\xi = \frac{e^x}{x} + \int_{-\infty}^{x} \frac{e^{\xi}}{\xi^2} d\xi$$

$$u = 1/\xi^2 \qquad u' = \frac{-2}{\xi^3}$$

$$v = e^{\xi} \qquad v' = e^{\xi} d\xi$$

$$Ei(x) = \frac{e^x}{x} + \left[\frac{e^{\xi}}{\xi^2}\right]_{-\infty}^{x} + 2\int_{-\infty}^{x} \frac{e^{\xi}}{\xi^3} d\xi$$

$$Ei(x) = \frac{e^x}{x} + \frac{e^x}{x^2} + \dots = \frac{e^x}{x}\left(1 + \frac{1}{x} + \dots\right)$$

$$Ei(x) = \frac{e^x}{x}\left(1 + \frac{1}{x} + \dots + (n+1)! \int_{-\infty}^{x} \frac{e^{\xi}}{\xi^{n+2}} d\xi\right) \quad \text{as} \quad x \to -\infty$$

$$Ei(x) \sim \frac{e^x}{x}\left(1 + \frac{1}{x} + \dots + \frac{n!}{x^n} + \dots\right) \quad \text{as} \quad x \to -\infty$$

(b) $I(x) = \int_0^x e^{-t^2} dt$ as $x \to +\infty$

$u = e^{-t^2}$ $\qquad u' = (-2t)\, e^{-t^2}$

$v = t$ $\qquad v' = dt$

$$I(x) = t e^{-t^2} \Big|_0^x + 2 \int_0^x t^2 e^{-t^2} dt$$

$$= x e^{-x^2} + 2 \int_0^x t^2 e^{-t^2} dt$$

$u = e^{-t^2}$ $\qquad u' = (-2t)\, e^{-t^2}$

$v = \frac{t^3}{3}$ $\qquad v' = t^2 dt$

$$I(x) = x e^{-x^2} + 2 \left[e^{-t^2} \cdot \frac{t^3}{3} \Big|_0^x + \frac{2}{3} \int_0^x t^4 e^{-t^2} dt \right]$$

$$I(x) = x e^{-x^2} + \frac{2}{3} x^3 e^{-x^2} + \dots$$

$$= x e^{-x^2} \left(1 + \frac{2}{3} x^2 + \dots \right).$$

$t^2 = w$

$2t\, dt = dw$

$dt = \frac{dw}{2t} = \frac{dw}{2\sqrt{w}}$

$$I(x) = \frac{1}{2} \int_0^x w^{-1/2} e^{-w} dw$$

$$= \frac{1}{2} \Gamma(1/2) \sim \frac{\sqrt{\pi}}{2}$$

Q3.

$$Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt \qquad x \to \infty$$

$$u = 1/t \qquad u' = -1/t^2$$

$$v = \sin t \qquad v' = \cos t \, dt$$

$$Ci(x) = \frac{\sin t}{t}\Big|_x^\infty + \int_x^\infty \frac{\sin t}{t^2}\,dt \qquad u = 1/t^2 \qquad u' = \frac{-2}{t^3}$$

$$v = -\cos t \qquad v' = \sin t\, dt$$

$$Ci(x) = \frac{\sin t}{t}\Big|_x^\infty + \left(-\frac{\cos t}{t^2}\right)\Big|_x^\infty - 2\int_x^\infty \frac{\cos t}{t^3}\,dt$$

$$u = t^{-3} \qquad u' = \frac{-3}{t^4}$$

$$Ci(x) = \frac{\sin x}{x} - \frac{\cos x}{x^2} - 2\left[\frac{\sin t}{t^3}\Big|_x^\infty + 3\int_x^\infty \frac{\sin t}{t^4}\,dt\right]$$

$$v = \sin t \qquad v' = \cos t\, dt$$

$$= \frac{\sin x}{x} - \frac{\cos x}{x^2} - 2\frac{\sin x}{x^3} - 6\left[-\frac{\cos t}{t^4} - 4\int_x^\infty \frac{\cos t}{t^5}\,dt\right]$$

$$= \frac{\sin x}{x} - \frac{\cos x}{x^2} - 2\frac{\sin x}{x^3} + 6\frac{\cos x}{x^4} + 24\left[\frac{\sin t}{t^5} + 5\int_\infty^x \frac{\sin t}{t^6}\,dt\right]$$

$$= \sin x\left(\frac{1}{x} - \frac{2}{x^3} + \frac{2t}{x^5} + \cdots\right) + \cos x\left(-\frac{1}{x^2} + \frac{6}{x^4} - \frac{30}{x^6} + \cdots\right)$$

$$= \sin x\left(-\frac{1}{x} + \frac{2!}{x^3} + \cdots\right) + \cos x\left(\frac{1}{x^2} - \frac{3!}{x^4} + \cdots\right)$$

