Theory of Thin Aerofoil in Incompressible Flow

Exercise 7 (part b) - Page 109

The methods used in this note are useful in analysing two-dimensional inviscid incompressible fluid flow.

- Question 7. (a) The flow past a corner shown in Figure 2.12 is just a deflected version of problem 6. Since we are dealing with a steady incompressible inviscid flow the solution reminds the same. As before this problem could be solved by using the pressure definition

  $$p_1 = \frac{1}{\pi}\int_0^b \frac{Y_+'(\zeta)}{(\zeta - x)} d\zeta.$$

  however, we could determine pressure without using the definition above. Instead we seek the pressure in the form

  $$p_1 + iv_1 = \big(C_r + iC_i\big)\ln z + iD,$$

  which leads to

  $$\begin{aligned} p_1 &= C_r \ln r - C_i\theta, \\ v_1 &= C_r\theta + C_i \ln r + D, \end{aligned}$$

  where $z = re^{i\theta}$. The flow is inviscid thus we can use impermeability conditions. First, if we consider the upstream of the flow where $\theta = 0$ and second condition is found when we go downstream of the flow which takes $\theta = -\pi$. Note that v_1 is the vertical velocity and at $\theta = 0$ it simply is zero.

  $$v_1 = (C_r\theta + C_i \ln r + D)\Big|_{\theta=0} = 0,$$

  which leads to $C_i = D = 0$.

  $$\begin{aligned} \frac{v}{u} &= \frac{\epsilon v_1}{1+\epsilon u_1} = \epsilon\theta_0, \\ v_1 &= \theta_0, \end{aligned}$$

  Now considering the second condition

  $$v_1 = C_r\theta\Big|_{\theta=-\pi} = \theta_0,$$

  this gives $C_r = -\frac{\theta_0}{\pi}$. This leads to

  $$p_1 = -\frac{\theta_0}{\pi}\ln r.$$

  (b) Considering this analysis using conformal mapping method with no restriction on the deflection angle θ shows

  $$\bar{V}(z) = u - iv = Az^{\frac{\theta}{\pi-\theta}} + \dots \quad \text{as} \quad z \to 0.$$

where A-real constant. Our task is to show $\bar{V}(z)$ reduces to

$$p_1 = -\frac{\theta_0}{\pi}\ln z + \ldots,$$

for the case where $\theta \ln z \ll 1$. Remeber that $p = \epsilon p_1$

$$p = \frac{\epsilon\theta}{\pi}\ln z + \ldots\ .$$

We can show that

$$\ln\big(\bar{V}(z)\big) = \ln A - \frac{\theta \ln z}{\theta - \pi},$$

We separate the real and imaginary part and using the fact that $u_1 = -p_1$. We find that

$$\ln\big(\bar{V}(z)\big) = \ln\big|\big(\bar{V}(z)\big)\big| + i\phi,$$
$$\ln\big(\bar{V}(z)\big) = u - iv = 1 + \epsilon u_1 - iv_1,$$

the real part could be expressed as

$$\ln\big|\big(\bar{V}(z)\big)\big| = \sqrt{(1+\epsilon u_1)^2 + (\epsilon v_1)^2} \approx 1 + \epsilon u_1,$$

thus

$$1 + \epsilon u_1 = \ln A - \frac{\epsilon\theta_0}{\epsilon\theta_0 - \pi}\ln z,$$

we can choose $\ln A = 1$.

$$\epsilon u_1 = -\frac{\epsilon\theta_0}{\epsilon\theta_0 - \pi}\ln z \approx \frac{\epsilon\theta_0}{\pi}\ln z,$$

Assuming $\theta \ln z \ll 1$, the equation for velocity obtained using the conformal mapping method reduces to the solution obtained from the solution for pressure .

- Question 8. Let $z = re^{i\theta}$

$$p_1 + iv_1 = (A_r + iA_i) + \big(B_r + iB_i\big)r^{\lambda}e^{i\lambda\theta},$$

separating the real and imaginary parts such

$$p_1 = A_r + r^{\lambda}\big(B_r\cos\lambda\theta - B_i\sin\lambda\theta\big),$$
$$v_1 = A_i + r^{\lambda}\big(B_i\cos\lambda\theta + B_r\sin\lambda\theta\big),$$

