# Lecture 2 (Chapter 9)

## Calculus of Parametric Curves

*Dr M. Loan*
*Department of Physics, SCUPI*

# Learning Objectives

- *Determine derivatives and equations of tangents for parametric curves.*
- *Find the area under a parametric curve.*
- *Use the equation for the arc length of a parametric curve.*
- *Apply the formula for surface area to a volume generated by a parametric curve.*

# Derivatives of Parametric Equations

- *Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus.*
- *If we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?*
- *Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve* $x(t), y(t)$ *then we should be able to use calculus to find the speed of the ball at any given time.*
- *Furthermore, we should be able to calculate just how far that ball has travelled as a function of time.*

# Derivatives of Parametric Equations

- *Parametric equations express a relationship between the variables x and y. Therefore, it makes sense to ask about* $dy/dx$*, the rate of change of y with respect to x at a point on a parametric curve.*
- *Once we know how to compute* $dy/dx$*, it can be used to determine slopes of lines tangent to parametric curves.*
- *Consider the parametric equations* $x = f(t), y = g(t)$ *on an interval on which both f and g are differentiable.*
- *The Chain Rule relates the derivatives* $dx/dt, dy/dt$*, and* $dy/dx$*.*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

*provided* $dx/dt \neq 0$*.*

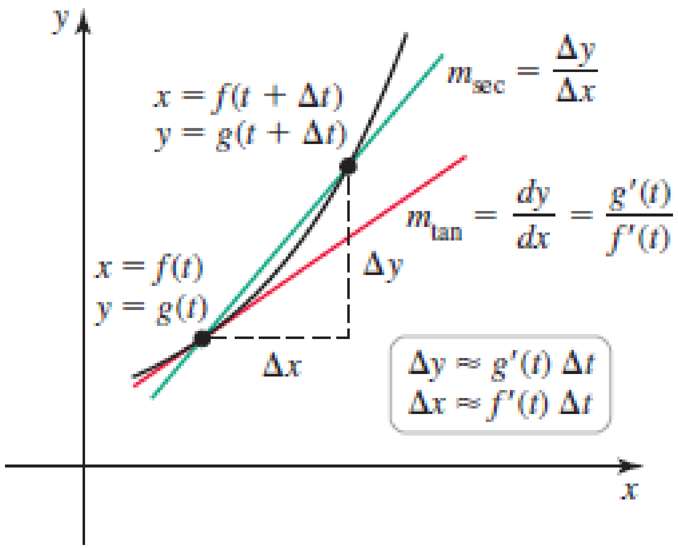
# Derivatives of Parametric Equations

*Theorem - Tangents*

*Let $x = f(t)$ and $y = g(t)$, where $f$ and $g$ are differential on an interval $[a, b]$. Then the slope of the line tangent to the curve at the point corresponding to t is*

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \qquad -(1)$$

*provided $f'(t) \neq 0$.*




- *The curve has a horizontal tangent when $dy/dt = 0$, provided that $dx/dt \neq 0$, and it has a vertical tangent when $dx/dt = 0$, provided that $dy/dt \neq 0$.*

- *If both $dx/dt = 0$ and $dy/dt = 0$, then we would need to use other methods to determine the slope of the tangent.*

# Derivatives of Parametric Equations

*It is useful to consider the second derivative. (Why exactly?)*

*This can be found by replacing $y$ with $dy/dx$ in Eq. (1):*

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

*Note that*

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

# Derivatives of Parametric Equations

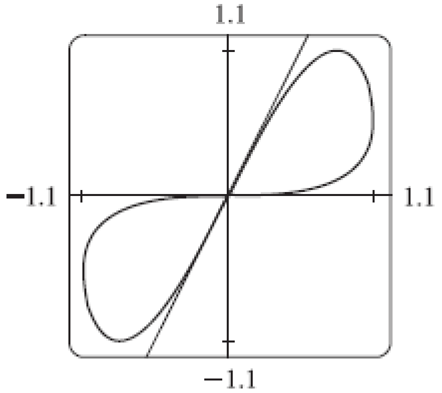
*Q. 8. Ex – 9.2. Find the equations of the tangents to the curve*

$$x = \sin t, \qquad y = \sin(t + \sin t)$$

*At the origin. Graph the curve and the tangents.*

*Solution:*

$$\frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = \cos(t + sint)\,(1 + cost)$$




*Therefore,*

$$\frac{dy}{dx} = (\sec t + 1)\cos(t + \sin t)$$

*There are two tangents at point $(0,0)$, since both $t = 0$ and $t = \pi$ correspond to the origin.*

