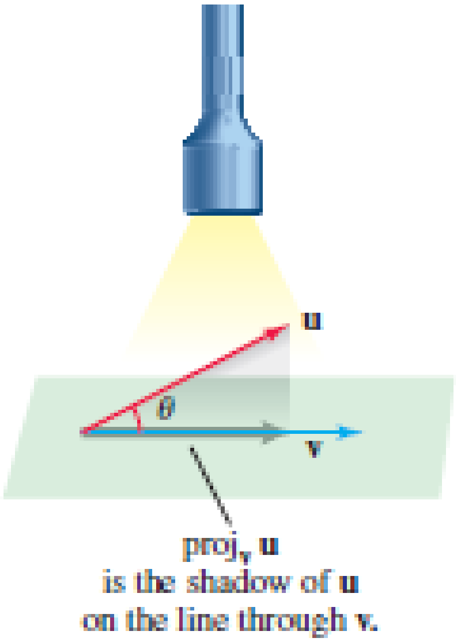
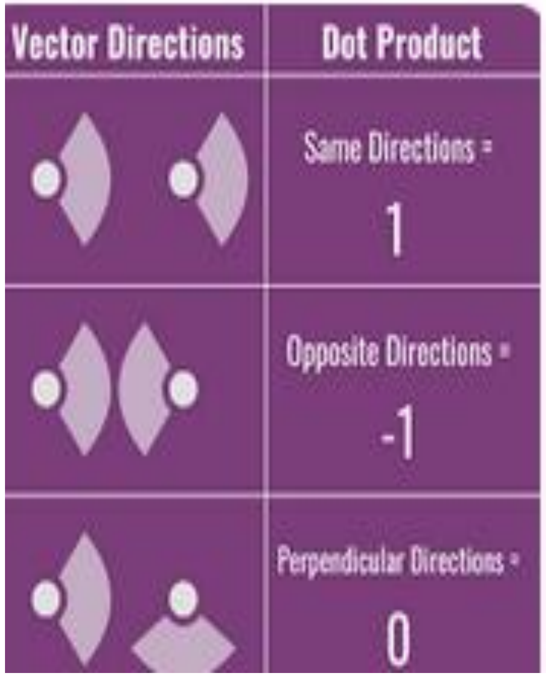
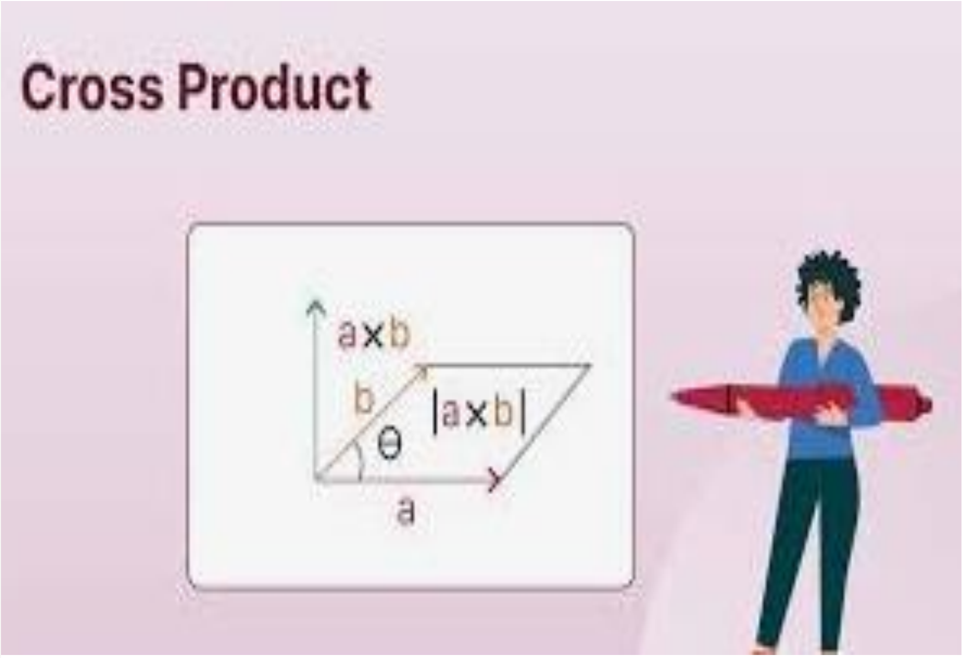
Lecture 7 - Chapter 10 – Sec. 10.3
Scalar & Vector Products







