Adiabatic invariance and the WKB methods

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In this note I focus on Adiabatic invariance and the WKB methods to solve differential equations. I have shown how to deal with secular terms in solutions while using asymptotic methods. In some cases dealing with secular terms is not as straight forward as we have seen so far. The examples below demonstrate when and how to apply Adiabatic invariance and the WKB methods. We apply these methods when simply setting the secular terms to zero is impossible because it would not satisfy the given initial condition hence we require to adapt a different strategy to solve such problems.

- Question 1. We seek the solution in the following form

$$y(\tau, \epsilon) = e^{\delta_0 s_0(\tau)+\delta_1 s_1(\tau)+\dots}, \tag{1}$$

the first and second derivatives of (1) are expressed as

$$y' = \left(\delta_0 s_0' + \delta_1 s_1' + + \dots\right) e^{\delta_0 s_0(\tau)+\delta_1 s_1(\tau)+\dots}, \tag{2}$$

$$y'' = \left(\delta_0^2 s_0'^2 + 2\delta_0\delta_1 s_0' s_1' + \delta_0 s_0'' + \dots\right) e^{\delta_0 s_0(\tau)+\delta_1 s_1(\tau)+\dots}, \tag{3}$$

where $\delta_0 = \frac{1}{\epsilon}$ and $\delta_1 = 1$. By substituting (1) and its derivates into the problem, we find the leading order terms are

$$s_0'^2 = \pm\mu(\tau)^2,$$

$$s_0^{(1)} = i\int_0^\tau \mu(\zeta)d\zeta, \quad s_0^{(2)} = -i\int_0^\tau \mu(\zeta)d\zeta,$$

and the next leading order term is

$$s_1 = -\frac{s_0''}{2s_0'}, \quad s_1^{(1)} = s_1^{(2)} = -\frac{1}{2}\ln\left[\mu(\tau)\right].$$

The general solution to y is expressed as

$$y = c_1 y^{(1)}(\tau) + c_2 y^{(2)}(\tau),$$

where

$$y^{(1)}(\tau) = e^{\frac{i}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta - \frac{1}{2}\ln\left[\mu(\tau)\right]}, \quad y^{(2)}(\tau) = e^{\frac{-i}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta - \frac{1}{2}\ln\left[\mu(\tau)\right]},$$

separating the imaginary and real part

$$y^{(1)}(\tau) = e^{\frac{i}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta}\ e^{\ln\left[\mu(\tau)\right]^{-\frac{1}{2}}}, \quad y^{(2)}(\tau) = e^{\frac{-i}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta}\ e^{\ln\left[\mu(\tau)\right]^{-\frac{1}{2}}},$$

using these expression the solution may be written as

$$y = \frac{1}{\sqrt{\mu(\tau)}}\left[A\sin\left(\frac{1}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta\right) + B\cos\left(\frac{1}{\epsilon}\int_0^\tau \mu(\zeta)d\zeta\right)\right].$$

- Question 2. (a) To show the exact general solution

$$y = c_1 x^{1/2+\sqrt{\lambda^2+1/4}} + c_2 x^{1/2-\sqrt{\lambda^2+1/4}},$$

is the solution to the given equation we just need to substitute the solution above into the equation to show that it satisfies the equation.

(b) Now to approximate the solution for the same equation we shall use WKB where the solution is sough in the form (1)-(3) where $\delta_0 = \lambda$, $\delta_1 = 1$ and the independent variable is denoted by x

$$\left(\lambda^2 s_0'^2 + 2\lambda s_0' s_1' + \lambda s_0''\right) = \lambda^2 x^{-2},$$

this leads to

$$s_0 = \pm \ln(x) + c_0,$$

and

$$s_1 = -\frac{s_0''}{2s_0'} = -\frac{1}{2}\ln(s_0') + c_1.$$

Focusing on $s_0 = \ln(x) + c_0$

$$y^{(1)} = e^{\lambda \ln(x) - \frac{1}{2}\ln(x^{-1}) + c} = \tilde{c}\ x^{\lambda+\frac{1}{2}},$$

the general solution may be expressed as

$$y = \tilde{c}_1 y^{(1)} + \tilde{c}_2 y^{(2)} = \tilde{c}_1 x^{\lambda+\frac{1}{2}} + \tilde{c}_2 x^{-\lambda+\frac{1}{2}}, \tag{4}$$

(c) To compare the exact and the approximated solutions we can substitute the approximated solution into the problem. Doing that shows that the approximated solution does not satisfy the equation, thus we may rewrite the approximated solution as

$$y^{(1)} = \tilde{c}\ x^{\frac{1}{2}+\sqrt{\lambda^2+\frac{1}{4}-\frac{1}{4}}} = \tilde{c}\ x^{\frac{1}{2}+\sqrt{\tilde{\lambda}^2+\frac{1}{4}}},$$

- Question 3. Again we seek in the solution in the form

$$y(x,\lambda) = e^{\lambda s_0(x) + s_1(x) + s_1(x) + \dots}, \tag{5}$$

By deriving the fist and second derivative of (5) and substituting them into the problem we find that

$$\lambda^2 s_0'^2 + \lambda\left(s_0'' + 2s_0's_1'\right) + \left(s_1'' + s_1'^2 + 2s_0's_2'\right) + \lambda^{-1}\left(s_2'' + 2s_1's_2'\right) = \lambda^2 x^2 + \frac{\lambda}{x},$$

the leading order term $O(\lambda^2)$

$$\begin{aligned} s_0'^2 &= x^2, & s_0 &= \pm\frac{x^2}{2} + c_0, \\ s_0'' + 2s_0's_1' &= \frac{1}{x}, & s_1 &= \pm\left(\frac{1}{2} + \ln(x)\right) + c_1, \\ s_1'' + s_1'^2 + 2s_0's_2' &= 0, & s_2^{(1)} &= \frac{3}{16x^2} - \frac{1}{4x^3} + \frac{1}{32x^4} + c_2, \end{aligned}$$

Consequently, we find that

$$y^{(1)}(x) = \tilde{c} e^{\lambda\frac{x^2}{2} + \left(\frac{1}{2x} - \ln(x)\right) + \lambda^{-1}\left(\frac{3}{16x^2} - \frac{1}{4x^3} + \frac{1}{32x^4}\right)},$$

similarly one can derive $y_1^{(2)}(x)$ and the general solution is defined as

$$y = \tilde{c}_1 y^{(1)} + \tilde{c}_2 y^{(2)}.$$

As examples above show, WKB theory deals with problems in which there's a global breakdown in the solution as $\epsilon \to 0^+$. With an exception of boundary layer theory WKB deals with linear problems in quantum theory, acoustic and slowly-modulated waves or oscillations like the so called Tollmien–Schlichting wave.

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.