*The tangent corresponding to t=0 has slope 2 and its equation is $y = 2x$.*

*The tangent corresponding to $t = \pi$ has slope 0, so it is the x-axis.*

# Derivatives of Parametric Equations

*Find the tangent to the cycloid*

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta)$$

*at the point where $\theta = \pi/3$.*

*(b) At what points is the tangent horizontal? When is it vertical?*

*Solution:*

*Slope of the tangent line is*

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \sin\theta/(1 - \cos\theta)$$

*When $\theta = \pi/3$, we have*

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \qquad y = \frac{r}{2}$$

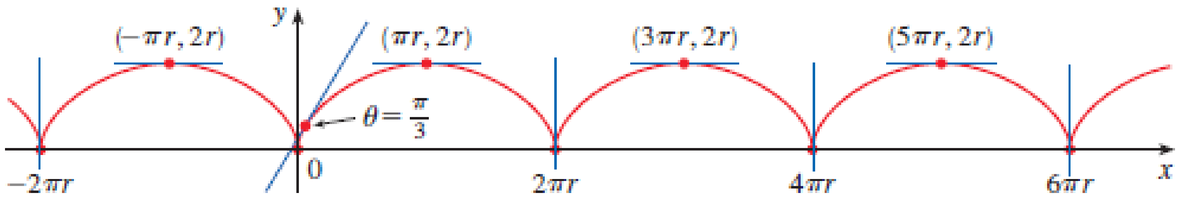
*and*

$$\frac{dy}{dx} = \frac{\sin\pi/3}{1 - \cos\pi/3} = \sqrt{3}$$

# Derivatives of Parametric Equations

Solution (Contd.): *Equation of the tangent is*

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)$$




*or*

$$\sqrt{3}\, x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

*(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $sin\theta = 0$ and $1 - cos\theta \neq 0$, that is $\theta = (2n-1)\pi, n$ is an integer. The corresponding point on the cycloid is $((2n-1)\pi r, 2r)$.*

*When $\theta = 2\pi n$, both $dx/d\theta$ and $dy/d\theta$ are zero. It appears from the graph that there are vertical tangents at those points.*

***Follow-up: Use l'Hospital's Rule to verify that tangents are indeed vertical when $\theta = 2n\pi$***

# Equation of a Tangent to Parametric Curve

*Q. 26. Ex. 9.2: Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter*

$$x = \sin \pi t, \qquad y = t^2 + t; \qquad (0,2)$$

*Solution*:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = (2t + 1)/(\pi \cos \pi t)$$

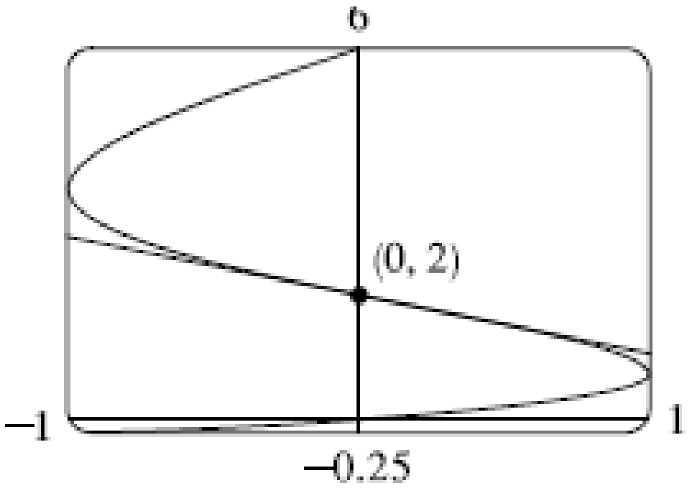
*Value of t corresponding to the point (0,2):*

*solve* $y = 2 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$ *or* $t = 1.$

*Either value gives* $dy/dx = -3/\pi$*. So an equation of the tangent is*

$$y - 2 = -\frac{3}{\pi}(x = 0)$$

$$y = -\frac{3}{\pi}x + 2$$



# Practice Problem 1 – 9.2

*At what point(s) on the curve*

$$x = 2t^3, \qquad y = 1 + 4t - t^2$$

*Does the tangent line have slope ½?*

# Homework 1 – 9.2

***Q. 23. Ex. 9.2***

*(a) Find the slope of the tangent line to the trochoid*

$$x = r\theta - d\sin\theta, \qquad y = r - d\sin\theta$$

*in terms of* $\theta$.

*(b) Show that if* $d < r$*, then the trochoid does not have a vertical tangent.*

***Q. 24. Ex. 9.2***

*(a) Find the slope of the tangent to the astroid*

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

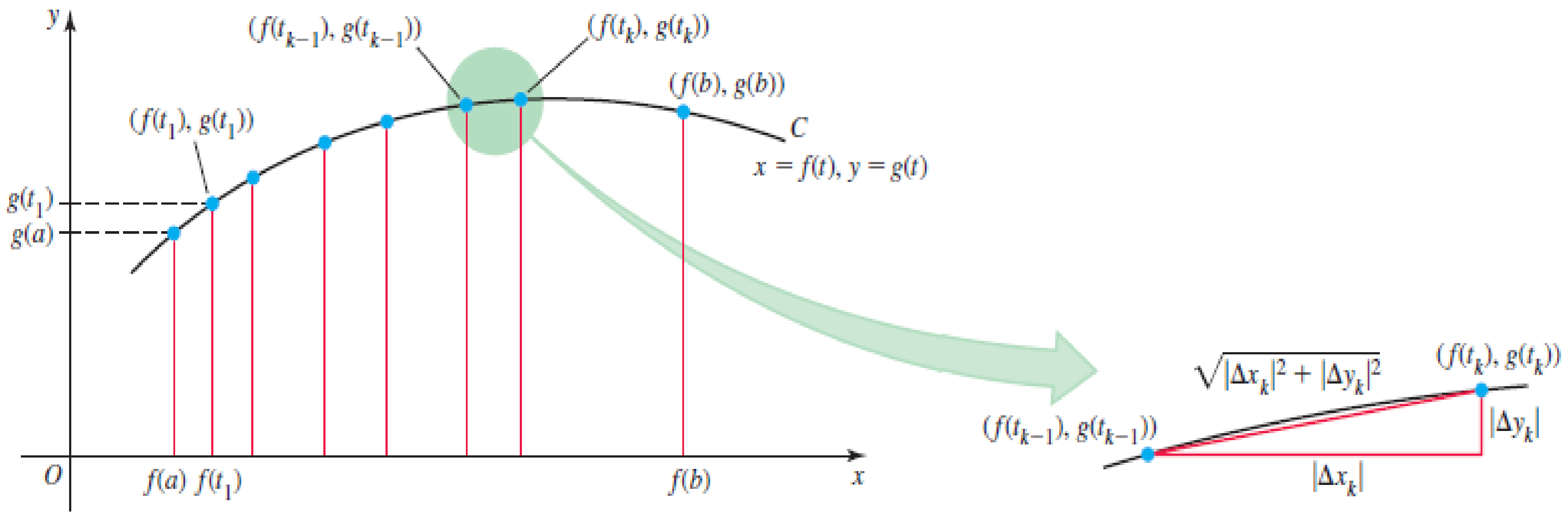
*(b) At what points is the tangent horizontal or vertical?*

*(c) At what points does the tangent have slope 1 or -1?*

# Arc Length for Curves defined by Parametric Equations

***Definition:*** *Consider the curve described by the parametric equations* $x = f(t)$, $y = g(t)$, *where* $f'$ *and* $g'$ *are continuous, and the curve is traversed once for* $a \le t \le b$. *The* ***arc length*** *of the curve between* $(f(a), g(a))$ *and* $(f(b), g(b))$ *is*

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)} \;\; dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} \;\; dt$$



# Length of a Curve

**Q. 47. Ex. 9.2** *Find the distance travelled by a particle with position $(x, y)$ as t varies in the given time interval. Compare with the length of the curve.*

$$x = \sin^2 t, \qquad y = \cos^2 t, \qquad 0 \le t \le 3\pi$$

*Solution:*

$$\frac{dx}{dt} = 2\sin t\cos t, \qquad \frac{dy}{dt} = -2\cos t\sin t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t$$

*Distance travelled by the particle is*

$Distance = \int_0^{3\pi} \sqrt{2}\,|\sin 2t\,|\,dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\,dt$ *(using symmetry)*

$Distance = 6\sqrt{2}.$

*The full curve is transversed as t goes from 0 to $\pi/2$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely transversed as t goes from 0 to $\pi/2$. Thus,* $L = \int_0^{\pi/2} \sin 2t\,dt = \sqrt{2}$ *as above.*

# Homework 2 – 9.2

*A projectile is fired from the point (0, 0) with an initial velocity of* $v_0/s$ *at an angle* $\alpha$ *above the horizontal. If we assume that air resistance is negligible then the position (in meters) of the projectile after t seconds is given by the parametric equations:*

$$x = (v_0 \cos\alpha)t, \qquad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

