Lecture 18 (Chapter 11 – Sec. 11.6)

Direction Derivatives & Gradient Vector

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Need for the Directional Derivatives

- *Partial derivatives tell us a lot about the rate of change of a function on its domain. However, they do not directly answer some important questions.*

- *Suppose you are standing at a point* $(a, b, f(a, b))$ *on the surface* $z = f(x, y)$.

- *The partial derivatives* f_x *and* f_y *tell you the rate of change (or slope) of the surface at that point in the directions parallel to the x-axis and y-axis, respectively.*

- *But you could walk in an infinite number of directions from that point and find a different rate of change in every direction*

Need for the Directional Derivatives

. With this observation in mind, we pose several questions.

- *Suppose you are standing on a surface and you walk in a direction other than a coordinate direction—say, northwest or south-southeast. What is the rate of change of the function in such a direction?*

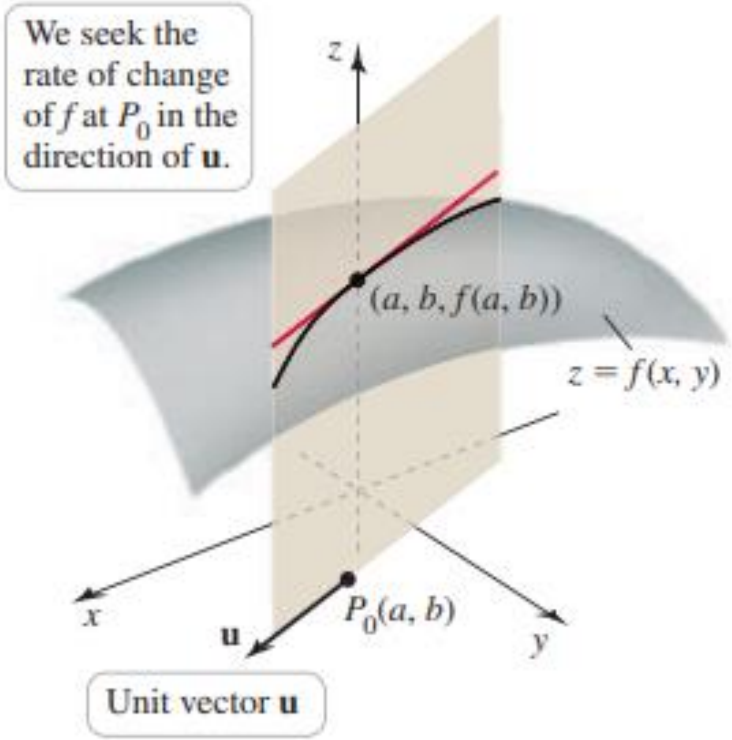
- *Suppose you are standing on a surface and you release a ball at your feet and let it roll. In which direction will it roll?*

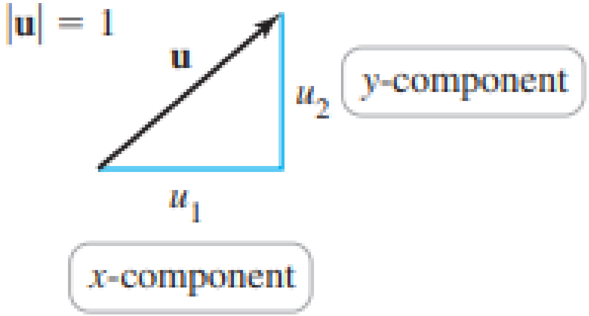
- *If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the steepest path?*

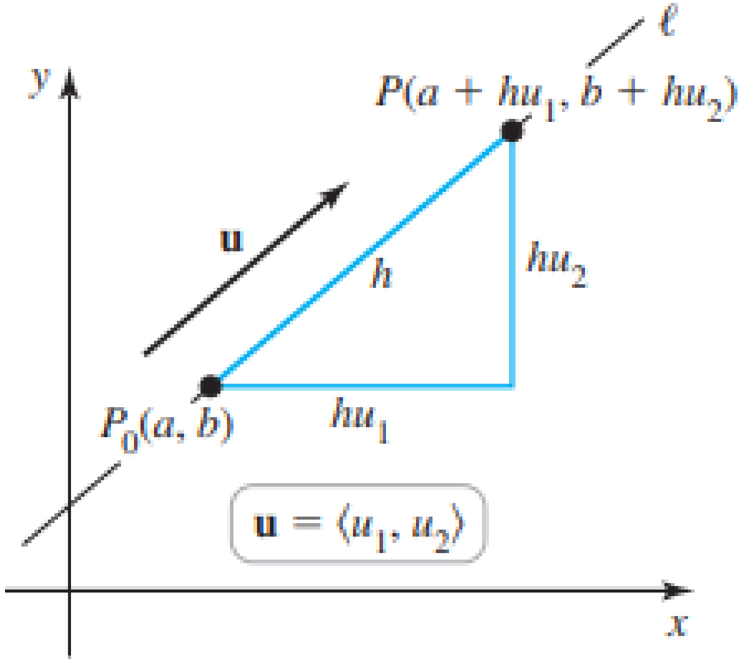
These questions are answered by introducing the directional derivative, followed by one of the central concepts of calculus—the gradient.

