Asymptotic problems of fluid dynamics - Fluid Dynamics II

Exercise 4 - Page 63

We have focused on asymptotic expansions and the method of matched asymptotic up to now. In some case we may encounter secular terms as the independent variable tends to infinity and this causes the entire procedure by the method of matched asymptotic to become invalid. To deal with such situation we shall employ the method of multiple scales. We use this method to find solutions that present the physical process characterised by two distinct time scales. Exercises below present examples where the secular terms appear. In this work we shall show how one can deal with them in details.

- Question 1. First we try the method of matched asymptotic

  $$y(t,\epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2).$$

  by substituting (1) into the IVP and we find the leading order terms are $O(1)$

  $$\frac{d^2y_0(t)}{dt^2} + y_0(t) = 0,$$
  $$y_0(0) = 0, \quad \frac{dy_0(0)}{dt} = 1,$$

  the general solution is

  $$y_0(t) = A\cos t + B\sin t.$$

  considering the initial conditions we find that $A = 0$ and $B = 1$.

  $$y_0(t) = \sin t.$$

  the second term $O(\epsilon)$

  $$\frac{d^2y_1}{dt^2} + y_1 = \sin^2 t \ \cos t,$$
  $$y_1(0) = 0, \quad \frac{d^2y_1(0)}{dt^2} = 0,$$

  The solution to y_1 is

  $$y_1(t) = \frac{1}{32}\bigg((-1+32c_1)\cos t + \cos 3t + 4(t+8c_2)\sin t\bigg).$$

  Using the initial conditions we find that $c_1 = c_2 = 0$.

  $$y_1(t) = \frac{1}{32}\bigg(-\cos t + \cos 3t + 4t\sin t\bigg).$$

  Using the method of matched asymptotic, we find the final solution to be

  $$y(t,\epsilon) = \sin t + \frac{\epsilon}{32}\bigg(-\cos t + \cos 3t + 4t\sin t\bigg) + O(\epsilon^2).$$

  According to the asymptotic expansion the second term must be small as compared to the first term. this requirement is ensured by smallness of ϵ. However, as $t \to \infty$, the secular term $t \sin t$ becomes very large, violating our initial assumption for asymptotic expansion which is a proper ordering of the terms in the final asymptotic solution. Thus using the method of matched asymptotic expansion becomes invalid at this instance. Let's now employ the method of multiple scales. Using this method we seek the solution in the form

$$y_n(t, \epsilon) = Y_n(t^*, \tilde{t}),$$

where y_n is the coefficients of the asymptotic expansions, t^* is referred to as the fast variable and $\tilde{t}$ is the slow variables, for the detailed definition please see Ruban (2015). Let's seek the solution in the form

$$y(t, \epsilon) = Y_0(t^*, \tilde{t}) + \epsilon Y_1(t^*, \tilde{t}) + O(\epsilon^2), \tag{1}$$

where

$$t^* = t(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots), \quad \tilde{t} = \epsilon t. \tag{2}$$

in this example we shall let the fast variable be written $t^* = t$. Differentiating (1) gives

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial t^*} + \epsilon\left(\frac{\partial Y_1}{\partial t^*} + \frac{\partial Y_0}{\partial \tilde{t}}\right) + O(\epsilon^2), \tag{3}$$

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^{*2}} + \epsilon\left(\frac{\partial^2 Y_1}{\partial t^{*2}} + 2\frac{\partial^2 Y_0}{\partial \tilde{t}\partial t^*}\right) + O(\epsilon^2). \tag{4}$$

The substitution of the derivatives above into the problem yields to

$$O(1): \quad \frac{d^2Y_0(t^*)}{dt^{*2}} + Y_0(t^*) = 0,$$

the general solution is

$$Y_0(t^*, \tilde{t}) = A_0(\tilde{t})\cos t^* + B_0(\tilde{t})\sin t^*.$$

Considering the initial conditions we find that $A_0(0) = 0$ and $B_0(0) = 1$. The second order term $O(\epsilon)$ is

$$\frac{d^2Y_1(t^*)}{dt^{*2}} + Y_1(t^*) =$$
$$\left(2A_0'(\tilde{t}) - \frac{A_0(\tilde{t})B_0^2(\tilde{t})}{4} - \frac{A_0^3(\tilde{t})}{4}\right)\sin t^* + \left(-2B_0'(\tilde{t}) + \frac{A_0^2(\tilde{t})B_0(\tilde{t})}{4} + \frac{B_0^3(\tilde{t})}{4}\right)\cos t^*,$$

where

$$\sin^3 t^* = \frac{3}{4}\sin t^* - \frac{1}{4}\sin 3t^*, \quad \cos^3 t^* = \frac{3}{4}\cos t^* + \frac{1}{4}\cos 3t^*. \tag{5}$$

