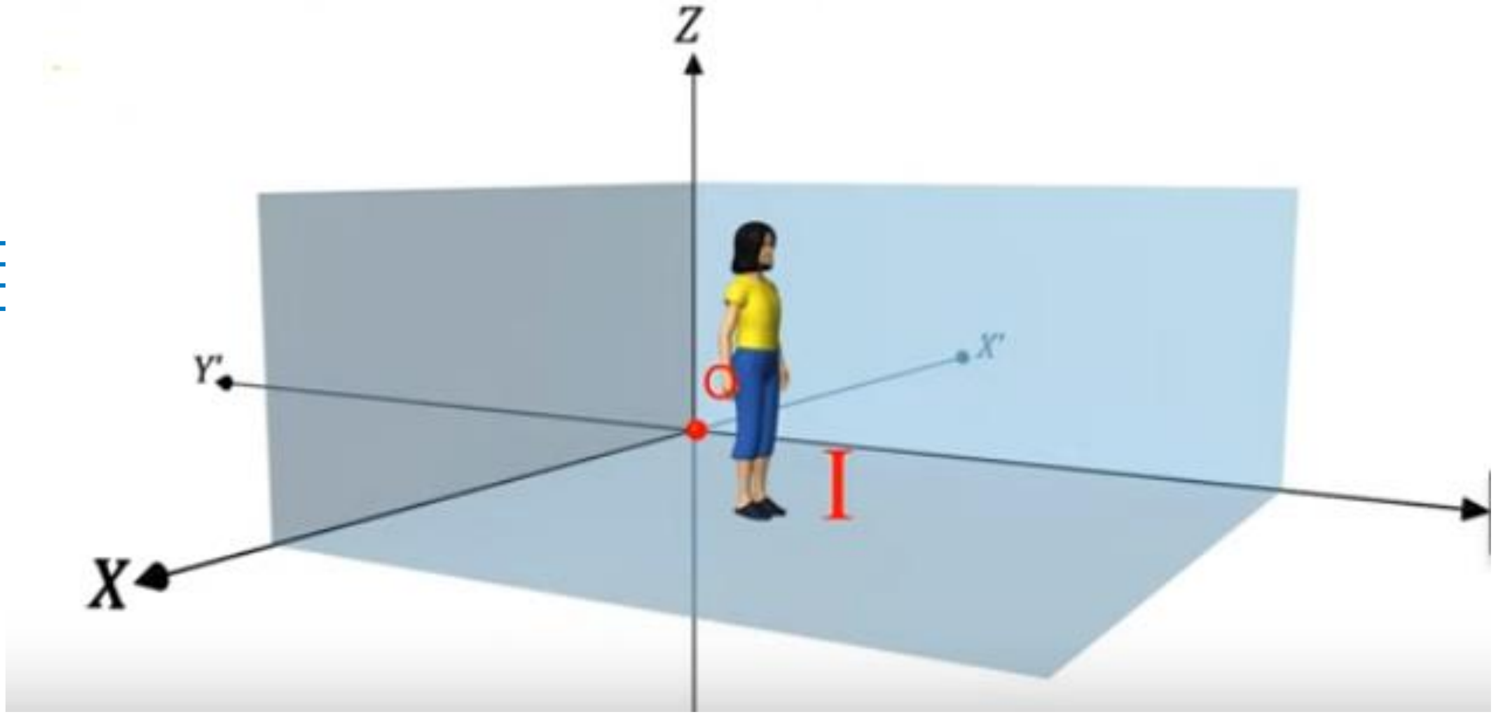
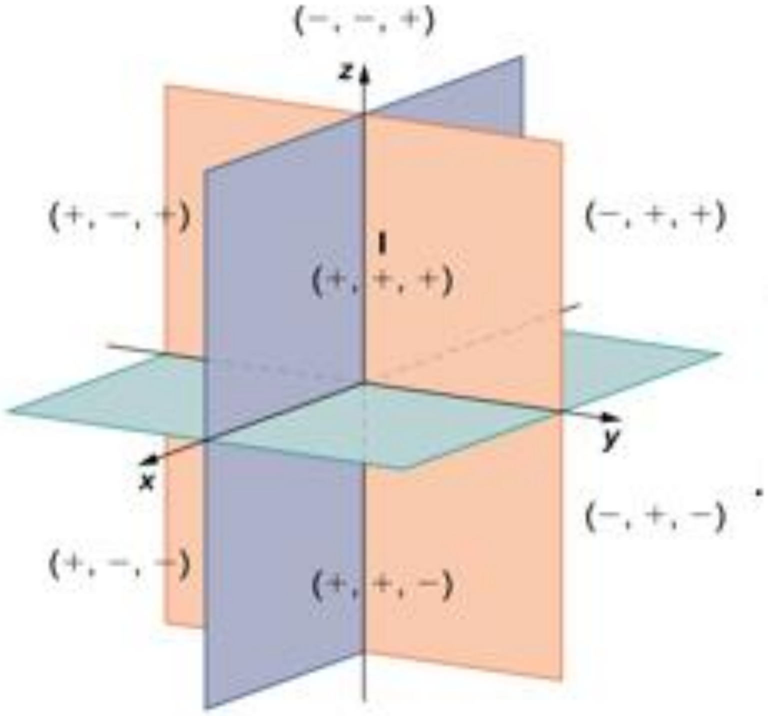
Lecture 5 - Chapter 10 – Sec. 10.1