Using the boundary condition we are able to find A and B. At $\theta = 0$ we have $v_1 = 0$ which means

$$v_1 = A_i + r^{\lambda}\big(B_i\cos\lambda\theta + B_r\sin\lambda\theta\big) = 0,$$

which means $A_i = B_i = 0$. At $\theta = \pi$

$$v_1 = (r^{\lambda}B_r\sin\lambda\theta)\Big|_{\theta=\pi} = \theta_0,$$

this gives $B_r = \frac{\theta_0 r^{-\lambda}}{\sin(\lambda\pi)}$. Therefore we find

$$p_1 = A_r + \frac{\theta_0}{\sin(\lambda\pi)}\cos(\theta\lambda) = A_r + \frac{\theta_0\Re\{z\}}{\sin(\lambda\pi)}.$$

- Question 9. We know that $u_1 = -p_1$ and pressure by definition is

$$u_1 = -p_1 = \frac{-1}{2\pi}\int_0^1 \frac{Y_+'(\zeta) - Y_-'(\zeta)}{\zeta - x} d\zeta \pm \sqrt{\frac{1-x}{x}}\left[\alpha_* - \frac{1}{2\pi}\int_0^1 \sqrt{\frac{\zeta}{\zeta - 1}}\frac{Y_+'(\zeta) + Y_-'(\zeta)}{\zeta - x} d\zeta\right].$$

Since we are dealing with a parabolic shape the first term becomes zero since $Y_+'(\zeta) = -Y_-'(\zeta)$.

$$u_1 = \pm\sqrt{\frac{1-x}{x}}\left[\alpha_* - \frac{1}{2\pi}\int_0^1 \sqrt{\frac{\zeta}{\zeta - 1}}\frac{Y_+'(\zeta) + Y_-'(\zeta)}{\zeta - x} d\zeta\right].$$

Note that $(1-x)^{1/2} = 1 - \frac{1}{2}x + \dots$ as $x \to 0_+$. Also the integrand may be written as

$$\sqrt{\frac{\zeta}{\zeta - 1}}[Y_+'(\zeta) + Y_-'(\zeta)](\zeta - x)^{-1} \approx \frac{1}{\sqrt{\zeta(\zeta - 1)}}[Y_+'(\zeta) + Y_-'(\zeta)] \quad \text{as} \quad x \to 0_+,$$

which leads to

$$u_1 = \pm\frac{\alpha_*}{\sqrt{x}} \mp \frac{1}{2\pi\sqrt{x}}\int_0^1 \frac{Y_+'(\zeta) + Y_-'(\zeta)}{\sqrt{\zeta(\zeta - 1)}} d\zeta.$$

The surface on the upper side is defined as $Y = \sqrt{2X}$ and by substituting this expression into

$$\hat{V} = V_\infty \frac{Y + k}{\sqrt{Y^2 + 1}},$$

we find that

$$\hat{V} = V_\infty \frac{\sqrt{2X} + k}{\sqrt{2X + 1}} = \frac{1 + \frac{k}{2X}}{\sqrt{1 + (2X)^{-1}}} = 1 + \frac{k}{\sqrt{2X}} + O(x^{-1}),$$

Now using the definition

$$\begin{aligned} \frac{\hat{V}}{V_\infty} &= \frac{\sqrt{\hat{u}^2 + \hat{v}^2}}{V_\infty}, \\ &= \frac{\sqrt{V_\infty^2(1 + \epsilon u_1)^2 + (V_\infty \epsilon v_1)^2}}{V_\infty}, \\ &= \sqrt{1 + 2\epsilon u_1 + \dots}, \\ &\approx 1 + \epsilon u_1 + \dots \ . \end{aligned}$$

Our task is to find k and to do this we substitute $u_1(x, 0) = \frac{\chi}{\sqrt{x}}$ into $\frac{\hat{V}}{V_\infty} = 1 + \epsilon u_1$ which leads to

$$\frac{\hat{V}}{V_\infty} = 1 \pm \epsilon\frac{\chi}{\sqrt{x}} + O(\epsilon^2),$$

comparing this expression with

$$\frac{\hat{V}}{V_\infty} = 1 + \epsilon \frac{k}{\sqrt{2X}} + O(X^{-1}),$$

given that $X = \frac{x}{\epsilon^2}$ we find k to be

$$k = \sqrt{2}\Bigg[\alpha_* - \frac{1}{2\pi}\int_0^1 \Bigg] \frac{Y'_+(\zeta) + Y'_-(\zeta)}{\sqrt{\zeta(\zeta - 1)}} d\zeta \Bigg].$$

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.