Note that we ignored terms that include $\cos 3t^*$ and $\sin 3t^*$ because only the terms $\sin t^*$ and $\cos t^*$ produce secular terms as shown in the straight forward asymptotic expansion. Remember that these secular terms cause issues which is the solution is not uniform. Consequently to get rid of these two secular terms we let

$$2A_0'(\tilde{t}) - \frac{A_0(\tilde{t})B_0^2(\tilde{t})}{4} - \frac{A_0^3(\tilde{t})}{4} = 0,$$
$$-2B_0'(\tilde{t}) + \frac{A_0^2(\tilde{t})B_0(\tilde{t})}{4} + \frac{B_0^3(\tilde{t})}{4} = 0,$$

where $A_0(0) = 0$ and $B_0(0) = 1$. The general solutions of $A_0(\tilde{t})$ and $B_0(\tilde{t})$ are

$$A_0'(\tilde{t}) = \frac{2c_1}{\sqrt{-\tilde{t} - \tilde{t}c_1^2 - 8c_2}}, \quad B_0'(\tilde{t}) = \frac{2c_1}{\sqrt{-\tilde{t} - \tilde{t}c_1^2 - 8c_2}},$$

using the initial conditions we find that $c_1 = 0$ and $c_2 = -1/2$. Therefore

$$A_0'(\tilde{t}) = 0, \quad B_0'(\tilde{t}) = \frac{2}{\sqrt{4 - \tilde{t}}}.$$

By substituting the results above into (1)

$$y(t, \epsilon) = -2\frac{\sin\left[(t + \dots)\right]}{(4 - \epsilon t)^{1/2}} + O(\epsilon).$$

- Question 2. Substituting expressions (1)-(5) into the problem yields to

$$\frac{d^2 Y_0}{\partial t^{*2}} + Y_0 + \epsilon\left(\frac{d^2 Y_1}{dt^{*2}} + Y_1 + 2\frac{\partial^2 Y_0}{\partial t^* \partial \tilde{t}}\right) = \epsilon(1 - Y_0^2)\frac{\partial Y_0}{\partial t^*}.$$

By considering the first order terms, $O(1)$, and the initial conditions we find that

$$Y_0(t^*, \tilde{t}) = A_0(\tilde{t})e^{it^*} + \bar{A}_0(\tilde{t})e^{-it^*}$$

and $A_0(0) = \bar{A}_0(0) = a/2$. Note that $\bar{A}_0(\tilde{t})$ is the complex conjugate of $A_0(\tilde{t})$. Now we turn our attention to the second order terms $O(\epsilon)$

$$\begin{aligned}\frac{d^2 Y_1}{dt^{*2}} + Y_1 &= (1 - Y_0^2)\frac{\partial Y_0}{\partial t^*} - 2\frac{\partial^2 Y_0}{\partial t^* \partial \tilde{t}},\\ \frac{d^2 Y_1}{dt^{*2}} + Y_1 &= \left(i2A_0' - iA_0 + iA_0^2\bar{A}_0\right)e^{it^*} + \dots.\end{aligned}$$

now we let

$$2A_0' - A_0 + A_0^2\bar{A}_0 = 0,$$

by letting $A_0(\tilde{t}) = R(\tilde{t})e^{i\theta(\tilde{t})}$ the equation above with its initial condition become

$$\begin{aligned}&2(Re^{i\theta})' - Re^{i\theta} + R^3e^{i\theta} = 0, \qquad R(0) = a/2,\\ &\theta'(\tilde{t}) = 0, \qquad \theta(0) = 0\end{aligned}$$

where $|R(\tilde{t})e^{i(\tilde{t})}|^2 = R^2$. The solution to this system is

$$\begin{aligned}&2(Re^{i\theta})' - Re^{i\theta} + R^3e^{i\theta} = 0, \qquad R(0) = a/2,\\ &\theta'(\tilde{t}) = 0, \qquad \theta(0) = 0\end{aligned}$$

The solution to $R(\tilde{t})$ is

$$R(\tilde{t}) = \frac{e^{\tilde{t}/2}}{(e^{\tilde{t}} + e^{2c_1})^{1/2}} = \frac{e^{\tilde{t}/2}}{(\frac{4}{a^2} - 1 + e^{\tilde{t}})^{1/2}} \tag{6}$$

where $c_1 = \ln\left(\frac{4}{a^2} - 1\right)^{1/2}$. The solution above leads us

$$A_0(\tilde{t}) = \frac{e^{\tilde{t}/2}}{(\frac{4}{a^2} - 1 + e^{\tilde{t}})^{1/2}}. \tag{7}$$