Geometry of Space





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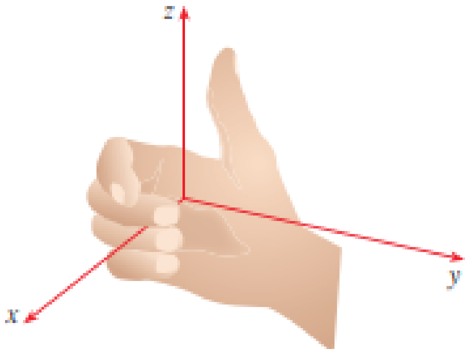
Learning Objectives

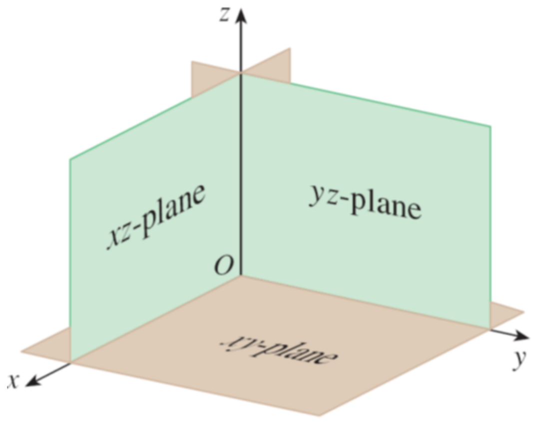
- *Plot points in three-dimensional space.*
- *Find the distance between two points in three-dimensional space.*
- *Write the equation of a sphere centred at a given point with a given radius.*
- *Find symmetric points about: a point, a line, or a plane.*
- *Parameterize a line segment.*
- *Find the midpoint of a line segment.*

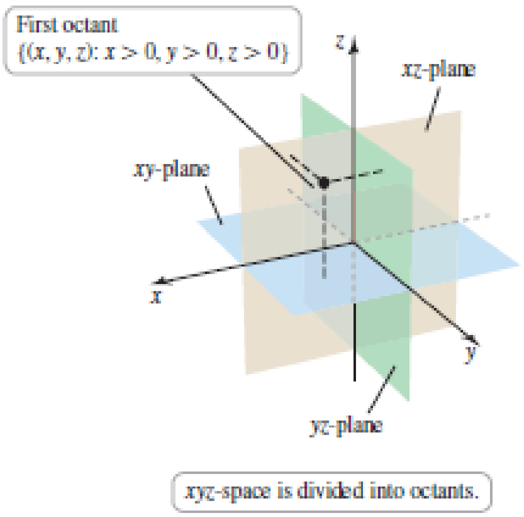
3D Space Geometry

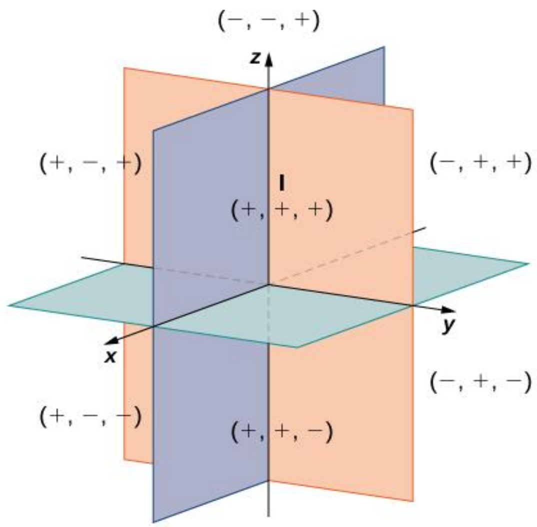
- *The three coordinate axes determine the three* ***coordinate planes.***
- *These three coordinate planes divide space into eight parts, called* ***octants****. The* ***first octant****, in the foreground, is determined by the positive axes*.
- *You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point* O.