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Learning Objectives

- *Evaluate the dot product of two vectors.*
- *Interpret the dot product geometrically.*
- *Find the projection and component of projection of one vector onto another.*
- *State and prove the Schwartz inequality.*
- *Evaluate the cross product of two vectors.*
- *Interpret the cross product geometrically.*
- *Define scalar triple product.*
- *Use the scalar triple product to find the volume of a parallelepiped.*

Dot or Scalar Product

- *The physical meaning of The dot product, also called the scalar product, is* ***a measure of how closely two vector quantities align/overlap,*** *in terms of the directions they point.*

- *The measure is a scalar number (single value) that can be used to compare the two vectors and to understand the impact of repositioning one or both of them*

- *The dot product is used to determine the angle between two vectors.*

- *It is also a tool for calculating projections—the measure of how much of a given vector lies in the direction of another vector.*

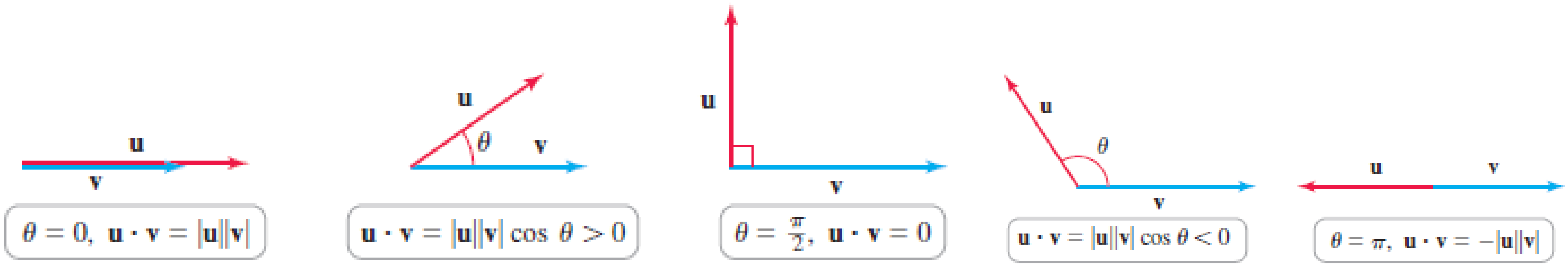
Two Forms of the Dot Product

DEFINITION - Dot Product

Given two nonzero vectors $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in two or three dimensions, their* ***dot product*** *is*

$$\boldsymbol{u}.\boldsymbol{v} = |\,\boldsymbol{u}\,||\,\boldsymbol{v}\,|\cos\theta$$

where θ *is the angle between* **u** *and* **v** *with* $0 \leq \theta \leq \pi$. *If* **u** *= 0 or* **v** *= 0, then* **u** *#* **v** *= 0, and* θ *is undefined.*



What is a Vector?

- *The dot product of two vectors is itself a scalar.*

- *Two special cases immediately arise:*

  u *and* ***v*** *are parallel* ($\theta = 0$ *or* $\theta = \pi$) *if and only if* $\boldsymbol{u}.\boldsymbol{v} = \pm|\,\boldsymbol{u}\,||\,\boldsymbol{v}\,|$

  u *and* ***v*** *are perpendicular* $(\theta = \pi/2)$ *if and only if* $\boldsymbol{u}.\boldsymbol{v}$*=0.*

- *The second case gives rise to the important property of orthogonality*

DEFINITION - Orthogonal Vectors

Two vectors ***u*** *and* ***v*** *are* ***orthogonal*** *if and only if* $\boldsymbol{u}.\boldsymbol{v}$ *= 0. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.*

Dot Product as Components of Vectors

- *Computing a dot product in this manner requires knowing the angle θ between the vectors.*
- *Often the angle is not known; in fact, it may be exactly what we seek.*
- *For this reason, we present another method for computing the dot product that does not require knowing θ.*

Definition of the Dot Product *If* $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ *and* $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$*, then the* ***dot product*** *of* ***a*** *and* ***b*** *is given by*

$$a \cdot b = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product is also referred as to ***scalar product*** *or* ***inner product****.*

Example – Q. 23. Ex. 10.3

Find a unit vector that is orthogonal to both $\hat{\imath} + \hat{\jmath}$ *and* $\hat{\imath} + \hat{k}$

Solution:

