Large Aspect Ratio Wing

Exercise 11 - Page 163

Up to now we only considered two-dimensional flows past aerofoils. However in reality one has to deal with a three-dimensional flows. If a wing has infinite span meaning if the span of a wing is large compared to an average chord of the wing section the the two-dimensional aerofoil theory us a valid leading-order approximation to the flow description.

The Cartesian coordinate system $(\hat{x}, \hat{y}, \hat{z})$ where $\hat{x}$-axis is parallel to the free-stream velocity vector and the $\hat{z}$ is measured along the wing span. For convenient we place the $(\hat{x}, \hat{y})$-plane in the plane of symmetry of the flow.

Aspect ratio of a wing is defined as the wing span l divided by the average chord of the wing section L which is $\lambda = \frac{l}{L}$. Aircraft wings are designed in such a way that the lift force

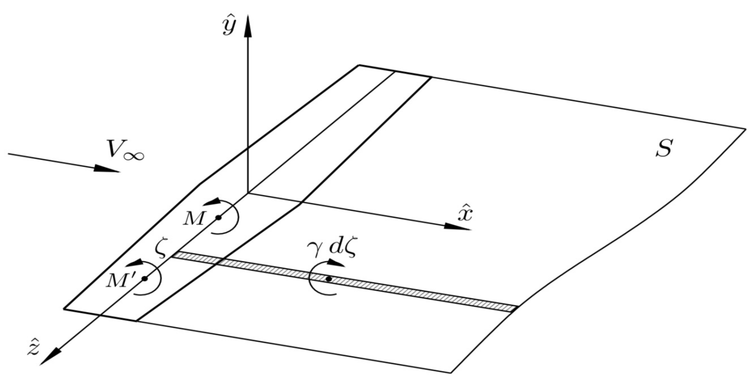

Figure 1: Vortex sheet behind a wing, Ruban (2015).

changes in the spanwise direction and completely disappear at the wings tip. There is a pressure difference on the upper and lower sides of the wing but as the fluid approaches the trailing edge, the pressure difference disappears. Then according to Bernoulli equation the velocities V_+, V_+ immediately above the trailing edge should become equal however due to inertial the velocity vectors remain at an angle with one another. This discontinuity in the velocity vector persists further downstream and its modelled by a vortex sheet. In this note we study the influence of velocity vector discontinuity, namely, the vortex sheet on the flow field.

- Question 1.

$$
\begin{aligned}
\frac{d\hat{y}}{d\hat{x}} = \hat{v} = &\int_{-l}^{l} \frac{d\gamma/d\zeta}{\zeta - \hat{z}} d\zeta, \\
&= \frac{d\gamma/d\zeta}{2\pi} \int_{0}^{l} \frac{1}{\zeta - \hat{z}} d\zeta, \\
&= \frac{d\gamma/d\zeta}{2\pi} \ln\left(\zeta - \hat{z}\right)\Big|_0^l,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\frac{d\hat{y}}{d\hat{x}} &= \frac{d\gamma/d\zeta}{2\pi}\left(\ln\frac{l}{z} - \ln\frac{1}{z}\right), \\
\frac{\hat{y}}{l} &\sim \frac{\ln\lambda}{\lambda},
\end{aligned}
$$

where $\lambda = l/L$

- Question 2. The velocity induced by the vortex line is calculated as

$$V = \int_0^\infty \frac{[\tilde{\gamma} \times r]}{4\pi |r|^3} d\hat{x},$$

by definition $\tilde{\gamma} \times r = |\tilde{\gamma}||r| \sin\theta$

$$V = \frac{|\tilde{\gamma}|}{4\pi} \int_0^\infty \frac{1}{|r|^2} d\hat{x},$$

we express $|r|^2 = r^2 = \hat{x}^2 + \hat{z}^2$

$$\begin{aligned} V &= \frac{|\tilde{\gamma}|}{4\pi} \int_0^x \frac{d\hat{x}}{\hat{x}^2 + \hat{z}^2}, \\ &= \frac{|\tilde{\gamma}|}{4\pi} \left[\frac{1}{\hat{z}} \tan^{-1} \left(\frac{\hat{x}}{\hat{z}} \right) \right], \\ &= \frac{|\tilde{\gamma}|}{4\pi \hat{z}}. \end{aligned}$$

$\hat{x} = \frac{\pi}{4} \hat{z}$.

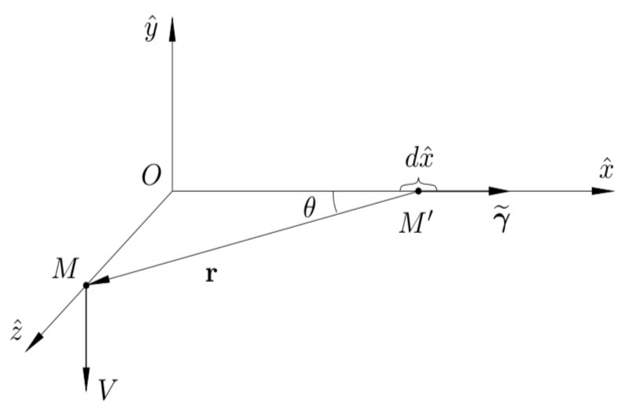

Figure 2: Graphical illustration of the Biot-Savart formula, Ruban (2015).

- Question 3. The lift and induced drag forces for a wing with the following distribution of the circulation along the wing span are calculated as

$$\Gamma(\hat{z}) = \Gamma_0 \frac{l + \hat{z}}{\epsilon},$$

Lift force is calculated

$$\hat{L} = -\rho V_\infty \int_{-l}^{-l+\epsilon} \Gamma_0 \frac{l + \hat{z}}{\epsilon} d\hat{z} = -\frac{\rho V_\infty \Gamma_0 \epsilon}{2},$$

Drag force is calculated as

$$\hat{D} = -\frac{\rho \Gamma_0}{\epsilon} \int_{-l}^{-l+\epsilon} \big(l + \hat{z}\big) v(\hat{z}) d\hat{z}.$$

Questions

The most iconic British fighter plane called The Spitfire has an elliptic shape which is represented by

$$L(\hat{z}) = L(0)\sqrt{1 - \frac{\hat{z}^2}{l^2}}\ ,$$

which is derived by the theory of large aspect ratio wing originally developed in Germany.

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.