Simulation on the Blackboard



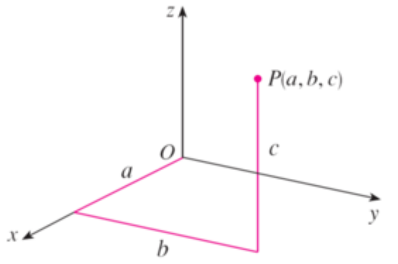




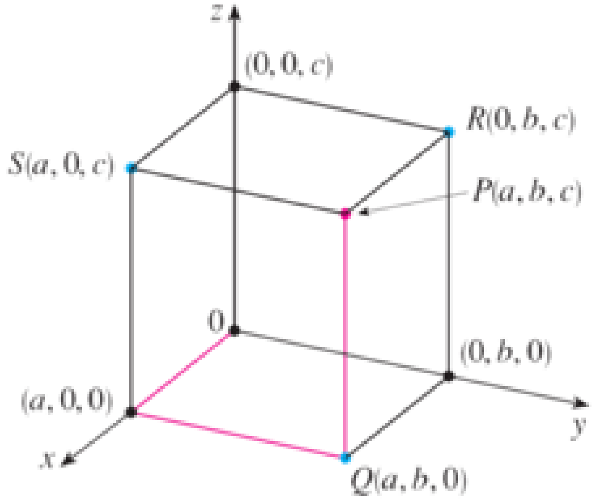


3D Space Geometry

- *Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis.*



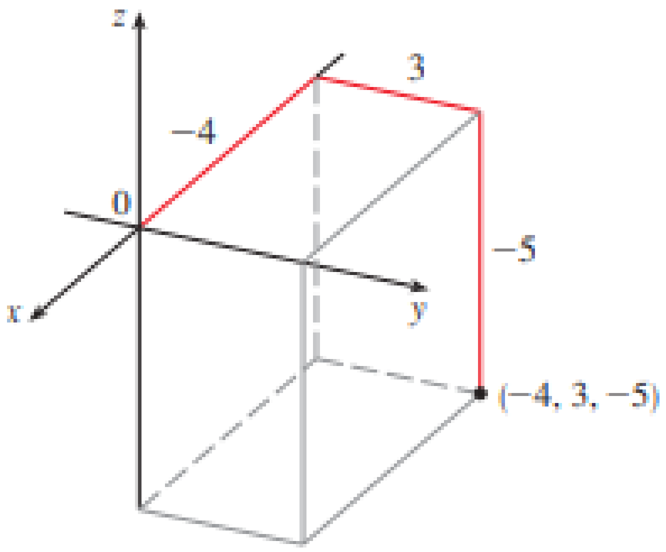
- *If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the* ***projection*** *of P onto the xy-plane.*
- *Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.*

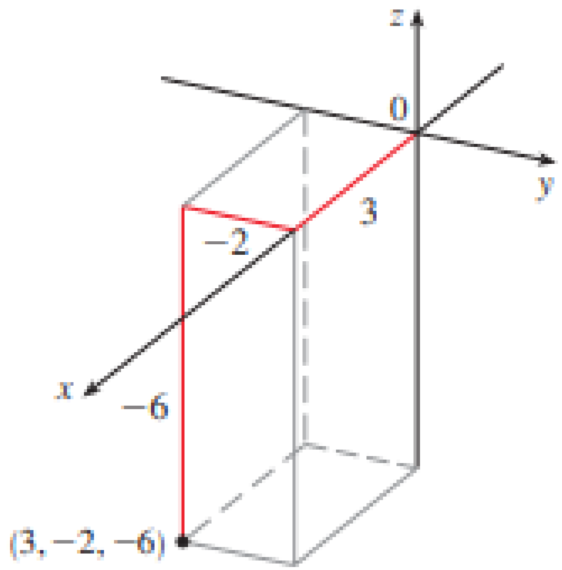


3D Space Geometry

- *The Cartesian product* $\mathcal{R} \times \mathcal{R} \times \mathcal{R} = \{(x, y, z) | x, y, z \in \mathcal{R}\}$ *is the set of all ordered triples of real numbers and is denoted by* $\mathcal{R}^3$
- *It is called a* ***three-dimensional rectangular coordinate system***.

