The method of Strained coordinates and Renormalisation

Exercise 5 - Page 71

In this note I shall focus on the method of strained coordinates and renormalisation where the latter is the revised version of the former. The reason I need to consider these two methods is that the straight forward asymptotic method may not produce a uniform solutions. In general there are two examples where the match asymptotic method is invalid. First, the amplitude of oscillations does not vary with t, in an oscillatory process. In this case we may apply a simplified version of multiple scale where the solution is only dependent on the fast time scale only. Second, when there is a singularity in the solution and to deal with this situation we may apply the strained coordinate method.

- Question 1. We employ the strained coordinates method. We present the asymptotic solution of the problem in the form

$$y(t,\epsilon) = y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2), \tag{1}$$

where τ is the strained coordinate defined as

$$\tau = t\big(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots\big). \tag{2}$$

By substituting (1)-(2) into the problem we find the leading order terms are $O(1)$

$$\frac{d^2 y_0(\tau)}{d\tau^2} + y_0(\tau) = 0,$$
$$y_0(0) = 0, \quad \frac{dy_0(0)}{d\tau} = 1,$$

the general solution is

$$y_0(\tau) = A\cos\tau + B\sin\tau.$$

considering the initial conditions we find that $A = 0$ and $B = 1$.

$$y_0(\tau) = \sin\tau.$$

the second order term $O(\epsilon)$

$$\frac{d^2 y_1}{d\tau^2} + y_1 = y_0\left(\frac{dy_0}{d\tau}\right)^2 - 2\omega_1\frac{d^2 y_0}{d\tau^2} = \sin\tau\ \cos^2\tau + 2\omega_1\sin\tau,$$
$$= \left(\frac{1}{4} + 2\omega_1\right)\sin\tau + \frac{1}{4}\sin 3\tau,$$

$$y_1(0) = 0, \quad \frac{d^2 y_1(0)}{d\tau^2} = 0,$$

To avoid secular term we need to let $\frac{1}{4} + 2\omega_1 = 0$ which leads to $\omega_1 = -1/8$. The solution to y_1 is

$$y_1(t) = \frac{1}{32}\Big[5\sin\tau - 4\cos^2\tau\sin\tau - \cos 4\tau\sin\tau - 2\cos\tau\sin 2\tau + \cos\tau\sin 4\tau\Big].$$

The final solution is expressed as

$$y(t,\epsilon) = t + \frac{\epsilon}{32}\left[t - 4t\left(1 - \frac{t^2}{2}\right)^2 - t\Big(1 - 8t^2\Big) + 8t\big(1 - \frac{t^2}{2}\big)\right] + O(\epsilon^2).$$

- Question 2. By substituting (1)-(2) into the problem we find the leading order terms are $O(1)$

$$\frac{d^2y_0(\tau)}{d\tau^2} + y_0(\tau) = 0,$$
$$y_0(0) = 1, \quad \frac{dy_0(0)}{d\tau} = 0,$$

the general solution is

$$y_0(\tau) = A\cos\tau + B\sin\tau.$$

considering the initial conditions we find that $A = 1$ and $B = 0$.

$$y_0(\tau) = \cos\tau.$$

the second order term $O(\epsilon)$

$$\begin{aligned}\frac{d^2y_1}{d\tau^2} + y_1 &= y_0 - y_0^3 - 2\omega_1\frac{d^2y_0}{d\tau^2},\\ &= (1 + 2\omega_1)\cos\tau + \frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau,\\ &= (7/4 + 2\omega_1)\cos\tau + \frac{1}{4}\cos 3\tau.\end{aligned}$$

To avoid secular term we need to let $\frac{7}{4} + 2\omega_1 = 0$ which leads to $\omega_1 = -7/8$. The solution to y_1 is

$$y_1(t) = \frac{1}{32}\Big[3\cos\tau - 4\cos^3\tau + \cos\tau\cos 4\tau + 2\sin\tau\sin 2\tau + \sin\tau\sin 4\tau\Big].$$

The final solution is expressed as

$$y(t,\epsilon) = \big(1 - t^2/2\big) + \frac{\epsilon}{32}\Big[3 + \frac{59}{8}t^2 + 4\big(1 - t/2\big)^2 + 4\big(1 - t/2\big)^3\Big] + O(\epsilon^2).$$

