Q1. $I(\lambda) = \int_0^1 \frac{x e^{\lambda x}}{\sqrt{(1+x^4)}} dx \sim \frac{1}{\sqrt{2}} \frac{e^{\lambda}}{\lambda} - \frac{e^{\lambda}}{\sqrt{2}\lambda^2}$ [mathematica]

(a) The integrand has its max value at its end point so we are able to use integration by parts.

$$I(x) = \int_0^1 x e^{\lambda x} (1+x^4)^{-1/2} dx = \int_0^1 x e^{\lambda x} \Big[\underbrace{1 - \frac{x^4}{2} + \frac{3x^8}{8} - \frac{5x^{16}}{16} + \cdots}_{\psi(x)} \Big] dx$$

$$I(x) \sim \int_0^1 x e^{\lambda x} dx$$

$u = x \quad u' = 1$

$v = \frac{e^{\lambda x}}{\lambda} \quad v' = e^{\lambda x} dx$

$$\sim \frac{x}{\lambda} e^{\lambda x} \Big|_0^1 - \frac{1}{\lambda} \int_0^1 e^{\lambda x} dx$$

$$\sim \frac{e^{\lambda}}{\lambda} - \frac{1}{\lambda^2} e^{\lambda x} \Big|_0^1 \sim \frac{e^{\lambda}}{\lambda} - \frac{e^{\lambda}}{\lambda^2} + \frac{1}{\lambda^2} + \cdots$$

$(1+x^4)^{-1/2} \sim 1 + \cdots$ expanding around $x_0 = 0$

$(1+x^4)^{-1/2} \sim \frac{1}{\sqrt{2}} + \cdots$ expanding around $x_0 = 1$

$I(\lambda) \sim \frac{e^{\lambda}}{\lambda}$ or $I(\lambda) \sim \frac{e^{\lambda}}{\sqrt{2} \cdot \lambda}$

The max point is at endpoint $x = 1$ thus we choose

$$I(\lambda) \sim \frac{e^{\lambda}}{\sqrt{2}\lambda}. \quad \square$$

(b) $I(\lambda) = \int_0^1 t(1-t^2)^\lambda \, dt = \int_0^1 t \, e^{\lambda \ln(1-t^2)} \, dt \sim \int_0^1 t \, e^{\lambda[-t^2 - \frac{t^4}{2} + \cdots]} \, dt$

Max point $\ln(1-t^2)$ is at $t_0 = 0$

$$I(\lambda) \sim \int_0^1 t \, e^{-\lambda t^2} \, dt \qquad \begin{matrix} w = t^2 \\ dw = 2t \, dt \end{matrix} \Rightarrow \begin{matrix} t = \sqrt{w} \\ dt = \frac{dw}{2t} \end{matrix}$$

$$\sim \int_0^1 t \, e^{-\lambda w} \frac{dw}{2t} = \frac{1}{2} \int_0^1 e^{-\lambda w} \, dw$$

$$= \frac{1}{2} \left[\frac{e^{-\lambda w}}{-\lambda} \right]_0^1 = \frac{1}{2} \left[\frac{e^{-\lambda}}{-\lambda} + \frac{1}{\lambda} \right]$$

$$I(\lambda) \sim \frac{1}{2\lambda}$$

(c) $I(\lambda) = \int_0^{\pi/4} \tan t \, e^{\lambda \cos t} \, dt \sim \int_0^{\pi/4} (t + \dots) e^{\lambda(1 - t^2/2! + \dots)} \, dt$

$$\sim \int_0^{\pi/4} t \, e^{\lambda - \frac{\lambda t^2}{2}} \, dt = e^{\lambda} \int_0^{\pi/4} t \, e^{-\lambda t^2/2} \, dt$$

$$t^2 = \omega \Rightarrow 2t\,dt = d\omega \Rightarrow dt = \frac{d\omega}{2t}$$

$$t = \sqrt{\omega}$$

$$e^{\lambda} \int_0^{\pi/4} t \, e^{-\frac{\lambda}{2}\omega} \cdot \frac{d\omega}{2t} = \frac{e^{\lambda}}{2} \int_0^{\infty} e^{-\frac{\lambda}{2}\omega} \, d\omega$$

$$= \frac{e^{\lambda}}{\cancel{2}} \cancel{2} \frac{e^{-\lambda\omega}}{\lambda} \Big|_0^{\infty} = \frac{e^{\lambda}}{\lambda}$$

Q_2.

$$I(t) = \int_0^\infty \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) e^{-tx^2}\, dx = \int_0^\infty e^{-tx^2} - \frac{1}{2}\int_0^\infty x^2 e^{-tx^2}$$

$$-\frac{1}{2}\int_0^\infty e^{2\ln x - tx^2}\, dx \qquad x = s/\sqrt{t} \qquad dx = \frac{ds}{\sqrt{t}}$$

$$= -\frac{1}{2}\int_0^\infty e^{2\ln\left(\frac{s}{\sqrt{t}}\right) - s^2} \frac{ds}{\sqrt{t}} = -\frac{1}{2\sqrt{t}} \int_0^\infty e^{2\ln s - 2\ln\sqrt{t} - s^2}\, ds$$

$$= \frac{-1}{2t^{3/2}} \int_0^\infty e^{2\ln s - s^2}\, ds = \frac{-1}{2t^{3/2}} \int_0^\infty s^2 e^{-s^2}\, ds$$

$$z - 1 = 1/2 \to z = 1/2 + 1$$
$$z = 3/2$$

$$s^2 = w \Rightarrow 2s\, ds = dw$$

$$= \frac{-1}{2t^{3/2}} \int_0^\infty s^2 e^{-w} \frac{dw}{2s} = \frac{-1}{4t^{3/2}} \int_0^\infty w^{1/2} e^{-w}\, dw = \frac{-1}{4t^{3/2}}\Gamma(3/2)$$

$$= \frac{-1}{4t^{3/2}} \cdot \frac{\sqrt{\pi}}{2} = \frac{-\sqrt{\pi}}{8t^{3/2}}.$$

$$I(\lambda) \sim \frac{1}{2}\sqrt{\frac{\pi}{t}} - \frac{\sqrt{\pi}}{8t^{3/2}}.$$

$\square$

Q3 $I_n(r) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} e^{r\cos\theta} \cdot \cos(n\theta)\, d\theta$

2 term expansions

let $t = 1 - \cos\theta$, $t = \frac{1}{2}\theta^2 + \dots$, $\left.\begin{array}{l}\cos(n\theta) \\ \sin\theta = \sqrt{1-\cos^2\theta}\end{array}\right\}$ in powers of t.

$\cos\theta = 1 - t$

$-\sin\theta\, d\theta = -dt \Rightarrow (2t - t^2)^{1/2} d\theta = dt$, $t = 1 - \cos\theta = 1 - 1 + \frac{\theta^2}{2!} + \dots = \frac{\theta^2}{2}$

$$I_n(r) = \frac{1}{\pi}\int_0^{\pi} \frac{e^{r(1-t)}}{(2t-t^2)^{1/2}}\left(1 - \frac{n^2\theta^2}{2} + \dots\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{(1 - n^2 t)}{(2t - t^2)^{1/2}} e^{r(1-t)}\, dt = \underbrace{\frac{1}{\pi}\int_0^{\pi} \frac{e^{r(1-t)}}{(2t - t^2)^{1/2}}\, dt}_{①} - \underbrace{\frac{n^2}{\pi}\int_0^{1} \frac{t\, e^{r(1-t)}}{(2t-t^2)^{1/2}}\, dt}_{②}$$

$$① = \frac{1}{\pi}\int_0^{\pi} \frac{e^{r(1-t)}}{\sqrt{2t}}\left(1 - \frac{t}{2}\right)^{-1/2} dt \sim \frac{1}{\sqrt{2}\,\pi}\int_0^{\pi} t^{-1/2} e^{r(1-t)}\left(1 + \frac{t}{4} + \dots\right) dt$$

$$\sim \frac{e^r}{\pi\sqrt{2}}\int_0^{\pi} t^{-1/2} e^{-rt}\, dt \qquad t = \frac{s}{r} \qquad dt = \frac{ds}{r} \quad (*)$$

$$\sim \frac{e^r}{\pi\sqrt{2}}\int_0^{\infty} \left(\frac{s}{r}\right)^{-1/2} e^{-s} \frac{ds}{r} \sim \frac{e^r}{\pi\sqrt{2r}}\int_0^{\infty} s^{-1/2} e^{-s}\, ds$$

$(*)\ \dfrac{e^r(1/4)}{\sqrt{2\pi r}} \dfrac{1}{2r}$

$$\sim \frac{e^r}{\pi\sqrt{2r}}\Gamma(1/2) = \frac{e^r\sqrt{\pi}}{\pi\sqrt{2r}} = \frac{e^r}{\sqrt{2\pi r}}$$

$$② \ \frac{-n^2}{\pi}\int_0^{\pi} \frac{t}{\sqrt{2t}}\left(1 - \frac{t}{2}\right)^{-1/2} e^{r(1-t)}\, dt \sim \frac{-n^2}{\pi\sqrt{2}}\int_0^{\infty} t^{1/2} e^{r(1-t)}\, dt \qquad \text{as } r \to \infty$$

$$\sim \frac{-n^2}{\pi\sqrt{2r}} e^r \cdot \frac{\sqrt{\pi}}{2} = \frac{-n^2 e^r}{\sqrt{\pi r}\; 2^{3/2}} + \dots \frac{e^r(1/4)}{\sqrt{2\pi r}}\frac{1}{2r} = \frac{e^r}{\sqrt{2\pi r}}\left(\frac{-(n^2 - 1/4)}{2r}\right)$$