- *As numerical illustrations, the points (−4, 3, −5) and (3, −2, −6) are plotted in*



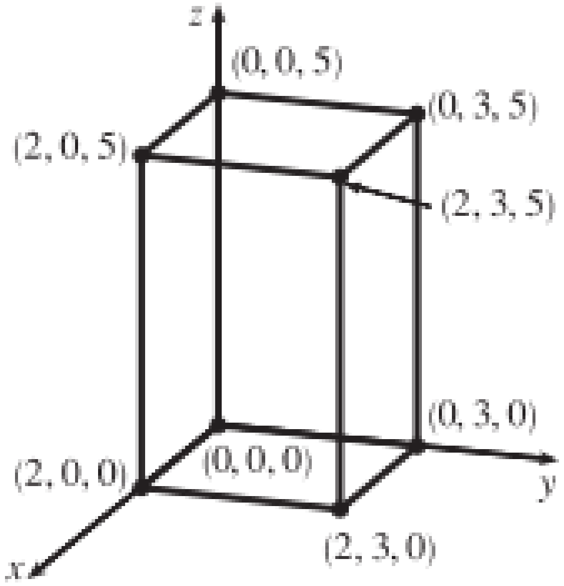


Example – Q. 4. Ex. 10.1

What are the projections of the point (2,3,5) on the xy-, yz-, and zx- planes? Draw a rectangular box with the origin and (2,3,5) as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box.

Solution:

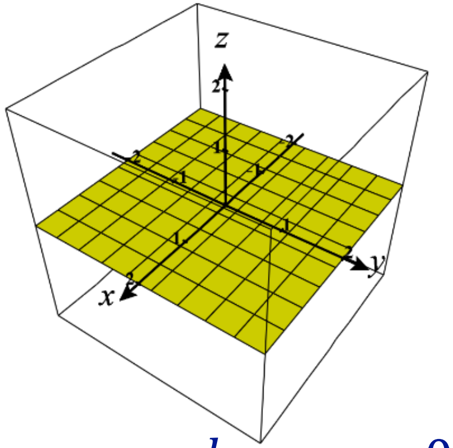
The projection of $(2,3,5)$ *onto the* xy- *plane is* $(2,3,0)$; *onto the* yz- *plane,* $(0,3,5)$; *onto the* zx- *plane,* $(2,0,5)$.



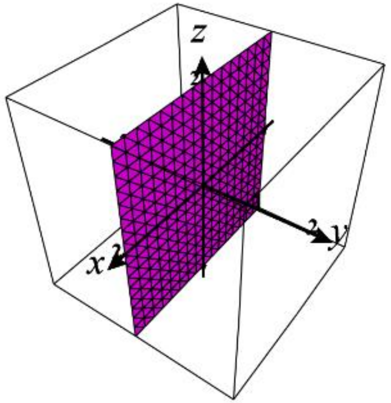
Surfaces and Solids

- *In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathcal{R}^2$*
- *In three-dimensional analytic geometry, an equation in x, y, and z represents a surface in $\mathcal{R}^3$*
- *The xy-plane consists of all points in xyz-space that have a z-coordinate of 0. Therefore, the xy-plane is the set $\{(x, y, z)|z = 0\}$; it is represented by the equation $z = 0$. Similarly, the xz-plane has the equation $y = 0$, and the yz-plane has the equation $x = 0$.*

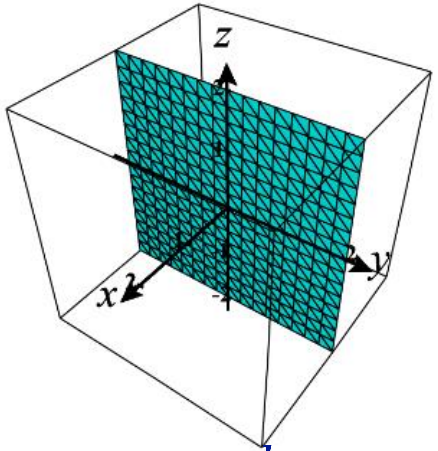
[3D Applet]