First, we define the vector whose unit vector is orthogonal to the given vectors. Let

$$\boldsymbol{a} = \hat{\imath}\, a_1 + \hat{\jmath}\, a_2 + \hat{k}\, a_3$$

Since the unit vector is in the direction of the vector in cartesian coordinates, therefore, we evaluate $\boldsymbol{a} \cdot (\hat{\imath} + \hat{\jmath})$ *and* $\boldsymbol{a} \cdot (\hat{\imath} + \hat{k})$

$$\boldsymbol{a} \cdot (\hat{\imath} + \hat{\jmath}) = 0 \Rightarrow a_1 + a_2 = 0$$

$$\boldsymbol{a} \cdot \left(\hat{\imath} + \hat{k}\right) = 0 \Rightarrow a_1 + a_3 = 0$$

$$\Rightarrow a_1 = -a_2 = -a_3$$

Furthermore $\boldsymbol{a}$ *is to be a unit vector, so*

$$1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2 \Rightarrow a_1 = \pm 1/\sqrt{3}$$

Thus $a = \frac{1}{\sqrt{3}}\,\hat{\imath} - \frac{1}{\sqrt{3}}\,\hat{\jmath} - \frac{1}{\sqrt{3}}\,\hat{k}$ *and* $a = -\frac{1}{\sqrt{3}}\,\hat{\imath} + \frac{1}{\sqrt{3}}\,\hat{\jmath} + \frac{1}{\sqrt{3}}\,\hat{k}$ *are two such unit vectors.*

Example – Ex. 10.3

Find the unit vector that makes an angle of 60^o with $\boldsymbol{v} = \langle 3, 4 \rangle$.

Solution: Let the unit vector be

$$u = \langle a, b \rangle$$

Since it makes an angle of 60^o with $\boldsymbol{v} = \langle 3, 4 \rangle$,

$$u \cdot v. = |\, u \,||\, v \,| \cos 60 \Rightarrow 3a + 4b \Rightarrow\ 3a + 4b = (1)(5)\frac{1}{2} = 5/2$$

This implies that

$$b = \frac{5}{8} - \frac{3}{4}a$$

Since $|\, u \,| = \sqrt{a^2 + b^2} = 1 \Rightarrow a^2 + b^2 = 1$ and

$$100a^2 - 60a\ - 39 = 0 \Rightarrow a = \frac{3 \pm 4\sqrt{2}}{10}$$

Example – Ex. 10.3

This yields

$$b = \frac{4 - 3\sqrt{3}}{10} \quad if\ a = \frac{3 + 4\sqrt{2}}{10} \qquad and$$

$$b = \frac{4 + 3\sqrt{3}}{10} \quad if\ a = \frac{3 - 4\sqrt{2}}{10}$$

Thus, the two unit vectors are

$$\left\langle \frac{3 + 4\sqrt{2}}{10}, \frac{3 - 3\sqrt{3}}{10} \right\rangle \approx \langle 0.9928, \qquad -0.1196 \rangle$$

and

$$\left\langle \frac{3 - 4\sqrt{2}}{10}, \frac{3 + 3\sqrt{3}}{10} \right\rangle \approx \langle -0.3928, \qquad -0.9196 \rangle$$

Example – Q. 28, Ex. 10.3

Find the acute angles between the curves

$$y = \sin x, \qquad y = \cos x, \qquad 0 \leq x \leq \pi/2$$

at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

Solution

First find the point of intersection between the two curves

$$six = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4 \quad (0 \leq x \leq \pi/4)$$

Thus, the point of intersection is $\left(\pi/4, \sqrt{2}\,/2\right)$.

To find the slopes of the tangent lines, we have

$$\frac{d}{dx}\sin x \mid_{\pi/4} = \cos x \mid_{\pi/4} = \sqrt{2}/2$$

$$\frac{d}{dx}\cos x \mid_{\pi/4} = -\sin x \mid_{\pi 4} = -\sqrt{2}/2$$

Example – Q. 28, Ex. 10.3

Vectors parallel to the tangent lines are

$$\left\langle 1, \sqrt{2}\,/2 \right\rangle \qquad \textit{and}\ \langle 1, -\sqrt{2}\,/2 \rangle$$

and the angle between them is given by

$$\cos\theta = \frac{\left\langle 1, \sqrt{2}\,/2 \right\rangle \cdot \langle 1, -\sqrt{2}\,/2 \rangle}{\mid \left\langle 1, \sqrt{2}/2 \right\rangle \mid \langle 1, -\sqrt{2}\,/2 \rangle \mid}$$

$$= \frac{1}{3}$$

$$\theta = 70.5^{o}$$

Note: For any real number m, the vector $(1, m)$ determines a line of slope m through the origin:

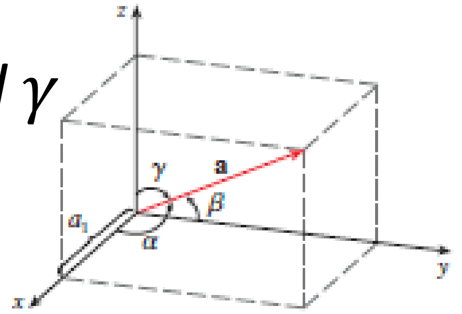
Homework 1 – Ex. 10.3

(a) For what values of b are the vectors $\langle -6, b, 2\rangle$ *and* $\langle b, b^2, b\rangle$ *orthogonal?*

(b) Find the acute angle between the curves at their points of intersection

$$5x - y = 8, \qquad x + 3y = 15$$

Example – Q. 26. Ex. 10.3



The ***direction angles*** *of a nonzero vector* ***a*** *are the angles α, β, and γ (in the interval $[0, \pi]$ that* ***a*** *makes with the positive x-, y-, and z-axes, respectively.*

The cosines of these direction angles, $\cos\alpha, \cos\beta$, and $\cos\gamma$, are called the ***direction cosines*** *of the vector* ***a****.*

$$\frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

which implies that the direction cosines of ***a*** *are the components of the unit vector in the direction of* ***a****.*

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \qquad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \qquad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

Example – Ex. 10.3

Find the direction angles of the vector $\boldsymbol{a} = \langle \mathbf{1}, \mathbf{2}, \mathbf{3} \rangle$.

Solution:

Since

$$| a | = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

direction cosines of ***a*** *are*

$$\cos\alpha = \frac{1}{\sqrt{14}}, \qquad \cos\beta = \frac{2}{\sqrt{14}}, \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = 74^o, \qquad \beta = 58^o, \qquad \gamma = 37^o$$

Homework 2 – Ex. 10.3

If a vector has direction angles $\alpha = \pi/4$, $\beta = \pi/3$, *find the third direction angle .*

Properties of the Dot Product

THEOREM

Suppose $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$ *are vectors and let c be a scalar.*

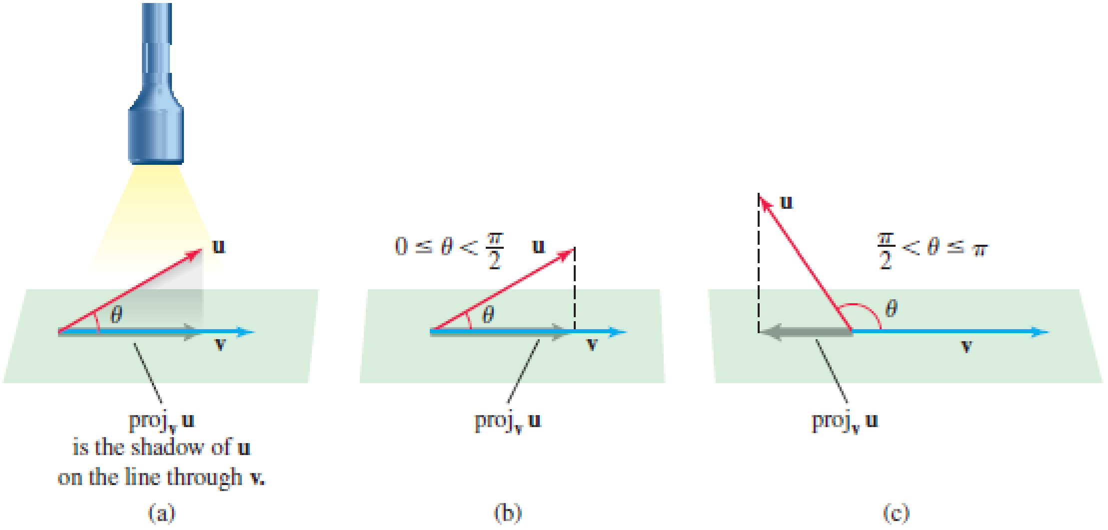
1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ *Commutative property*

