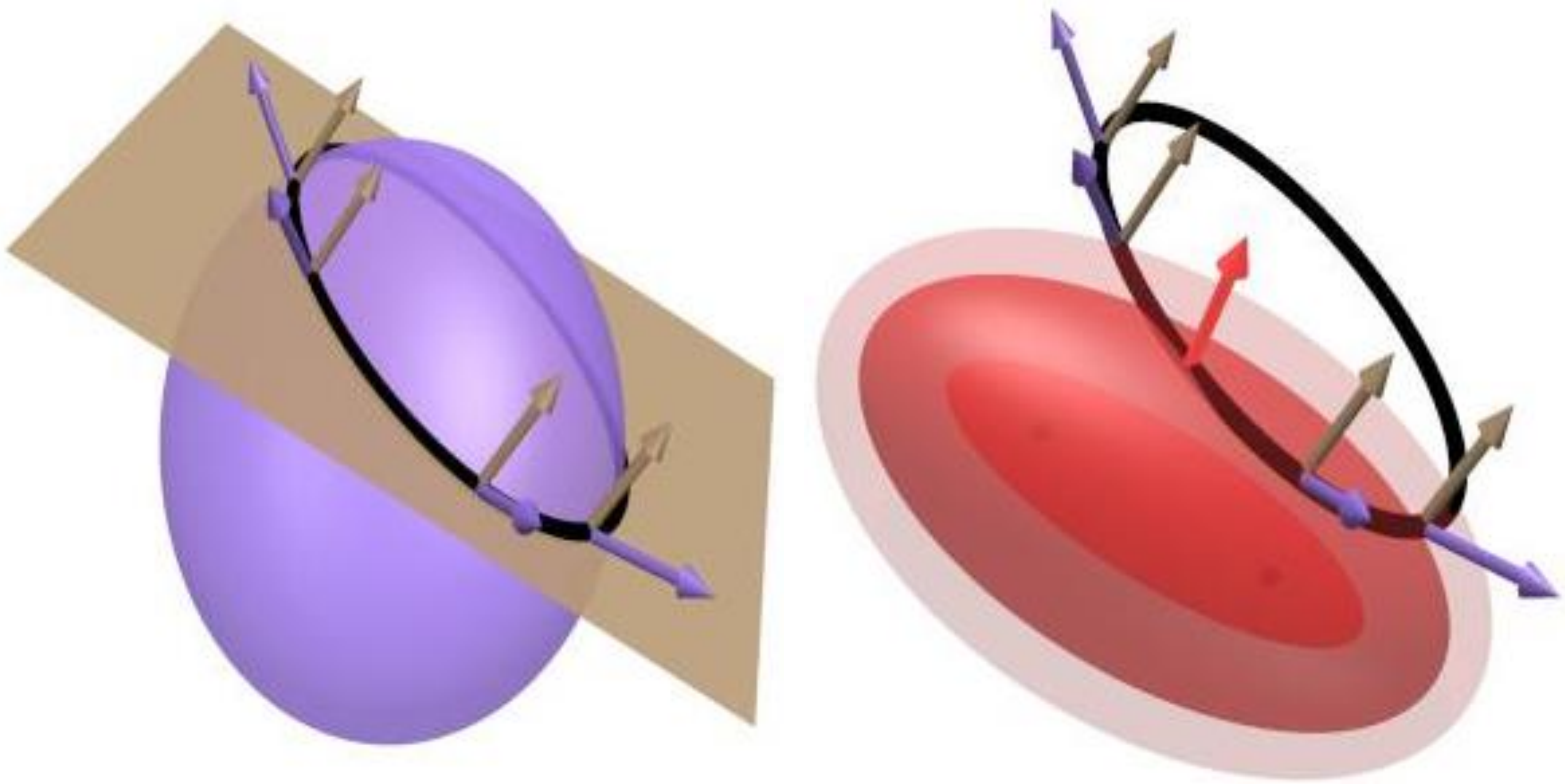
Chapter 11 – Sec. 11.8

Constrained Optimization - Lagrange Multipliers



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Learning Objectives

- *What is constrained optimization and why do we need it?*
- *What are Lagrange's multipliers and how do we use them to optimize a given objective function*
- *Lagrange's multipliers for a two-variable function*
- *Applications*

Why Constrained Optimization??

Imagine you are asked to design a rectangular box without a lid that is to be made from $12\ m^2$ *of cardboard. How would you choose your dimensions to find the maximum volume of such a box???*

Lagrange Multipliers – One Constraint

- *Lagrange's method for maximizing or minimizing a general function* $f(x, y, z)$ *subject to a constraint (or side condition) of the form* $g(x, y, z) = k$.

- *It's easier to explain the geometric basis of Lagrange's method for functions of two variables.*

- *So we start by trying to find the extreme values of* $f(x, y)$ *subject to a constraint of the form* $g(x, y) = k$.

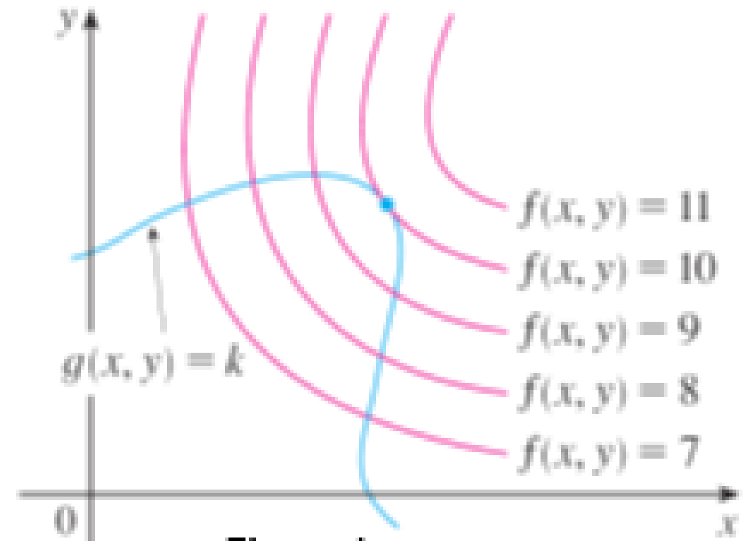
- *In other words, we seek the extreme values of* $f(x, y)$ *when the point* (x, y) *is restricted to lie on the level curve* $g(x, y) = k$.

Lagrange Multipliers – One Constraint

Fig. shows this curve together with several level curves of f

These have the equations $f(x, y) = c$, where c = 7, 8, 9..

- *To maximize f (x, y) subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.*



- *It appears from Fig. that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)*

Lagrange Multipliers – One Constraint

This means that the normal lines at the point (x_0, y_0) *where they touch are identical. So the gradient vectors* $\nabla f(x_0, y_0)$ *and* $\nabla g(x_0, y_0)$ *are parallel; that is,*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some scalar λ.

- *This kind of argument also applies to the problem of finding the extreme values of* $f(x, y, z)$ *subject to the constraint* $g(x, y, z) = k$.
- *Thus the point* (x, y, z) *is restricted to lie on the level surface S with equation* $g(x, y, z) = k$.

Lagrange Multipliers – One Constraint

Therefore, if $g(x, y, z) \neq 0$*, there is a number* λ *such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \, \nabla g(x_0, y_0, z_0)$$

The number λ *is called a* ***Lagrange multiplier****.*

Lagrange Multipliers – One Constraint

The procedure is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Lagrange Multipliers

If we write the vector equation $\nabla f = \lambda \nabla g$ *in terms of components, then the equations in step (a) become*

$$f_x = \lambda g_x,\ \ f_y = \lambda g_y,\ \ f_z = \lambda g_z,\quad g(x, y, z) = k$$

- *This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.*

Lagrange Multipliers

For functions of two variables the method of Lagrange multipliers is similar to the method just described.

To find the extreme values of $f(x, y)$ *subject to the constraint* $g(x, y) = k$*, we look for values of* x, y*, and* λ*, such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0), \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad g(x, y) = k$$

Example – Lagrange's Method

Find the extreme values of the function

$$f(x, y) = x^2 + 2y^2$$

on the circle $x^2 + y^2 = 1$.

Solution:

The optimization problem is:

Optimize the objective function:

$$f(x, y) = x^2 + 2y^2$$

Subjected to the constraint

$$g(x, y) = x^2 + y^2 = 1$$

Solution

Using Lagrange multipliers, we solve the equations
$\nabla f(x, y) = \lambda \nabla g(x, y)$, *and* $g(x, y) = 1$

These can be written as:

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad g(x, y) = 1$$
$$2x = 2x\lambda, \qquad 4y = 2y\lambda, \qquad x^2 + y^2 = 1$$
$$2x(1 - \lambda) = 0, \qquad \Rightarrow x = 0, \qquad or \quad \lambda = 1$$

- *If* $x = 0$, *then* $y = \pm 1$.
- *If* λ *= 1, then* $y = 0$ *so, then* $x = \pm 1$.

Solution (Contd.)

Therefore, f has possible extreme values at the points

$$(0,1), (0,-1), (1,0), (-1,0)$$

Evaluating f at these four points, we find that:

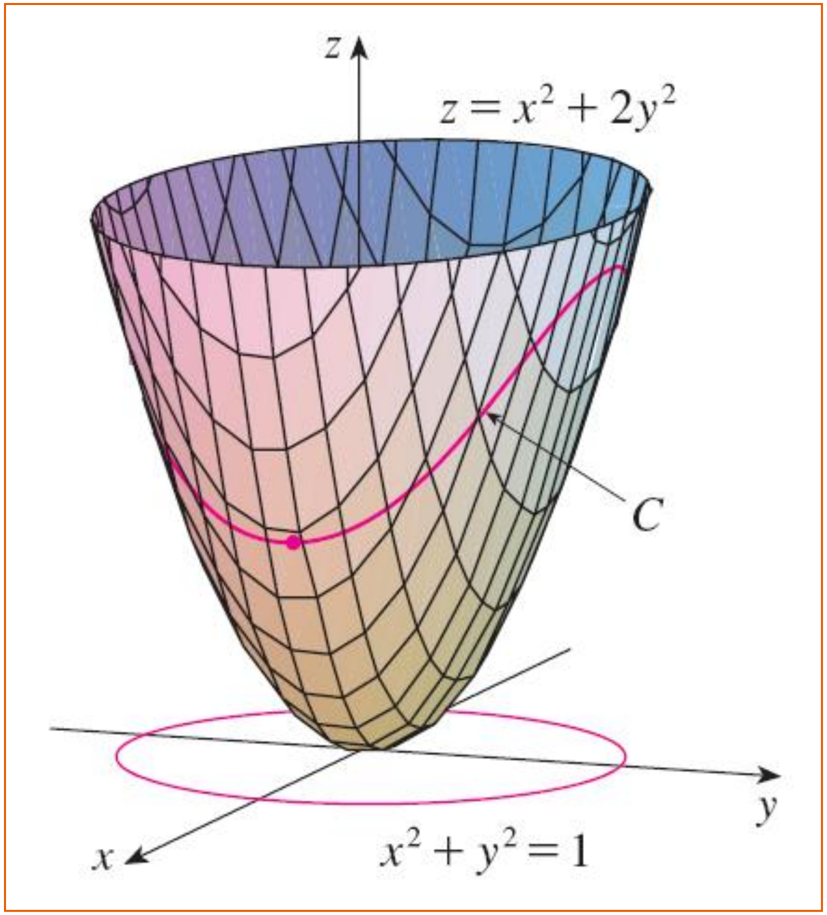
$$f(0,1) = 2, \qquad f(0,-1) = 2,$$
$$f(1,0) = 1, \qquad f(-1,0) = 1$$

Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ *is:* $f(0, \pm 1) = 2$

The minimum value is: $f(\pm 1, 0) = 1$.

Solution (Contd.)

Checking with the figure, we see that these values look reasonable.



Example

Find the maximum and minimum values of the objective function

$$f(x, y) = 2x^2 + y^2 + 2,$$

where x and y lie on the ellipse C given by

$$g(x, y) = x^2 + 4y^2 - 4 = 0 \Rightarrow x^2 + 4y^2 = 4.$$

Solution

Using Lagrange multipliers, we solve the equations

$\nabla f(x, y) = \lambda \nabla g(x, y),$ *and* $g(x, y) = 4$

These can be written as:

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad g(x, y) = 4$$

$$4x = 2x\lambda, \qquad 2y = 8y\lambda, \qquad x^2 + 4y^2 = 4$$

$$2x(2 - \lambda) = 0, \qquad \Rightarrow x = 0, \qquad or \quad \lambda = 2$$

Solution (Contd.)

- *If* $x = 0$, *then* $y = \pm 1$.
- *If λ = 2, then* $y = 0$ *so, then* $x = \pm 2$.

Therefore, f has possible extreme values at the points

$$(0,1), (0,-1), (2,0), (-2,0)$$

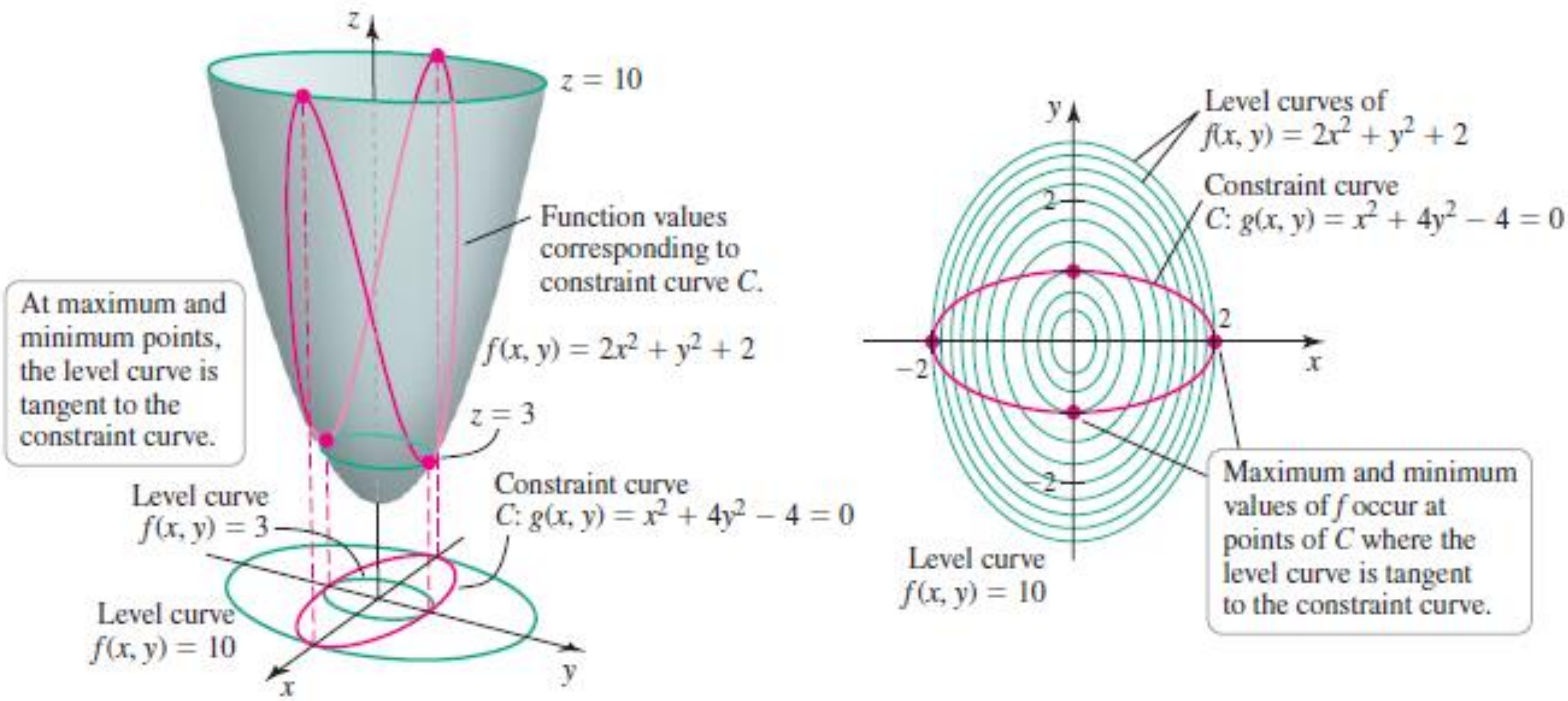
Evaluating f at these four points, we find that:

$$f(0,1) = 3, \qquad f(0,-1) = 3,$$
$$f(2,0) = 10, \qquad f(-2,0) = 10$$

Therefore, the maximum value of f on the ellipse $x^2 + 4y^2 = 4$ *is:* $f(\pm 2,0) = 10$

The minimum value is: $f(0, \pm 1) = 3$.

Solution (Contd.)



Application – Designing a Box

A rectangular box without a lid is to be made from $12\ m^2$ of cardboard. Find the maximum volume of such a box.

Solution:

Let x, y and z be the length, width, and height, respectively, of the box in meters.

Then we wish to maximize the objective function

$$V(x, y, z) = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$.

Solution (Contd.)

This gives the equations

$v_x = \lambda g_x,\ V_y = \lambda g_y,\ V_z = \lambda g_z,$ *and* $2xz + 2yz + xy = 12$

which become

$yz = \lambda(2z + y), \quad xz = \lambda(2z + x), \quad xy = \lambda(2x + 2y).$

Solve the simultaneous system of three equations:

$xyz = \lambda(2xz + xy),\ xyz = \lambda(2yz + xy)$ *and*

$xyz = \lambda(2xz + xy).$

We observe that $\lambda \neq 0$ *because* $\lambda = 0$ *would imply*

$yz = xz = xy = 0$ *and this would contradict*

$$2xz + 2yz + xy = 12$$

Solution (Contd.)

Therefore, from first two equation, above, we have

$$2xz + xy = 2yz + xy \Rightarrow xz = yz \Rightarrow z(x - y) = 0$$

But $z \neq 0$ *(since z = 0 would give V = 0), so* $x = y$.

From last two equations, we have

$$2yz + xy = 2xz + 2yz \Rightarrow 2xz = xy$$

$$x(2z - y) = 0$$

which gives $2z = y$ *(since x* $\neq$ *0).*

If we now put $x = y = 2z$ *in constraint function, we get*

$$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = \pm 1$$

Solution (Contd.)

Since x, y *and* z *are all positive, we therefore have*

$$x = 2, \qquad y = 2, \qquad z = 1$$

Then $V = 2 \times 2 \times 1 = 4$*, so the maximum volume of the box is* $4\ m^3$.

Example – Q. 10. Ex 11.8

Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = e^{xyz}$$

subject to the given constraint

$$2x^2 + y^2 + z^2 = 24$$

Solution: $\nabla f = \lambda \nabla g$ *implies*

$$\langle yz\, e^{xyz}, xze^{xyz}, xye^{xyz} \rangle = \langle 4\lambda x, 2\lambda y, 2\lambda z \rangle$$

Then

$$yze^{xyz} = 4\lambda x, \qquad xze^{xyz} = 2\lambda y, \qquad xye^{xyz} = 2\lambda z$$

And $2x^2 + y^2 + z^2 = 24.$

If any of x, y, z *or* λ *is zero, then the first three equations imply that two of the variables* x, y, z *must be zero*

Solution

If $x = y = z = 0$*, it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are*

$$\left(\pm 2\sqrt{3}, 0{,}0\right), \left(0, \pm 2\sqrt{6}, 0\right), \left(0{,}0, \pm 2\sqrt{6}\right)$$

all with the f-value of $e^0 = 1$.

If none of x, y, z*, and* λ *is zero, then from the first three equations we have,*

$$\frac{4\lambda x}{yz} = e^{xyz} = \frac{2\lambda y}{xz} = \frac{2\lambda z}{xy} \Rightarrow \frac{2x}{yz} = \frac{y}{zx} = \frac{z}{xy}$$

This gives $2x^2z = y^2z \Rightarrow 2x^2 = y^2$ *and* $xy^2 = xz^2 \Rightarrow y^2 = z^2$.

Substituting into fourth equation, we have

Solution

Substituting into fourth equation, we have

$$y^2 + y^2 + y^2 = 24 \Rightarrow y^2 = 48 \Rightarrow y = \pm 2\sqrt{2}$$

So $x^2 = 4 \Rightarrow x = \pm 2$ *and* $z^2 = y^2 \Rightarrow z = \pm 2\sqrt{2}$

This gives possible points

$$\left(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2}\right) \quad (all\ combinations)$$

The value of f is e^{16} *when all coordinates are positive or exactly two are negative, and the value is* e^{-16} *when all are negative or exactly one of the coordinates is negative.*

Thus, the maximum of f subject to the constraint is e^{16} *and the minimum is* e^{-16}

Homework 1 – Ex. 11.8

Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

$$f(x,y) = xye^{-x^2-y^2}, \qquad 2x - y = 0$$

$$f(x,y,z) = 2x = 2y + z, \qquad x^2 + y^2 + z^2 = 9$$

Example – Q. 55. Ex. 11.8

A grain silo is to be built by attaching a hemispherical roof and a flat floor onto a circular cylinder. Use Lagrange multipliers to show that for a total surface area S, the volume of the silo is maximized when the radius and height of the cylinder are equal.

Solution: Let r and h are the radius and height of the silo. The problem reduces to maximizing the function

$$V(r,h) = \pi r^2 h + \frac{2}{3}\pi r^3$$

subject to the constraint

$$g(r,h) = 2\pi rh + \pi r^2 + (4\pi r^2)2 = 2\pi rh + 3\pi r^2 = S.$$

Then $\nabla V = \lambda \nabla g$

$$\Rightarrow \langle 2\pi r\, h + 2\pi r^2, \pi r^2 \rangle = \langle 2\lambda\pi h + 6\lambda\pi r, 2\lambda\pi r \rangle$$

Solution (Contd.)

So the three equations are

$$2\pi rh + 2\pi r^2 = 2\lambda\pi h + 6\lambda\pi r$$

$$\pi r^2 = 2\lambda\pi r$$

$$2\pi r\, h + 3\pi r^2 = S$$

The second equation implies $r = 2\lambda[r \neq 0]$.

Substituting $r = 2\lambda$ *into the first equation gives*

$$2\pi(2\lambda)h + 2\pi(2\lambda)^2 = 2\lambda\pi h + 6\lambda\pi(2\pi)$$

$$\Rightarrow 4\pi\lambda h + 8\pi\lambda^2 = 2\lambda\pi h + 12\,\pi\lambda^2$$

$$\Rightarrow 2\pi\lambda h = 4\pi\lambda^2 \Rightarrow h = 2\lambda$$

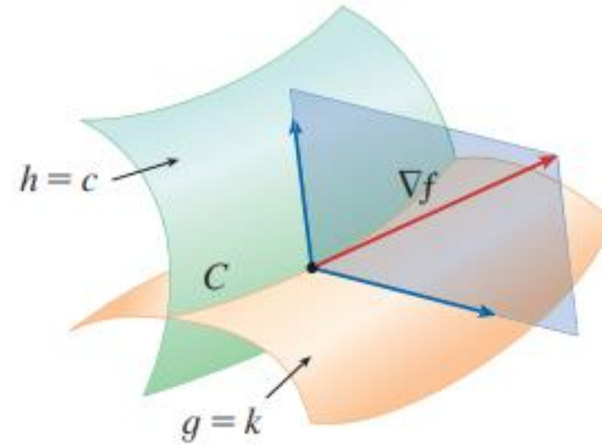
Thus, $r = 2\lambda = h$, *and the volume of the silo is maximized, subject to a given surface area, when the radius and height are equal.*

Solution (Contd.)

Find the maximum and minimum volumes of a rectangular box whose surface area is $1500\ cm^2$ *and whose total edge length is 200 cm*

Lagrange Multipliers – 2 Constraints

- *Suppose now that we want to find the maximum and minimum values of a function* $f(x, y, z)$ *subject to two constraints (side conditions) of the form* $g(x, y, z) = k$ *and* $h(x, y, z) = c$.



  - *Geometrically, this means that we are looking for the extreme values of f when* (x, y, z) *is restricted to lie on the curve of intersection C of the level surfaces* $g(x, y, z) = k$ *and* $h(x, y, z) = c$.

Solution (Contd.)

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$.

∇f *is orthogonal to C at P.*

But we also know that ∇g *is orthogonal to* $g(x, y, z) = k$ *and* ∇h *is orthogonal to* $h(x, y, z) = c$*, so* ∇g *and* ∇h *are both orthogonal to C.*

This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ *is in the plane determined by* $\nabla g(x_0, y_0, z_0)$ *and* $\nabla h(x_0, y_0, z_0)$.

So there are numbers and (both called Lagrange multipliers) such that

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)}$$

Solution (Contd.)

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, x, λ *and* μ.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Solution (Contd.)

Find the maximum value of the function

$$f(x, y, z) = x + y + 3z$$

on the curve of the intersection of the plane $x - y + z = 1$ *and the cylinder* $x^2 + y^2 = 1$.

Solution: The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, *so we solve the equations*

$$f_x = \lambda g_x + \mu h_x \Rightarrow \quad 1 = \lambda + 2x\mu$$

$$f_y = \lambda g_y + \mu h_y \Rightarrow 2 = -\lambda + 2y\mu$$

$$f_z = \lambda g_z + \mu h_z \Rightarrow 3 = \lambda$$

$$x - y + z = 1, \qquad x^2 + y^2 = 1$$

Substituting $\lambda = 3$, *we get*

$$2x\mu = -2 \Rightarrow x = -1/\mu$$

Similarly, $y = 5/(2\mu)$.

Solution (Contd.)

Substituting in $x^2 + y^2 = 1$, *gives*

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1 \Rightarrow \mu = \pm\sqrt{29}/2$$

Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$. *From* $z = 1 - x + y$, *we have*

$$z = 1 \pm 7/\sqrt{29}$$

The corresponding values of the function are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right)$$
$$= 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 \pm \sqrt{29}$.

Example – Q. 57. Ex. 11.8

The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution: *We need to find the extreme values of function defining an ellipse*

$$f(x, y, z) = x^2 + y^2 + z^2$$

subjected to the conditions

$$g(x, y, z) = x + y + 2x - 2 = 0 \qquad \textit{and}$$

$$h(x, y, z) = x^2 + y^2 - z = 0$$

$$\nabla f = \langle 2x, 2y, 2z \rangle,$$

$$\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle, \qquad \mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$$

$$\Rightarrow 2x = \lambda + 2\mu x, \qquad 2y = \lambda + 2\mu y, \qquad 2z = 2\lambda - \mu \;\; \&$$

$$x + y + 2z = 2, \qquad x^2 + y^2 - z = 0$$

Solution (Contd.)

From the first two equations, $2(x-y)=2\mu(x-y) \Rightarrow \mu = 1$ *and* $\lambda = 0$.

Substituting this in (3) gives $z=-1/2$. *The constrained equations yield* $x+y-3=0$ *and* $x^2+y^2+1/2=0$

The last Eq. cannot be true, so this case gives no solution.

Therefore, we must have $x=y$.

Thus last two equations yield, $x = 1/2,\ or\ x=-1$, *which give*

$$y=\frac{1}{2}, z=\frac{1}{2}, \qquad and\ y=-1, z=2$$

Thus two points are $(1/2, 1/2, 1/2)$ *and* $(-1,-1,2)$.

Thus $f(1/2,1/2,1/2) = 3/4$ *and* $f(-1,-1,2) = 6$. *Therefore* $(1/2,1/2,1/2)$ *is the point on the ellipse nearest the origin and* $(-1,-1,2)$ *the farthest from the origin.*

Homework 2 – Q. Ex. 11.8

Use Lagrange multipliers to prove that the rectangle with the maximum area that has a given perimeter p is a square.

Homework 3 – Q. Ex. 11.8

Use Lagrange multipliers to prove that the triangle with a maximum area that has a given perimeter p is equilateral. (Hint: Use heron's formula for the area

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

Where $s = p/2$ *and* x, y, z *are the lengths of the sides.)*