Modelling of gravitational force between a football and a golf ball

Along side electromagnetism, the strong and weak nuclear forces, gravity is one of the fundamental forces in the universe. It is known that all the objects in the universe attract each other including two balls in a room. Albeit, we do not see balls suddenly get pulled toward each other without an external force. The reason is that gravitational force is very weak and frictional forces prevent such phenomenon. In this work, I assume a hypothetical scenario where the two balls are in a long frictionless room with no other objects. Also, the friction force between the balls and the floor is zero. Under such circumstances we should observe that the balls move toward each other due to the gravitational force until they collide. Further, I assume that the balls have uniform density, where the density only depends on radius and not the angles. The parameters M, m and R denote mass of football, mass of golf ball and the distance between them respectively.

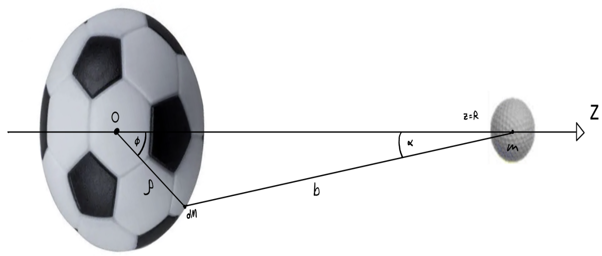

Figure 1: The geometric representation of the balls.

The football is not a point but its roughly a sphere and here I am setting myself the question of calculating the gravitational attraction of a football on a point particle which I could think of as the golf ball. I can simply use the Newton's triple integral to model the gravitational force, even though every atom in the football pulling on every atom on the golf ball from different distances. The football is very big compared to the golf ball, and all football atoms are at different distances from the atoms in the golf ball. Some at the back of the football and some are at the front of the football that are closer to the golf ball.

Although football is not a point, one can calculate the gravitational pull of all the different particles in the football and add them up to show that the net gravitational attraction of the football is exactly as if all of its mass were concentrated at the perfect geometrical centre of the football. Which means one can replace the football with a single point at the centre.

Based on the geometric representation of the balls shown in Figure 1 the force is defined as

$$dF = \frac{GmdM}{b^2},$$

where G is a constant. b is the distance between dM and m points, dM is a mass element and it is pulling on the other mass m. The value of b changes depending on which mass element I refer to.

Think of each point of the football as a little box which mathematically could be defined as volume element. Then add up all the boxes and each have gravitational pulling on the mass of the golf ball. The dM exerts a force on m. There's only one force in z-direction which is an arrow between m-o points. Centre of masses. Hence, there is zero net forces in x and y directions and only in z direction we see a net force

$$dF_x = 0, \quad dF_y = 0, \quad dF_z = \frac{GmdM}{b^2}\cos\alpha.$$

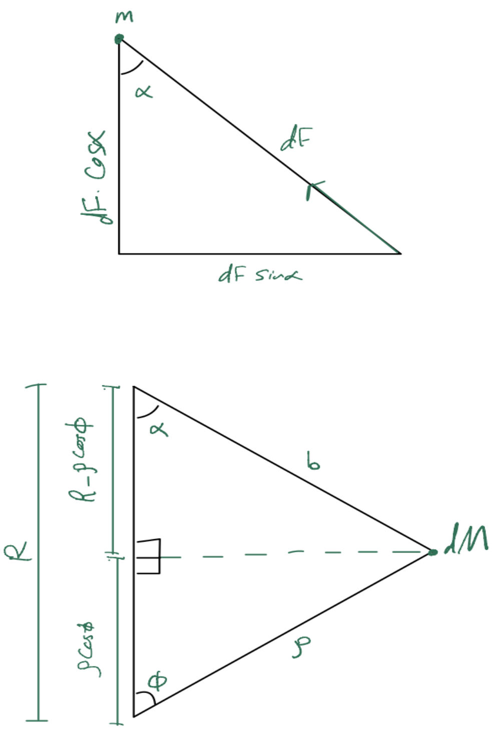

Figure 2: This is the detailed representation of the force between the balls in Figure 1.