Q5

$$Ai(z) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{z^{1/4}} e^{-\frac{2}{3}z^{3/2}} + O\left(1/z^{3/2}\right)$$

$z = x+iy$ for positive real-values $z \longrightarrow x$

$$I(x) = \frac{1}{2\sqrt{\pi}} \int_0^x \frac{\cos(-2/3\,\xi^{3/2})}{\xi^{1/4}} d\xi$$

$\frac{1}{4} + \frac{1}{2} = \frac{1+2}{4} = 3/4$

$\xi = \left(-\frac{3}{2}w\right)^{2/3}$

$$w = -\frac{2}{3}\xi^{3/2} \Rightarrow dw = -\xi^{1/2} \cdot d\xi \Rightarrow d\xi = -\frac{dw}{\xi^{1/2}}$$

$$I(x) = \frac{1}{2\sqrt{\pi}} \int_0^x \frac{\cos(w)}{\xi^{1/4}} \left(-\frac{1}{\xi^{1/2}}\right) \cdot dw = \frac{-1}{2\sqrt{\pi}} \int_0^x \frac{\cos w}{\xi^{3/4}} \cdot dw$$

$$I(x) = \frac{-1}{2\sqrt{\pi}} \int_0^x \frac{\cos w}{\left(-\frac{3}{2}\right)^{2/3 \cdot 3/4} \left(w^{2/3 \cdot 3/4}\right)} \cdot dw$$

$$= \underbrace{\frac{-1}{2\sqrt{\pi}} \frac{1}{\left(-\frac{3}{2}\right)^{1/2}}}_{C} \int_0^x \frac{\cos w}{w^{1/2}} dw$$

$u = w^{-1/2}$ $\quad u' = \frac{-1}{2w^{3/2}}$

$v = \sin w$ $\quad v' = \cos w \cdot dw$

$$= C\left[\frac{\sin w}{w^{1/2}} \Big|_0^x + \frac{1}{2} \int \frac{\sin w}{w^{3/2}} dw \right]$$

$u = w^{-3/2}$ $\quad u' = -\frac{3}{2} w^{-5/2}$

$v = -\cos w$ $\quad v' = \sin w\, dw$

$$= C\left[\frac{\sin x}{x^{1/2}} + \frac{1}{2}\left(-\frac{\cos w}{w^{3/2}} \Big|_0^x - \frac{3}{2} \int \frac{\cos w}{w^{5/2}} dw \right)\right.$$

$$= C\left[\frac{\sin x}{x^{1/2}} - \frac{\cos x}{2x^{3/2}} + \cdots \right]$$

$$\int_0^\infty$$

$$\xi^{3/4} = \left(-\frac{3}{2}w\right)^{2/3 \cdot 3/4}$$
$$= \left(-\frac{3}{2}w\right)^{1/2}$$

$$Ai(x) = \int_0^\infty \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\xi^{1/4}} \cos\left(-\frac{2}{3}\xi^{3/2}\right) d\xi = I(r)$$

$$w = -\frac{2}{3}\xi^{3/2} \Rightarrow dw = -\xi^{1/2} d\xi \Rightarrow d\xi = -\xi^{-1/2} \cdot dw$$

$$\xi = \left(-\frac{3}{2}w\right)^{2/3}$$

$$\frac{1}{4} + \frac{1}{2} = \frac{1+2}{4} = \frac{3}{4}$$

$$I(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{\cos(w)}{\xi^{1/4}} (-\xi)^{-1/2} dw$$

$$= \frac{-1}{2\sqrt{\pi}} \int_0^\infty \frac{\cos w}{\left(-\frac{3}{2}\right)^{1/2} w^{1/2}} \cdot dw = \frac{-1}{i\sqrt{6\pi}} \int_0^\infty \frac{\cos w}{w^{1/2}} \cdot dw$$

$$u = w^{-1/2} \qquad u' = -\frac{1}{2} w^{-3/2}$$

$$v = \sin w \qquad v' = \cos w \cdot dw$$

$$= \frac{i}{\sqrt{6\pi}} \left[\frac{\sin w}{w^{1/2}}\Big|_0^\infty + \frac{1}{2}\int_0^\infty \frac{\sin w}{w^{3/2}} \right]$$

$$u = w^{-3/2} \qquad u' = -\frac{3}{2} w^{-5/2}$$

$$v = -\cos w \qquad v' = \sin w$$

$$= \frac{i}{\sqrt{6\pi}} \left[\frac{\sin x}{x^{1/2}} + \frac{1}{2}\left[-\frac{\cos w}{w^{3/2}} - \frac{3}{2}\int_0^\infty \frac{\cos w}{w^{5/2}} \right) \right]$$

