Theory of Thin Aerofoil in Incompressible Flow

Exercise 7 - Page 105

The methods used in this note are useful in analysing two-dimensional inviscid incompressible fluid flow.

- Question 1. Consider $y = \epsilon \sin(kx)$. (a) We need to consider the leading order terms where the solutions of a two-dimensional inviscid incompressible fluid flow is sought in the form

  $$u = 1 + \epsilon u_1(x,y) + \dots, \quad v = \epsilon v_1(x,y) + \dots, \quad p = \epsilon p_1(x,y) + \dots,$$

  the terms with $O(\epsilon)$

  $$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x}, \quad \frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}, \quad \frac{\partial u_1}{\partial x} = -\frac{\partial v_1}{\partial y}.$$

  (b) using the first and last equations above we find that

  $$\frac{\partial v_1}{\partial y} = \frac{\partial p_1}{\partial x},$$

  then we differentiate it w.r.t x to get

  $$\frac{\partial^2 v_1}{\partial y \partial x} = -\frac{\partial^2 p_1}{\partial x^2},$$

  now we need to differentiate the second equation w.r.t y which allows us to express pressure as

  $$\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} = 0. \tag{1}$$

  (c)

  $$p_1 = -\int \frac{\partial v_1}{\partial x} dy,$$

  we seek the solution for pressure in the form $p_1 = f(y)\sin(kx)$ and substitute this into (1) which yields to

  $$f''(y) - k^2 f(y) = 0,$$

  and the general solution $f(y) = c_0 e^{ky} + c_1 e^{-ky}$. Since we need to satisfy $f(y) \to 0$ as $y \to \infty$ we have to choose $c_0 = 0$. Therefore the pressure solution is

  $$p_1 = c_1 e^{-ky} \sin(kx),$$

  if $v_1(x, 0) = y' = k\cos(kx)$ thus $\frac{\partial v_1}{\partial x} = -k^2 \sin(kx)$ and

  $$p_1 = k^2 c_1 e^{-ky} \int_0^y \sin(kx) d\zeta = k^2 e^{-ky} y \sin(kx) + c(x).$$

- Question 2. (a) Assuming the aerofoil is symmetric w.r.t the x-axis and angel of attack is zero. Then we can express $Y'_+(\zeta) = -Y'_-(\zeta)$ and pressure

$$p_1 = \frac{1}{\pi}\int_0^b \frac{Y'_+(\zeta)}{(\zeta - x)}d\zeta.$$

(i) The aerofoil surface is a semi-infinite parabolic aerofoil where $Y_\pm = \pm\sqrt{x}$ where $x \in [0,\infty)$. We find the pressure distribution along the aerofoil to be

$$p_1 = \frac{1}{\pi}\int_0^\infty \frac{1}{2}\frac{d\zeta}{\zeta^{1/2}(\zeta - x)} = \frac{-1}{\pi\sqrt{x}}\tanh^{-1}\left(\sqrt{\frac{\zeta}{x}}\right)\Bigg|_0^\infty = -\frac{1}{2\sqrt{x}}, \tag{2}$$

(i) The aerofoil surface is a semi-infinite parabolic aerofoil where $Y_\pm = \pm\sqrt{x(1-x)}$ where $x \in [0,1]$. We find the pressure to be

$$\begin{aligned} p_1 &= \frac{1}{2\pi}\int_0^1 \frac{1}{(\zeta - x)}\frac{(1-2\zeta)}{\sqrt{\zeta(1-\zeta)}}d\zeta \\ &= \frac{1}{2\pi}\int_0^1 \frac{1}{(\zeta - x)}\frac{1}{\sqrt{\zeta(1-\zeta)}}d\zeta - \frac{1}{2\pi}\int_0^1 \frac{1}{(\zeta - x)}\frac{2\zeta}{\sqrt{\zeta(1-\zeta)}}d\zeta \end{aligned}$$

the first integral can be divided into intervals $\zeta \in [0,\delta]$ and $\zeta \in [\delta,1]$. The integral corresponding the latter interval is transcendentally small and goes to zero and the integral corresponding to the former interval is equal to (2) thus

$$p_1 = \frac{-1}{2\sqrt{x}} - \frac{1}{\pi}\int_0^1 \frac{1}{(\zeta - x)}\frac{\zeta}{\sqrt{\zeta(1-\zeta)}}d\zeta$$

the second integral is solved to be

$$\begin{aligned} p_1 &= \frac{-1}{2\sqrt{x}} - \frac{1}{\pi}\left[\frac{-\pi}{\sin(3\pi/2)}\left(1 + \sqrt{\frac{x}{1-x}}\cos(3\pi/2)\right)\right]_0^1, \\ p_1 &= -\frac{1}{2\sqrt{x}} - 1. \end{aligned}$$

(b)

$$p_1 = \frac{1}{\pi}\int_0^\infty \frac{1}{2}\frac{d\zeta}{\zeta^{1/2}(\zeta - x)} = \frac{-1}{\pi\sqrt{x}}\tanh^{-1}\left(\sqrt{\frac{\zeta}{x}}\right)^{-1}\Bigg|_0^\infty = -\frac{1}{2\sqrt{x}},$$

the impermeability condition on the aerofoil surface is

$$v_1\Big|_{y=\pm 0} = \pm\frac{1}{2\sqrt{x}} + \dots,$$

seeking the solution $f(z) = p_1 + iv_1$. To C in $f(z) = \frac{C}{\sqrt{z}}$. Over the upper side of the aerofoil

$$\frac{C}{\sqrt{z}} = \frac{C_r + iC_i}{\sqrt{r}e^{i\vartheta/2}} = -\frac{1}{2\sqrt{r\cos\vartheta}} + i\frac{1}{2\sqrt{r\cos\vartheta}}$$

$$\frac{C}{\sqrt{z}} = \frac{1}{\sqrt{r}}\left[\Big(C_r\cos\vartheta/2 + C_i\sin\vartheta/2\Big) + i\Big(C_i\cos\vartheta/2 - C_i\sin\vartheta/2\Big)\right],$$

from the expression above we find that

$$C_i = \frac{\cos\vartheta/2 - \sin\vartheta/2}{2\sqrt{\cos\vartheta}}, \quad C_r = -\frac{1}{\cos\vartheta/2}\left(C_i\sin\vartheta/2 + \frac{1}{2\sqrt{\cos\vartheta}}\right).$$

- Question 3. (a) we define $y'_+ - y'_- = -\epsilon 3x^{1/2}$ thus we find the lift coefficient to be

$$\begin{aligned} C_L &= 2\pi\alpha + 6\epsilon \int_0^1 \sqrt{\frac{\zeta}{1-\zeta}}\,\sqrt{\zeta}d\zeta, \\ &= 2\pi\alpha + 6\epsilon\Big((-2/3)(1-x)^{1/2}(2+x)\Big)_0^1, \\ &= 2\pi\alpha + 8\epsilon. \end{aligned}$$

α denotes the angle of attack.

(b) we define $y'_+ - y'_- = \epsilon 2(1-2x)$

$$\begin{aligned} C_L &= 2\pi\alpha - 4\epsilon \int_0^1 \left(\sqrt{\frac{\zeta}{1-\zeta}} - 2\frac{\zeta^{3/2}}{\sqrt{1-\zeta}}\right)d\zeta, \\ &= 2\pi\alpha - 4\epsilon\left[\sqrt{\frac{1}{1-x}}\left(\sqrt{x}(x-1) + \sqrt{x-1}\tanh^{-1}\frac{1}{\sqrt{(x-1)/x}}\right)\right] \\ &+ 8\epsilon\left[-\frac{1}{4}\sqrt{-x(x-1)}(3+2x) + \frac{3}{2}\cot^{-1}\left(\frac{-1+\sqrt{1-x}}{\sqrt{x}}\right)\right], \\ &= \pi(2\alpha - 3\epsilon). \end{aligned}$$

- Question 4. To find the pressure distribution on the surface of an infinitely thin aerofoil whose upper and lower sides are given by $y_+(x) = y_-(x) = \epsilon x^{3/2}$

$$\begin{aligned} \frac{\hat{p} - p_\infty}{\rho V_\infty^2} &= \frac{1}{2\pi}\int_0^1 \frac{y_+(\zeta) - y_-(\zeta)}{\zeta - x}d\zeta \pm \sqrt{\frac{1-x}{x}}\left[-\alpha + \frac{1}{2\pi}\int_0^1 \sqrt{\frac{1-\zeta}{\zeta}}\frac{y_+(\zeta)+y_-(\zeta)}{\zeta - x}d\zeta\right], \\ &= \mp\alpha\sqrt{\frac{1-x}{x}} \pm \frac{3\epsilon}{2\pi}\sqrt{\frac{1-x}{x}}\left[-2\sqrt{1-\zeta} + \frac{2x\tan^{-1}\sqrt{\frac{1-\zeta}{x-1}}}{\sqrt{x-1}}\right]_{\zeta=0}^{\zeta=1}, \\ &= \mp\alpha\sqrt{\frac{1-x}{x}} \pm \frac{3\epsilon}{2\pi}\sqrt{\frac{1-x}{x}}\left[1 - \frac{x\tan^{-1}\frac{1}{\sqrt{x-1}}}{\sqrt{x-1}}\right]. \end{aligned}$$

- Question 5. The surface function is defined as $y_\pm = \pm\epsilon x^\alpha + \dots$ as $x \to 0_+$.

(a) first we find the asymptotic behaviour of the integral

$$\begin{aligned} p_1 &= \frac{1}{\pi}\int_0^1 \frac{Y'_+(\zeta)}{\zeta x}d\zeta, \\ &= \frac{1}{\pi}\int_0^\delta \frac{Y'_+(\zeta)}{\zeta x}d\zeta + \frac{1}{\pi}\int_\delta^1 \frac{Y'_+(\zeta)}{\zeta x}d\zeta, \end{aligned}$$

the second integral is transcendentally small as $x \to 0_+$. The solution to the first integral is found by letting $s = \zeta/x$

$$x^{\alpha-1}\int_0^\infty \frac{s^{s-1}}{s-x}ds = -x^{\alpha-1}\pi\cot(\alpha\pi),$$

therefore the pressure

$$p_1 = -\epsilon\alpha x^{\alpha-1}\pi\cot(\alpha\pi), \quad \text{as} \quad x \to 0_+.$$

(b) To represent the solution in the form

$$p_1 + iv_1 = Cz^{\alpha-1} + \dots \quad \text{as} \quad z \to 0,$$

we need to define $z = re^{i\vartheta}$ while using the impermeability condition on the aerofoil $\vartheta = 0$ and the flow symmetry condition on the x-axis upstream of the aerofoil nose $\vartheta = \pi$. At the upper side of the aerofoil surface $\vartheta = 0$

$$\begin{aligned} p_1 + iv_1 &= (C_r + iC_i)r^{\alpha-1}e^{i(\alpha-1)\vartheta}, \\ -\epsilon\alpha(r\cos\vartheta)^{\alpha-1}\pi\cot(\alpha\pi) &\pm i\epsilon\alpha(r\cos\vartheta)^{\alpha-1} = (C_r + iC_i)r^{\alpha-1}. \end{aligned}$$

which yields to $C_r = -\epsilon\alpha\cot(\alpha\pi)$ and $C_i = \epsilon$. Similarly at the lower side of the surface

$$p_1 + iv_1 = -(C_r + iC_i)r^{\alpha-1}e^{i\pi\alpha},$$

$$-\epsilon\alpha(r\cos\vartheta)^{\alpha-1}\cot(\alpha\pi) - i\epsilon\alpha(r\cos\vartheta)^{\alpha-1}\Big|_{\vartheta=\pi} = -r^{\alpha-1}(C_r + iC_i)(\cos\pi\alpha + i\sin\pi\alpha).$$

- Question 6. Consider a thin symmetric aerofoil with a wedge-shaped trailing edge where the angle between the upper and lower side of the aerofoil at the trailing edge is $\theta = \epsilon 2\theta_0$. By geometry

$$\frac{dy}{dx} = \tan\left(\frac{\theta}{2}\right) = \tan\left(\epsilon\theta_0\right) \approx \epsilon\theta_0,$$

thus the pressure is expressed as

$$p_1 = \frac{1}{\pi}\int_0^1 \frac{\epsilon\theta_0}{\zeta - x}d\zeta = \frac{\epsilon\theta_0}{\pi}\Big[\ln(1-x) - \ln(-x)\Big] = -\frac{\epsilon\theta_0}{\pi}\ln(-x),$$

note that the term $\ln(1-x)$ goes to zero as $x \to 0_-$. By impermeability condition $v_1 = y' = \epsilon\theta_0$. Thus, we seek the solution near the trailing edge in the form

$$\begin{aligned} p_1 + v_1 &= -\frac{\epsilon\theta_0}{\pi}\ln(-x) + i\epsilon\theta_0, \\ &= -\frac{\epsilon\theta_0}{\pi}\Big(\ln(x) + \ln(-1)\Big) + i\epsilon\theta_0, \\ &= -\frac{\epsilon\theta_0}{\pi}\Big(\ln(x) + i\pi\Big) + i\epsilon\theta_0, \\ &= -\frac{\epsilon\theta_0}{\pi}\ln(x) + 0i, \end{aligned}$$

since the solution is valid

$$\big(C_r + iC_i\big)\ln z = -\frac{\epsilon\theta_0}{\pi}\ln(x) + 0i,$$

which leads to $C_r = -\epsilon\frac{\theta_0}{\pi}\ln(x)$ and $C_i = 0$

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.