Directional Derivatives

- $(a, b, f(a, b))$ *is a point on the surface* $z = f(x, y)$ *and let* u *be a unit vector in the xy-plane.*
- *Our aim is to find the rate of change of f in the direction u at* $P(a, b)$.
- *In general, this rate of change is neither* f_x *nor* f_y *unless* $u = \langle 1,0 \rangle$ *or* $u = \langle 0,1 \rangle$*, but it turns out to be a combination of* $f_x(a, b)$ *and* $f_y(a, b)$.





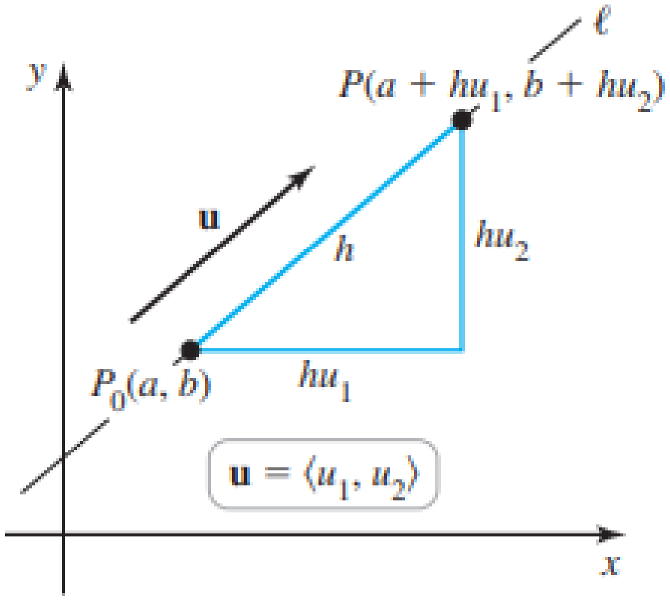


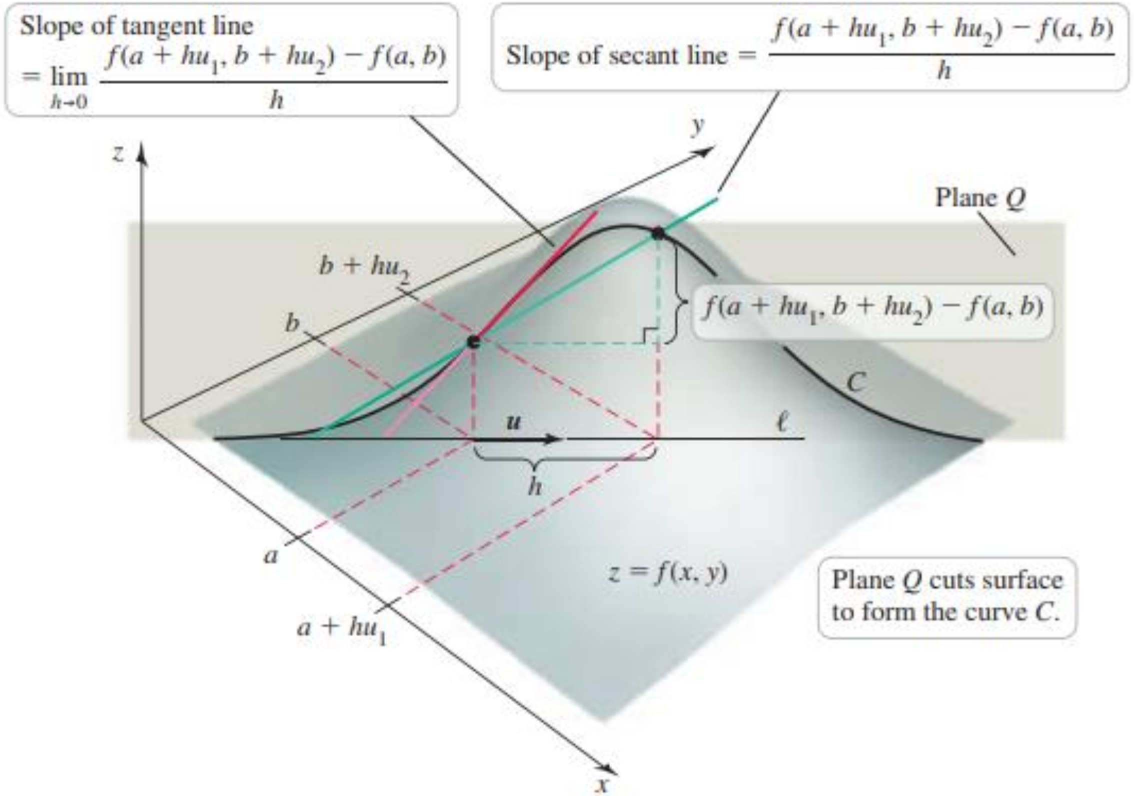
Directional Derivatives

Unit vector $u = \langle u_1, u_2 \rangle$*; its x- and y-components are* u_1 *and* u_2*, respectively.*

The derivative we seek must be computed along the line l *in the* xy*-plane through* P_0 *in the direction of u.*

A neighbouring point P, which is h units from P_0 *along* l*, has coordinates* $P(a + hu_1, b + hu_2)$.





Directional Derivatives

Now imagine the plane Q perpendicular to the xy-plane, containing l. This plane cuts the surface $z = f(x, y)$ in a curve C.

Consider two points on C corresponding to P_0 and P; they have z-coordinates $f(a, b)$ and $f(a + hu_1, b + hu_2)$.

The slope of the secant line between these points is

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

The derivative of f in the direction of u is obtained by letting $h \to 0$; when the limit exists, it is called the directional derivative of f at (a, b) in the direction of u.

It gives the slope of the line tangent to the curve C in the plane Q.

Setting $u_2 = 0$ & ignoring the second variable gives

$$\lim_{h \to 0} \frac{f(a + hu_1) - f(a)}{h}.$$