- Question 3. First we use straight forward asymptotic expansion and this gives us

$$\frac{d^2y_0(t)}{dt^2} + y_0(t) = 0,$$
$$\frac{d^2y_1(t)}{dt^2} + y_1(t) = \cos t - \cos^3 t,$$

Using the initial conditions we find that

$$y_0(t) = \cos t,$$
$$y_1(t) = \frac{1}{32}\big(-\cos t + \cos 3t\big) + \frac{t}{8}\sin t,$$

$$y(t,\epsilon) = \cos t + \frac{\epsilon}{8}\Big[\frac{\cos 3t - \cos t}{4} + t\sin t\Big] + O(\epsilon^2).$$

The secular term is $t\sin t$. To deal with this term we shall use the renormalisation with

$$t = \tau + \epsilon f_1(\tau) + \dots \ . \tag{3}$$

We substitution (3) directly into the solution obtained by the straight forward asymptotic expansion which yields to

$$y = \cos\big[\tau + \epsilon f_1(\tau) + \dots\big] + \frac{\epsilon}{8}\Bigg[\frac{\cos\big[3(\tau + \epsilon f_1(\tau) + \dots)\big] - \cos\big[\tau + \epsilon f_1(\tau) + \dots\big]}{4} +$$
$$\big(\tau + \epsilon f_1(\tau) + \dots\big)\sin\big[\tau + \epsilon f_1(\tau) + \dots\big]\Bigg] + O(\epsilon^2).$$

Keeping $O(1)$ and $O(\epsilon)$ terms gives us

$$y(t,\epsilon) = \big(1 - \tau^2/2\big) + \epsilon\tau\big(\tau/4 - f_1\big) + O(\epsilon^2).$$

To avoid y_1 being a secular term as $\tau \to \infty$ we have to choose

$$f_1 = \tau/4.$$

We have the sought solution in the form

$$y(t,\epsilon) = \big(1 - \tau^2/2\big) + O(\epsilon^2),$$

where

$$t = \tau + \epsilon\frac{\tau}{4} + \dots\ .$$

- Question 4. (a) Substituting (1)-(2) into the problem gives

$$\frac{d^2 y_0(\tau)}{d\tau^2} + y_0(\tau) = 0.$$

Using the given initial conditions we find that the first order is

$$\begin{aligned} y(t,\epsilon) &= \cos\tau + O(\epsilon) = \cos\big[t(1 + \epsilon\omega_1 + \dots)\big], \\ &= \cos\big[t - 3/8\epsilon t + \dots\big] = 1 - \frac{(t - 3/8t\epsilon)^2}{2} + \dots, \\ &= 1 - \frac{t^2}{2} + O(\epsilon). \end{aligned}$$

The solution above is the first leading order term and to get the second order term we need to solve for y_1 which is done in part (b) with renormalisation method.

(b) We obtain the solution for the same problem using straight forward asymptotic expansions with the renormalisation method. The first and second order equations give us

$$y_0(t) = \cos t,$$

$$y_1(t) = \frac{1}{32}\Bigg[-8\cos t + 8\cos^5 t + 12t\sin t + 8\sin t\sin 2t + \sin t\sin 4t\Bigg].$$

now we use renormalisation directly where the variable $t = \tau + \epsilon f_1(\tau) + \dots$ is substituted into $y = y_0 + \epsilon y_1 + O(\epsilon^2)$ and this yields to

$$y = \cos(\tau + \epsilon f_1) + \frac{\epsilon}{32}\Bigg[-8\cos(\tau + \epsilon f_1) + 8\cos^5(\tau + \epsilon f_1)$$
$$+ 12(\tau + \epsilon f_1)\sin(\tau + \epsilon f_1) + 8\sin(\tau + \epsilon f_1)\sin 2(\tau + \epsilon f_1) + \sin(\tau + \epsilon f_1)\sin 4(\tau + \epsilon f_1)\Bigg],$$

only focusing on $O(1)$ and $O(\epsilon)$ while expanding cos and sin with respect to τ

$$y = \left(1 - \frac{\tau^2}{2}\right) + \frac{\epsilon}{8}\left(- 2 + 9\tau^2 + 2(1-\tau^2/2)^5 - 8\tau f_1\right),$$

