Supersonic Flows Past Thin Aerofoils

Exercise 12 - Page 174

- Question 1. We could argue that the perturbations produced by the aircraft flying with supersonic speed are confined within a mach cone therefore the perturbations $g(\eta)$ outside the cone must be zero.

$$u_1 - p_1 = \phi(y) = \phi(\epsilon Y) \approx 0,$$
$$h_1 - p_1 = \psi(y) = \psi(\epsilon Y) \approx 0,$$

φ does not change along the streamlines $\varphi(x, y) = f(\xi)$

$$v_1 = \frac{\partial \varphi}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} = -\beta f' = Y'_+(x),$$

$$f = -\frac{Y_+(x)}{\beta} + c,$$

we disregard c which leads to

$$\varphi(x, y) = f(\xi) = -\frac{Y_+(x)}{\beta},$$

where $\xi = x - \beta y$.

- Question 2.

$$p = -\frac{\partial \varphi}{\partial x} = \frac{Y_+(x)}{\beta},$$
$$p(x, 0_+) = \frac{Y_+(x)}{\sqrt{M_\infty^2 - 1}}.$$

We know that

$$\epsilon Y'_+(x) = \tan\theta(x) = \theta(x) + \dots,$$
$$\epsilon Y'_-(x) = -\tan\theta(x) = -\theta(x) + \dots,$$

this the pressure may be expressed as

$$\hat{p} = p_\infty + \rho_\infty V_\infty^2 \frac{\theta}{\sqrt{M_\infty^2 - 1}}.$$

- Question 3.

$$\Gamma_1 = \int_0^1 \sqrt{\frac{\zeta}{1-\zeta}}\Big[Y'_+(\zeta) + Y'_-(\zeta)\Big] d\zeta,$$

The lift force is defined as

$$\begin{aligned}\hat{L} &= \int_0^L \big[\hat{p}(\hat{x}, 0_-) - \hat{p}(\hat{x}, 0_+)\big]\, d\hat{x},\\ &= \rho_\infty V_\infty^2 L \int_0^1 \big[p(x, 0_-) - p(x, 0_+)\big]\, dx,\\ &= \frac{\rho_\infty V_\infty^2 L}{\beta} \int_0^1 \big[Y_-'(x) - Y_+'(x)\big]\, dx,\\ &= \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} 2h.\end{aligned}$$

- Question 4. (a) To calculate the Drag force, consider the geometry of the force acting on the upper and lower sides of the surface

$$\begin{aligned}d\hat{D}_+ &= \hat{F}.\theta = -\hat{p}_- dl\theta,\\ \hat{D}_+ &= \int \hat{p}_+ \theta dl dx = \int \hat{p}_+ y_+'(x) dx.\end{aligned}$$

Total drag

$$\hat{D} = D_+ + D_- = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^L \Big[\hat{p}_+ y_+'(\hat{x}) + \hat{p}_- y_-'(\hat{x})\Big] d\hat{x},$$

Note that $\hat{p}_+ = \epsilon Y_+'(x) = \tan\theta = \theta$

$$\hat{D} = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^L \left[\left(\frac{d\hat{y}_+}{d\hat{x}}\right)^2 + \left(\frac{d\hat{y}_-}{d\hat{x}}\right)^2\right] d\hat{x}.$$

(b)

$$\begin{aligned}&\frac{d\hat{y}_+}{d\hat{x}} = -\alpha + \theta_+, \quad \frac{d\hat{y}_-}{d\hat{x}} = -\alpha + \theta_-\\ &\left(\frac{d\hat{y}_+}{d\hat{x}}\right)^2 + \left(\frac{d\hat{y}_-}{d\hat{x}}\right)^2 = 2\alpha^2 + \theta_+^2 + \theta_-^2.\end{aligned}$$

We calculate the drag force as

$$\hat{D} = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^L \big(2\alpha^2 + \theta_+^2 + \theta_-^2\big) d\hat{x}.$$

(c) When the angle of attack is fixed, the minimal drag is produced by a flat plate.