*where* $g = 9.8\, m/s^2$ *is the acceleration due to gravity.*

*(a) Find the speed of the projectile when it hits the ground.*

*(b) Find the speed of the projectile at its highest point.*

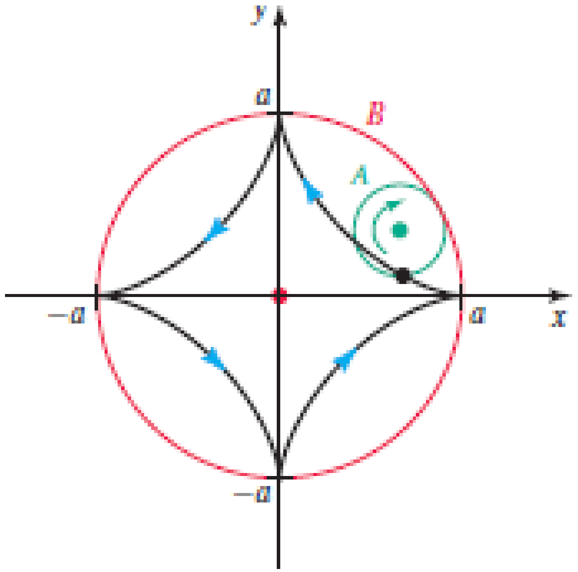
# Total Length of an Astroid

*Q. 50 – Ex. 9.2: Find the total length of the astroid*

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta, \qquad a > 0\,.$$

*Solution Method 1:*

*The path of a point on circle A with radius $a > 0$ that rolls on the inside of circle B with radius a is an **astroid** or a **hypocycloid**.*




$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= (-3a\,cos^2\theta\ sin\ \theta)^2 + (3a\sin^2\theta\ cos\ \theta)^2$$

$$= 9a^2\sin^2\theta\cos^2\theta$$

*The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \le \theta \le \pi/2$.*

# Total Length of an Astroid

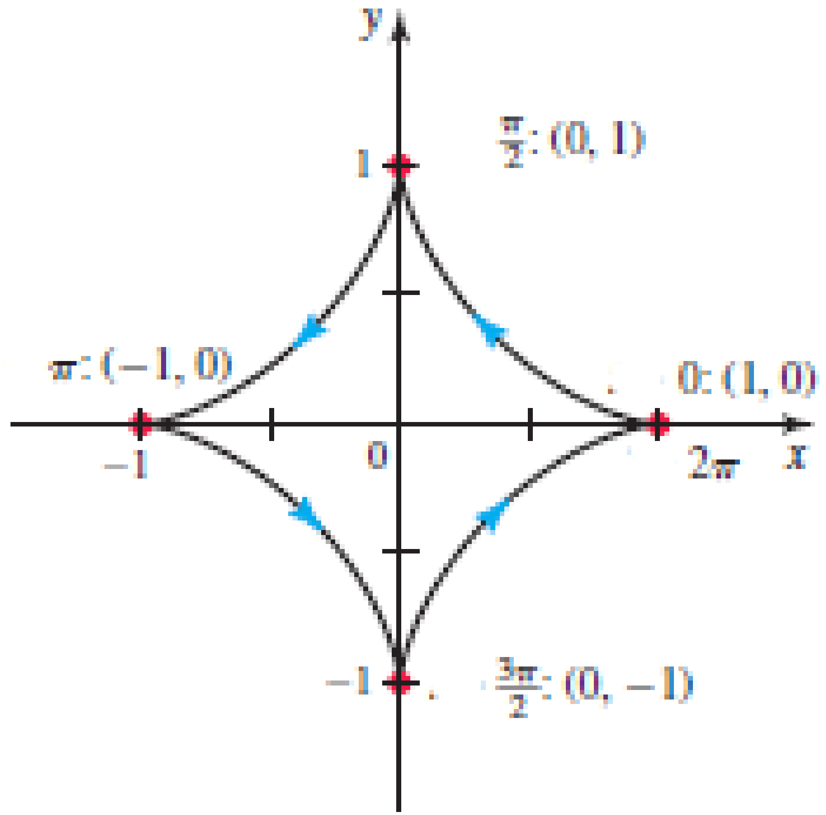
*Solution (Contd.)*

*Thus*

$$L = 4\int_0^{\pi/2} 3a\ sin\theta cos\theta\ d\theta$$

$$L = 6a$$

*The length of the entire hypocycloid is 6 units.*



# Practice Problem 1 – 9.2

***Method 2- Using Cartesian Parametrization***

***Here*** $x = a\cos^3\theta, \quad y = a\sin^3\theta$

*Eliminating* $\theta$ *we get,* $x^{2/3} + y^{2/3} = a^{2/3}$

*Arc length of astroid is given by,* $L = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$

*Differentiating equation* $x^{2/3} + y^{2/3} = a^{2/3}$ *with respect to x,*

$$dy/dx = -\left(\frac{y}{x}\right)^{\frac{1}{3}} = -\tan\theta$$

*Given that* $x = a\cos^3\theta \Rightarrow dx = -3a\cos^2\theta\sin\theta d\theta$

*Since the graph has four-fold symmetry, therefore*

$$L = -3a\int_0^{\pi/2} \sqrt{1+\tan^2\theta}\ \cos^2\theta\sin\theta\ d\theta = -3a\int_0^{\pi/2} cos\theta\sin\theta\ d\theta = 1.5a$$

*Thus, the length of the astroid is S = 4L = 6a.*

# Area Under a Parametric Curve

*Consider the non-self-intersecting plane curve defined by the parametric equations*

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

*and assume that is differentiable.*

*The area under this curve is given by*

$$\boxed{A = \int_a^b y(t)x'(t)dt}$$

# Area Under a Parametric Curve

*Find the area under the curve of the cycloid defined by the equations*

$$x(t) = t - \sin t, \qquad y(t) = 1 - \cos t, \qquad o \le t \le 2\pi.$$

*Solution:* *Using* $A = \int_a^b y(t)x'(t)dt$ *, we have*

$$A = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\, dt$$

$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt$$

$$= \int_0^{2\pi} \left[\frac{3}{2} - 2cost + \frac{1}{2}\cos(2t)\right] dt$$

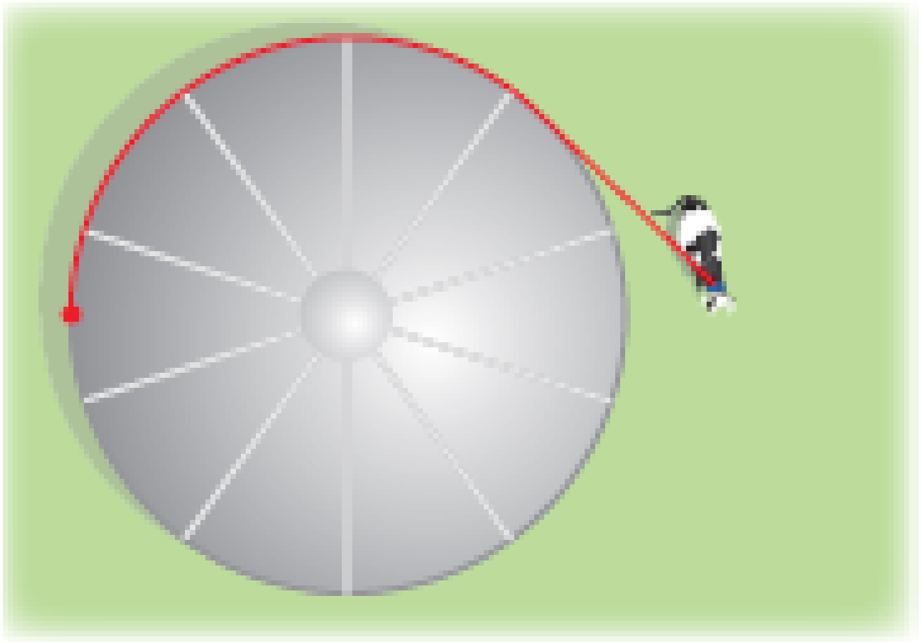
$$= 3\pi$$

*Follow-up: (i) Show that the surface area of a sphere of radius r is* $4\pi r^2$.
*(ii) Find the area under the curve of the hypocycloid defined by the equations*

$$x(t) = 3cost + \cos(3t)\,, \qquad y = 3sint - \sin(3t)\,, \qquad 0 \le\ t \le \pi$$

# Area Under a Curve

*Q. 53. Ex. 9.1: A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the grazing area available for the cow.*



# Area Under a Curve

*Solution – Method 1:*

*Strategy: Length of the rope = unknown??*

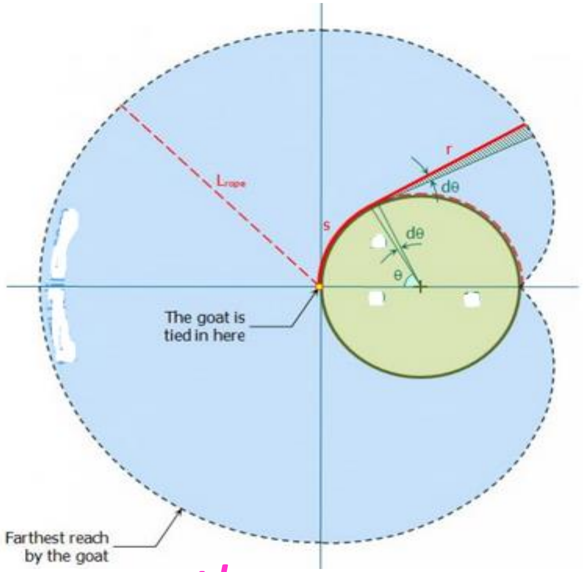
*We know that the rope can reach the opposite end of the silo.*

*Let $l$ be the length of the rope, then $l = \frac{1}{2} \times 2\pi r = \pi r$.*

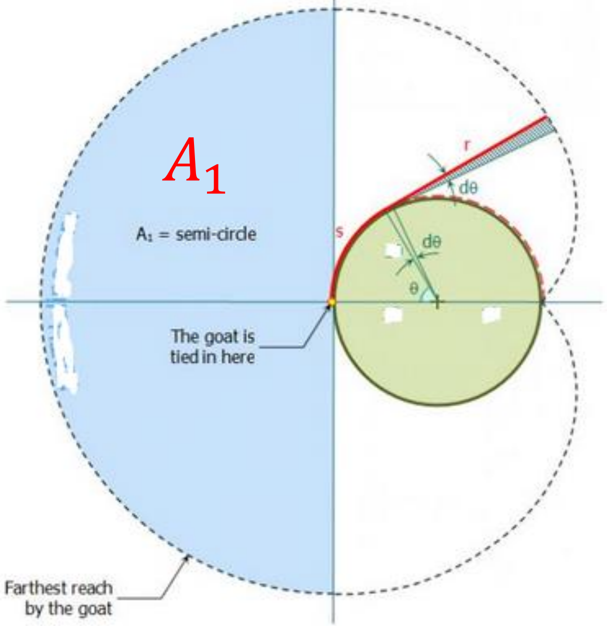
*Find area: We split the area into two parts:*

1. *The semicircle on the left of the diagram*
2. *The circle involute on the right of the diagram*

*Area of semicircle $A_1 = \frac{1}{2} \times \pi(\pi r)^2 = \frac{1}{2}\pi^3 r^2$*




The area the cow can graze



# Homework 3 – 9.2

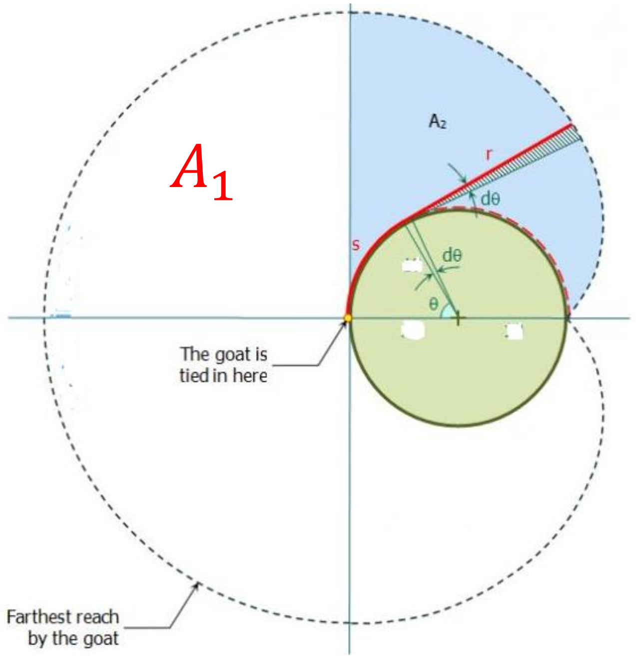
*Area of circle involute: First find the length of the unwrapped rope.*

$$= l - s = \pi r - r\theta = r(\pi - \theta)$$

*Area of circle involute: First find the length of the unwrapped rope.*

$$= l - s = \pi r - r\theta = r(\pi - \theta)$$

*Find area of the circle involute:*




$$A_2 = \left| \frac{1}{2}\int_0^{\pi} r^2 d\theta \right| = \left| \frac{1}{2}\int_0^{\pi} r^2(\pi - \theta)^2 d\theta \right|$$

$$= \left| -\frac{\pi^3 r^2}{6} \right| = \frac{\pi^3 r^2}{6}$$

*Now, we have the tools to find the total grazing area.*

$$A = A_1 + 2A_2$$

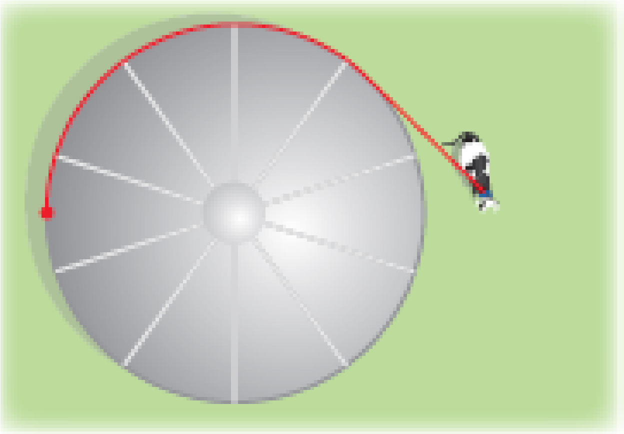
$$A = \frac{1}{2}\pi^3 r^2 + 2\left(\frac{\pi^3 r^2}{6}\right)$$

$$A = \frac{5}{6}\pi^3 r^2$$

# Area Under a Curve

*Q. 53. Ex. 9.1: A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the grazing area available for the cow.*

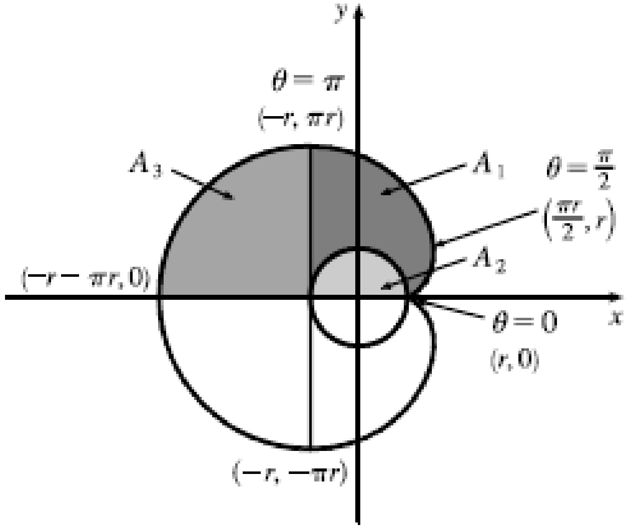
*Solution - Method 2:*



*If the cow walks with the rope taut, it traces out the portion of the involute corresponding to the range* $0 \leq \theta \leq \pi$*, arriving at the point* $(-r, \pi r)$ *when* $\theta = \pi$*.*

*With the rope now fully extended, the cow walks in a semicircle of radius* $\pi r$*, arriving at* $(-r, -\pi r)$*.*
*Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. This corresponds to the range* $-\pi \leq \theta \leq 0$*.*

# Area Under a Curve




*Solution (Contd.)*

*Referring to the figure, we note that the total grazing area is* $2(A_1 + A_3)$.

$A_3$ *is 1/4th of the area of a circle of radius* $\pi r$. *Therefore,*

$$A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2.$$

*To find* $A_1 + A_2$, *we note the rightmost point of the involute is* $\left(\frac{\pi}{2}r, r\right)$. $\left(\because \frac{dx}{d\theta} = 0\right.$ *when* $\theta = 0$ *or* $\left.\theta = \frac{\pi}{2}\right)$.

$\theta = 0$ *corresponds to the cusp at* $(r, 0)$ *and* $\theta = \frac{\pi}{2}$ *corresponds to* $\left(\frac{\pi}{2}r, r\right)$.

*The leftmost point of the involute is* $(-r, \pi r)$. *Thus,*

$$A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx + \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$$

# Area Under a Curve

*Solution (Contd.)* *Now*

$$ydx = r(sin\theta - \theta cos\theta)r\theta cos\theta d\theta.$$

*Integrate*

$$\left(\frac{1}{r^2}\right)\int ydx = -\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)sin\theta cos\theta - \frac{1}{6}\theta^2 + \frac{1}{2}\theta + C$$

$$A_1 + A_2 = r^2[-\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)sin\theta cos\theta - \frac{1}{6}\theta^2 + \frac{1}{2}\theta]\,\Big|_{\pi}^{0}$$

$$= r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

*Therefore,* $A_1 = r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right) - A_2 = \frac{1}{6}\pi^3 r^2$