To avoid the secular term, f_1 could be chosen to be

$$f_1 = \frac{9\tau^2 - 2}{8\tau}.$$

- Question 5. First we use straight forward asymptotic expansion and this gives us

$$t^2 \frac{d^2 y_0(t)}{dt^2} + y_0(t) = 0,$$
$$t^2 \frac{d^2 y_1(t)}{dt^2} + y_1(t) = -y_0 \frac{dy_0(t)}{dt},$$

Using the initial conditions we find that

$$y_0(t) = e^{1/t},$$
$$y_1(t) = -t^{-2}\left[e^{1/t}\big(e^{1/t} - 2te^{1/t} - t^2 e + 2t^2 e^{1/t}\big)\right],$$

There is a singularity at $t = 0$. The straight forward asymptotic solution $y(t) = y_0(t) + \epsilon y_1(t) + \dots$ shows that a singularity point develops at $t = 0$ and each subsequent term proves to be more singular than the previous one. Therefore we shall introduce a new variable such

$$t = \tau + \epsilon f_1(\tau) + \dots, \quad x^{-1} = \frac{1}{\tau} - \epsilon\frac{f_1}{\tau^2} + \dots, \quad x^{-2} = \frac{1}{\tau^2} - \epsilon 2\frac{f_1}{\tau^3} + \dots. \tag{4}$$

Now we substitute (4) into the y_0 and y_1 which leads y to be

$$y(t,\epsilon) = e^{1/\tau} - \epsilon e^{1/\tau}\frac{f_1}{\tau^2} - \epsilon\left(\frac{e^{2/\tau}}{\tau^2} - 2\frac{e^{2/\tau}}{\tau} - e^{1+1/\tau} + 2e^{2/\tau}\right) + O(\epsilon^2).$$

From the solution above we can let

$$f_1 = e^{1/\tau}\big(-1 + 2\tau - 2\tau^2\big),$$

and therefore we find that

$$y(t,\epsilon) = e^{1/\tau} + \epsilon e^{1+1/\tau} + O(\epsilon^2).$$

- Question 6. To show that

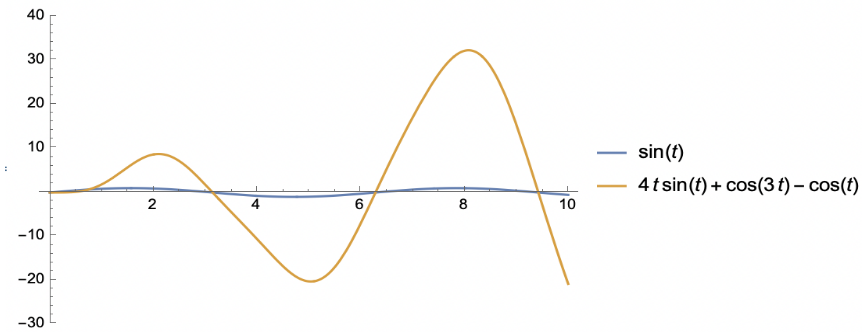
$$y = -\frac{x}{\epsilon} + \left(x^2/\epsilon^2 + 2/\epsilon + 1\right)^{1/2},$$

is the solution to the problem we can simply substitute into the equations and expect the left and right hand sides hold. This is done by differentiating y which is

$$\frac{dy}{dx} = -\frac{1}{\epsilon} + \frac{1}{2}\left(\frac{2x}{\epsilon^2}\right)\left(x^2/\epsilon^2 + 2/\epsilon + 1\right)^{-1/2},$$

The outer solution may be presented as $y = 1/x + \epsilon\left(1/2x - 1/2x^3\right)$. By substitution of the scaled strained coordinate $s = \epsilon^{1/2}S$ and $\frac{1}{S} = -X + \sqrt{2 + X^2}$ we can show that inner solution where $x = \epsilon^{1/2}X$ where $X = O(1)$ as $\epsilon \to 0$ confirms that leading order terms of the inner solution is

$$y = -X + \big(2 + X^2\big)^{1/2}.$$



Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.