$$u_1 \underbrace{\lim_{h \to 0} \frac{f(a + hu_1) - f(a)}{hu_1}}_{f'(a)} = u_1 f'(a).$$

Graphing a Parametrically Defined Curve

The key is to define a function that is equal to f along the line l through (a,b) in the direction of the unit vector $u = \langle u_1, u_2 \rangle$. The points on l satisfy the parametric equations

$$x = a + su_1, \qquad y = b + su_2$$

$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

Notice that $dx/ds = u_1, \;\; dy/ds = u_2$

$$D_{\mathbf{u}} f(a, b) = g'(0) = \left(\frac{\partial f}{\partial x} \underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y} \underbrace{\frac{dy}{ds}}_{u_2} \right) \Bigg|_{s=0} \quad \text{Chain Rule}$$

$$= f_x(a, b)u_1 + f_y(a, b)u_2 \quad s = 0 \text{ corresponds to } (a, b).$$

$$= \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle. \quad \text{Identify dot product.}$$

Theorem – Directional Derivative

THEOREM **Directional Derivative**

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle.$$

Example – Q. 8 – Ex. 11.6

Consider the paraboloid $z = f(x, y) = 1/4(x^2 + 2y^2) + 2$. Let P_0 be the point (3, 2) and consider the unit vectors

$$u = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle, \qquad v = \langle 1/\sqrt{2}, -3/\sqrt{2} \rangle$$

(a). Find the directional derivative of f at P_0 in the directions of u and v.

(b). Graph the surface and interpret the directional derivatives.

Solution: *(a) We need to evaluate f_x and f_y at (3,2) first.*

$$f_x = \frac{x}{2}, \qquad \Rightarrow f_x(3,2) = \frac{3}{2} \qquad f_y = y \Rightarrow \ f_y(3,2) = 2$$

