Fluid Dynamics II - Anatoly I. Ruban

Exercise 3 - Page 53

- Question1: assuming the solution has the following form

$$y(x;\epsilon) = y_0(x) + \epsilon y_1 x + O(\epsilon^2). \tag{1}$$

by substituting (1) into the IVP and we find the leading order terms are $O(1)$

$$\frac{d^2 y_0(x)}{dx^2} + y_0(x) = 0,$$
$$y_0(0) = 0, \quad \frac{d^2 y_0(0)}{dx^2} = \frac{2}{\sqrt{3}},$$

solution to the IVP is

$$y_0(x) = \frac{2}{\sqrt{3}} \sin x.$$

the second term $O(\epsilon)$

$$\frac{d^2 y_1(x)}{dx^2} + y_1(x) = y_0^3 - y_0,$$
$$y_1(0) = 0, \quad \frac{d^2 y_1(0)}{dx^2} = 0,$$

which yields to

$$y_1(x) = \frac{2}{\sqrt{27}} \sin 3x.$$

Composing the final solution is

$$y(x, \epsilon) = \frac{2}{\sqrt{3}} \sin x + \frac{2}{3\epsilon} \sin 3x.$$

The analytical solution above matches the numerical solution for $\epsilon = 0.001$ obtained using Mathematica

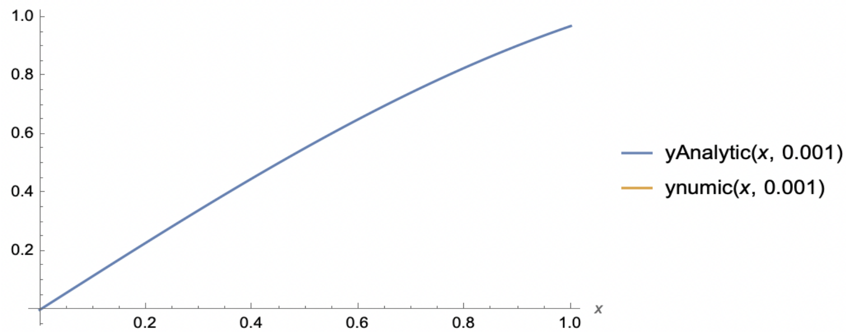

Figure 1: Numerical vs Asymptotic solutions - Question 1.

Note that the code for the numerical solutions are provided in a different file.

- Question 2: The general solution is found by assuming $y = e^{\lambda x}$ and then substitute this solution into the differential equation. As a result we find that

$$\epsilon\lambda^2 + \lambda + 1 = 0, \quad \lambda_{1,2} = -\frac{1}{2\epsilon} \pm \sqrt{\frac{1}{4\epsilon^2} - \frac{1}{\epsilon}},$$

the general solution has the following form

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

A and B are constant that are found using initial conditions and matching inner and outer solutions. Using the initial condition we find the final solution is

$$y = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}}.$$

- Question 3: First we find the outer solution by assuming the solution has the form (1). By substituting (1) into the IVP and we find the leading order terms are $O(1)$

$$y_0(x) = \cos x,$$

Now we turn attention to the inner solution. Let $X = \frac{x}{\delta}$ and we find the balanced dominant terms when $\delta = \epsilon$. The inner solution is denoted by $Y(X)$ and the dominant terms are

$$\frac{dY}{dX} + Y = 0,$$

and the solution is

$$Y(X) = e^{-X} + 1,$$

the final solution is

$$y(x;\epsilon) = \cos x + e^{-x/\epsilon} + \dots.$$

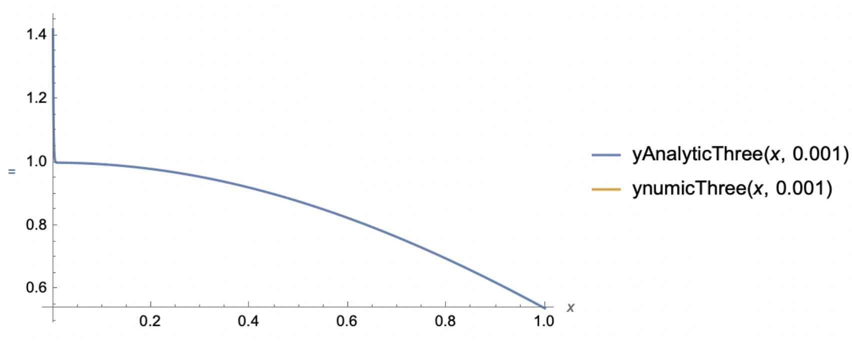

Figure 2: Numerical vs Asymptotic solutions - Question 3.

- Question 4: We find the outer solution to be $y_0 = -1$. The location of the boundary layer is defined as $x = x_0 + \epsilon^\alpha X$ and consider two cases (i) $x_0 = 0$ and (ii) $x_0 = 1$. To get the balanced dominant terms we need to let $\alpha = 1/2$. For case (i) using initial condition $y(0) = 0$ we find the solution to be

$$Y_0^L = (1 - B)e^{x/\sqrt{\epsilon}} + Be^{-x/\sqrt{\epsilon}} - 1,$$

by matching the inner and outer solutions as $x \to \infty$, we find $B = 1$. For case (ii) using initial condition $y(1) = 2$

$$Y_0^R = (3 - B)e^{(x-1)/\sqrt{\epsilon}} + Be^{-(x-1)/\sqrt{\epsilon}} - 1.$$

Matching the inner solutions with the outer solution

$$\lim_{X\to\infty} (1 - B)e^{x/\sqrt{\epsilon}} + Be^{-x/\sqrt{\epsilon}} - 1 = \lim_{x\to 0} -1 \quad \to B = 1$$
$$\lim_{X\to-\infty} (3 - B)e^{(x-1)/\sqrt{\epsilon}} + Be^{-(x-1)/\sqrt{\epsilon}} - 1 = \lim_{x\to 0} -1 \quad \to B = 0,$$

we find that $B = 1$ from left and $B = 0$ from right side. By looking at the inner solution we can see that correct values for B eliminate the secular terms which cause the solutions to grow to infinity. In our analysis we do not want keep the terms that grow exponentially.

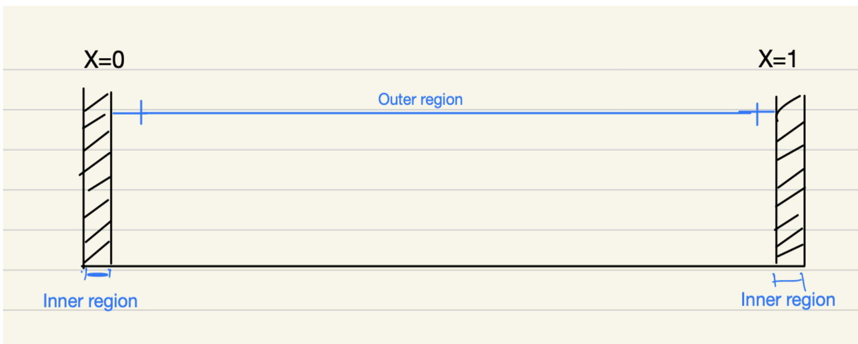

Figure 3: Schematic of different regions for Question 4.

- Question 5: First we find the outer solution by assuming the solution has the form (1). By substituting (1) into the IVP and we find the leading order terms are $O(1)$. The boundary layer is at $x_0 = 1$ thus we use the i.c $y_0(0) = 1$ which yields to $c = 1$. The outer solution is

$$y_0(x) = \frac{2}{x^2 + 2}.$$

The inner solution is found by letting $x = 1 + \epsilon^\alpha X$ and $\alpha = 1$. The leading order term for the inner region are

$$\frac{d^2Y}{dX^2} - \frac{dY}{dX} = 0.$$

and the solution to this equation is $Y(X) = B(e^X - 1)$. We find the constant B by matching the inner and outer solution

$$\lim_{x\to 1} \frac{2}{x^2 + 2} = \lim_{X\to-\infty} B(e^X - 1).$$

This leads to $B = -2/3$. Consequently we find the composite solution to be

$$y(x; \epsilon) = \frac{2}{x^2 + 2} - \frac{1}{3} - \frac{2}{3}e^{(x-1)/\epsilon} + \dots.$$

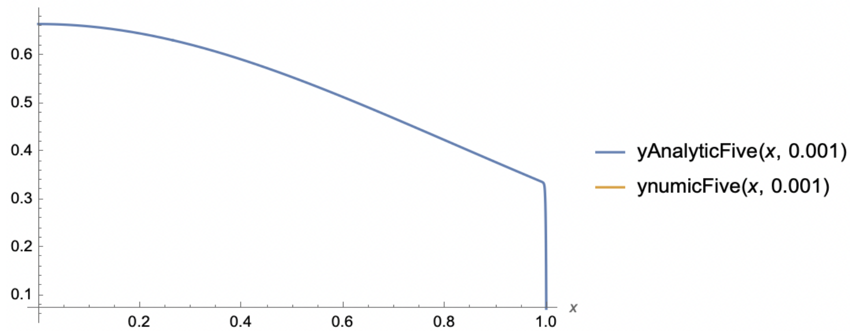

Figure 4: Numerical vs Asymptotic solutions - Question 5.

- Question 6: First we find the outer solution by assuming the solution has the form (1). By substituting (1) into the IVP and we find the leading order terms are $O(1)$ with the initial condition $y_0(1) = 1$.

$$(x-2)\frac{dy_0}{dx} + y_0 = 0$$

and the solution is

$$y_0 = e^{1/2+1/(x-2)}$$

The inner solution is found be letting $x = 1 + \epsilon X$. This yields to the dominant terms to be

$$\frac{d^2Y}{dX^2} - \frac{dY}{dX} = 0$$

Considering the i.c. $Y(0) = 1$ gives the following solution

$$Y(X) = 1 + B(e^X - 1).$$

We find the constant B by matching the inner and outer solution as

$$\lim_{X\to-\infty} 1 + B(e^X - 1) = \lim_{x\to 1} \frac{2}{x-2}.$$

This leads to $B = 3$. The final solution is

$$y_{\text{com}} = \frac{2}{x-2} + 2e^{(x-1)/\epsilon} - 4.$$

- Question 7: Outer solution is

$$y_0 = -\ln\left(\frac{x+1}{2}\right)$$

Considering the i.c. $Y(0) = 0$ and setting $x = \epsilon X$ gives the following inner solution

$$Y_{XX} + 2Y_X = 0, \quad \to \quad Y(X) = B(e^{-2X} - 1)$$

$$\lim_{X\to-\infty} B(e^{-2X} - 1) = \lim_{x\to 0} -\ln\left(\frac{x+1}{2}\right).$$

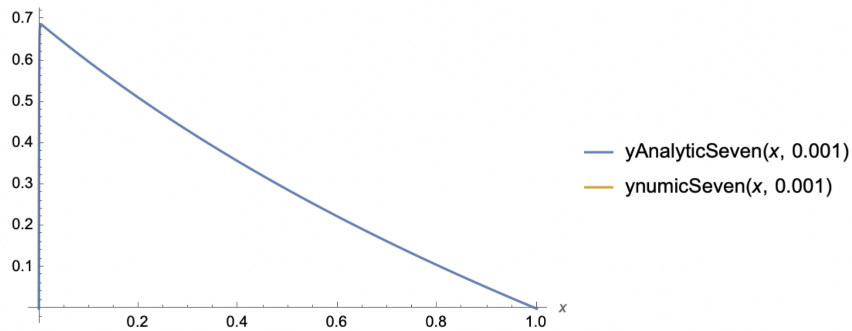

Figure 5: Numerical vs Asymptotic solutions - Question 5.

gives $B = -\ln(2)$. Consequently we find the final solution to be

$$y(x;\epsilon) = -\ln(x+1) + \ln 2\ (1 - e^{-2x/\epsilon}) + \dots.$$

- Question 8: Outer solution is

$$y_0 = e^{2\sqrt{x}}$$

where $y_0(1) = e^2$. Considering the i.c. $Y(0) = 0$ and setting $x = \epsilon^{2/3}X$ gives the following inner solution

$$Y_{XX} + \big(X^{1/2} - \frac{1}{2X}\big)Y_X = 0,$$
$$Y(X) = B_1 - e^{-\frac{2}{3}X^{3/2}+B_0}.$$

Using the initial condition $Y(0) = 0$ gives us $B_1 = e^{B_0}$ and the following

$$Y(X) = e^{B_0} - e^{-\frac{2}{3}X^{3/2}+B_0}$$

By matching the inner and outer solutions

$$\lim_{X\to\infty} e^{B_0} - e^{-\frac{2}{3}X^{3/2}+B_0} = \lim_{x\to 0} e^{2\sqrt{x}}.$$

gives $B_0 = 0$. The outer solution is

$$Y(X) = 1 - e^{-\frac{2}{3}X^{3/2}}.$$

Consequently we find the final solution to be

$$y(x;\epsilon) = 1 - e^{-\frac{2}{3}(x/\epsilon)^{3/2}} + e^{2\sqrt{x}}.$$

- Question 9:
  Outer (left) solution is

$$y_0^L = e^{x^2/2}$$

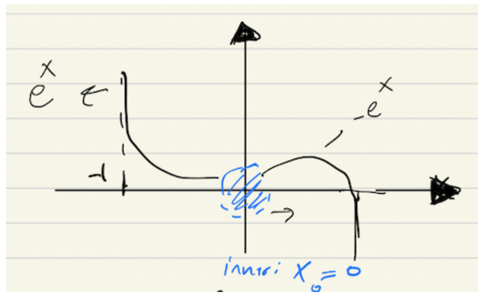

Figure 6: Schematic of different regions for Question 9.

and outer (right) solution is

$$y_0^R = -e^{x^2/2}$$

where $y_0(1) =$. Considering the i.c. $Y(0) = 0$ and setting $x = \epsilon^{2/3} X$ gives the following inner solution

$$Y_{XX} + XY_X = 0 \quad \rightarrow \quad Y(X) = B_1 + B_2\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right).$$

$\mathrm{erf}(0) = 0$, $\mathrm{erf}(-\infty) = -1$ and $\mathrm{erf}(\infty) = 1$. From left

$$\lim_{X\to-\infty} B_1^L = \lim_{x\to-1} 1.$$

from right

$$\lim_{X\to\infty} B_1^R = \lim_{x\to1} -1.$$

Questions

The questions to the solutions are posed by Ruban (2015).

References

Ruban, A. I. (2015), 'Fluid dynamics. part 2, asymptotic problems of fluid dynamics'.