$$u = w^{-5/2} \qquad u' = -\frac{5}{2} w^{-7/2}$$

$$v = \sin w \qquad v' = \cos w$$

$$= \frac{i}{\sqrt{6\pi}} \left[\frac{\sin x}{x^{1/2}} - \frac{\cos x}{2x^{3/2}} - \frac{3}{4}\left(\frac{\sin w}{w^{5/2}} - \int_0^\infty \frac{\sin w}{w^{7/2}} \right) \right]$$

$$= \frac{i}{\sqrt{6\pi}} \left[\frac{\sin x}{x^{1/2}} - \frac{\cos x}{2x^{3/2}} - \frac{3}{4} \frac{\sin x}{x^{5/2}} + \cdots \right]$$

Q6.

$$F(x) = \frac{\int_0^x e^{y^2} \cdot dy}{e^{x^2}} = \frac{\infty}{\infty} \quad \text{as } x \to \infty$$

$$f(x) = \int_0^x e^{y^2} \cdot dy \Rightarrow f'(x) = e^{x^2}$$

$$g(x) = \frac{e^{x^2}}{x} \rightarrow g'(x) = \frac{2x^2 e^{x^2} - e^{x^2}}{x^2} = \frac{e^{x^2}}{x^2}(2x^2 - 1)$$

$$\Phi(x) = x F(x) = \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{e^{x^2}}{\frac{e^{x^2}(2x^2-1)}{x^2}}$$

$$\Phi(x) = x F(x) = \frac{x^2}{2x^2 - 1} \overset{\text{div'd by } x^2}{=} \frac{1}{2 - 1/x^2} \approx 1/2 \quad \text{as } x \to +\infty$$

$$\Phi(x) = \lim_{x \to +\infty} x F(x) \approx 1/2$$

$$\lim_{x \to +\infty} F(x) \approx 1/2x$$

$$F(x) = e^{-x^2} \int_0^x e^{y^2} dy \quad \text{as } x \to \infty$$

$$\approx \frac{1}{2x}$$

$$f(x) = \int_0^x e^{s^2}\,ds \qquad f'(x) = e^{x^2}$$

$$g(x) = \frac{e^{x^2}}{x^2} \qquad g'(x) = \frac{2x^3 e^{x^2} - 2x e^{x^2}}{x^4}$$

$$\phi(x) = x^2 F(x) = \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{x^4 e^{x^2}}{2x e^{x^2}(x^2-1)}$$

$$x^2 F(x) = \lim_{x\to\infty} \frac{x^3}{-2(1+x^2)} = \lim_{x\to\infty} \frac{x^3}{-2x^2(1+x^{-2})}$$

$$x^2 F(x) = \lim_{x\to\infty} \frac{x}{-2(1+x^{-2})}$$

$$F(x) = \lim_{x\to\infty} \frac{1}{-2x}(1+x^{-2})^{-1} = \frac{1}{-2x}\left(1 - x^{-2} + \cdots\right)$$

$$= \frac{-1}{2x} + \frac{1}{2x^3} + \cdots$$

Q7. $y' + 2xy = 1$ g.s: $y = e^{my}$, $y' = m \cdot e^{my}$

$y(0) = a$

$$y = e^{-x^2}\left(a + \int_0^x e^{s^2}\,ds\right)$$

Solution: $I(x) = e^{2\int x\,dx} = e^{x^2}$ → multiple by DE:

$$e^{x^2} y' + 2x e^{x^2} y = e^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2} y\right) = e^{x^2} \Rightarrow y e^{x^2} = \int e^{x^2}\,dx + C$$

$$y = e^{-x^2}\left[C + \int_0^x e^{s^2}\,ds\right] \Rightarrow \text{applying i.C:}$$

$$y(0) = a = e^0\left[C + \cancel{\int_0^0}\,\right] \Rightarrow \boxed{C = a}$$

$$\therefore\quad y = e^{-x^2}\left[a + \int_0^x e^{s^2}\cdot ds\right]$$

Using Q6 solution we find:

$$\lim_{x\to\infty} y \approx 0 + \frac{1}{2x} + \cdots$$

Final solution: $y' = 1 - 2xy$ as $x \Rightarrow \infty$

$$y' = 1 - 2x\left(\frac{1}{2x} + \frac{1}{2x^3} + \cdots\right) = 1 - 1 - \frac{1}{x^2} + \cdots$$

$$y' \sim x^{-2} + \cdots \quad \text{as} \quad x \Rightarrow \infty. \quad \square$$