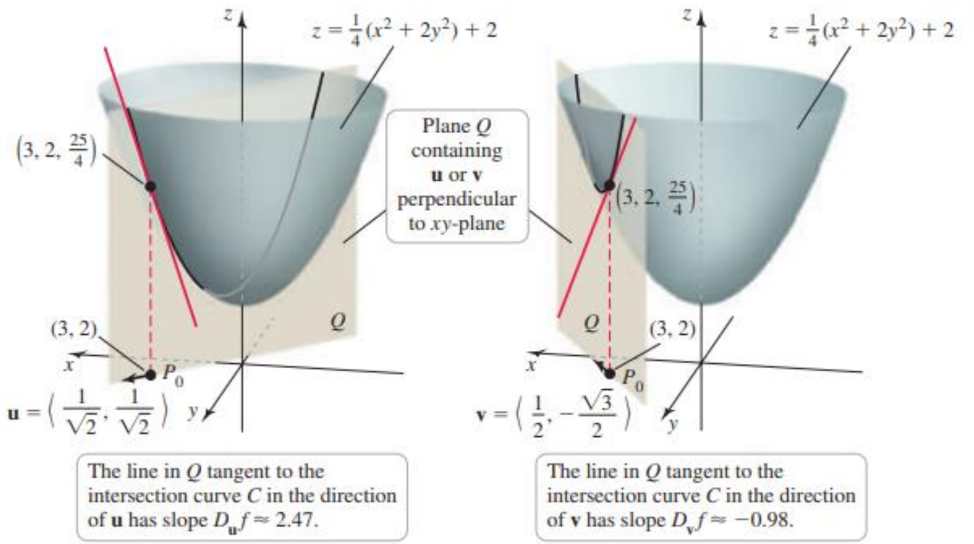
The directional derivatives in the direction of u and v are

$$D_u f(3,2) = \langle f_x(3,2), f_y(3,2) \rangle \cdot \langle u_1, u_2 \rangle = 2.47$$
$$D_v f(3,2) = \langle f_x(3,2), f_y(3,2) \rangle \cdot \langle v_1, v_2 \rangle = -0.98$$

Example – Q. 6 – Ex. 11.6

In the direction of u, the directional derivative is approximately 2.47. Because it is positive, the function is increasing at (3, 2) in this direction.

Equivalently, if Q is the vertical plane containing u, and C is the curve along which the surface intersects Q, then the slope of the line tangent to C is approximately 2.47.



In the direction of v, the directional derivative is approximately -0.98. Because it is negative, the function is decreasing in this direction.

In this case, the vertical plane Q contains v and again C is the curve along which the surface intersects Q; the slope of the line tangent to C is approximately -0.98.

Gradient Vector

- *Directional derivatives give the rate of change of a function in a particular direction*
- *The question now arises of which of such directions represents the maximum rate of increase at a point – Enter Gradient.*
- *The gradient of a function points in the direction of the steepest ascent/descent at a point*

DEFINITION Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

Gradient Vector

- *With the definition of the gradient, the directional derivative of f at (a,b) in the direction of the unit vector **u** can be written*

$$D_u f(x,y) = \nabla f(a,b) \cdot u$$

- *The gradient satisfies sum, product, and quotient rules analogous to those for ordinary derivatives*
- *It is important to remember (but easy to forget) that* $\nabla f(a,b)$ *lies in the same plane as the domain of f.*

Properties of the Gradient Vector Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

Example – Q. 11 – Ex. 11.6

Given the function

$$f(x,y,z) = y^2 e^{xyz}, \qquad P(0,1,-1), \qquad u = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle$$

(a) Find the gradient of f .

(b) Evaluate the gradient at the point P.

*(c) Find the rate of change of f at P in the direction of the vector **u**.*

Solution:

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$:

$$= \langle y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy) \rangle$$

$$= \langle y^3 z e^{xyz}, (xy^2 z + 2y)e^{xyz}, xy^3 e^{xyz} \rangle$$

(b) $\nabla f(0,1,-1) = \langle -1, 2, 0 \rangle$

(c) $D_{\mathbf{u}} f(0,1,-1) = \nabla f(0,1,-1) \cdot \mathbf{u} := \langle -1, 2, 0 \rangle \cdot \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

Interpretation of Gradient Vector

*We have seen that the directional derivative of f at f(a,b) in the direction of the unit vector **u** is* $D_u f(a,b) = \nabla f(a,b) \cdot u$.

