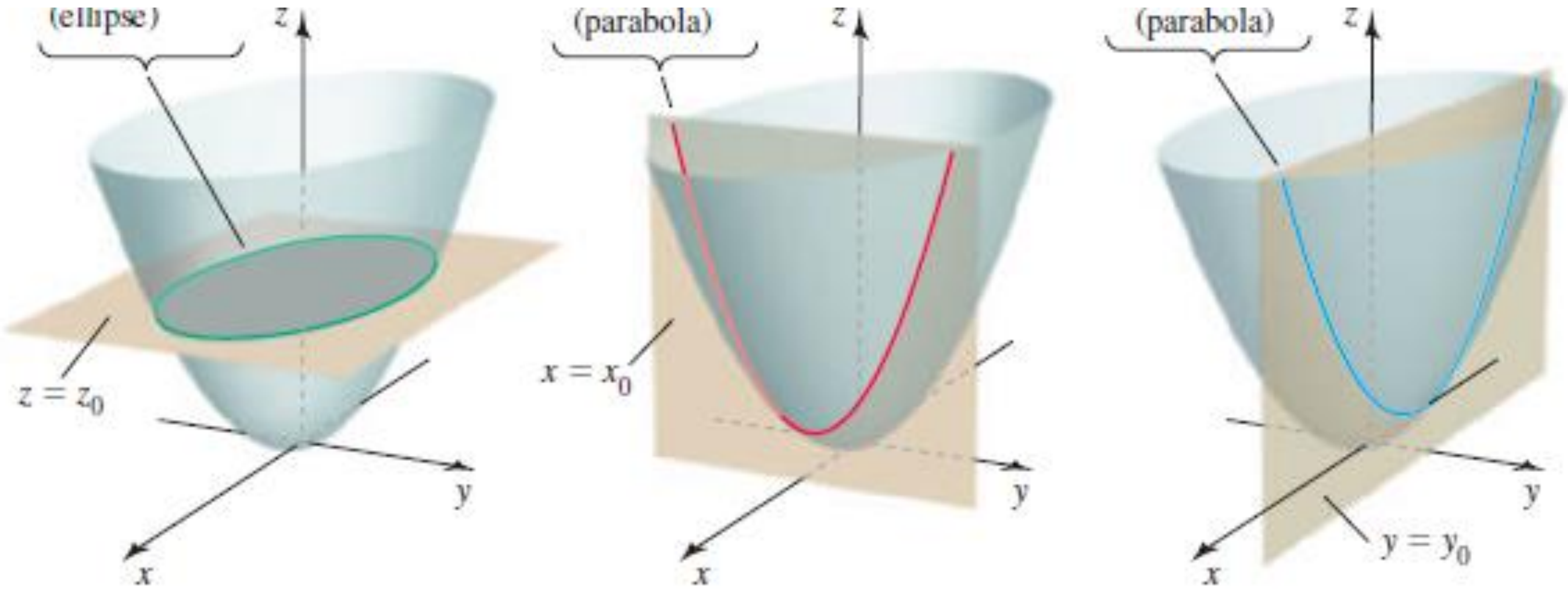
Lecture 9 - Chapter 10 – Sec. 10.6

Cylinders & Quadric Surfaces



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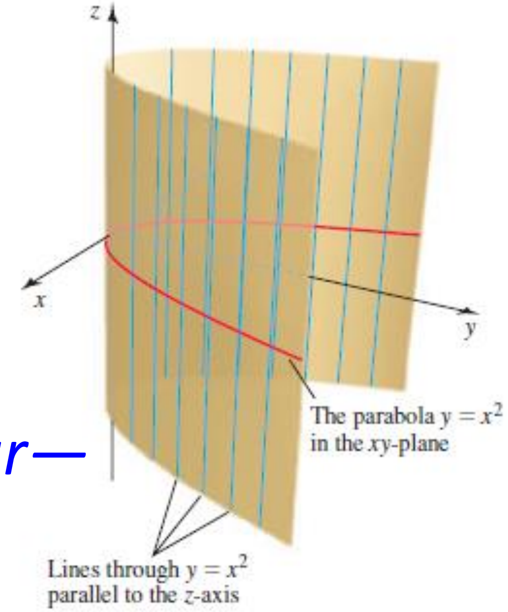
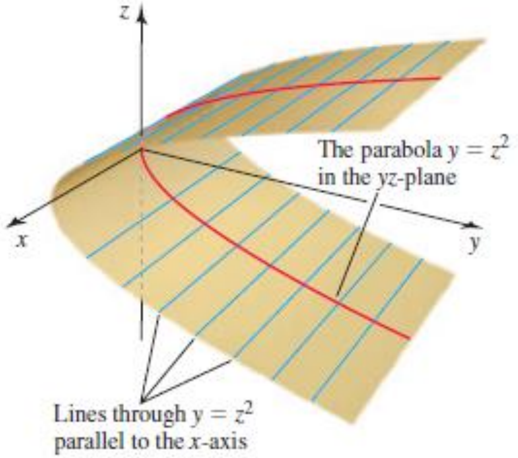
Learning Objectives