Here using expressions in terms of α is not convenient because it depends on dM, so I want express all calculations in terms of ϕ

$$b^2 = R^2 + \rho^2 - 2R\rho\cos\phi,$$

$$\frac{R - \rho\cos\phi}{b} = \cos\alpha,$$

thus

$$dF_z = \frac{Gm\cos\alpha}{b^2}\, dM = \frac{Gm(R - \rho\cos\phi)}{(R^2 + \rho^2 - 2R\rho\cos\phi)^{3/2}}\, dM.$$

Remember the football is assumed to be homogeneous sphere (uniform density) and this means the mass is just the density times the volume

$$dM = \delta dV = \text{density} * \text{volume}.$$

The volume element is defined as

$$\begin{aligned} dV &= r d\theta \; d\rho \; \rho d\phi, \\ &= \rho \sin\phi d\theta \; d\rho \; \rho d\phi, \\ &= \rho^2 \sin\phi d\phi \; d\rho \; d\theta, \end{aligned}$$

where the coordinates are

$$\begin{aligned} x &= r\cos\theta = (\rho\sin\phi)\cos\theta \\ y &= \rho\sin\theta = (\rho\sin\phi)\sin\theta, \\ z &= \rho\cos\phi. \end{aligned}$$

Volume in the force may be expressed as

$$dF_z = \frac{Gm(R-\rho\cos\phi)}{(R^2+\rho^2-2R\rho\cos\phi)^{3/2}}\delta\underbrace{\rho^2\sin\phi d\rho d\theta}_{\text{volume}}.$$

dF_z is the amount of force produced by the small volume element. Now we need to add up all the force contributions over the entire football which is done by the triple integral. In other words, the net force in z-direction is the triple integral of all infinitesimal forces taken over the whole football

$$F_z = \iiint_{football} dF_z = \int_{\phi=0}^{\pi}\int_{\rho=0}^{a}\int_{\theta=0}^{2\pi}\frac{Gm(R-\rho\cos\phi)}{(R^2+\rho^2-2R\rho\cos\phi)^{3/2}}\delta\rho^2\sin\phi\ d\theta d\rho d\phi.$$

where a is the radius of the football.

$$F_z = 2\pi Gm\delta\int_{\phi=0}^{\pi}\int_{\rho=0}^{a}\frac{\rho^2\sin\phi\ (R-\rho\cos\phi)}{(R^2+\rho^2-2R\rho\cos\phi)^{3/2}}\ d\rho d\phi.$$

Instead of directly integrating with respect to ϕ we could use the substitution $u = R^2+\rho^2-R\rho\cos\phi$ and $u_{1,2} = (R\pm\rho)^2$

$$\int_{u_1}^{u_2}\frac{u+R^2-\rho^2}{4R^2\rho u^{3/2}}du = \frac{2}{R^2}.$$

Now integrating with respect to ρ

$$F_z = \frac{4\pi Gm\delta}{R^2}\int_0^a\rho^2 d\rho = \overbrace{\left(\frac{4\pi a^3}{3}\delta\right)}^{\text{football volume = M}}\frac{Gm}{R^2} = \frac{G\ m\ M}{R^2}.$$

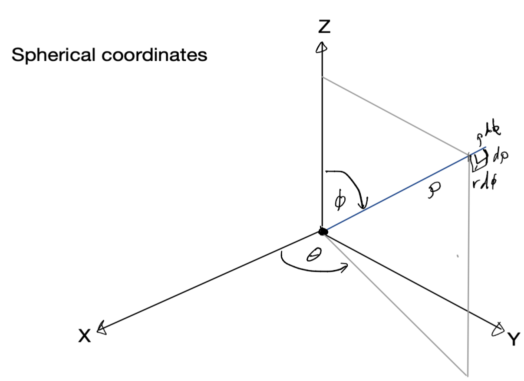

Figure 3: The spherical coordinates.

I restated the problem by assuming the football has a uniform density and then I added up the contributions of all the atoms one spherical shell by another starting at the centre and then expanding out till we get to the outer radius of the football. I wanted to show that the football's gravitational pull on a point mass (golf ball) outside of itself is the same as if all its mass were concentrated at the centre. To do this task I assumed the Newton's inverse square law. This law states that the force due to a mass M on another mass m where they are separated by a distance R then the force is $F = \frac{GmM}{R^2}$. We assumed this law applies to every pair of points on the football and the golf ball. The idea that you can replace all the mass by putting it at the geometrical centre of the configuration (football) of all the little masses, thats not an obvious property of a mass, it only works for uniformly distributed masses. Note that such force would not formed on the earth due to frictional forces and thus two objects would be affect each other by exerting gravitational force. A realistic scenario would be the gravitational force of the sun on the earth.