$xy - plane, z = 0$



$xz - plane, y = 0$



$yz - plane, x = 0$

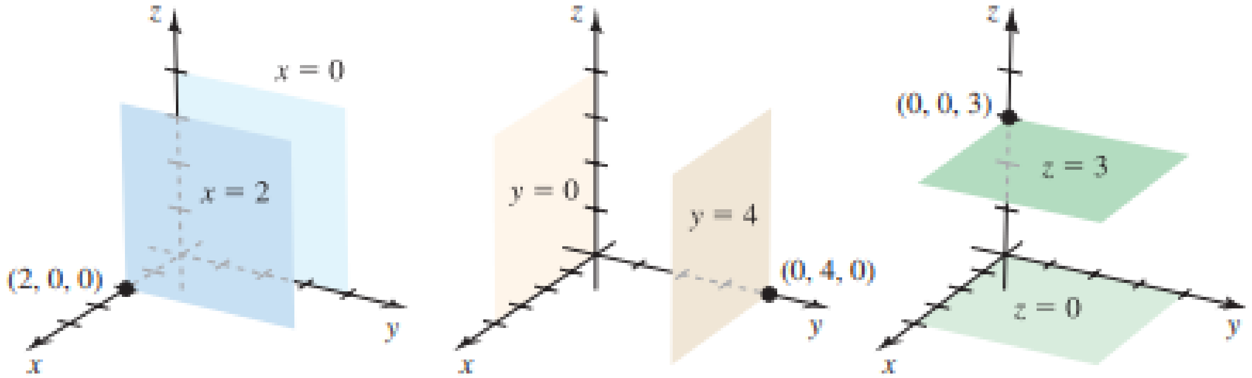
Surfaces and Solids

- ***Planes parallel to one of the coordinate planes***

The equation $x = a$ describes the set of all points whose x-coordinate is a and whose y and z-coordinates are arbitrary; this plane is parallel to and a units from the yz-plane.

Similarly, the equation $y = b$ describes a plane that is everywhere b units from the xz-plane, and $z = c$ is the equation of a horizontal plane c units from the xy-plane

[***3D Applet here***]



Follow-up: What does the equation $x = -4$ represent in $\mathcal{R}^2$? What does it represent in $\mathcal{R}^3$? Illustrate with sketches.

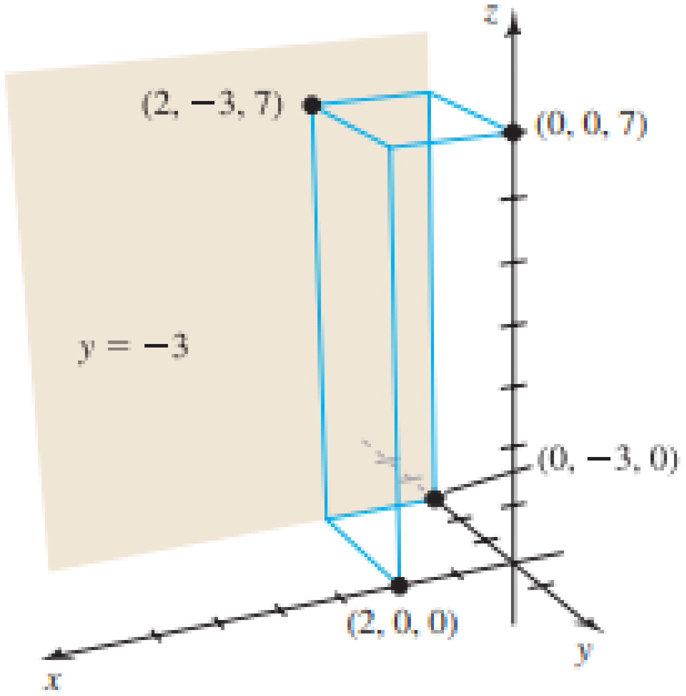
Example

Sketch the plane parallel to the xz-plane through $(2, -3, 7)$ and find its equation.

Solution:

Points on a plane parallel to the xz-plane have the same y-coordinate.

Therefore, the plane passing through the point $(2, -3, 7)$ with a y-coordinate of -3 has the equation $y = -3$.



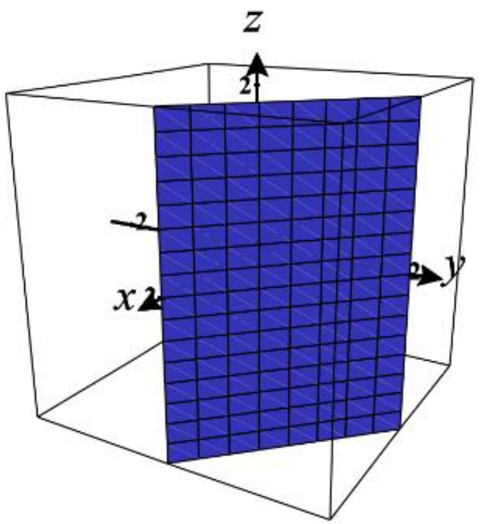
Surfaces and Solids

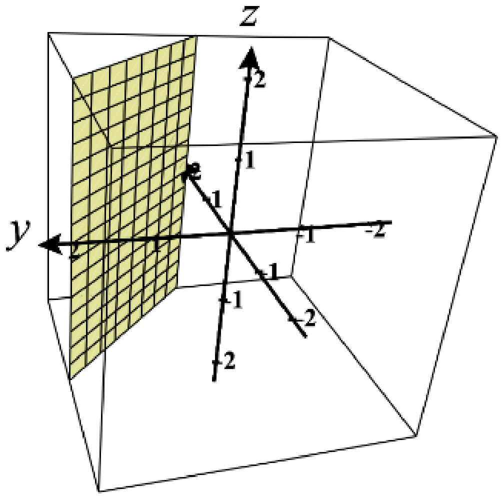
Q. 5. Ex. 10.1

Describe and sketch the surface in $\mathcal{R}^3$ represented by the equation $x + y = 2$.