- *Identify a cylinder as a type of three-dimensional surface.*
- *Recognize the main features of ellipsoids, paraboloids, and hyperboloids.*
- *Use traces to draw the intersections of quadric surfaces with the coordinate planes.*

Cylinders

- *In the context of three-dimensional surfaces, the term cylinder refers to a surface that is parallel to a line - i.e., parallel to one of the coordinate axes.*
- *Equations for such cylinders are easy to identify: The variable corresponding to the coordinate axis parallel to the cylinder is missing from the equation*
- *For example, in $\mathbb{R}^3$, the equation $y = x^2$ does not include z, which means that z is arbitrary and can take on all values.*
- *Therefore, $y = x^2$ describes the cylinder consisting of all lines parallel to the z-axis that pass through the parabola $y = x^2$ in the xy-plane.*





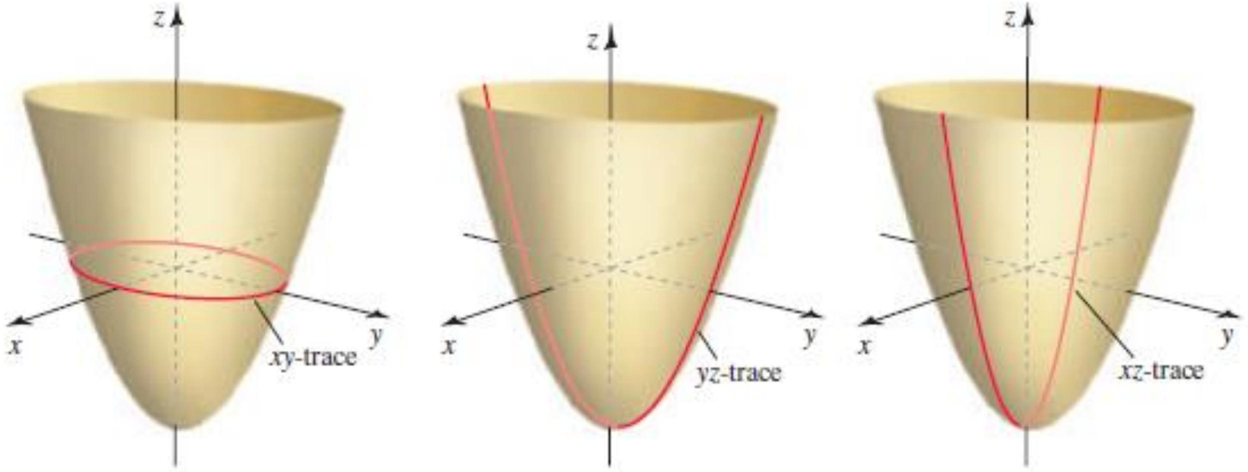
Graphing surfaces—and cylinders in particular—is facilitated by identifying the traces of the surface

Traces of a Surface

***DEFINITION* Trace**

*A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the **xy-trace**, the **yz-trace**, and the **xz-trace***

*A **cylinder** is a surface that consists of all lines (**rulings, traces**) that are parallel to a given line and pass through a given plane curve.*



Follow-up: To which coordinate axis in $\mathbb{R}^3$ is the cylinder $z - 2\log x$ parallel? To which coordinate axis in $\mathbb{R}^3$ is the cylinder $y = 4z^2 - 1$ parallel?