2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ *Associative property*

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ *Distributive property*

Vector Projections

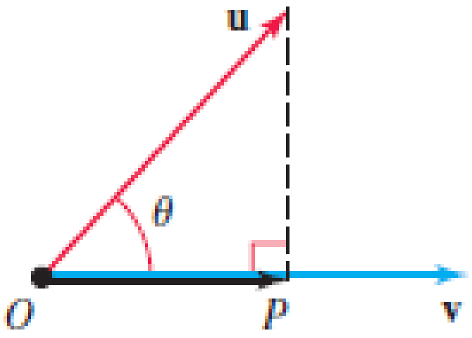
- *Given vectors **u** and **v**, how closely aligned are they? That is, how much of **u** points in the direction of **v**? This question is answered using projections.*
- *The projection of the vector **u** onto a nonzero vector **v**, denoted $proj_v u$, is the "**shadow**" cast by **u** onto the line through **v**.*
- *The projection of **u** onto **v** is itself a **vector**; it points in the same direction as **v** if the angle between **u** and **v** lies in the interval $0 \leq \theta \leq \pi/2$.*
- *It points in the direction opposite that of **v** if the angle between **u** and **v** lies in the interval $\pi/2 \leq \theta \leq \pi$.*



Vector Projection

- *If* $0 \leq \theta \leq \pi/2$*, then* $proj_v\boldsymbol{u}$ *has length* $|\ \boldsymbol{u}\ |\cos\theta$ *and points in the direction of the unit vector* $\boldsymbol{v}/|\ \boldsymbol{v}\ |$*. Therefore,*

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{|\mathbf{u}|\cos\theta}_{\text{length}}\underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$



$0 \leq \theta < \frac{\pi}{2}$

$\mathrm{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta > 0$

We define the ***scalar component*** *of* ***u*** *in the direction of* ***v*** *to be*

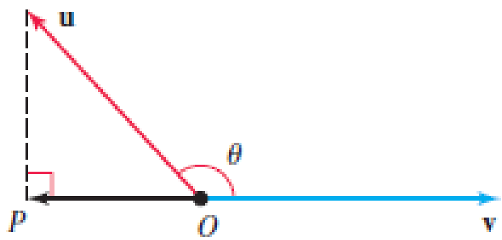
$$scal_v\boldsymbol{u} = |\ u\ |\cos\theta$$

In this case, $scal_v\boldsymbol{u}$ *is the length of* $proj_v\boldsymbol{u}$*.*

- *If* $0 \leq \theta \leq \pi/2$*, then* $proj_v\boldsymbol{u}$ *has length* $-|\ \boldsymbol{u}\ |\cos\theta$ *(which is positive) and points in the direction of* $-\boldsymbol{v}/|\ \boldsymbol{v}\ |$*.*

Therefore,

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{-|\mathbf{u}|\cos\theta}_{\text{length}}\underbrace{\left(-\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$