Using properties of the dot product, we have

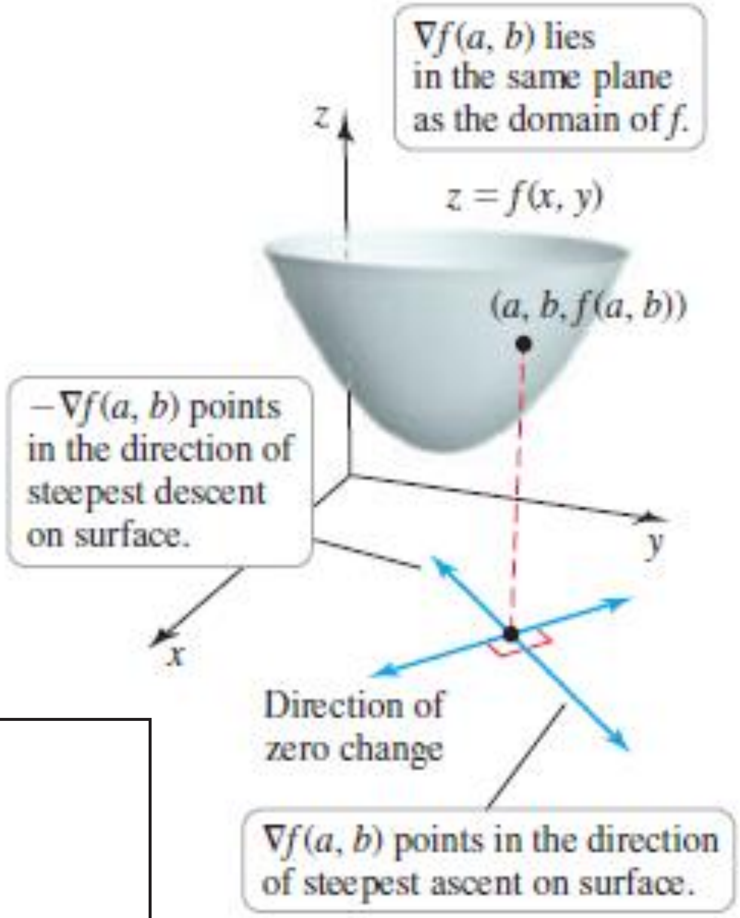
$$D_u f(a,b) = |\nabla f(a,b)||u| \cos\theta$$
$$= |\nabla f(a,b)| cos\theta, \qquad |u| = 1$$

where θ *is the angle between* ∇f *and u.*

- *At* $\theta = 0$, $D_u f(a,b)$ *has its maximum value and f has its greatest rate of increase when f(a, b) and **u** point in the same direction.*
- *When* $\cos\theta = 1$, *the actual rate of increase is* $D_u f(a,b) = |\nabla f(a,b)|$
- *Similarly, when* $\theta = \pi$, *f has its greatest rate of decrease when f(a, b) and **u** point in opposite directions and the actual rate of decrease is* $D_u f(a,b) = -|\nabla f(a,b)|$

Interpretation of Gradient Vector

- *The gradient* $|\nabla f(a, b)|$ *points in the direction of steepest ascent at (a, b), while* $-|\nabla f(a, b)|$ *points in the direction of steepest descent.*



THEOREM **Directions of Change**

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Direction Derivatives in Three Dimensions

DEFINITION Directional Derivative and Gradient in Three Dimensions

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h},$$

provided this limit exists.

The gradient of f at the point (x, y, z) is the vector-valued function

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}. \end{aligned}$$

Example – Q. 14. Ex. 11.6

Find the directional derivative of

$$f(x, y, z) = xy + yz + zx$$

at P(1,-1,3) in the direction of Q(2,4,5)

Solution: *The unit vector in the direction of* $\overrightarrow{PQ} = \langle 1, 5, 2\rangle is$

$$u = \frac{1}{\sqrt{30}}\langle 1, 5, 2\rangle$$

The gradient of the function is

$$\nabla f(x, y, z) = \langle y + z, x + z, y + x\rangle$$
$$\nabla f(1, -1,3) = \langle 2, 4,0\rangle$$

Therefore,

$$D_u f(1, -1,3) = \nabla f(1, -1,3) \cdot u = \frac{1}{\sqrt{30}}\langle 2,4,0\rangle \cdot \langle 1,5,2\rangle$$
$$D_u f(1, -1,3) = 22/\sqrt{30}$$

Electric Field as the Gradient of Potential

Electrical potential difference measured over a path C is given by

$$V = \int_C \vec{E}(r) \cdot dl$$

Where E(r) is the electric field strength at each point r on C.

The contribution of an infinitesimal length of the integral to the total integral is given by

$$dV \;-\; -E(r) \cdot dl \qquad (1)$$

$$dl = i\ dx + j\ dy + k\ dz$$

For a scalar function, including V(r), we have

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial y}dy$$

Note the above relationship is not specific to electromagnetics; it is simply mathematics.