Solution:

The equation $x + y = 2$ represents the set of all points in $\mathcal{R}^3$ where $y = 2 - x$. This is the set $\{x, x - 2, z \mid x \in \mathcal{R}, z \in \mathcal{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x, z = 0$.



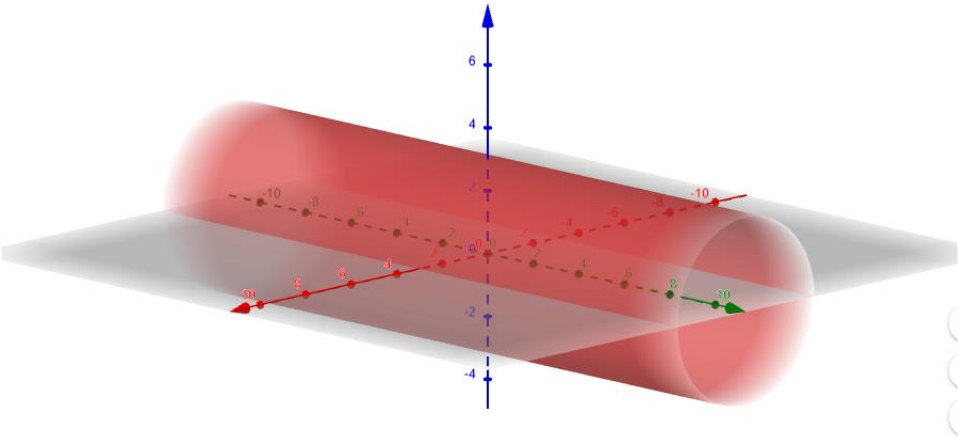


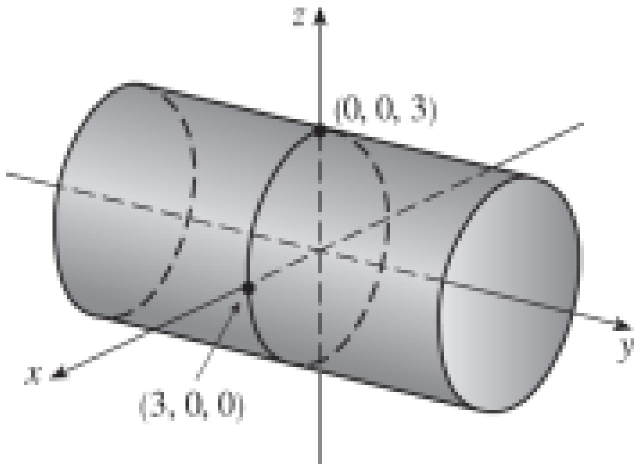
Surfaces and Solids

Describe and sketch the surface in $\mathcal{R}^3$ represented by the equation $x^2 + z^2 = 9$

Solution:

The equation $x^2 + z^2 = 9$ has no restrictions on y and the x and z coordinates satisfy the equation for a circle of radius 3 with centre on the origin. Thus the surface $x^2 + z^2 = 9$ consists of all possible vertical circles (parallel to the xy-plane) $x^2 + z^2 = 9,\ y = k$, and is therefore a circular cylinder with radius 3 whose axis is the y-axis





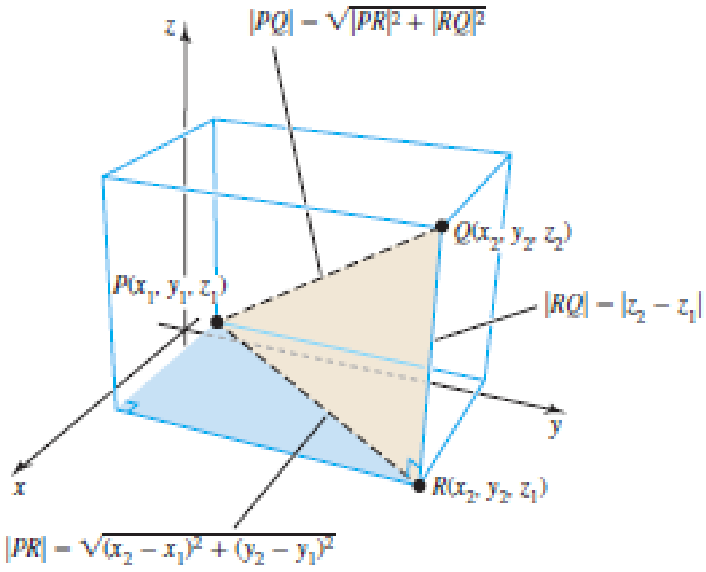
3D Space Geometry

Distance Formula in Three Dimensions

$R(x_2, y_2, z_1)$ *is an auxiliary point that has the same* z*-coordinate as P and the same* x*- and* y*-coordinates as Q.*
The distance $|\, PQ \,|$ *between the points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is given by*