$$\begin{aligned}\hat{D}_{min} &= \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^L 2\alpha^2 d\hat{x},\\ &= \frac{2\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \alpha^2 L.\end{aligned}$$

- Question 5.

$$C_L = \frac{\hat{L}}{(1/2)\rho_\infty V_\infty^2},$$

The lift force over the flat plate is defined as $\hat{L} = L_{AB} + L_{BC}$

$$\begin{aligned}\hat{L} &= \frac{2\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}}\big[h_{AB} + h_{BC}\big],\\ &= \frac{2\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}}\big[3\sqrt{1-\alpha^2} + \sqrt{1-\delta^2}\big].\end{aligned}$$

The drag force is calculated as

$$\begin{aligned}\hat{D} &= \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}}\left[\int_0^{3L/4}\big(y_+'^2 + y_-'^2\big)d\hat{x} + \int_{3L/4}^{L}\big(y_+'^2 + y_-'^2\big)d\hat{x}\right],\\ \hat{D} &= 2\frac{\rho_\infty V_\infty^2 L}{\sqrt{M_\infty^2 - 1}}\alpha^2.\end{aligned}$$

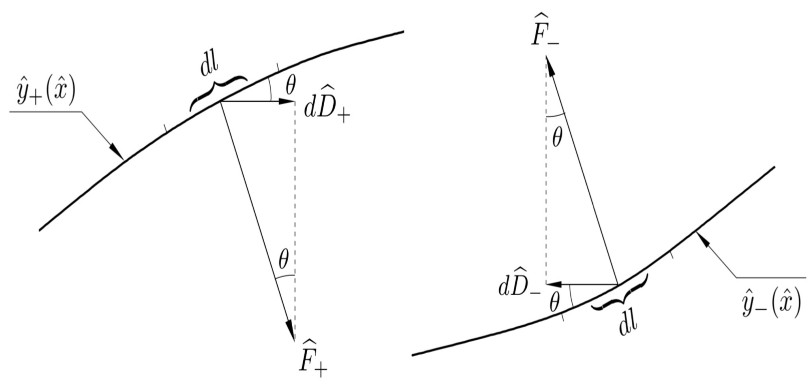

Figure 1: Calculation of the drag force. Ruban (2015).

- Question 6.

$$\begin{aligned}&p = p_{\text{shock wave}} + p_{\text{wall}},\\ &p_{\text{wall}} = \theta,\\ &p_{\text{shock wave}} = p^+_{\text{shock wave}} - p^-_{\text{shock wave}} = \theta_0(\hat{x} - \hat{x}_0) - \big[-\theta_0(\hat{x} - \hat{x}_0)\big],\end{aligned}$$

thus we find

$$p = p_\infty + \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}}\big[\theta + 2H\theta_0(\hat{x} - \hat{x}_0)\big],$$

where H is Heaviside step function

$$H = \begin{cases} 1 & \text{if } \hat{x} > 0 \\ 0 & \text{if } \hat{x} < 0. \end{cases}$$

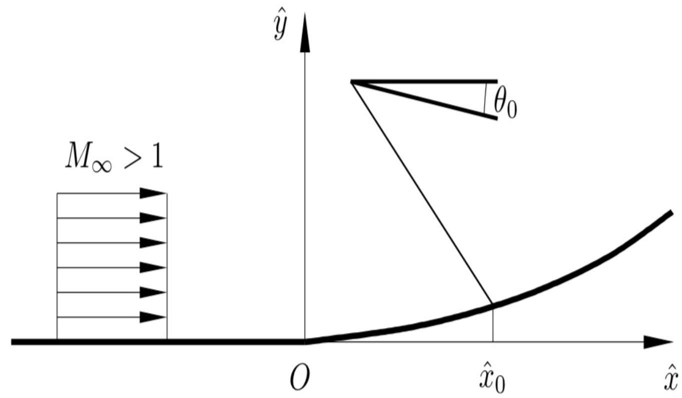

Figure 2: Supersonic flow with an impinging shock wave. Ruban (2015).

Questions

The questions are found in Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.