Electric Field as the Gradient of Potential

Also note that

$$dx = dl \cdot i, \qquad dy = dl \cdot j, \qquad dz = dl \cdot k$$

Therefore

$$dV = \frac{\partial V}{\partial x}(dl \cdot i) + \frac{\partial V}{\partial y}(dl \cdot j) + \frac{\partial V}{\partial y}(dl \cdot k)$$

$$dV = \left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)V\right] \cdot dl \qquad (2)$$

Comparing Eqs. (1) and (2), we have

$$E = -\left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)V\right]$$

and $E(r) = -\nabla V(r)$

The electric potential due to a charged sphere is given by $V = q/4\pi\epsilon_0 r$, find the electric field at r.

Applications of Gradient Descent in Machine Learning

- *To reduce a model's cost function, machine learning practitioners frequently employ the gradient descent optimization procedure.*
- *It entails incrementally changing the model's parameters in the direction of the cost function's steepest decline.*
- *A free machine learning package called* ***TensorFlow*** *has built-in support for gradient descent optimization.*
- *For determining a function's minimal value, an iterative optimization process called gradient descent is performed. For training machine learning models, it is frequently employed.*
- *The approach works by incrementally changing a model's parameters in the direction of the cost* ***function's steepest descent with respect to those parameters****.*
- *The cost function, which is a mathematical function, calculates the discrepancy between the model's projected and actual outputs.*

Applications of Gradient Descent in Machine Learning

Mathematically speaking, the generic update rule for gradient descent is:

$$\theta = \theta \ - \alpha \nabla J(\theta)$$

- θ *is the parameter vector that has to be optimized.*
- *The size of the step in each iteration is determined by the learning rate, which is α.*
- *The cost function J's gradient vector,* $\nabla J(\theta)$, *shows the cost function's steepest descent in relation to.*

- *Iteratively updating until a minimum of J is attained is the algorithm's aim.*
- *An essential hyperparameter that affects convergence stability and speed is the learning rate.*
- *The method may overshoot the minimum and fail to converge if the learning rate is too high.*
- *The approach may take a very long time to converge if the learning rate is too low.*

Example – Q. 15. Ex. 11.6

Find the maximum rate of change of f at a given point and the direction in which it occurs.

$$f(x, y) = \sin(xy0, \qquad (1,0),$$

Solution

$$\nabla f(x, y) = \langle y\cos(xy), x\cos(xy)\rangle$$

$$\nabla f(1, 0) = \langle 0, 1\rangle$$

Thus the maximum rate of change is

$$|\nabla f(1,0)| = 1$$

in the direction $\langle 0, 1\rangle$.

Homework 1 – Ex. 11.6

Find the maximum rate of change of

$$T(x, y, z) = \tan(x + 2y + 3z)$$

At $(-5, 1,1)$ *and the direction in which it occurs.*

Example – Q. 21. Ex. 11.6

Find all points at which the direction of greatest rate of change of the function

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

is $i + j$.

Solution: The direction of greatest rate of change is given by the gradient of the function

$$\nabla f(x,y) = (2x-2)i + (2y-4)j$$

We need to find all the points (x,y) *where* $\nabla f(x,y)$ *is parallel to* $i + j$.

$$(2x-2)i + (2y-4)j = c(i+j)$$

$$\Rightarrow c = 2x - 2, \qquad and \quad c = 2y - 4$$

$$\Rightarrow y = x + 1$$

So at all points on the line $y = x + 1$, *the direction of greatest rate of change of f is* $i + j$.

Homework 2 – Ex. 11.6

Find the directions in which the directional derivative of

$$f(x,y) = x^2 + \sin(xy)$$

At the point (1,0) has the value 1.

Example – Q. 26. Ex. 11.6

Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.05x^2 - 0.01y^2,$$

where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. *The positive x-axis points east and the positive y-axis points north.*

a) *If you walk due south, will you start to ascend or descend? At what rate?*

b) *If you walk northwest, will you start to ascend or descend? At what rate?*

c) *In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin.*

Example – Q. 26. Ex. 11.6

Solution:

$$\nabla f(x,y) = \langle -0.01x, -0.02\,y \rangle$$
$$\nabla f(60,40) = \langle -0.6, -0.8 \rangle$$

(a) Due south is in the direction of the unit vector $u = -j$ *and*

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$
$$D_u f(60,40) = \nabla f(60,40) \cdot \langle 0, -1 \rangle = 0.8$$

Thus, if you walk due south from (60, 40, 966), you will ascend at a rate of 0.8 vertical metres per horizontal metre.