Graphing Cylinders – Q. 3-8, Ex. 10.6

Sketch the graphs of the following cylinders in $\mathbb{R}^3$. Identify the axis to which each cylinder is parallel. (a) $x^2 + 4y^2 = 16$ *(b)* $x - \sin z = 0$

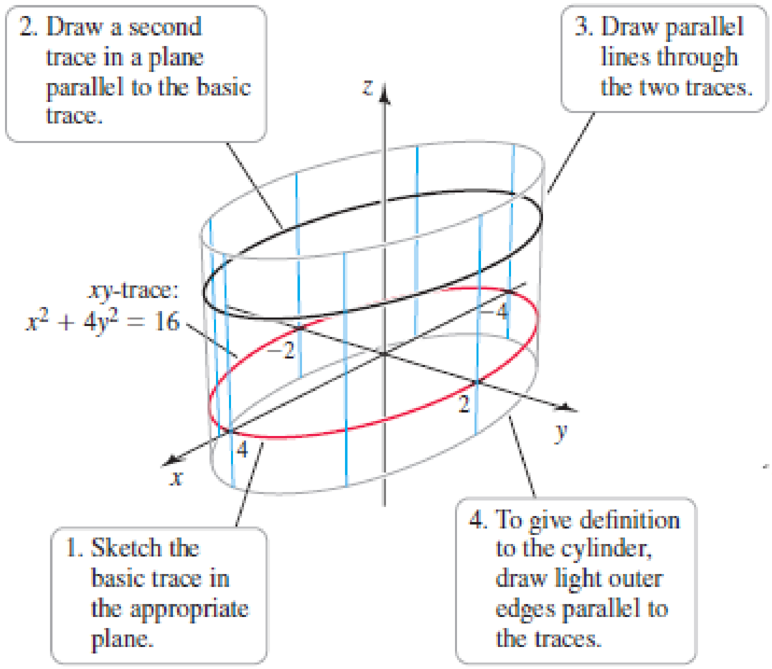
Solution: (a) As an equation in $\mathbb{R}^3$, the variable z is absent. Therefore, z assumes all real values and the graph is a cylinder consisting of lines parallel to the z-axis passing through the curve $x^2 + 4z^2 = 16$ in the xy-plane.

You can sketch the cylinder in the following steps.

1. *Rewrite the given equation as*

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

We see that the trace of the cylinder in xy-plane (the xy-trace) is an ellipse. We begin by drawing this ellipse.