*So, the gazing area is*

$$2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$$

# Surface Area Generated by a Parametric Curve

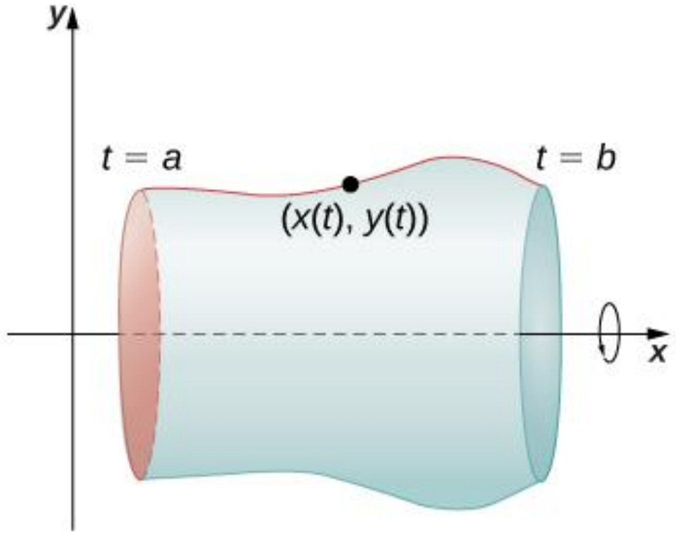
- *Recall the problem of finding the surface area of a volume of revolution.*
- *In Curve Length and Surface Area, you derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the $x$-axis:*

$$S = 2\pi \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2} \; dx$$

- *We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t),\ \ y = y(t)$, for $a \leq t \leq b$ around the x-axis*

*The analogous formula for a parametrically defined curve is*

$$\boxed{S = 2\pi \int_a^b y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} \; dt}$$



# Surface Area Generated by a Parametric Curve

*Find the surface area of a sphere of radius r centred at the origin.*

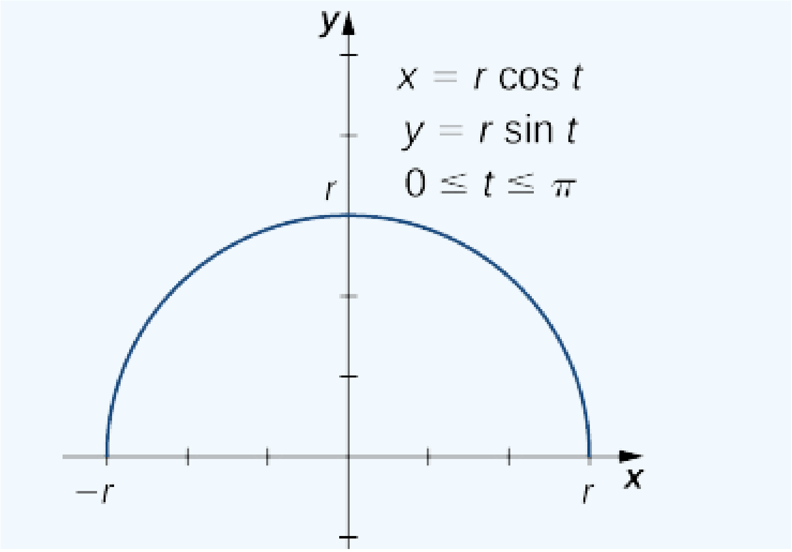
*Solution:* *Start with the curve defined by the equations*

$$x = rcost, \qquad y = rsint, \qquad 0 \le t \le \pi$$

*This generates an upper semicircle of radius r centred at the origin*

*When this curve is revolved around the $x$ −axis, it generates a sphere of radius $r$.*
*To calculate the surface area of the sphere, we use*

$$S = 2\pi \int_a^b y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\ dt$$




***A semicircle generated by parametric equations.***

# Surface Area Generated by a Parametric Curve

*Solution (Contd.)*

$$S = 2\pi \int_0^{\pi} rsint\sqrt{(-rsint)^2 + (rcost)^2}\ dt$$

$$= 2\pi \int_a^{\pi} rsint\sqrt{(-rsint)^2 + (rcost)^2}\ dt$$

$$= 2\pi \int_0^{\pi} rsint\sqrt{(-rsint)^2 + (rcost)^2}\ dt$$

$$= 2\pi \int_0^{\pi} r^2 sint\ dt = 4\pi r^2$$

# Homework 3 – 9.2

*Find the exact area of the surface obtained by rotating the following curve about the x-axis.*

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta, \qquad 0 \le \theta \le \pi/2$$

*Answer:* $\frac{6}{5}\pi a^2$