Unsteady Flow Past Thin Aerofoil

Exercise 10 - Page 151

- Question 1.

$$\bar{V}(z) = \bar{V}(x+iy) = -\frac{\gamma_0}{2\pi i}\int_{-\infty}^{\infty}\frac{d\zeta}{z-\zeta}d\zeta,$$
$$= -\frac{\gamma_0}{2\pi i}\int_{-\infty}^{\infty}\frac{z-\zeta}{(z-\zeta)^2}d\zeta.$$

The real part of $\bar{V}(z)$ may be expressed as

$$u = -\frac{\gamma_0}{2\pi i}\int_{-\infty}^{\infty}\frac{y}{(x-\zeta)^2+y^2}d\zeta,$$
$$u = -\frac{\gamma_0}{2\pi i}\left[\frac{1}{y}\tan^{-1}\left(\frac{x-\zeta}{y}\right)\right]_{-\infty}^{\infty},$$
$$u = \frac{\gamma_0}{2},$$

for $y > 0$. Similarly we find $u = -\frac{\gamma_0}{2}$ for $y < 0$.

- Question 2. We consider the contour of intergration encloses the aerofiol and the starting vortex. According to Calvin Circulation theorem circulation would not change with time. The flow is steady and therefore we let the derivatives w.r.t t to be zero.

$$\int_1^{1+t}\sqrt{\frac{\zeta}{\zeta-1}}\gamma(t,\zeta)d\zeta = \int_0^1\sqrt{\frac{\zeta}{\zeta-1}}\left[\frac{\partial}{\partial t}(Y_+ + Y_-) + \frac{\partial}{\partial\zeta}(Y_+ + Y_-)\right]d\zeta - \Gamma_1,$$

reduces to

$$\int_1^{1+t}\sqrt{\frac{\zeta}{\zeta-1}}\gamma(t,\zeta)d\zeta = \int_0^1\sqrt{\frac{\zeta}{\zeta-1}}\left[\frac{\partial}{\partial\zeta}(Y_+ + Y_-)\right]d\zeta - \Gamma_1,$$

We know if at an instant t the distribution of γ alone the wake is known the at the next instant $t+\delta t$ the distribution of γ simply shifts as a whole downstream through a distance that equals δt. We define the circulation at the second instant as

$$\Gamma_1' = \int_0^1\sqrt{\frac{\zeta}{\zeta-1}}\left[\frac{\partial}{\partial\zeta}(Y_+ + Y_-)\right]d\zeta.$$

Now we are able to calculate $\Gamma_1' - \Gamma_1$ as

$$\Gamma_1' - \Gamma_1 = \int_1^{1+t}\sqrt{\frac{\zeta}{\zeta-1}}\gamma(t,\zeta)d\zeta.$$

- Question 3.

$$\begin{aligned}
\frac{d\Gamma_1}{dt} &= \oint_c \frac{\partial}{\partial t}\big(u_1 dx + v_1 dy\big), \\
&= \oint_c \frac{\partial u_1}{\partial t} dx + \frac{\partial v_1}{\partial t} dy, \\
&= \oint_c -\frac{\partial p_1}{\partial x} dx - \frac{\partial u_1}{\partial x} dx - \frac{\partial p_1}{\partial y} dy - \frac{\partial v_1}{\partial x} dy, \\
&= \oint_c -\frac{\partial p_1}{\partial y} dy - \frac{\partial v_1}{\partial x} dy, \\
&= \oint_c \left(\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x}\right) dy = 0.
\end{aligned}$$

Given that the vorticity equation is zero. Consequently we find that $\frac{d\Gamma_1}{dt} = 0$ which means the circulation is steady and does not vary with time.

- Question 4. To find γ_0 we calculate the integral

$$\gamma_0 = \frac{1}{D(\omega)} \int_0^1 \sqrt{\frac{\zeta}{\zeta - 1}} \left[i\omega\big(Y_+ + Y_-\big) + \frac{d}{d\zeta}\big(Y_+ + Y_-\big)\right] d\zeta,$$

where $Y_\pm(t, x) = h_0$ where h_0 is constant.

$$\gamma_0 = \frac{i2\omega h_0}{D(\omega)} \int_0^1 \sqrt{\frac{\zeta}{\zeta - 1}} d\zeta,$$

This integral is solved by substitution $\eta^2 = 1 - \zeta$ and then $\eta = \sin\theta$

$$\begin{aligned}
\gamma_0 &= \frac{i4\omega h_0}{D(\omega)} \int_0^{\pi/2} \sqrt{\frac{1 + \cos 2\theta}{2}} d\theta, \\
&= \frac{i\pi\omega h_0}{D(\omega)}.
\end{aligned}$$

We need to calculate $D(\omega)$ as

$$\begin{aligned}
D(\omega) &= e^{i\omega} \int_0^\infty \left(\sqrt{\frac{\zeta}{\zeta - 1}} - 1\right) e^{i\omega\zeta} d\zeta - \frac{i}{\omega}, \\
&= e^{i\omega} \int_0^\infty \left(\sqrt{\frac{\zeta}{\zeta - 1}} - 1\right) e^{i\omega\zeta} d\zeta - \frac{i}{\omega}, \\
&= e^{i\omega} \left(\sqrt{\frac{\pi}{i\omega}} - \frac{1}{i\omega}\right) - \frac{i}{\omega}.
\end{aligned}$$

- Question 5. Majority of the aerofoils used in aircrafts have 10% thickness and for some fighter jets is 4%. When we talk about the least degeneration, the questions is what lower region we need to consider. We could let the flow over the aerofoil to be defined based on the geometry of the aerofoil shape which are $x \sim O(1)$ and $y \sim \epsilon Y$ and remember here we are considering an incompressible inviscid flow which means the Laplace equation must hold. However these scaling shows that the partial derivative with y is much larger than the one with respect to x which leads to only one dominant term and this contradicts the

Laplace equation (velocity potential ϕ). To avoid this degeneration, we let $x \sim y$ to make sure the Laplace equation holds in this region. The reason why we do not accept only one dominant term to be the governing equation is that, the solution be a linear and a trivial one and we would be able to satisfy the boundary condition in x-direction. Note that the Laplace equations hold any arbitrary shape in an incompressible flow.

Considering $x \sim y \sim 1$ and the continuity equation we find that $u \sim v$. Also to find the order of v we consider the impermeability

$$\begin{aligned} v &= \frac{\partial y_\pm}{\partial t} + u\frac{\partial y_\pm}{\partial x}, \\ v &\sim u, \end{aligned}$$

where $y_\pm = \epsilon Y_\pm(t^*, x)$ and $t^* = t$ St

$$v \sim \frac{\partial y}{\partial t} \quad v \sim \epsilon St, \quad u \sim \epsilon St.$$

We seek the solution of the flow in the form (2.4.73) asymptotic expansions. As a result of these asymptotic expansions we find an analytical complex function $f(z) = u_1 - iv_1$.

- Question 6. The auxiliary function $g(z) = \sqrt{z(z-1)}$ was introduced by Mstislav Keldysh. When we have a unknown function and if you multiply this unknown function with the mentioned auxiliary function then we are able to find the solution by applying the residue theorem. Our task is to calculate the following integral

$$f(z) = \frac{dh/dt^*}{i\pi\sqrt{z(z-1)}} \int_0^1 \frac{\sqrt{\zeta(1-\zeta)}}{\zeta - z} d\zeta,$$

applying Sokhotsky-Plemelji formula instead of the straight forward residue theorem as

$$\begin{aligned} F^+(\zeta_0) &= F(\zeta_0) + \frac{1}{2} f(\zeta_0), \\ &= \frac{dh/dt^*}{i\pi\sqrt{\zeta_0(\zeta_0-1)}} \frac{i}{2}\sqrt{\zeta_0(1-\zeta_0)} + \frac{dh/dt^*}{i\pi\sqrt{\zeta_0(\zeta_0-1)}} \int_0^1 \frac{\sqrt{\zeta(1-\zeta)}}{\zeta-\zeta_0} d\zeta, \end{aligned}$$

we find the solution for the integral as

$$\int_0^1 \frac{\sqrt{\zeta(1-\zeta)}}{\zeta-\zeta_0} d\zeta = -\pi\left(\zeta_0 - \frac{1}{2}\right),$$

hence

$$F^+(\zeta_0) = \frac{dh/dt^*}{i2\sqrt{\zeta_0(\zeta_0-1)}} \left[1 - 2\zeta_0 + \frac{i}{\pi}\sqrt{\zeta_0(1-\zeta_0)}\right].$$

When dealing with steady flow the starting vortex is moved to infinity. Starting vortex is important for unsteady flow because the aerofoil is changing its angle of attack, it shades vorticity all the time. When applying Calvin Circulation theorem, we should take a large enough contour such it encloses the circulation around the aircraft wings and the starting vortex that is left behind by the aircraft. The circulation around the wing that creates the lift force is equal to the starting vortex, they are the same amount in magnitude but negative circulation. This starting vortex is the reason why aircrafts would not take-off immediately one after another but have to wait around 3 minutes. Further, the starting vortex would not dissipate quickly however it moves away from the runway by wind.

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.