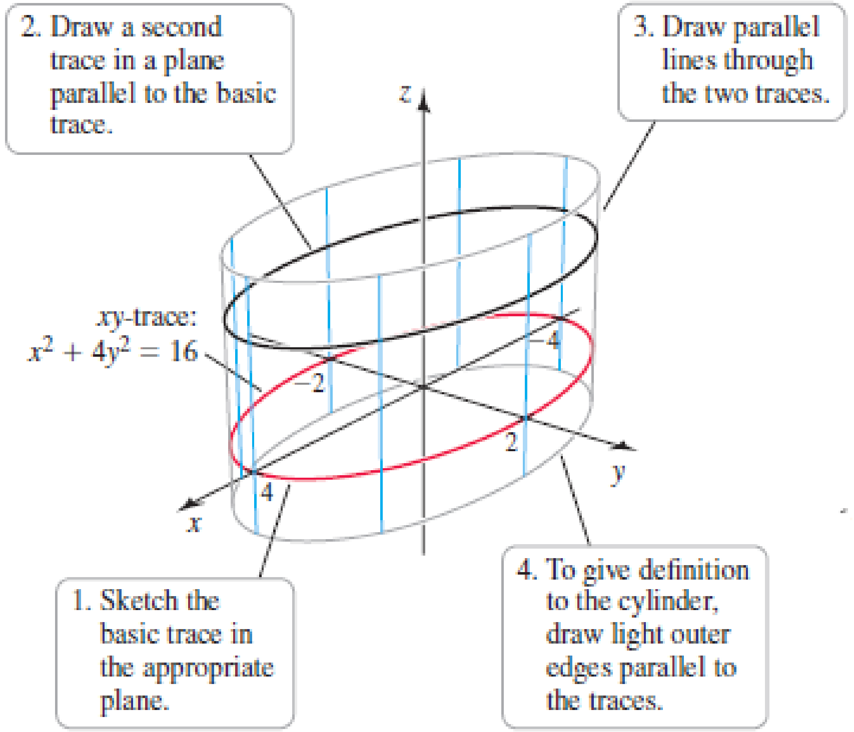


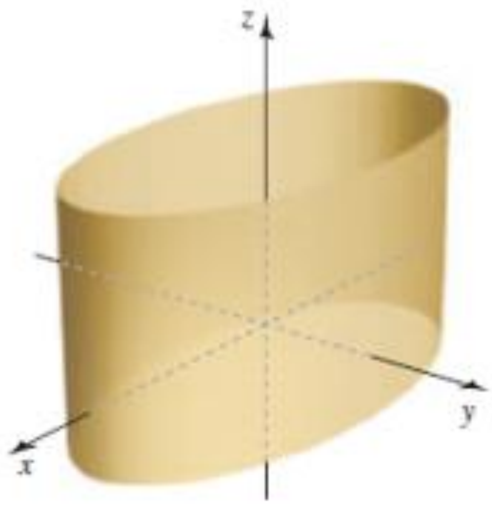
Graphing Cylinders – Q. 3-8, Ex. 10.6

2. Next draw a second trace (a copy of the ellipse in Step 1) in a plane parallel to the xy-plane.

3. Now draw the lines parallel to z-axis through the two traces to fill out the cylinder.

4. The resulting surface, called an elliptic cylinder, runs parallel to z-axis.





Elliptic cylinder

Graphing Cylinders – Q. 3-8, Ex. 10.6

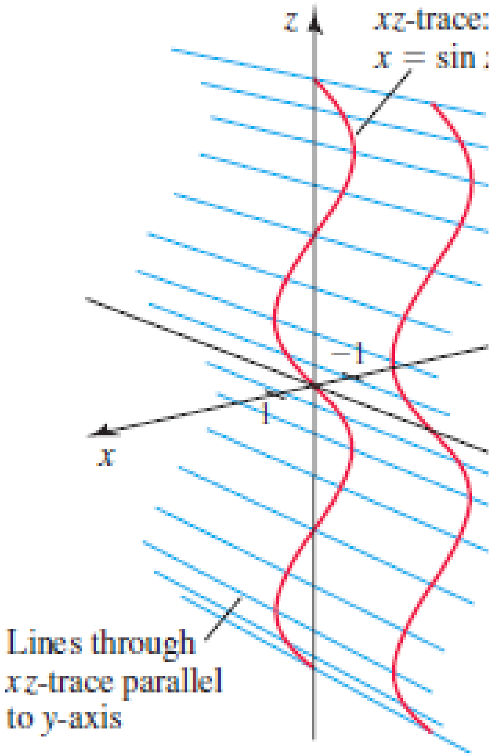
(b) As an equation in $\mathbb{R}^3$, $x - \sin z = 0$ *is missing the variable y. Therefore, y assumes all real values and the graph is a cylinder consisting of lines parallel to the y-axis passing through the curve* $x = sinz$ *in the xz-plane.*

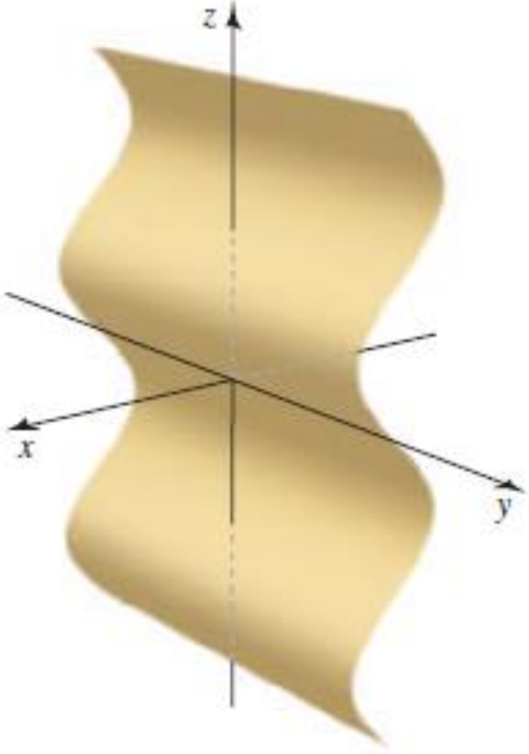
1. Graph the curve x = sin z in the xz-plane, which is the xz-trace of the surface.

2. Draw a second trace (a copy of the curve in Step 1) in a plane parallel to the xz-plane.

3. Draw lines parallel to the y-axis passing through the two traces.

The result is a cylinder, running parallel to the y-axis, consisting of copies of the curve x = sin z.