$\frac{\pi}{2} < \theta \leq \pi$

$\mathrm{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta < 0$

Projection Visualization

Visual Description of Dot Product and Orthogonal Projection

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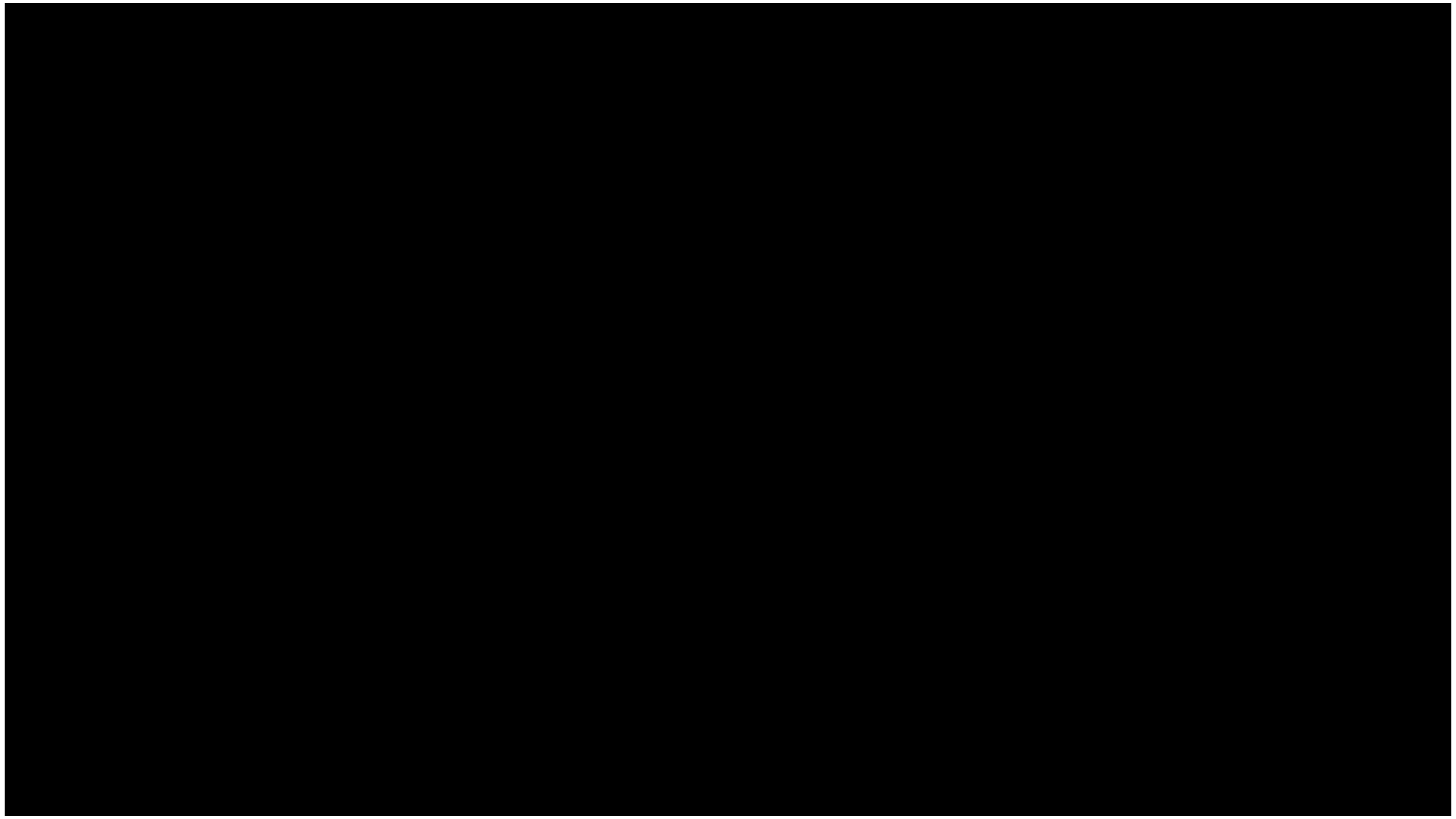


Scalar & Vector Projections

*The **scalar projection** of **u** onto **v** (also called the **component of u along v**) is defined to be the signed magnitude of the vector projection, which is the number,* $|\, u \,| \cos\theta$.

*Scalar projection of **u** onto **v***

$$scal_v u = \boldsymbol{u}.\frac{\boldsymbol{v}}{|\,\boldsymbol{u}\,|}$$

Vector projection of v onto u

$$proj_v u = \left(\frac{\boldsymbol{u}.\boldsymbol{v}}{|\,\boldsymbol{u}\,|}\right)\frac{\boldsymbol{u}}{|\,\boldsymbol{u}\,|}$$

$$= \frac{\boldsymbol{u}.\boldsymbol{v}}{|\,u\,|^2}\,\boldsymbol{u}$$

DEFINITION (Orthogonal) Projection of u onto v

The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

where the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

Example – Q. 32, Ex. 10.3

Distance from a Point to a Line - *Use a scalar projection to show that the distance from a point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is*

$$\frac{|\ ax_1 + by_1 + c\ |}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution: *Let point $P_2(x_2, y_2)$ lie on the line.*

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Then the distance from $P_1(x_1, y_1)$ to the line is the absolute value of the scalar projection of $\overrightarrow{\boldsymbol{P_1P_2}}$ onto $\boldsymbol{n} = \langle a, b \rangle$.

$$scal_n\left(\overrightarrow{P_1P_2}\right) = \frac{|\ \boldsymbol{n}\ \cdot \boldsymbol{P_1P_2}\ |}{|\ \boldsymbol{n}\ |} = \frac{|\ n \cdot \langle x_2 - x_1, y_2 - y_1\ |}{n}$$

Example – Q. 32, Ex. 10.3

$$scal_n(\overrightarrow{P_1P_2}) = \frac{|\ ax_2 - ax_1 + by_2 - by_1\ |}{\sqrt{a^2 + b^2}}$$

$$scal_n(\overrightarrow{P_1P_2}) = \frac{|\ ax_1 + by_1 + c\ |}{\sqrt{a^2 + b^2}}$$

(since $ax_2 + by_2 = -c$).

