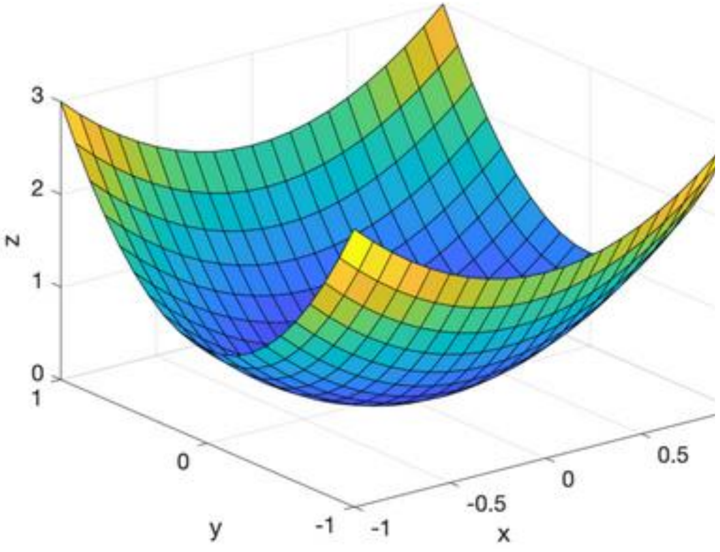
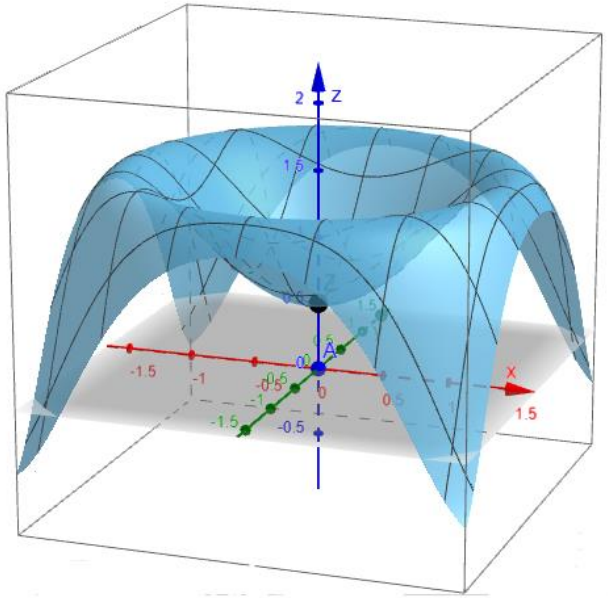
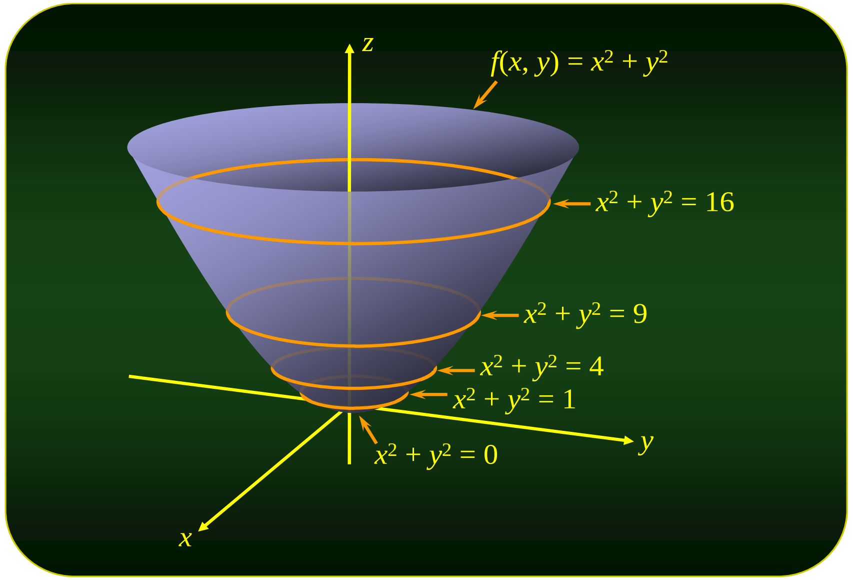
Lecture 13 - Chapter 11 – Sec. 11.1
Multivariable Functions







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Learning Objectives

- *Characteristics of a multivariable function*
- *Algebraic and graphical representations of a two-variable function*
- *Domain and range*
- *Contours and surfaces*
- *Applications*

Why do we need multivariable functions?

- *The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.*
  - *We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y).*
  - *We indicate this functional dependence by writing:*

$$T = f(x, y)$$

- *The electric field of an electromagnetic signal depends on its position x and time t*
  - *We indicate this functional dependence by writing:*

$$E = E(x, t)$$

- *The volume V of a circular cylinder depends on its radius r and its height h.*
  - *In fact, we know that $V = \pi r^2 h$.*
  - *We say that V is a function of r and h.*
  - *We write $V(r, h) = \pi r^2 h$.*

Domain and Range of a two variable Function

A real-valued function of two variables f, consists of a rule that assigns to each ordered pair of real numbers (x, y) *in a set D a unique (one and only one) real number denoted by* $z = f(x, y)$.

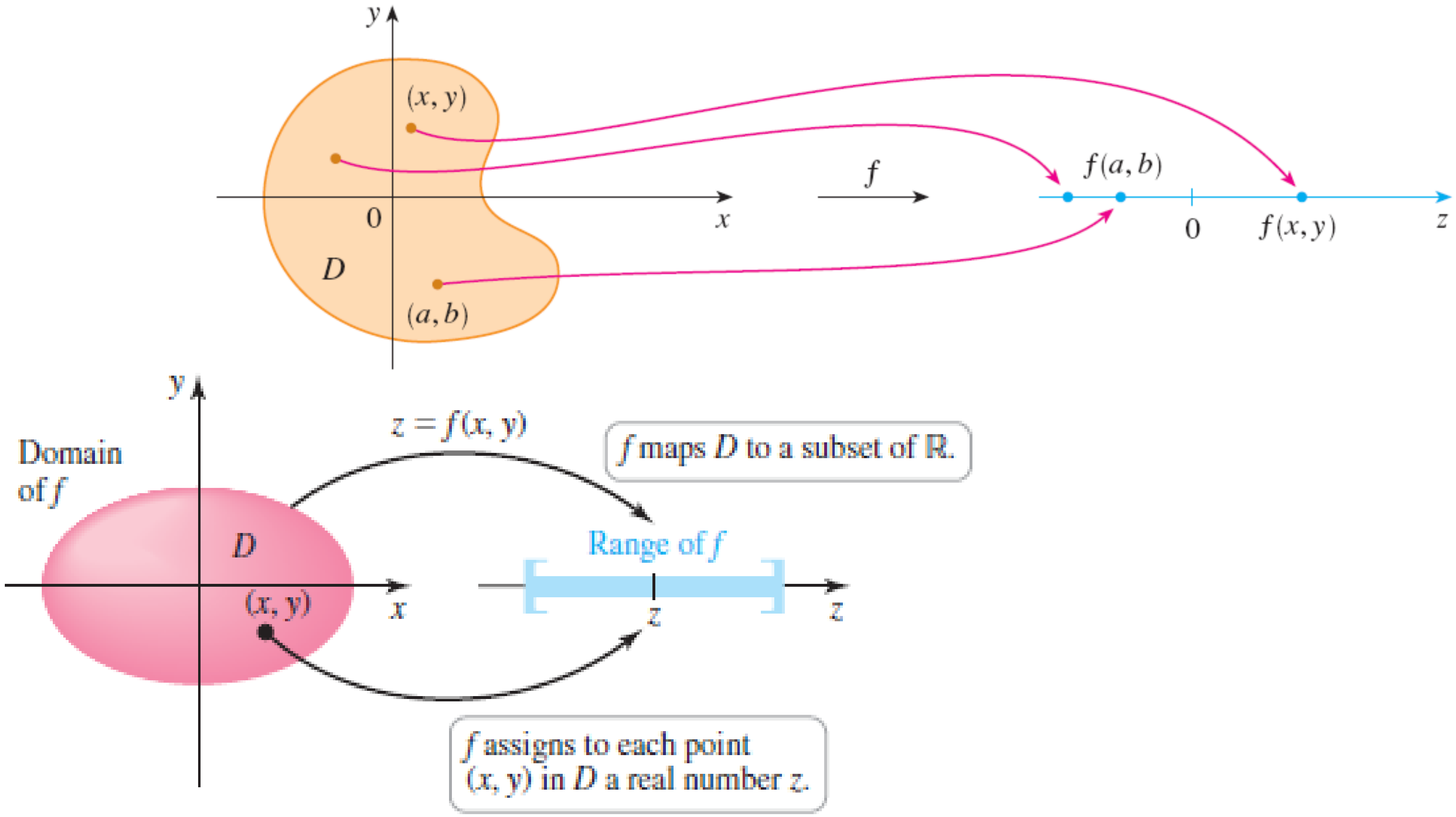
- *The set D is the domain of f.*
- *The **range** of f is the set of real numbers z that are assumed as the points* x_1, y_1 *vary over the domain*

$$\{f(x, y) | (x, y) \in D\}$$

- *Domain is a subset of* $\mathbb{R}^2$
- *Range is a subset of* $\mathbb{R}$

Domain and Range of a two variable Function

A function of two variables can be visualized by the arrow diagram where the domain D is represented as a subset of the xy-plane.



Example

For each of the following functions, evaluate f(3, 2) and find the domain. Sketch the graphs.

(a)

$$z = f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

(b)

$$w = f(x, y) = x \ln(y^2 - x)$$

Example - Solution

(a) The value of the function z *at* $(3,2)$ *is*

$$z = f(3,2) = \frac{\sqrt{6}}{2} = \sqrt{3/2}$$

To find the domain, we need to find a set (x, y) *for which z is well-defined.*

- *For z to be well-defined, the denominator must not be 0 and the quantity under the square root sign must be nonnegative.*

$$x \neq 1, \qquad and \quad x + y + 1 \geq 0$$

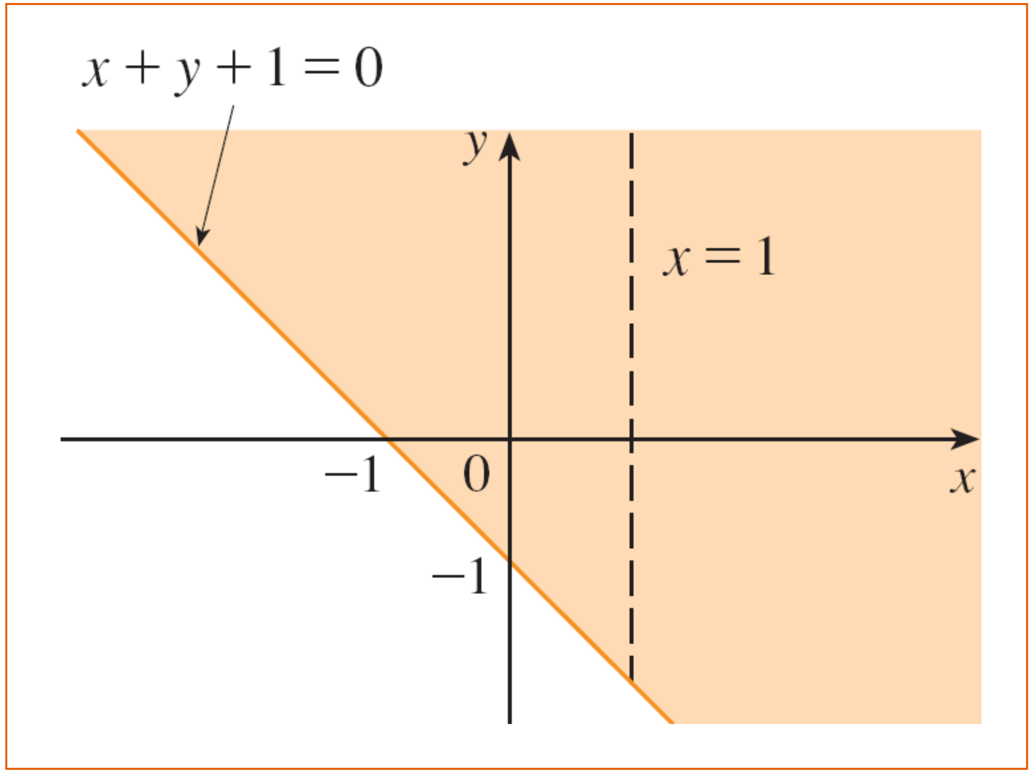
- *So, the domain of f is:*

$$D = \{(x, y) | x + y + 1 \geq 0, x \neq 1\}$$

Example - Solution

The inequality $x + y + 1 \geq 0, or\ y \geq -x - 1$*, describes the points that lie on or above the line* $y = -x - 1$.

- $x \neq 1$ *means that the points on the line x = 1 must be excluded from the domain.*



Example - Solution

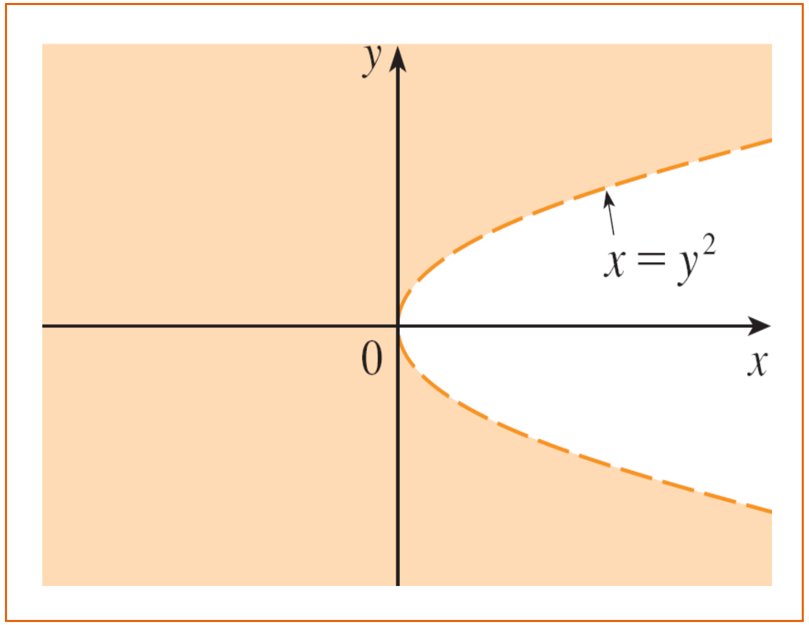
(b) The value of the function z at $(2,3)$ *is*

$$w = f(3,2) = 3 \times \ln(2^2 - 3) = 3 \ln 1 = 0$$

For w to be well-defined,

$$y^2 - x > 0, \Rightarrow x < y^2$$

• *So, the domain of w is:* $D = \{(x,y) | x < y^2\}$



Example

Find the domain of f if

$$z = f(x, y) = \ln(y - x) + x \sin\frac{y}{x}$$

Solution:

The above expression is well-defined as long as

$$y - x > 0 \Rightarrow y > x \text{ and } x \neq 0.$$

Therefore, the domain of the function,

$$Domain = \{(x, y) | y > x, x \neq 0\}$$

Example – Solution

Find the domain of the function

$$f(x, y) = \ln(9 - x^2 - 9y^2)$$

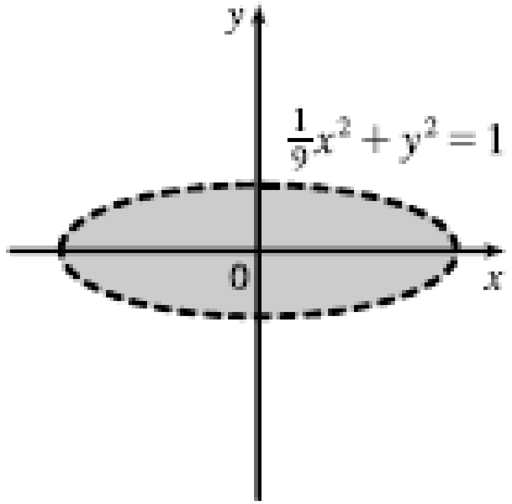
Solution

For the function f to be well-defined

$$9 - x^2 - 9y^2 > 0$$

$$x^2 + 9y^2 < 9$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{1} < 1$$



Domain: $\{(x, y) \in \mathbb{R}^2 | x^2 + 9y^2 < 9\}$

Example

Find the domain and range of the following functions. Then sketch a graph.

(a) $f(x, y) = 2x + 3y - 12$, (b) $g(x, y) = x^2 + y^2$

(c) $h(x, y) = \sqrt{1 + x^2 + y^2}$

Example - Solution

(a) Letting $z = f(x, y)$*, we have the equation*

$$z = 2x + 3y - 12 \Rightarrow 2x + 3y - z = 12$$

which describes a plane with a normal vector $< 2,3,-1 >$.

The domain consists of all points in $\mathbb{R}^2$*, and the range is* $\mathbb{R}$.

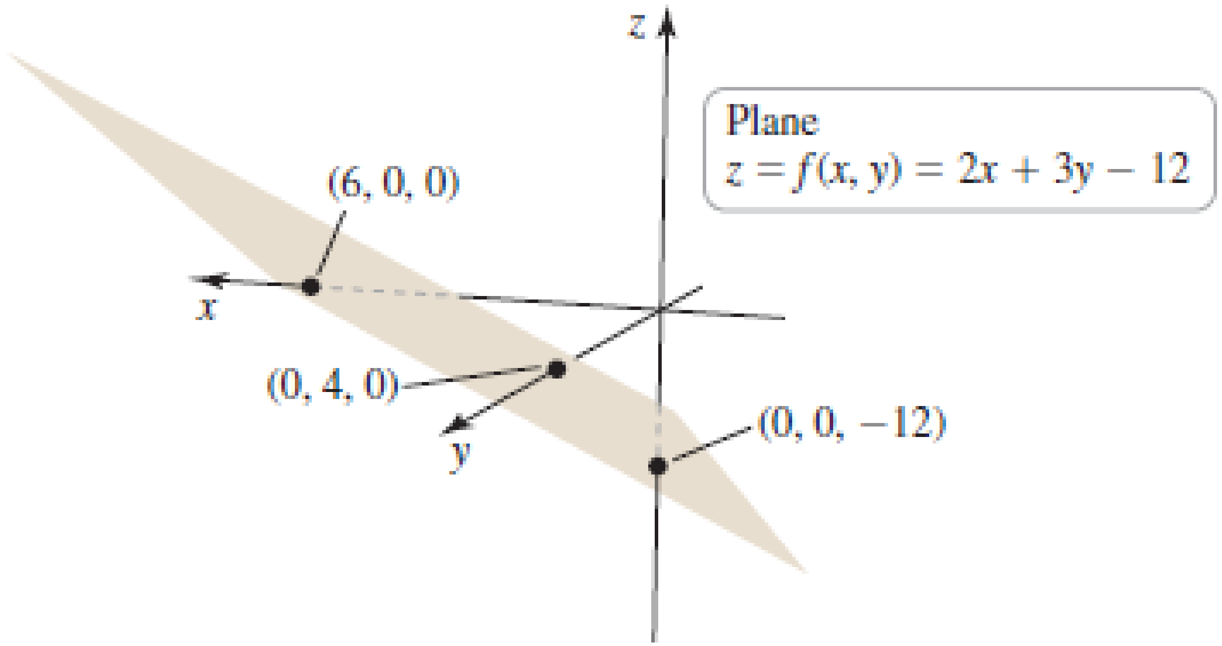
$$Domain = \{(x, y,) | (x, y) \in \mathbb{R}^2\}$$
$$Range = \{z; z \in \mathbb{R}\}$$

.

Example - Solution

Sketching the surface:

We sketch the surface by noting that the x-intercept is (6, 0, 0) (by setting $y = z = 0$*); the y-intercept is (0, 4, 0) and the z-intercept is (0, 0, -1).*



Example - Solution

(b) *Letting* $z = g(x, y)$*, we have the equation*

$$z = x^2 + y^2,$$

which describes a revolving parabola called an elliptic paraboloid that opens upward with vertex (0, 0).

The domain is $\mathbb{R}^2$ *and the range consists of all nonnegative real numbers.*

Example - Solution

(c) The domain of the function is $\mathbb{R}^2$ *because the quantity under the square root is always positive.*

Range: Since $1 + x^2 + y^2 \geq 1$*, so the range is*

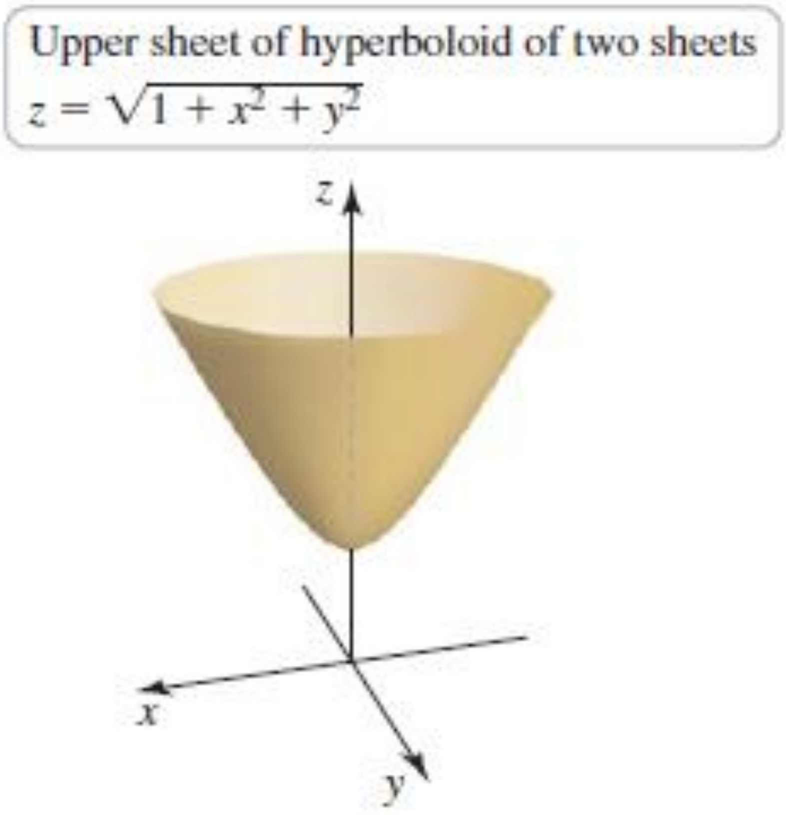
$$\{z; z \geq 1\}$$

Sketching the surface

$$z^2 = 1 + x^2 + y^2 \Rightarrow -x^2 - y^2 + z^2 = 1$$

This is the equation of a hyperboloid of two sheets that opens along the z-axis.

Because the range is $z: z \geq 1$*, the given function represents only the upper sheet of the hyperboloid.*



Example

Find the domain of the function

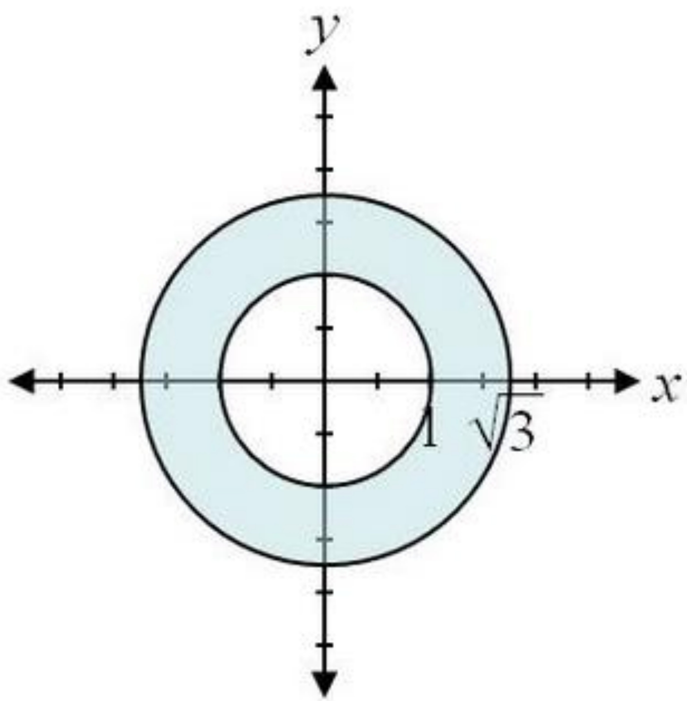
$$f(x, y) = \sin^{-1}(x^2 + y^2 - 2)$$

Solution:

Since we can only inverse sine of numbers betw
and -1, inclusive, therefore

$-1 \leq x^2 + y^2 - 2 \leq 1,$

$\Rightarrow 1 \leq x^2 + y^2 \leq 3.$



Homework

Find the domain and range of the following function

(a) $f(x,y) = \sqrt{(y-x)}\,\ln(y+x)$

(b) $g(x,y) = \frac{\sqrt{(y-x^2)}}{1-x^2}$

(c) $f(x,y) = x^2 e^{3xy}$

(d) $h(x,y) = \cos^{-1}(x^2 + 4y^2 - 4)$

Level Curves or Contours

So far, we have two methods for visualizing functions, arrow diagrams and graphs.

- *A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour curves, or level curves.*
- *The level curves of a function f of two variables are the curves with equations*

$$f(x, y) = k$$

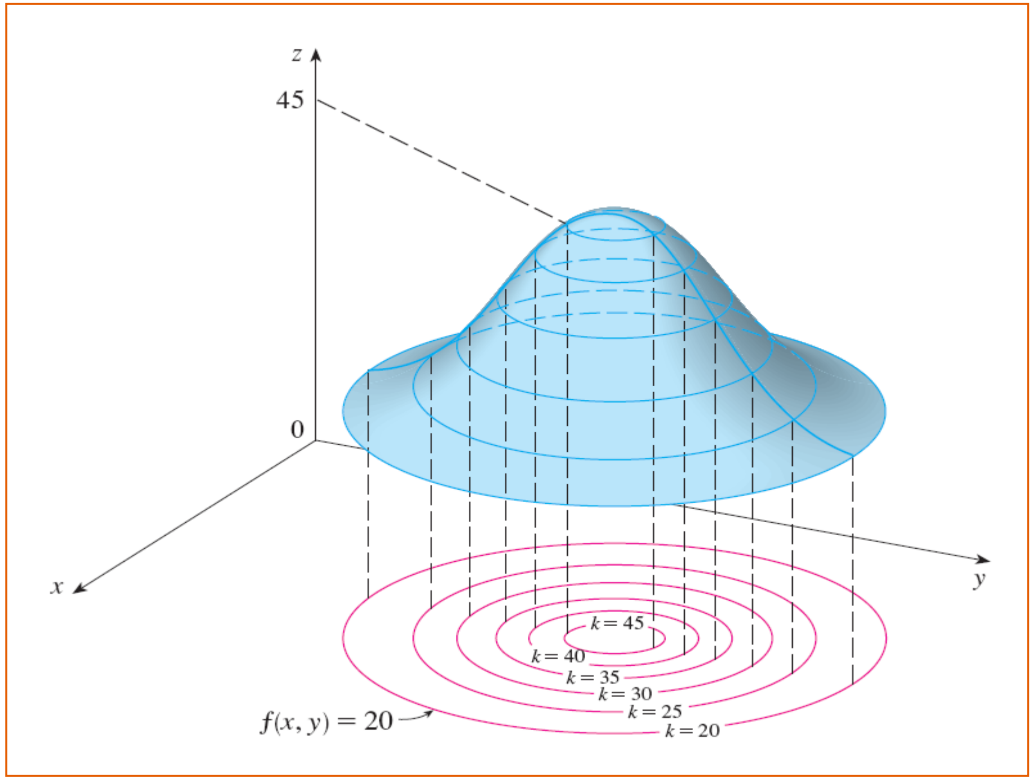
where k is a constant (in the range of f).

Level Curves or Contours

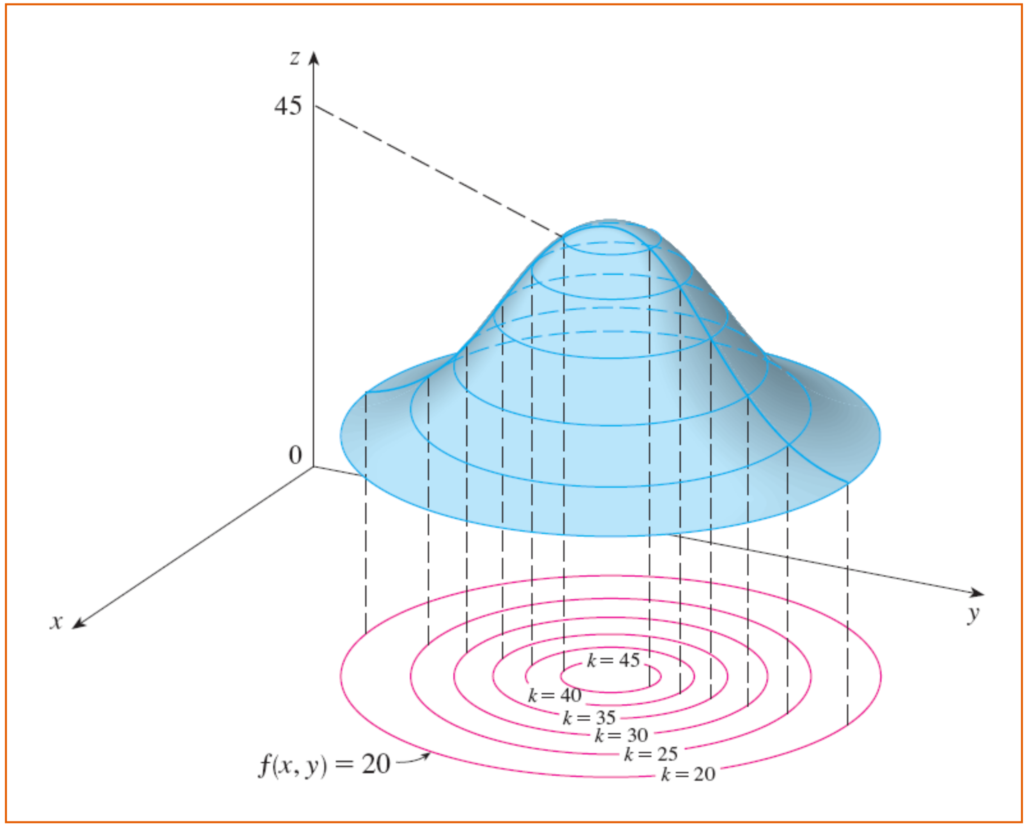
A level curve f(x, y) = k is the set of all points in the domain of f at which f takes on a given value k.

- *That is, it shows where the graph of f has height k.*

You can see from the figure the relation between level curves and horizontal traces.