$$|\, PQ \,| = \sqrt{|\, PR \,|^2 + |\, RQ \,|^2}$$

$$|\, PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



The ***midpoint*** *of the line segment joining* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$*, which is found by averaging the* x*-,* y*-, and* z*- coordinates.*

$$MD = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

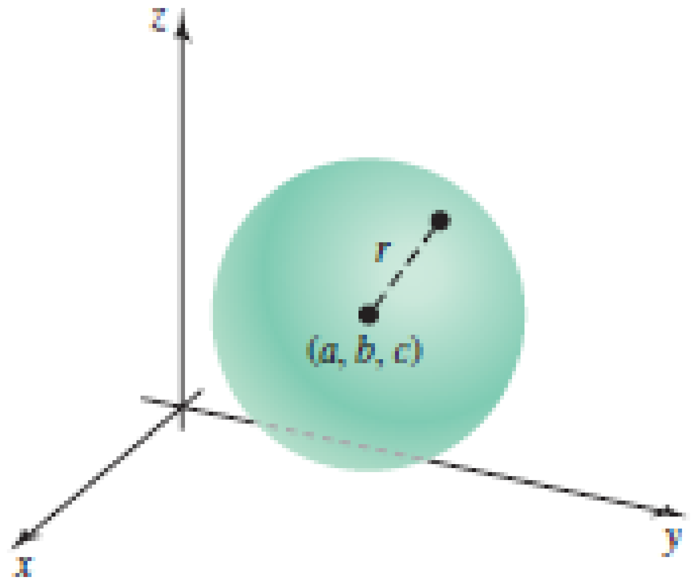
Equation of a Sphere

A sphere is the set of all points that are a constant distance r from a point (a, b, c); r is the radius of the sphere, and (a, b, c) is the centre of the sphere.
We now use the distance formula to translate these statements

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

*A **ball** centred at (a, b, c) with radius r is the set of points satisfying the inequality*

$$(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2$$



Sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
Ball: $(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2$

Practice Problem 1 – 10.1

a) Describe in words the region of $\mathcal{R}^3$ represented by the following inequality:

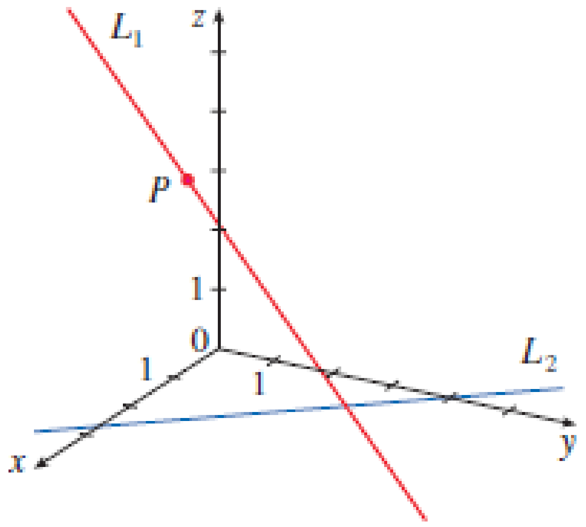
$$x^2 + y^2 + z^2 > 2z.$$

b) Write an inequality to describe the region consisting of all the points between (but not on) the sphere of radius r and R centred at the origin, where $r < R$.

(c) The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 onto the xy-plane. (In other words, the points on L_2 are directly beneath, or above, th points on L_1.)

i. Find the coordinates of the point P on the line L_1.

ii. Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz plane, respectively.



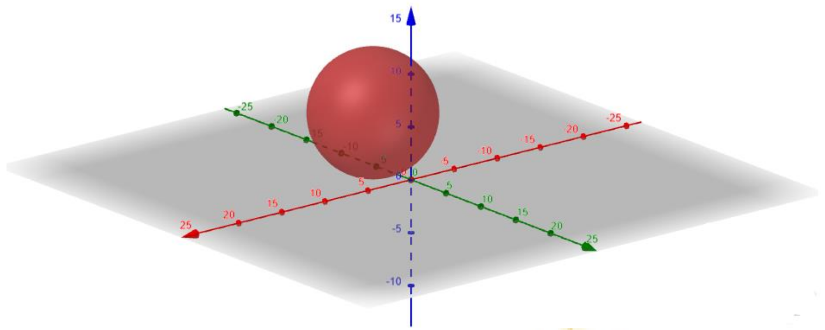
Example – Q. 19. Ex. 10.1

Find the equation of the spheres with centre (2, -3, 6) that touches
(a) the xy-plane, (b) the yz-plane, (c) the xz-plane.