The required distance is

$$\frac{|\ (3)(-2) + (-4)(3) + 5\ |}{\sqrt{3^2 + (-4)^2}}$$

$$= 13/5$$

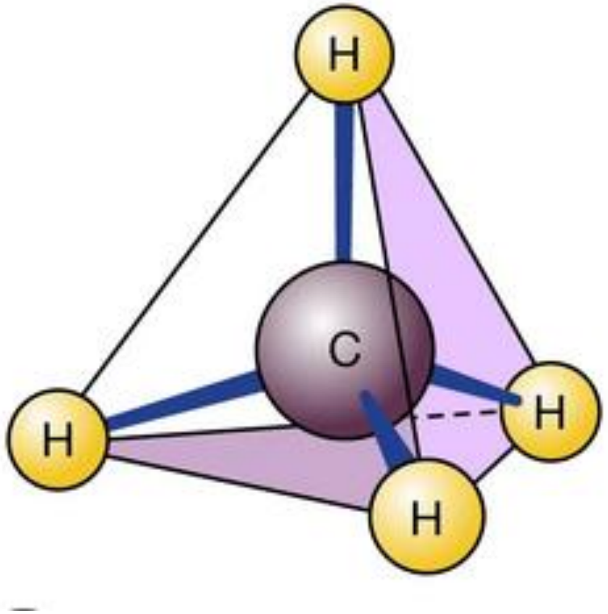
Homework 3 – Ex. 10.3

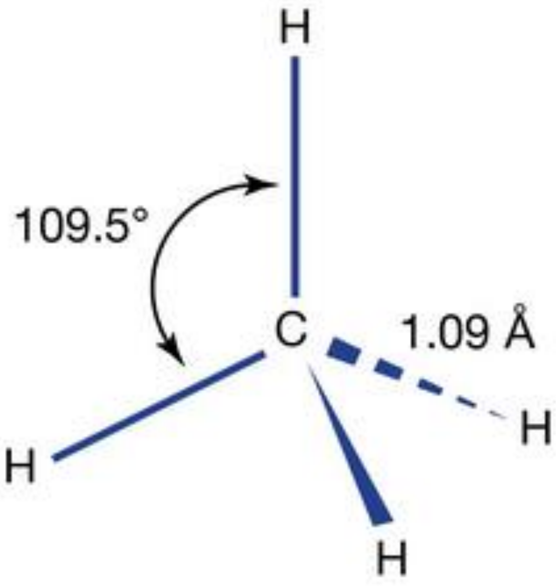
(a) Find the angle between a diagonal of a cube and one of it's edges.

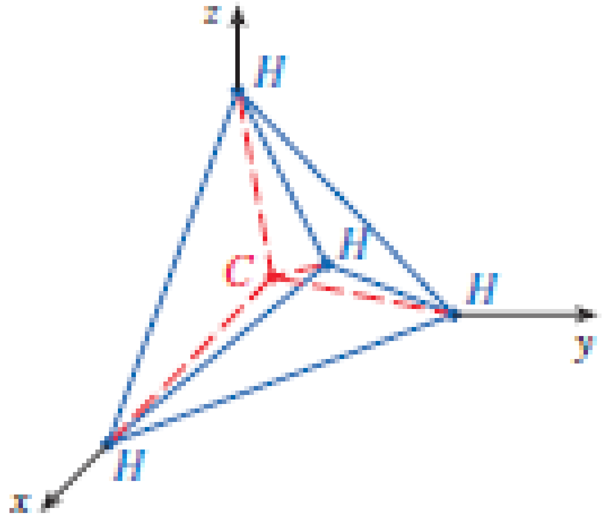
*(b) Show that if **u** + **v** and **u** - **v** are orthogonal, then the vectors **u** and **v** must have the same length.*

Example – Ex. 10.3

A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°.





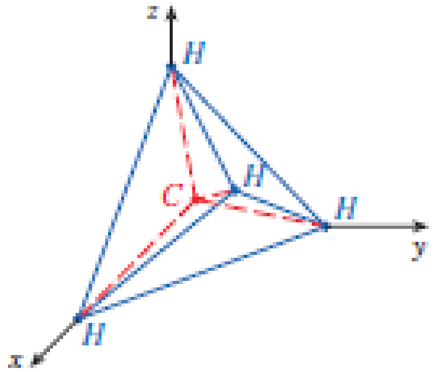


Example – Ex. 10.3

Solution:

Take the vertices of the tetrahedron to be the points (1, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 1). Then the centroid is $(1/2, 1/2, 1/2)$.

Consider the H—C—H combination consisting of the sole carbon atom and the two hydrogen atoms that are at (1,0,0) and (0, 1, 0) (or any H—C—H combination, for that matter).



Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are

$$\langle 1 - 1/2, 0 - 1/2, 0 - 1/2 \rangle = \langle 1/2, -1/2, -1/2 \rangle$$

and

$$\langle 0 - 1/2, 1 - 1/2, 0 - 1/2 \rangle = \langle -1/2, 1/2, -1/2 \rangle$$

Coordinates of centroid of a regular tetrahedron

$$x = \frac{x_1 + x_2 + x_3 + x_4}{4}$$
$$y = \frac{y_1 + y_2 + y_3 + y_4}{4}$$
$$z = \frac{z_1 + z_2 + z_3 + z_4}{4}$$

Example – Ex. 10.3

The bond angle, θ, is therefore given by

$$\cos\theta = \frac{\langle 1/2, -1/2, -1/2 \rangle \cdot \langle -1/2, 1/2, -1/2 \rangle}{|\langle 1/2, -1/2, -1/2 \rangle||\langle -1/2, 1/2, -1/2 \rangle|}$$

$$= -\frac{1}{3}$$

$$\theta = \cos^{-1}(-1/3)$$

$$\theta \approx 109.5^o$$

Homework 4 – Ex. 10.3

Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

Example – Q. 49, Ex. 10.3

Cauchy-Schwartz Inequality *Use the Theorem*

$$a \cdot b = |a||b|\cos\theta$$

to prove the Cauchy-Schwarz Inequality:

$$|a \cdot b| \leq |a||b|.$$

Solution: *The dot product is given by*

$$a \cdot b = |a||b|\cos\theta$$

Therefore,

$$|a \cdot b| = ||a||b|\cos\theta|$$

$$= |a||b||\cos\theta|$$

Since $|\cos\theta| \leq 1$,

$$|a \cdot b| = |a||b|\cos\theta| \leq |a||b|.$$

Note that we have equality in the case of $\cos\theta = \pm 1$, *so* $\theta = 0$ *or* $\theta = \pi$, *thus equality when a and b are parallel.*

Homework 5 – Ex. 10.3

Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

Calculating Dot Product in Parallels

- *The dot product between two arrays is the sum of the products. Consider the arrays*

  A=[1,2,3] and B=[4,5,6].
- *The dot product of these two arrays is 1x4 + 2x5 + 3x6 = 4+10+18 = 32. A C/C++ implementation of this example*

```
int main(){
  int a[3] = {1,2,3};
  int b[3] = {4,5,6};
  int i, sop=0;
  for (i = 0; i < 3; i++){
    sop+=a[i]*b[i];
  }
  printf("sop is: %d\n", sop);
  return 0;
}
```

- *To parallelize the dot product of two arrays over n elements and c cores we would do the following:*
- *Assign n/c elements of each array to each core.*
- *Each core will then calculate a local sum of products using the n/c elements assigned to it, and send it to the host.*
- *The host will sum all the local sums together to yield a final sum of products.*

Calculating Dot Product in Parallels

```
#include <stdio.h>
#include <stdlib.h>
#include "e-lib.h"
#include "common.h"

int main(void)
{
  unsigned *a, *b, *c, *d;
  int i;

  a     = (unsigned *) 0x2000;//Address of a matrix
  b     = (unsigned *) 0x4000;//Address of b matrix
  c     = (unsigned *) 0x6000;//Result
  d     = (unsigned *) 0x7000;//Done flag

  //Clear Sum
  (*(c))=0;

  //Sum of product calculation
  for (i=0; i<N/CORES; i++){
    (*(c)) += a[i] * b[i];
  }

  //Raising "done" flag
  (*(d)) = 0x00000001;

  //Put core in idle state
  __asm__ __volatile__("idle");
}
```