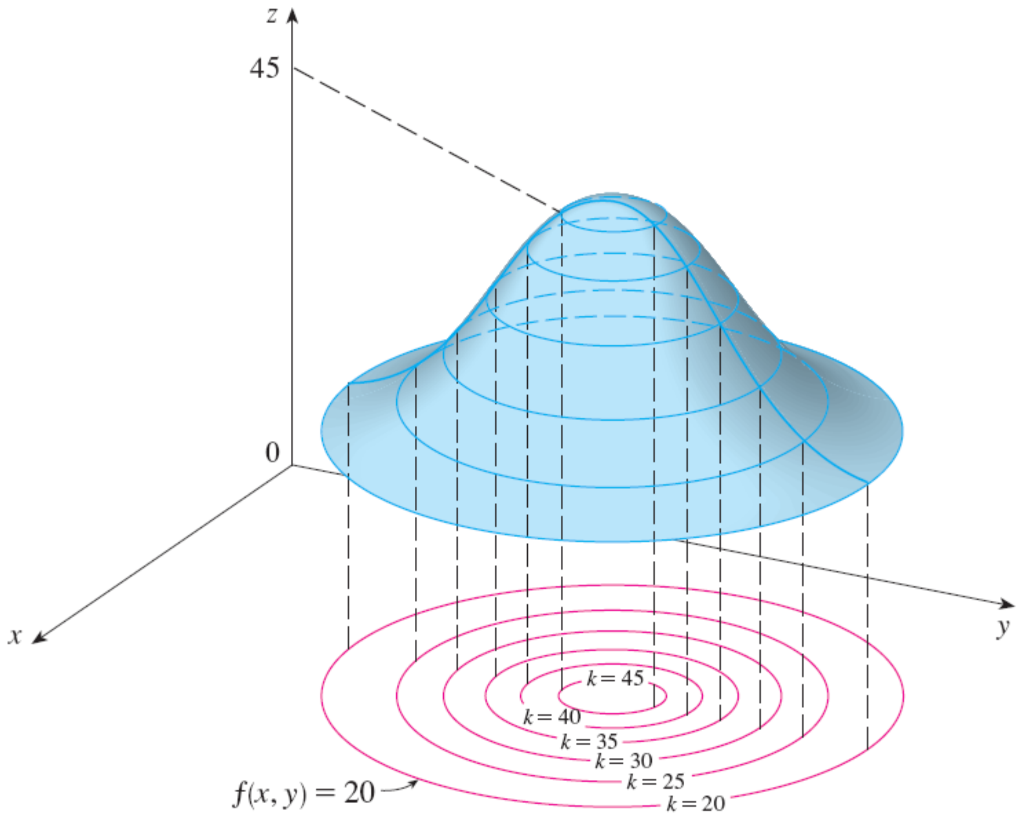
The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.



Level Curves or Contours

So, suppose you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height.

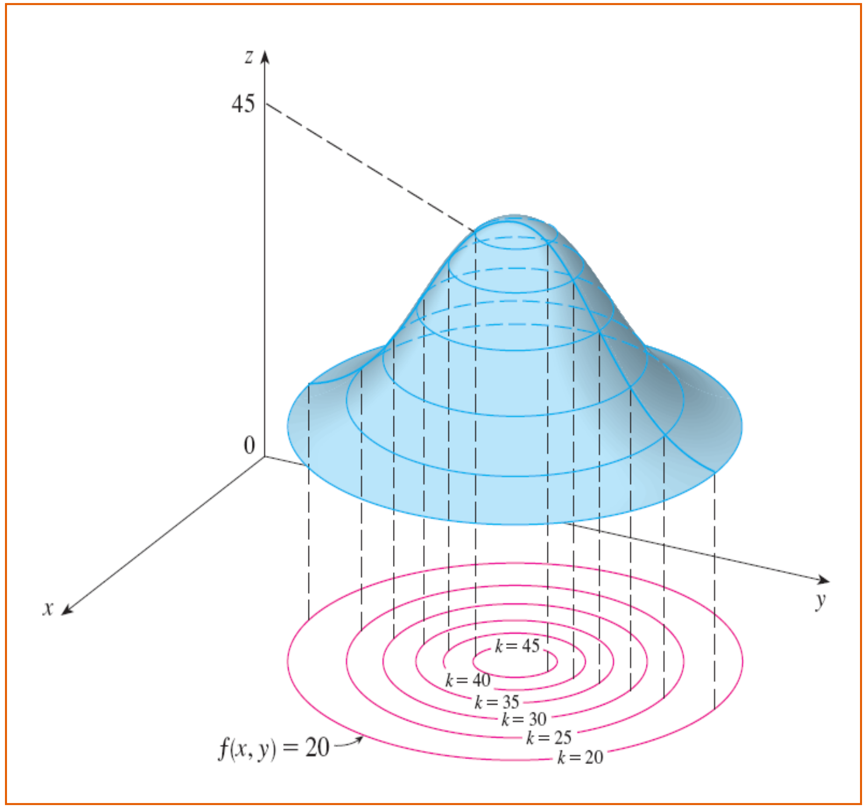
- *Then, you can mentally piece together a picture of the graph.*



Level Curves or Contours

The surface is:

- *Steep where the level curves are close together.*
- *Somewhat flatter where the level curves are farther apart.*



Example

Sketch level curves (or a contour map) of the function

$$f(x, y) = x^2 + y^2.$$

Solution

The function $f(x, y) = x^2 + y^2$ *is a revolving parabola called a* ***paraboloid***.

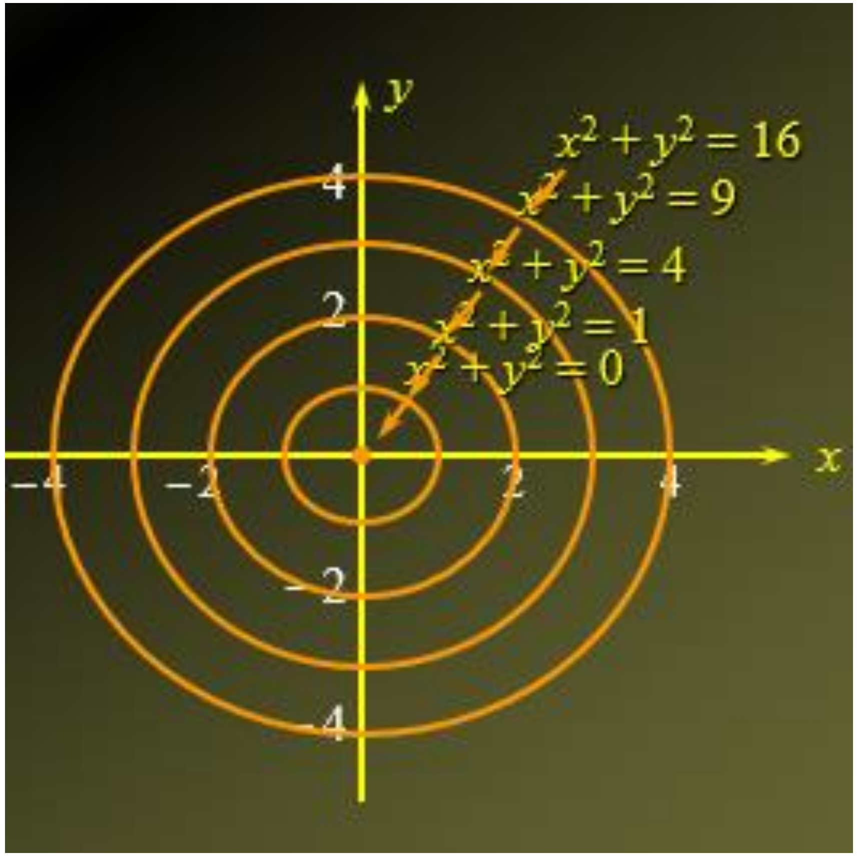
A level curve is the graph of the equation

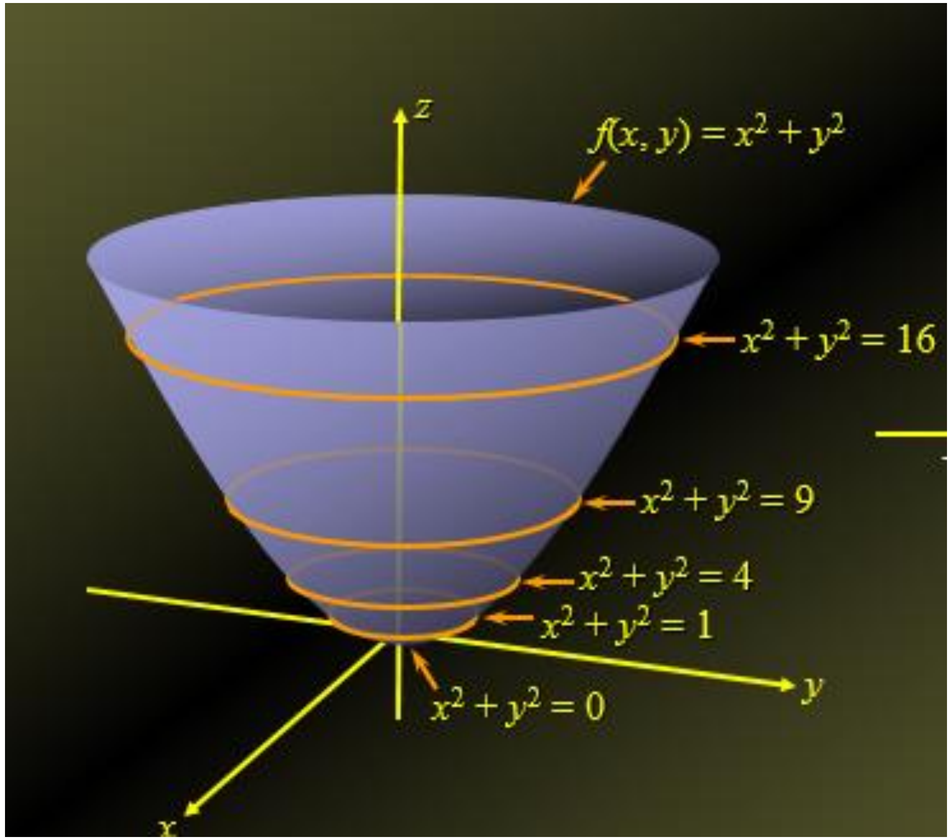
$$x^2 + y^2 = k,$$

which describes a circle with radius $\sqrt{k}$.

Taking different values of k we obtain the level curves of the function.

Example - Solution



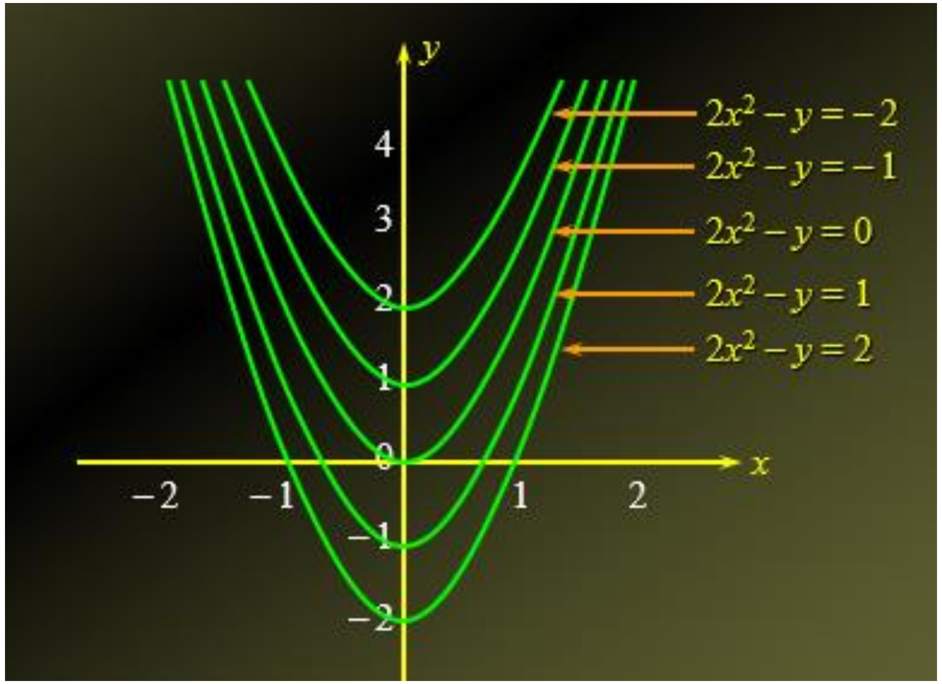


Example

Sketch level curves of the function $f(x,y) = 2x^2 - y$ *corresponding to* $z = -2, -1, 0, 1$ *and* $2.$

Solution

The level curves are the graphs of the equation $2x^2 - y = k$ *or for k = −2, −1, 0, 1, and 2:*



Example

Find and sketch the level curves of the following surface.

$f(x, y) = y - x^2 - 1,$

Solution

a. The level curves are described by the equation

$$y - x^2 - 1 = k$$

where k is a constant in the range of f.

For all values of k, these curves are parabolas in the xy-plane,

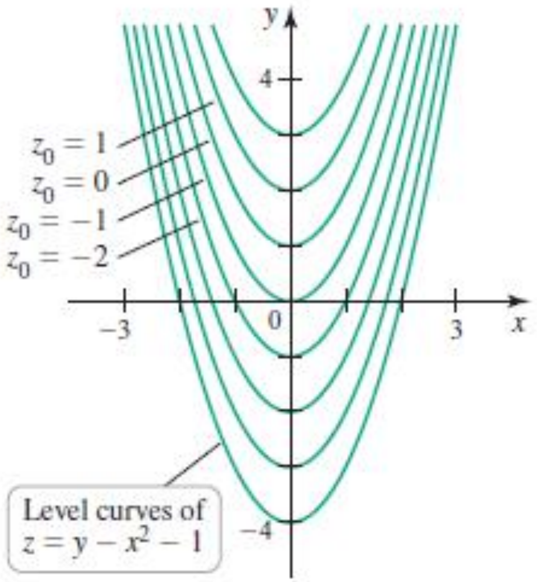
- *With k = 0, the level curve is the parabola* $y = x^2 + 1$*; along this curve, the surface has an elevation (z-coordinate) of 0.*
- *With k = -1, the level curve is* $y = x^2$*; along this curve, the surface has an elevation of -1.*

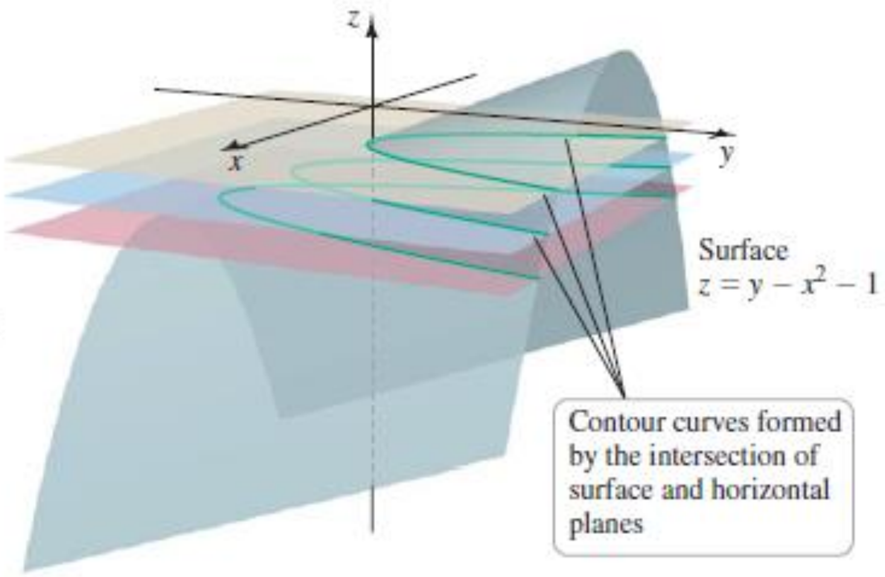
Example

• *With k=1, the level curve is* $y = x^2 + 2$*, along which the surface has an elevation of 1.*

The level curves form a family of shifted parabolas.

- *When these level curves are labelled with their z-coordinates, the graph of the surface* $z = f(x, y0$ *can be visualized*





Homework

Find and sketch the level curves of the following surfaces.

$$f(x,y) = 2 + \sin(\mathrm{x} + \mathrm{y}), \qquad f(x,y) = e^{x^2+y^2}$$

Application

Electric Potential Function due to a dipole: *The electric field at points in the xy-plane due to an electric dipole located at (0, 0) and (1, 0) is given by*

$$\Phi(x, y) = \frac{4}{\sqrt{x^2 + y^2}} + \frac{4}{\sqrt{(x-1)^2 + y^2}}$$

(a) For what values of x and y is the field Φ defined?

(b) Is the electric potential greater at (4, 2) or (2, 4)?

(c) Sketch the level curves.

Application - Solution

(a) *The domain of the potential field function contains all points in* $\mathbb{R}^2$ *for which denominator is not zero, i.e.,*

$$\sqrt{(x^2+y^2)((1-x)^2+y^2)} \neq 0 \quad \Rightarrow (x,y) \neq (0,0),(1,0)$$

Domain: $\{(x,y) \in \mathbb{R}^2 : x \neq 0,1, y \neq 0\}$

These are the points where the charges are located. As these points are approached, the potential function

becomes arbitrarily large.

The potential approaches zero as x or y increases.

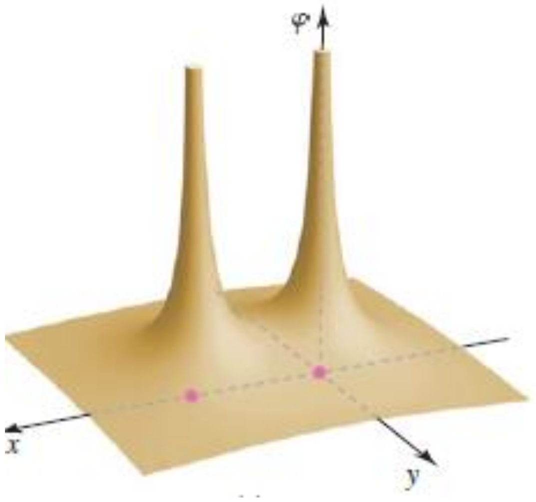
Application - Solution

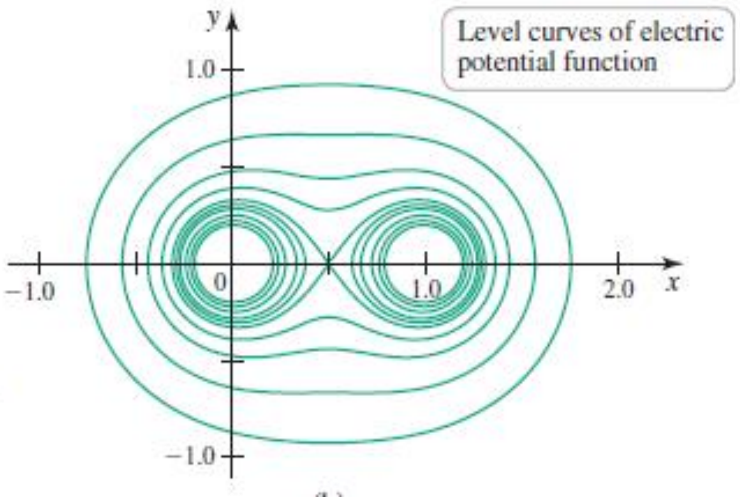
(b) To find where the potential is greater at the given points, we find the values of Φ *at (4,2) and (2,4):*

$$\Phi(4,2) = 2.003, \qquad \Phi(2,4) = 1.864$$

$\Phi(x, y)$ *is greater at (4,2).*

(c)The level curves of Φ *are closed curves, encircling either a single charge (at small distances) or both charges (at larger distances)*





Homework

Two capacitors of capacitance $x, and\ y$, respectively, are connected in series in an electrical circuit.

(a) Graph the effective capacitance function using the window $[0,5] \times [0,5] \times [0,5]$.

(b) Estimate the maximum value for the effective resistance for $0 < x \leq 2$ and $o < y \leq 2$.

(c) Explain what it means to say that the effective resistance function is symmetric in x and y.