Solution:

Centre of the sphere is $(2, -3, 6)$.

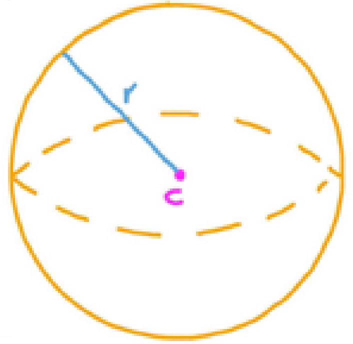
The sphere touches xy*-plane* $\Rightarrow$ x*-,* y*-coordinates are same but z is zero, i.e.,* $(2, -3, 0)$.



Therefore,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

gives $|\, r \,| = 11$*, and the equation of sphere is*



$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 11$$

Homework 1 – 10.1

(i) Find an equation of the sphere with centre (-3, 2, 5) and radius 4. What is the intersection of this sphere with the yz-plane?

(ii) Lengths of the diagonals of a box: What is the longest diagonal of a rectangular $2\ m \times 3\ m \times 4\ m$ *box?*

(iii) Determine whether the points P, Q, and R are collinear. If the points are collinear, determine which point lies between the other two points.

Example – Q. 37. Ex. 10.1

Find the distance between the spheres

$$x^2 + y^2 + z^2 = 4 \text{ and } x^2 + y^2 + z^2 = 4x + 4y + 4z - 11.$$

Solution: We need to be innovative about finding the distance between the spheres.

Let's first see what these spheres tell us:

$x^2 + y^2 + z^2 = 4$ *is a sphere with centre (0,0,0) and radius 2.*

$x^2 + y^2 + z^2 = 4x + 4y + 4z - 11 \Rightarrow (x-2)^2 + (y-2)^2 + (z-2)^2 = 1$ *is a sphere with centre (2,2,2) and radius 1.*

The (shortest) distance between the spheres is measured along the line segment connecting their centres. The distance between (0,0,0) and (2,2,2) is $2\sqrt{3}$.

Subtracting the radius of each circle, the distance between the spheres is

$$2\sqrt{3} - 3$$

Homework 2– 10.1

Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and

$$x^2 + y^2 + z^2 = 4.$$

Example – Q. 38. Ex. 10.1

Describe and sketch a solid with the following properties:

When illuminated by rays parallel to z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to x-axis, its shadow is an isosceles triangle.

Solution:

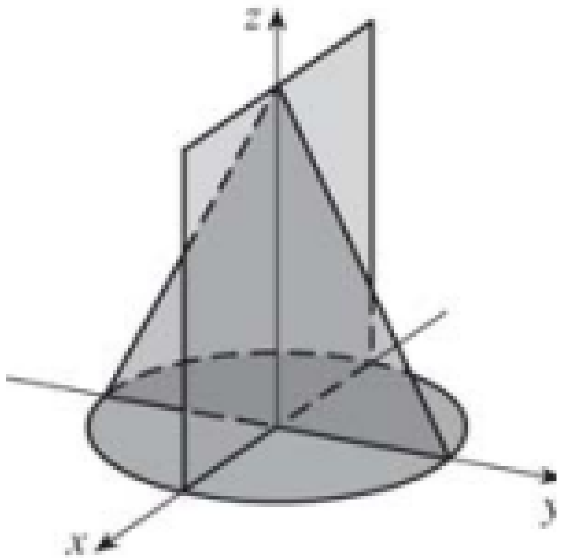
There are many different solids that fit the given description. However, what is the essential criteria for any such solid?

(i) The possible solid must have a circular horizontal cross-section at its top or at its base.

Consider a solid with a circular base in the xy-plane. The vertical cross-section through the centre of the base that is parallel to the xz-plane must be a square.

Example – Q. 38. Ex. 10.1

The vertical cross-section parallel to the yz-plane (perpendicular to the square) through the centre of the base must be a triangle with two vertices on the circle and the third vertex at the centre of the top side of the square.



i. *Draw the circular base and the vertical square first.*
ii. *Then draw a surface formed by line segments parallel to the yz-plane that connect the top of the square to the circle*

- *If we choose a circular cross-section at the top, this results in an inverted version of the solid described above.*