Test your Knowledge

(a) What does the equation $y = x^2$ *represent as a curve in* $\mathbb{R}^2$*?*
(b) What does it represent as a surface in $\mathbb{R}^3$*?*
(c) What does the equation $z = y^2$ *represent?*
(d) Describe and sketch the graph of $y = e^x$ *as a curve in* $\mathbb{R}^3$

Quadric Surfaces

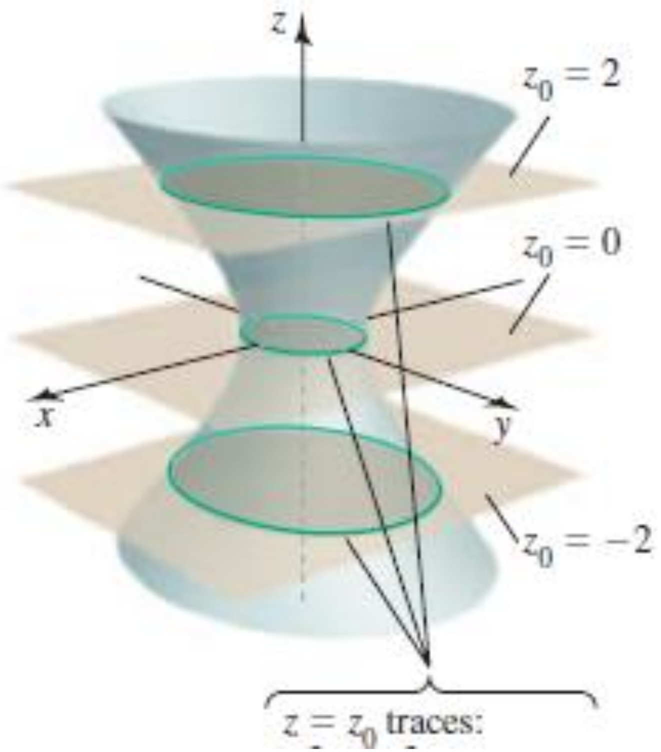
Quadric surfaces *are described by the general quadratic (second-degree) equation in three variables,* x, y *and* z*.*
The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Gx + Hy + Iz + J = 0$$

Where $A, B, C, \cdots I$ *are constants.*
By translation and rotation, it can be brought into one of the two standard forms

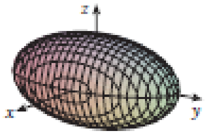
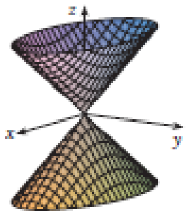
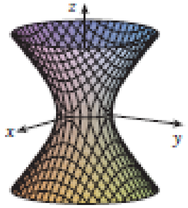
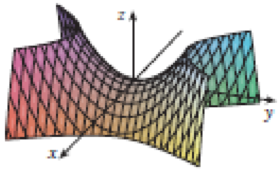
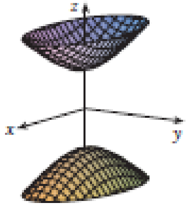
$$Ax^2 + By^2 + Cz^2 + j = 0 \qquad or$$
$$Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic section in the plane.



Test your Knowledge

Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example – Q. 9. Ex. 10.6

(a) Find and identify the traces of the quadric surface

$$x^2 + y^2 - z^2 = 1$$

and explain why the graph looks like the graph of the hyperboloid.

(b) If we change the equation in part (a) to

$$x^2 - y^2 + z^2 = 1$$

how is the graph affected?

(c) What if we change the equation in part (a) to

$$x^2 + y^2 + 2y - z^2 = 0?$$

Example – Q. 9. Ex. 10.6

Solution:

The traces of

$$x^2 + y^2 - z^2 = 1$$

in x=k are $y^2 - z^2 = 1 - k^2$*, a family of hyperbolas. (The hyperbolas are oriented differently for* $-1 < k < 1$ *than for* $k < -1$ *or* $k > 1$*.*

The traces in y=k are $x^2 - z^2 = 1 - k^2$*, a similar family of hyperbolas.*

The traces in z=k are $x^2 + y^2 = 1 + k^2$*, is a family of circles.*

For k=0, the trace in the xy-plane, the circle is of radius 1.

This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet.

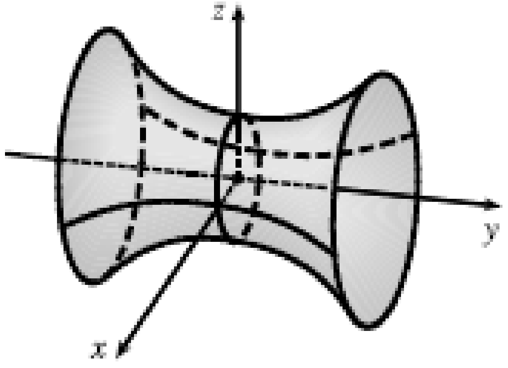
Example – Q. 9. Ex. 10.6

(b) If we change the equation

$$x^2 - y^2 + z^2 = 1$$

the shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis.