Q4

$$Ai(z) = \frac{1}{2\pi i} \int_C e^{\xi z - \xi^3/3} \, d\xi \qquad \xi = \sqrt{z}\, t \Rightarrow d\xi = \sqrt{z}\, dt$$

$$\xi^3 = z^{3/2} t^3$$

$$= \frac{\sqrt{z}}{2\pi i} \int_c e^{z^{3/2} t - \frac{z^{3/2}}{3} t^3} \, dt$$

$$= \frac{\sqrt{z}}{2\pi i} \int_c e^{z^{3/2}(t - t^3/3)} \, dt$$

$$g(t) = t - t^3/3$$

$$g'(t) = 1 - t^2 = 0 \Rightarrow t = \pm 1$$

$$g'' = -2t$$

$$g''(1) = -2, \quad g''(-1) = 2$$

$$g(1) = 2/3 \qquad g(-1) = -2/3$$

$$\sim \frac{\sqrt{z}}{2\pi i} \int_{C_{+1}} e^{z^{3/2}\left[\frac{2}{3} - (t-1)^2 + \cdots\right]} \, dt$$

$$(t-1) = re^{i\nu} \Rightarrow -1 = e^{-i\pi}$$

$$(t-1)^2 = r^2 e^{2i\nu} = r^2 \underbrace{\cos(2\nu) + i\sin(\nu)}_{=-1}$$

$$2\nu = \pi$$

$$\boxed{\nu = \pi/2}$$

$$dt = dre^{i\nu}$$

$$\boxed{\neq dr\, d\nu}$$

$$t - 1 = re^{i\nu}$$

$$dt = dre^{i\nu}$$

$$dt = e^{i\pi/2} dr$$

$$\sim \frac{\sqrt{z}}{2\pi i} e^{2/3 z^{3/2}} \int e^{-r^2} dr \,(e^{i\pi/2})$$

$$\sim \frac{\sqrt{z}}{2\pi i} e^{2/3 z^{3/2}} e^{i\pi/2} \cdot \sqrt{\pi}$$

$$\sim \frac{\sqrt{z}}{2\sqrt{\pi}} e^{2/3 z^{3/2}}$$

(*) write the note:

$$z = x + iy$$

We see that the airy function get's it value from $z = \pm 1$ which means gets no contribution from the Imaginary parts. Thus:

$$I_{t=1} + I_{t=-1} = \frac{1}{\sqrt{\pi}} x^{1/4} \cos\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4}\right) + \cdots \quad \text{as } x \to \infty$$

Q5.

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^{\zeta}\zeta^{-z}\,d\zeta \qquad \text{as} \qquad z \longrightarrow +\infty$$

$$= \frac{1}{2\pi i}\int e^{-z\ln\zeta+\zeta}\,d\zeta \qquad \zeta = zt \qquad d\zeta = z\,dt$$

$$= \frac{1}{2\pi i}\int_C e^{-z\ln(z)-z\ln(t)+zt} = \frac{z^{-z+1}}{2\pi i}\int e^{-z(\ln(t)-t)}\,dt$$

$f(z) = \ln t - t \rightarrow f(1) = -1$

$f' = \frac{1}{t} - 1 = 0 \Rightarrow \boxed{t=1}$

$f'' = -\frac{1}{t^2} \rightarrow f''(1) = -1$

$$= \frac{z^{-z+1}}{2\pi i}\int e^{-z\left[-1-\frac{(t-1)^2}{2}-\cdots\right]}\,dt = \frac{z^{1-z}}{2\pi i}e^{z}\int e^{\frac{z}{2}(t-1)^2}\,dt$$

$w = (t-1)^2 \qquad dw = 2(t-1)\,dt$

$2\int e^{\frac{z}{2}w}(t-1)^{-1}\,dt$

$2\sqrt{-1}\int e^{z/2\,w}(1+\dots)\,dt$

$2\sqrt{-1}\left.\frac{e^{z/2\,w}}{z/2}\right|_0^{\infty} =$

$$= \frac{z^{1-z}e^{z}}{\cancel{2}\pi \cancel{i}}\,\cancel{2i}\,\frac{\cancel{z}}{2}$$

$$= \frac{z^{2-z}e^{z}}{2\pi}$$

given there's no contribution from the imaginary

Q6:

In[ ]:= AsymptoticIntegrate$\left[\dfrac{\sqrt{t}}{t^{\wedge}3+1}\text{Cos}[l\,t],\ \{t,0,\text{Infinity}\},\ \{l,\text{Infinity},1\}\right]$

Out[ ]= $-\frac{1}{6}\pi\,\text{Cos}[l]+\frac{1}{3}\pi\,\text{Cos}\left[\frac{l}{2}\right]\text{Cosh}\left[\frac{\sqrt{3}\,l}{2}\right]-\frac{1}{6}\pi\,\text{Cosh}\left[\left(-i\,l^{3}\right)^{1/3}\right]-\frac{1}{3}i\pi\,\text{Cosh}\left[\frac{\sqrt{3}\,l}{2}\right]\text{Sin}\left[\frac{l}{2}\right]-\frac{1}{6}i\pi\,\text{Sin}[l]-\frac{1}{6}\pi\,\text{Sinh}\left[\left(-i\,l^{3}\right)^{1/3}\right]$