(b) Northwest is in the direction of the unit vector $u = \frac{1}{\sqrt{2}}(-i + j) = \frac{1}{\sqrt{2}}\langle -1, 1 \rangle$ *and*

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$
$$D_u f(60,40) = \nabla f(60,40) \cdot \frac{1}{\sqrt{2}}\langle 0, -1 \rangle = -0.14$$

Thus, if you walk due northwest from (60, 40, 966), you will descend at a rate of 0.14 vertical metres per horizontal metre.

Example – Q. 26. Ex. 11.6

(c)

$\nabla f(60,40) = \langle -0.6, -0.8 \rangle$ *is the direction of largest slope with a rate of ascent given by*

$$|\nabla f(60,40)| = \sqrt{(-0.6)^2 + (-0.8)^2}$$
$$= 1$$

The angle above the horizontal in which the path begins is given by

$$\tan\theta = 1$$

$$\theta = 45^o$$

Homework 3 – Ex. 11.6

The temperature at a point (x, y, z) *is given by*

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x, y, z in meters.

a. *Find the rate of change of temperature at the point P(2,-2,1) in the direction toward the point (3, -3, 3).*

b. *In which direction does the temperature increase fastest at P?*

c. *Find the maximum rate of increase at P*

Tangent Planes to Level Surfaces

- *Suppose S is a surface with equation*

$$F(x, y, z) = k$$

- *that is, it is a level surface of a function F of three variables, and let* $P(x_0, y_0, z_0)$ *be a point on S.*
- *Let C be any curve that lies on the surface S and passes through the point P. Then the curve C is described by a continuous vector function*

$$r(t) = \langle x(t), y(t), z(t) \rangle$$

- *Let* t_0 *be the parameter value corresponding to P; that is,*

$$r(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle$$

- *Since C lies on S, any point* $(x(t), y(t), z(t))$*, must satisfy the equation of S, that is*

$$F(x(t), y(t), z(t)) = k$$

Tangent Planes to Level Surfaces

- *If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to obtain*

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0$$

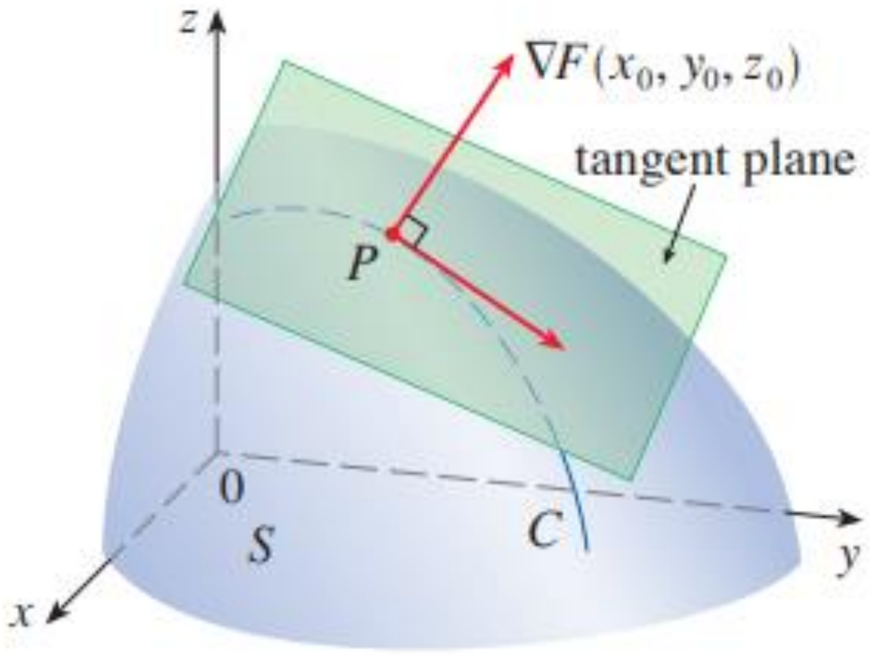
Since

$$\nabla F = i\frac{\partial F}{\partial x}+j\frac{\partial F}{\partial y}+k\frac{\partial F}{\partial z}, \qquad\qquad and$$

$$r'(t) = i\,x'(t)+jy'(t)+kz'(t)$$

Therefore

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt} \Rightarrow \nabla F\cdot r'(t)=0$$



Tangent Planes to Level Surfaces

- *In particular, when $t = t_0$, we have*

$$\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0$$

- *This equation implies that the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $r'(t)$ to any curve C on S that passes through P.*
- *If $\nabla F(x_0, y_0, z_0) \neq 0$, it is therefore natural to define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.*
- *Using the standard equation of a plane, we can write the equation of this tangent plane as*

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

Tangent Planes to Level Surfaces

- *The normal line to S at P is the line passing through P and perpendicular to the tangent plane.*

- *The direction of the normal line is therefore given by the gradient vector* $\nabla F(x_0, y_0, z_0)$*, and its symmetric equations are*

$$\boxed{\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}}$$

Example – Q. 36. Ex. 11.6

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

$$x^4 + y^4 + z^4 = 3x^2y^2z^2, \qquad (1,1,1)$$

Solution: *Let* $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. *Then*

$$x^4 + y^4 + z^4 = 3x^2y^2z^2$$

is the level surface $F(x,y,z) = 0$, *and*

$$\nabla F(x,y,z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \rangle$$

(a) $\nabla F(1,1,1) = \langle -2,-2,-2 \rangle$ *or equivalently* $\langle 1,1,1, \rangle$ *is a normal vector for the tangent plane at (1,1,1), so an equation of the tangent plane is*

$$1(x-1) + 1(y-1) + 1(z-1) = 0$$

$$x + y + z = 3$$

Homework 4 – Ex. 11.6

(b) The normal line has direction $\langle 1,1,1 \rangle$, *so parametric equations are*

$$x = 1 + t, \qquad y = 1 + t, \qquad z = 1 + t$$

and the symmetric equations are

$$x - 1 = y - 1 = z - 1$$

or

$$x = y = z.$$

Example – Q. 41. Ex. 11.6

Show that the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at the point (x_0, y_0, z_0) *can be written as*

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$$

Solution: Let

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

then $F(x,y,z)=1$ *is the level surface, and*

$$\nabla F(x_0,y_0,z_0)=\left\langle\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right\rangle$$

Thus, the equation of the tangent plane at (x_0, y_0, z_0) *is*

Example – Q. 41. Ex. 11.6

Thus, the equation of the tangent plane at (x_0, y_0, z_0) *is*

$$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 2(1) = 2$$

Since (x_0, y_0, z_0) *is a point on the ellipsoid. Hence*

$$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 1$$

is an equation of the tangent plane.

Example – Q. 46. Ex. 11.6

At what points does the normal line through the point (1,2,1) on the ellipsoid

$$4x^2 + y^2 + 4z^2 = 12$$

intersect the sphere

$$x^2 + y^2 + z^2 = 102?$$

Solution: *Th ellipsoid* $4x^2 + y^2 + 4z^2 = 12$ *is the level surface of*

$$F(x, y, z) = 4x^2 + y^2 + 4z^2$$

$$\nabla F(x, y, z) = \langle 8x, 2y, 8z \rangle \Rightarrow \ \nabla F(1,2,1) = \langle 8,4,8 \rangle = 4\langle 2,1,2 \rangle$$

Or equivalently $\langle 1,2,1 \rangle$ *is a normal vector to the surface.*

Thus, normal line to the ellipsoid at (1,2,1) is given by

$$x = 1 + 2t, \qquad y = 2 + t, \qquad z = 1 + 2t$$

Substituting into equation of the sphere gives

$$(1 + 2t)^2 + (2 + t)^2 + (1 + 2t)^2 = 102 \Rightarrow 3(t + 4)(3t - 8) = 0$$

Example – Q. 46. Ex. 11.6

$$\Rightarrow t = -4 \quad and \qquad t = 8/3$$

Thus, the line intersects the sphere when $t = -4$, *corresponding to the point*

$$(-7, -2, -7)$$

When $t = 8/3$, *the line intersects the sphere corresponding to the point*

$$(19/3, 14/3, 19/3)$$

Homework 5 - Ex. 11.6

Where does the helix

$$r(t) = \langle \cos \pi t, \sin \pi t, t \rangle$$

intersect the paraboloid

$$z = x^2 + y^2?$$

What is the angle of intersection between the helix and the paraboloid? (This is the angle between the tangent vector to the curve and the tangent plane to the paraboloid.)

Homework 6 - Ex. 11.6

Show that surfaces with equations $F(x,y,z) = 0$ *and* $G(x,y,z) = 0$ *are orthogonal at a point P where* $\nabla F \neq 0$ *and* $\nabla G \neq 0$ *if and only if*

$$F_x G_x + F_y G_y + F_z G_z = 0$$

at P.