Traces in y=k are circles, while traces in x=k and z=k are hyperbolas.

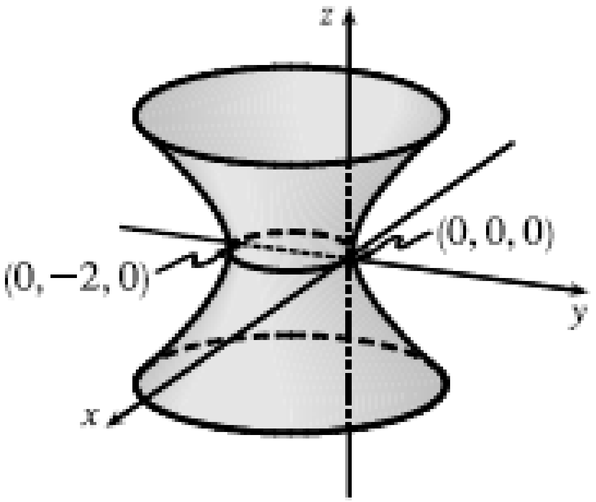


(c) Completing the square in

$$x^2 + y^2 + 2y - z^2 = 0$$

gives $x^2 + (y-1)^2 - z^2 = 1.$

The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.



Example – Q. 19. Ex. 10.6

Use traces to sketch and identify the surface $y = z^2 - x^2$.

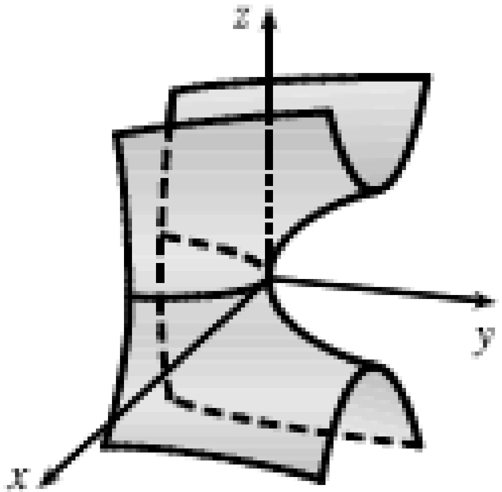
Solution:

The traces in x=k are the parabolas $y = z^2 - k^2$, opening in the positive y-direction.

The traces in y=k are k= $z^2 - x^2$, two intersecting lines when k=0 and a family of hyperbolas for $k \neq \ 0$ (note that the hyperbolas are oriented differently for k> 0 than for k< 0).

The traces in z=k are the parabolas $y = k^2 - x^2$ which open in the negative y-direction.

Thus the surface is a hyperbolic paraboloid centred at (0,0,0).



Sketch using MATLAB

Homework 1 – 10.6

Match the equation with its graph (labelled I–VIII). Give reasons for your choice.