At last, the final solution has the following form

$$\begin{aligned} y(t,\epsilon) &= \frac{e^{\tilde{t}/2}}{(\frac{4}{a^2}-1+e^{\tilde{t}})^{1/2}}e^{it^*}+c.c+O(\epsilon), \\ &= \frac{e^{\epsilon t/2}}{(\frac{4}{a^2}-1+e^{\epsilon t})^{1/2}}e^{it}+c.c+O(\epsilon). \end{aligned} \tag{8}$$

- Question 3. The secular terms are produced by the term e^{it^*} therefore we ignore any other terms such as e^{-it^*} and e^{i3t^*} .
  - (a) Substituting expressions (1)-(5) into the problem yields to

    $$\frac{d^2Y_0}{\partial t^{*2}}+Y_0+\epsilon\bigg(\frac{d^2Y_1}{dt^{*2}}+Y_1+2\frac{\partial^2 Y_0}{\partial t^*\partial\tilde{t}}+\frac{\partial Y_0}{\partial t^*}\bigg)=0.$$

    By considering the first order terms, $O(1)$, and the initial conditions we find that

    $$Y_0(t^*,\tilde{t})=A_0(\tilde{t})e^{it^*}+\bar{A}_0(\tilde{t})e^{-it^*},$$

    and $A_0(0)=\bar{A}_0(0)=1/2$. Note that $\bar{A}_0(\tilde{t})$ is the complex conjugate of $A_0(\tilde{t})$. Now we turn our attention to the second order terms $O(\epsilon)$

    $$\begin{aligned} \frac{d^2Y_1}{dt^{*2}}+Y_1 &= \frac{\partial Y_0}{\partial t^*}-2\frac{\partial^2 Y_0}{\partial t^*\partial\tilde{t}}, \\ \frac{d^2Y_1}{dt^{*2}}+Y_1 &= \big(-i2A_0'-iA_0\big)e^{it^*}+\big(2\bar{A}_0'+\bar{A}_0\big)e^{-it^*}. \end{aligned}$$

    To ensure the secular terms disappear we should let

    $$\begin{aligned} &i2A_0'+iA_0=0, \quad A_0(0)=1/2 \\ &A_0(\tilde{t})=\frac{1}{2}e^{-\tilde{t}/2}. \end{aligned}$$

    This leads to the final solution to be

    $$\begin{aligned} y(t,\epsilon) &= \frac{1}{2}e^{-\tilde{t}/2}e^{it^*}+c.c+O(\epsilon), \\ y(t,\epsilon) &= \frac{1}{2}e^{t(i-\epsilon)}+c.c+O(\epsilon). \end{aligned}$$

  - (b) Substituting expressions (1)-(5) into the problem yields to

    $$\frac{d^2Y_0}{\partial t^{*2}}+Y_0+\epsilon\bigg(\frac{d^2Y_1}{dt^{*2}}+Y_1+2\frac{\partial^2 Y_0}{\partial t^*\partial\tilde{t}}\bigg)=\epsilon\bigg(Y_0^2\frac{\partial Y_0}{\partial t^*}+Y_0\bigg).$$

    By considering the first order terms, $O(1)$, and the initial conditions we find that

    $$Y_0(t^*,\tilde{t})=A_0(\tilde{t})e^{it^*}+\bar{A}_0(\tilde{t})e^{-it^*},$$

    and $A_0(0)=-\bar{A}_0(0)=-i/2$. Now we turn our attention to the second order terms $O(\epsilon)$

    $$\frac{d^2Y_1}{dt^{*2}}+Y_1=\big(-i2A_0'+A_0+i2A_0^2\bar{A}_0-iA_0^2\bar{A}_0\big)e^{it^*}+\dots.$$

To ensure the secular terms disappear we should let

$$-i2A_0' + A_0 + i|A_0|^2 A_0, \quad A_0(0) = -i/2$$
$$A_0(\tilde{t}) = \frac{e^{c_1}}{(e^{\tilde{t}} - 3e^{2c_1})^{1/2}}, \quad c_1 = \ln i.$$

This leads to the final solution to be

$$y(t, \epsilon) = \frac{i}{(e^{\epsilon t} - 3)^{1/2}} e^{it} + c.c + O(\epsilon).$$

Questions

The questions are found in the book by Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.