$x^2 + 4y^2 + 9z^2 = 1$

$9x^2 + 4y^2 + z^2 = 1$

$x^2 - y^2 + z^2 = 1$

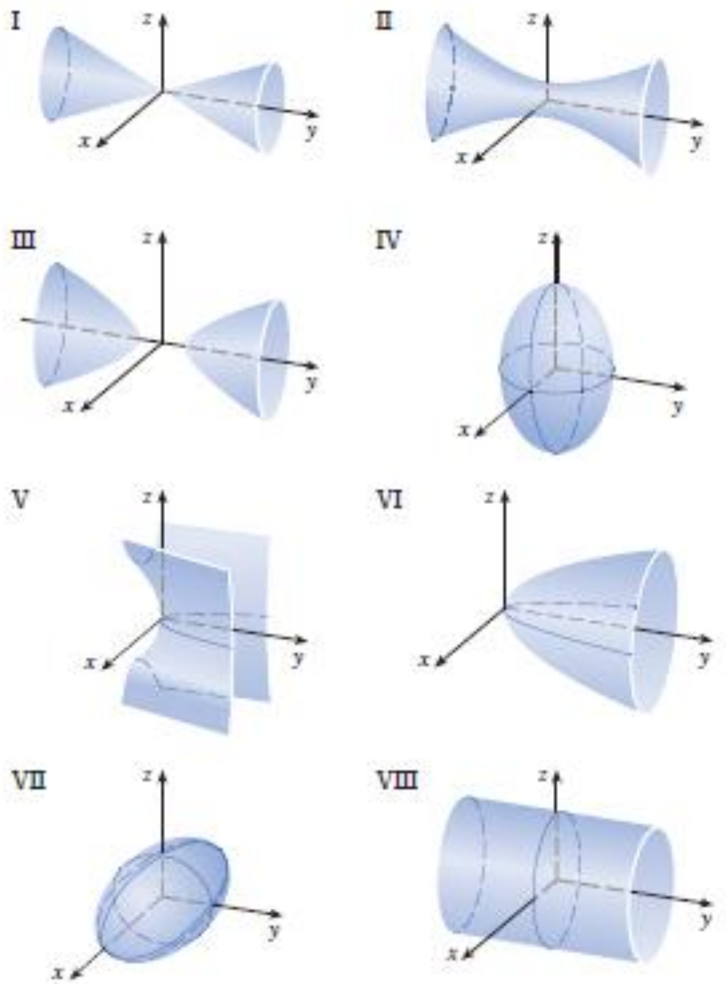
$-x^2 + y^2 - z^2 = 1$

$y = 2x^2 + z^2$

$y^2 = x^2 + 2z^2$

$x^2 + 2z^2 = 1$

$y = x^2 - z^2$



Example – Q. 9. Ex. 10.6

Reduce the equation to one of the standard forms, classify the surface, and sketch it.

$$4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$$

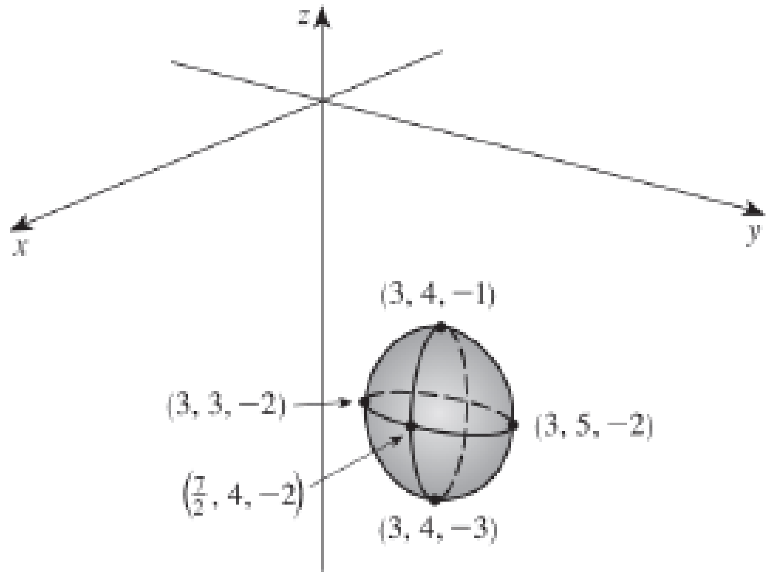
Solution:

Completing squares in all three variables gives

$$4(x^2 - 6x + 9) + (y^2 - 8y + 16) + (z^2 + 4z + 4) = -55 + 36 + 16 + 4$$

$$4(x-3)^2 + (y-4)^2 + (z+2)^2 = 1 \text{ or}$$

$$\frac{(x-3)^2}{1/4} + (y-4)^2 + (z+2)^2 = 1$$



This is an ellipsoid with centre (3, 4, −2).

Example – Q. 31. Ex. 10.6

Find an equation for the surface consisting of all points that are equidistant from the point (-1, 0, 0) and the plane x = 1. Identify the surface

Solution: *Let* $P = (x, y, z)$ *be an arbitrary point equidistant from (-1,0,0) and the plane x=1.*

Then the distance from P to (-1,0,0) is

$$\sqrt{(x+1)^2 + y^2 + z^2}$$

and the distance from P to the plane x=1 is

$$\frac{|x-1|}{\sqrt{1^2}} = |x-1|$$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

So $|x-1| = \sqrt{(x+1)^2 + y^2 + z^2} \Leftrightarrow -4x = y^2 + z^2$.

Thus, the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the –ve direction.

Homework 2 – Ex. 10.6

Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface