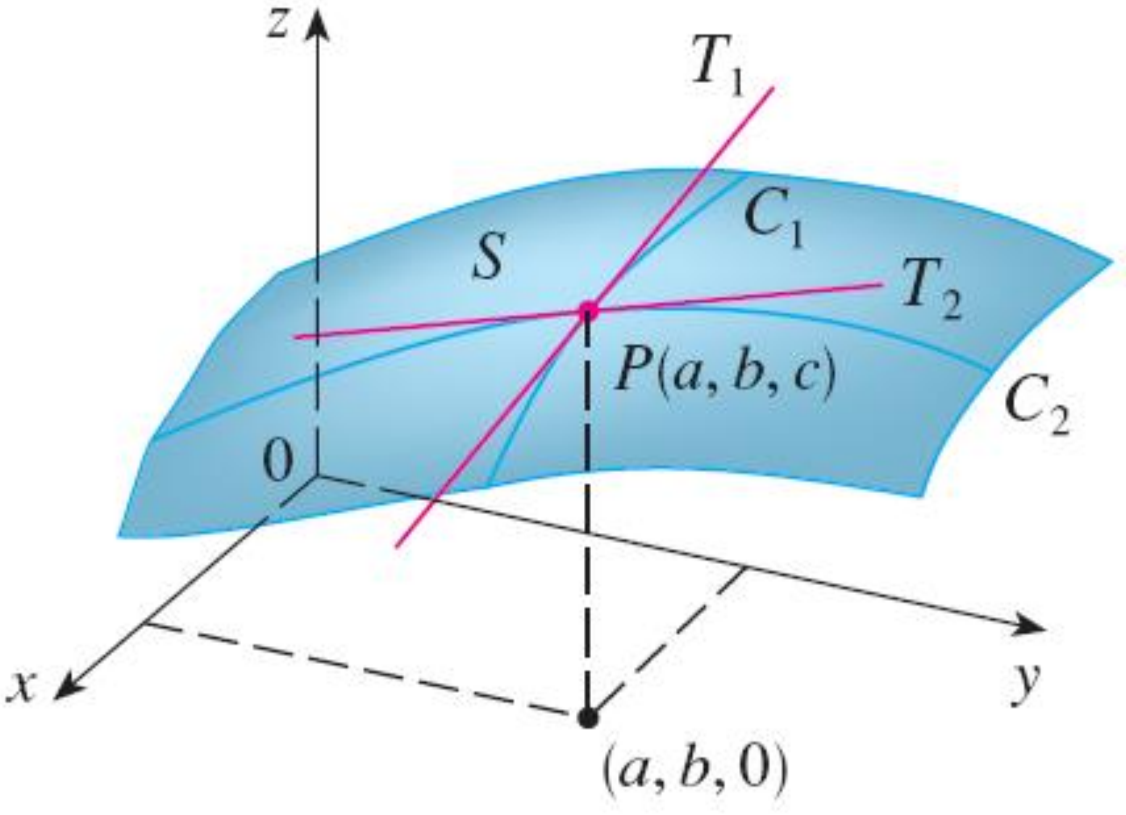
Lecture 15 - Chapter 11 – Sec. 11.3

Partial Derivatives



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Learning Objectives

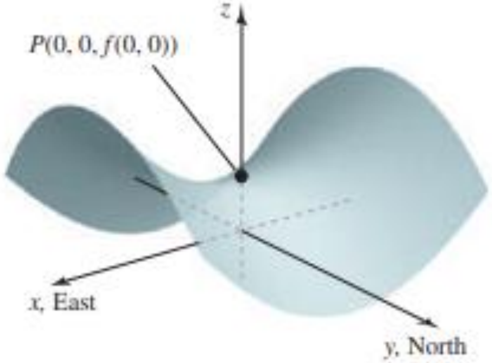
- *Partial derivatives of multivariable functions*
- *First partial derivative and it's geometric interpretation*
- *Second partial derivatives*
- *Clairaut's Theorem*
- *Applications*

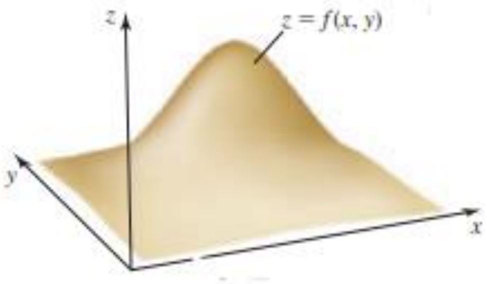
Partial Derivative

Suppose you are standing at the point $P(0,0,f(0,0))$, which lies on the pass or the saddle. The surface behaves differently depending on the direction in which you walk.

- *If you walk east (positive x-direction), the elevation increases and your path takes you upward on the surface.*
- *If you walk north (positive y-direction), the elevation decreases and your path takes you downward on the surface.*

In fact, in every direction you walk from the point P, the function values change at different rates. So how should the slope or the rate of change at a given point be defined?

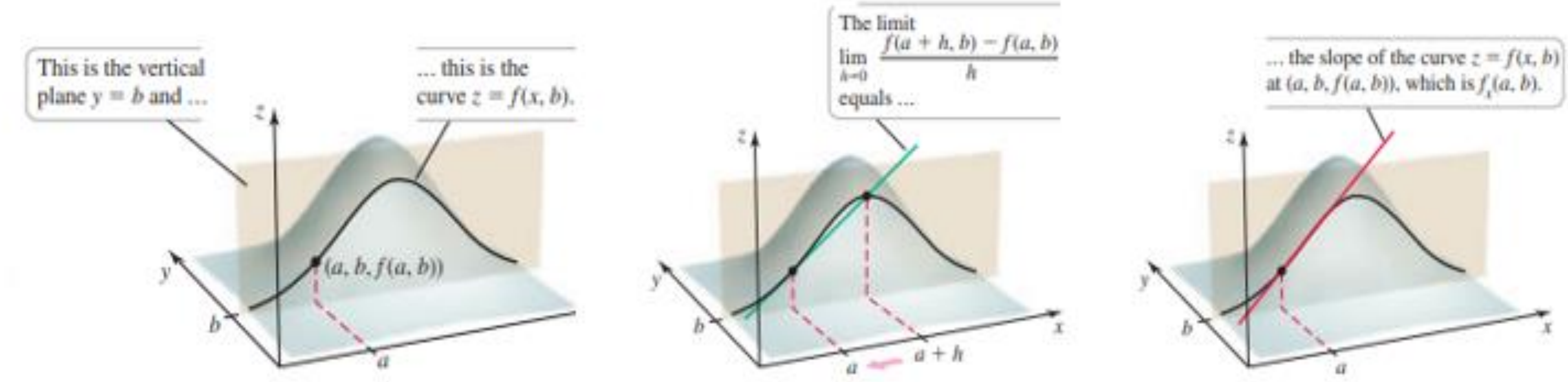




Partial Derivative

The answer to this question involves partial derivatives, which arise when we hold all but one independent variable fixed and then compute an ordinary derivative with respect to the remaining variable.

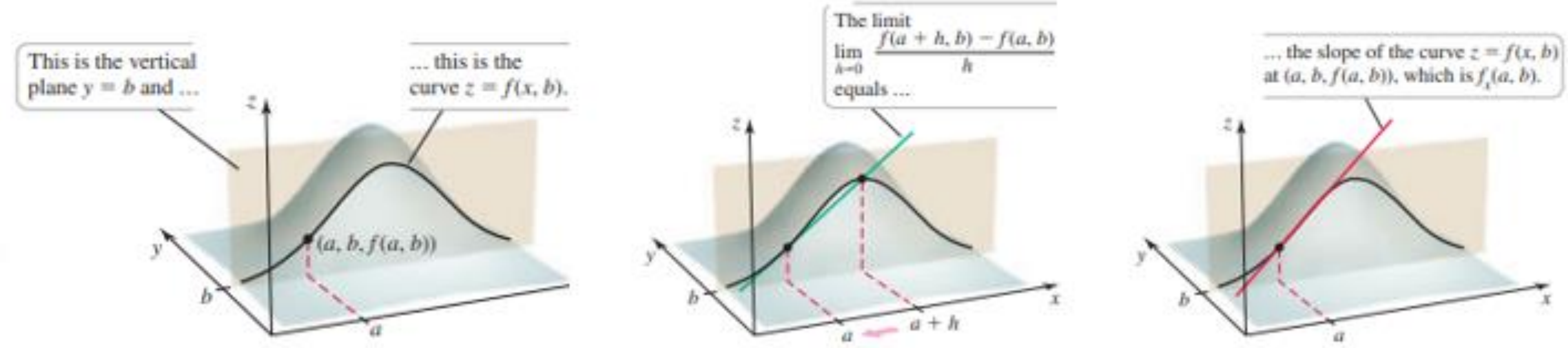
- *Suppose we move along the surface* $z = f(x, y)$*, starting at the point* $(a, b, f(a, b))$ *in such a way that* $y = b$ *is fixed and only* x *varies.*
- *The resulting path is a curve (a trace) on the surface that varies in the* x*-direction.*



Partial Derivative

- *This curve is the intersection of the surface with the vertical plane* $y = b$; *it is described by* $z = f(x, b)$, *which is a function of the single variable* x.
- *We know how to compute the slope of this curve: It is the ordinary derivative of* $f(x, b)$ *with respect to x.*
- *This derivative is called the* *partial derivative* *of f with respect to* x, *denoted* $\partial f \partial x$ *or* f_x. *When evaluated at* (a, b), *its value is defined by the limit*

$$\frac{\partial f}{\partial x} = f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, \qquad provided\ limit\ exists.$$

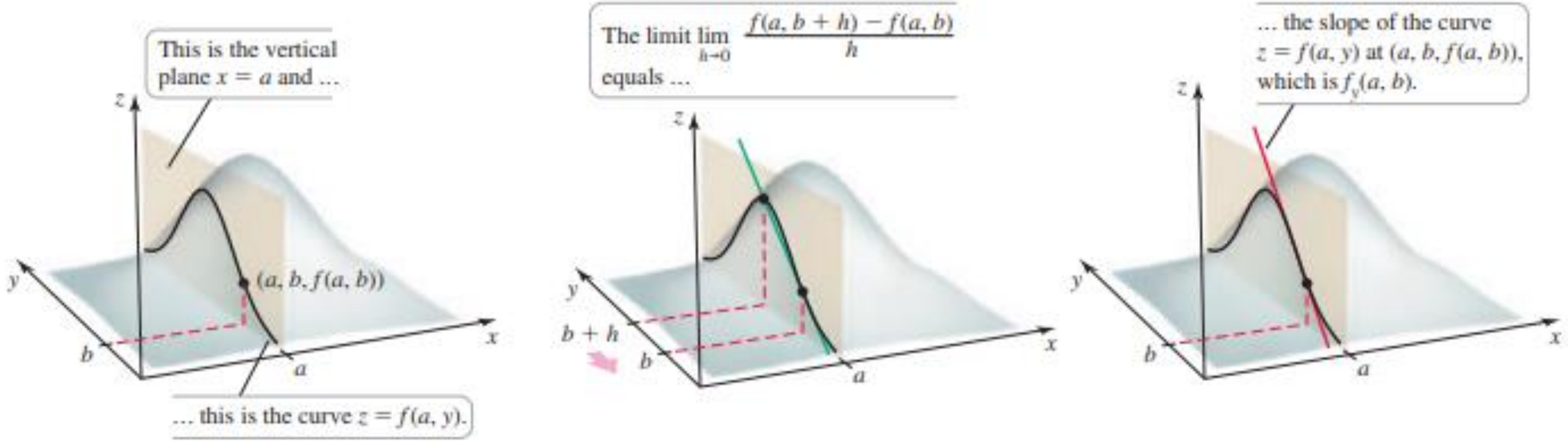


Partial Derivative

Similarly, we can move along the surface $z = f(x, y)$ *from the point* $(a, b, f(a, b))$ *in such a way that* $x = a$ *is fixed and only* y *varies.*

Now the result is a trace described by $z = f(a, y)$, *which is the intersection of the surface and the plane* $x = a$.

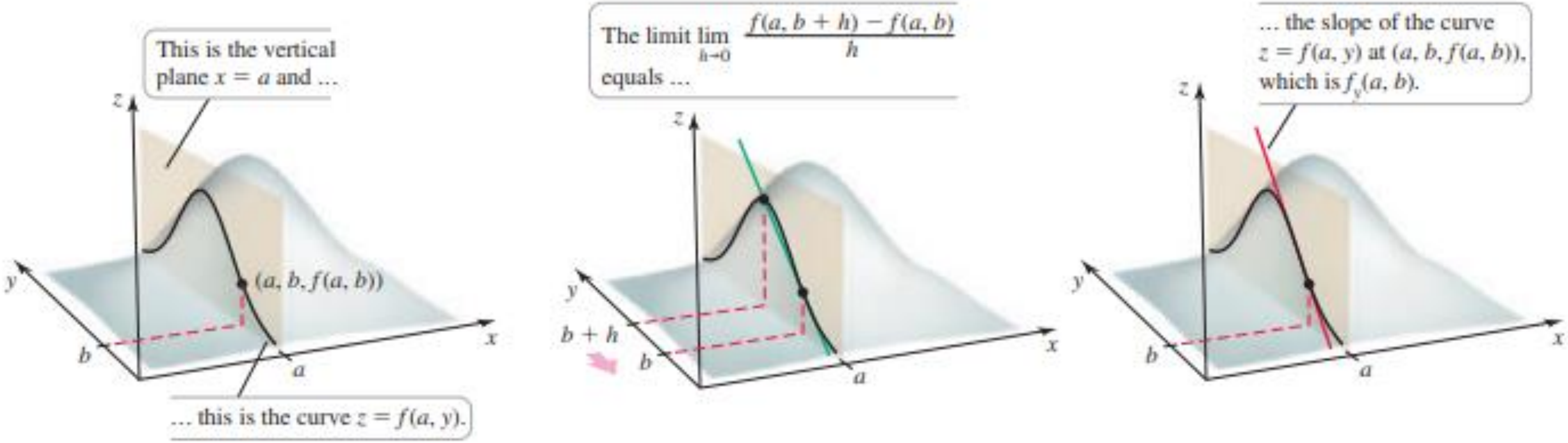
The slope of this curve at (a, b) *is given by the ordinary derivative of* $f(a, y)$ *with respect to* y.



Partial Derivative

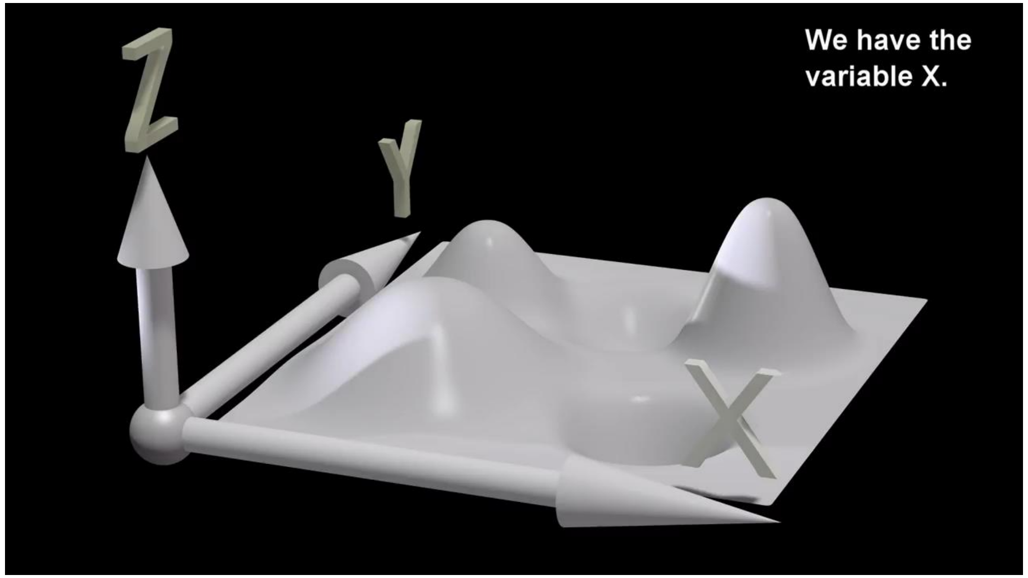
This derivative is called the partial derivative of f *with respect to* y*, denoted* $\partial f \delta y$*. When evaluated at* (a, b)*, it is defined by the limit*

$$\frac{\partial f}{\partial y} = f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}, \qquad \textit{provided limit exists.}$$



Partial Derivative - Simulations

Play Me



Definition – Partial Derivative

The partial derivative of f *with respect to* x *at the point* (a, b) *is*

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The partial derivative of f *with respect to* y *at the point* (a, b) *is*

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

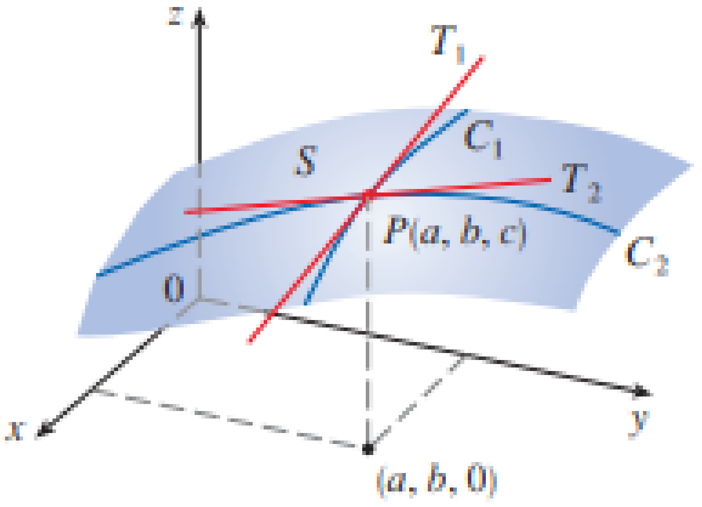
provided these limit exist.

Partial Derivative - Notations

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$



Notations for partial derivatives evaluated at a point (a, b):

$$f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{a,b} \quad and \quad f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{a,b}$$

Recall that f_x and f_y are the functions of x and y, while $f_x(a,b)$ and $f_y(a,b)$ are the value of the derivative at (a,b).

Example - Calculating Partial derivatives

If $z = f(x, y) = x \ln y + e^{2y} \tan x$, *find*

(i) *partial derivative of* z *with respect to* x *at* $(0,1)$.

(ii) *partial derivative of* z *with respect to* y *at* $(0,1)$

Solution: (i)

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = f_x = \frac{\partial}{\partial x}(x \ln y + e^{2y} \tan x)$$

$$f_x(x,y) = \ln y + e^{2y} \sec^2 x$$

Value of the derivative f_x *at* $(0,1)$ *is:*

$$f_x(1,0) = \left.\frac{\partial f}{\partial x}\right|_{0,1} = \ln(1) + e^{2(1)} \sec^2(0)$$

$$= 0 + (e^2)(1) = e^2$$

Example - Calculating Partial derivatives

(ii)

$$\frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} = f_y = \frac{\partial}{\partial y}(x\ln y + e^{2y}\tan x)$$

$$f_y(x,y) = x/y + 2e^{2y}\, tan\, x$$

Value of the derivative f_x at $(0,1)$ is:

$$f_y(1,0) = \frac{\partial f}{\partial x}\bigg|_{0,1} = 0 + 2e^2(0)$$

$$= 0$$

Example - Calculating Partial derivatives

If $z = f(x,y) = \frac{\sin(\pi x+2y)}{(1+x)}$, *find*

(i) *partial derivative of* z *with respect to* x *at* $(0,1)$.

(ii) *partial derivative of* z *with respect to* y *at* $(0,1)$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = f_x = \frac{\partial}{\partial x}\left(\frac{\sin(\pi x + 2y)}{(1+x)}\right)$$

$$f_x(x,y) = \frac{(1+x)\,\pi \times \cos(\pi x + 2y) - \sin(\pi x + 2y)}{(1-x)^2}$$

Value of the derivative f_x *at* $(0,1)$ *is:*

$$f_x(0,1) = \left.\frac{\partial f}{\partial x}\right|_{0,1} = \pi \cos 2 - sin2$$

Example - Calculating Partial derivatives

(ii)

$$\frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} = f_y = \frac{\partial}{\partial y}\left(\frac{\sin(\pi x + 2y)}{(1+x)}\right)$$

$$f_x(x,y) = \frac{2\cos(\pi x + 2y)}{(1+x)}$$

Value of the derivative f_y at $(0,1)$ is:

$$f_x(0,1) = \left.\frac{\partial f}{\partial x}\right|_{0,1} = 2cos2$$

Example – Q. 27. Ex. 11.3

Find the first partial derivatives of the function

$$h(x, y, z, t) = x^2 y \cos(z/t)$$

Solution: Since the function is a four-variable function, we have four first partial derivaives, thus we have

$$\partial h/\partial x = h_x = 2xy\cos\left(\frac{z}{t}\right)$$

$$\partial h/\partial y = h_y = x^2\cos\left(\frac{z}{t}\right)$$

$$\partial h/\partial z = h_z = -x^2 y\sin\left(\frac{z}{t}\right) \times \left(\frac{1}{t}\right) = -\frac{x^2 y}{t}\sin\left(\frac{z}{t}\right)$$

$$\frac{\partial h}{\partial t} = h_t = -x^2 y\sin\left(\frac{z}{t}\right) \times \left(-\frac{z}{t^2}\right) = \frac{x^2 yz}{t^2}\sin\left(\frac{z}{t}\right)$$

Example – Q. 29. Ex. 11.3

Find the first partial derivatives of the function

$$u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Solution: For each $i = 1, \cdots, n$,

$$u = \sqrt{\sum_i^n x_i^2}$$

$$\partial u/\partial x_i = u_{x_i} = \frac{1}{2}\left(\sum_i^n x_i^2\right)^{-1/2}\left(\sum_i^n 2x_i\right) = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

where $x_i = x_1 + x_2 + \cdots + x_n$

Fundamental Theorem (1) of Calculus

Let a be real, then

$$f(x,y) = \int_a^x g(t)dt - \int_a^y g(t)dt$$
$$= G(x) - G(y)$$

with

$$G(x) = \int_a^X g(t)dt$$

and $G'(x) = g(x)$. *Since* g *is continuous at* $\mathbb{R}$, *thus*

$$f_x(x,y) = G'(x) = g(x)$$
$$f_y(x,y) = G'(y) = -g(y).$$

Example – Q. 27. Ex. 11.3

Find the first partial derivatives of the function

$$F(x, y) = \int_y^x \cos(e^t)dt$$

Solution: The partial derivative with respect to x is given by

$$F_x(x, y) = \frac{\partial}{\partial x}\int_y^x \cos(e^t)dt = \cos(e^x)$$

where we have used the Fundamental Theorem (1) of Calculus

$$F_y(x, y) = \frac{\partial}{\partial y}\left[-\int_x^y \cos(e^t)dt\right] = -\frac{\partial}{\partial y}\int_x^y \cos(e^t)dt = -\cos(e^y)$$

Homework 1 – Ex. 11.3

Find the first partial derivatives of the following functions:

(a) $f(r,s) = (r+s)ln\,(r^2+s^2)$ *at (2,1)*

(b) $f(u,v) = (u-v)/(u^2+v^2)$ *at (1,0)*

(c) $g(\alpha,\beta) = \sin\alpha\ \cos\beta\ \tan\theta$ *at (0, π)*

(d) $f(x,y) = y^x$ *at (1,1)*

(e) $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

(f) $f(x,y) = \int_y^x \cos(t^2)dt$

Example – Q. 36. Ex. 11.3

Use the definition of partial derivatives as limit to find $f_x(x, y)$ *and* $f_y(x, y)$

$$f(x, y) = \frac{x}{x + y^2}$$

Solution:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{(x + h)/(x + h + y^2) - x/(x + y^2)}{h} \cdot \frac{(x + h + y^2)(x + y^2)}{(x + h + y^2)(x + y^2)}$$

$$f_x(x, y) = \lim_{h \to 0} \frac{y^2 h}{h(x + h + y^2)(x + y^2)} = \lim_{h \to 0} \frac{y^2}{(x + h + y^2)(x + y^2)}$$

As $h \to 0$,

$$f_x(x, y) = \frac{y^2}{(x + y^2)^2}$$

Example – Q. 36. Ex. 11.3

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x)/(x+(y+h)^2) - x/(x+y^2)}{h} \cdot \frac{(x+(y+h)^2)(x+y^2)}{(x+(y+h)^2)(x+y^2)}$$

$$f_x(x,y) = \lim_{h\to 0} \frac{h(-2xy - xh)}{h(x+(h+y)^2)(x+y^2)} = \lim_{h\to 0} \frac{-2xy - xh}{(x+(y+h)^2)(x+y^2)}$$

As $h \to 0$,

$$f_x(x,y) = \frac{-2xy}{(x+y^2)^2}$$

Implicit Differentiation

- *Given* $f(x, y)$*, one must assume that* $y = g(x)$ *or* $x = h(y)$ *in order to perform Implicit Differentiation.*
- *In other words, one must be a function of the other. If they are totally independent, Implicit Differentiation is not appropriate.*

- *With implicit differentiation, both variables are differentiated, but at the end of the problem, one variable is isolated (without any number being connected to it) on one side.*

- *On the other hand, with partial differentiation, one variable is differentiated, but the other is held constant.*

Example – Q. 42. Ex. 11.3

Use the implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$

$$\sin(xyz) = x + 2y + 3z$$

Solution: Partially differentiate with respect to x, we get

$$\frac{\partial}{\partial x}[\sin(xyz)] = \frac{\partial}{\partial x}(x + 2y + 3z)$$

$$\Rightarrow \cos(xyz) \cdot y\left(x\frac{\partial z}{\partial x} + z\right) = 1 + 3\frac{\partial z}{\partial x}$$

$$[xy\cos(xyz) - 3]\frac{\partial z}{\partial x} = 1 - yz\cos\,(xyz)$$

So,

$$\frac{\partial z}{\partial x} = \frac{1 - yz\cos\,(xyz}{xy\cos(xyz) - 3}$$

Example – Q. 42. Ex. 11.3

Partially differentiate with respect to y*, we get*

$$\frac{\partial}{\partial y}[\sin(xyz)] = \frac{\partial}{\partial y}(x + 2y + 3z)$$

$$\Rightarrow \cos(xyz) \cdot x\left(y\frac{\partial z}{\partial y} + z\right) = 2 + 3\frac{\partial z}{\partial y}$$

$$[xy\cos(xyz) - 3]\frac{\partial z}{\partial y} = 2 - xz\cos(xyz)$$

So,

$$\frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}$$

Thought for the Day

If the first partial derivatives represent the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$, what do second partial derivatives with respect to the same or cross variables represent?

Second Order Partial derivatives

- *A second derivative with respect to the **same variable discusses curvature**.*

- *A second **cross** partial derivative asks how the impact of one explanatory variable changes as another explanatory variable changes.*

For example: If Happiness = g(food, tv), then

$\partial^2 h / \partial f \partial tv$ *asks how watching more tv affects food's effect on happiness (or how food affects tv's effect on happiness). For example, watching TV may not increase happiness if someone is hungry.*

Second Order Partial derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example – Second Partial Derivatives

Find the second partial derivatives of the following functions at (1,0).

(a) $f(x, y) = xy\sin(xy)$ *(b)* $f(x, y) = e^{-2x^2+3xy}$

Solution:

$f_x = y\sin(xy) + xy^2\cos(xy)$,

$f_{xx} = 2y^2\cos(xy) - xy^3\sin(xy)$, $\quad f_{xx}\Big|_{1,0} = 0$

$f_{yx} = 3xy\cos(xy) + (1 - x^2y^2)\sin(xy)$, $\quad f_{yx}\Big|_{1,0} = 0$

$f_{yy} = -x^2y\sin(xy) - 2x^2\cos(xy)$, $\quad f_{yy}\Big|_{1,0} = 2$

$f_{xy} = (1 - x^2y^2)\sin(xy) + 3xy\cos(xy)$ $\quad f_{xy}\Big|_{1,0} = 0$

Clairaut's Theorem

Equality of Mixed Partial Derivatives

If a function f is defined on an open set D of $\mathbb{R}^2$ *and* f_{xy} *and* f_{yx} *are continuous throughout D, then*

$$f_{xy} = f_{yx} \quad or$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

for all points of D.

Homework 2 – Ex. 11.3

Find the second derivatives for the following function and verify the Clairaut's theorem in each case.

(a) $f(x, y) = \sin^2(mx + ny)$

(b) $z = g(x, y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(c) $v(x, y) = e^{xe^y}$

(d) $h(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$

Application of Clairaut's Theorem – Q.52

Verify that the conclusion of Clairaut's Theorem holds.

$$u = e^{xy} \sin y$$

Solution: *Applying Clairaut;s Theorem, we have* $u_{xy} = u_{yx}$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} = e^{xy} y \sin y = y e^{xy} \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}[y e^{xy} \sin y] = y e^{xy} \cos y + e^{xy} \sin y (xy + 1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = u_{xy} = e^{xy}[y \cos y + \sin y (xy + 1)]$$

Application of Clairaut's Theorem – Q.52

$$\frac{\partial u}{\partial y} = e^{xy} x \sin y + e^{xy} \cos y = e^{xy} (x \sin y + \cos y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}[e^{xy} (x \sin y + \cos y)] = y e^{xy}(x \sin y + \cos y) + e^{xy}(\sin y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = u_{yx} = e^{xy}[xy \sin y + y \cos y + \sin y]$$

$$\frac{\partial^2 u}{\partial x \partial y} = u_{yx} = e^{xy}[y \cos y + \sin y(xy + 1)]$$

Applications – Circuit Theory

Poisson Equation:

A partial differential equation that accommodates boundary conditions, and thereby facilitates the analysis of the scalar potential field (electrostatic potential, for example) in the vicinity of structures and spatially-varying material properties. In three dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\epsilon},$$

where ρ_V is volume charge density and ϵ is the permittivity.

Applications

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Laplace's equation

It governs ideal fluid flow, electrostatic potentials, and the steady state distribution of heat in a conducting medium.

- *It is basically Poisson equation in absence of charge (source-free region)*
- *In two dimensions, Laplace's equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

- *Solutions of this equation are called* ***harmonic functions***
- *They play a role in problems of heat conduction, fluid flow, and electric potential.*

Example – Laplace's Equation

Determine whether each of the following functions are harmonics, i.e., solutions of Laplace's equation

(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$

(c) $u = x^3 + 3xy^2$ (d) $u = \ln\sqrt{x^2 + y^2}$

(e) $u = \sin x \cosh y + \cos x \sinh y$

(f) $u = e^{-x}\cos y - e^{-y}\cos x$

Solution (a):

$$u_x = 2x, \qquad u_{xx} = 2, \qquad u_y = 2y, \qquad u_{yy} = 2$$

Substituting in the Laplace's equation, we get

$$u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$$

Therefore, $u = x^2 + y^2$ is not a solution of Laplace's equation and does not define a harmonics.

Example – Laplace's Equation

Solution (f):

$$u_x = -e^{-x}cosy + e^{-y}\sin x\,, \qquad u_{xx} = e^{-x}cosy + e^{-y}cosx$$

$$u_y = -e^{-x}siny + e^{-y}\cos x\,, \qquad u_{yy} = -e^{-x}cosy - e^{-y}cosx$$

Substituting the Laplace's equation, we get

$$u_{xx} + u_{yy} = e^{-x}cosy + e^{-y}cosx - e^{-x}cosy - e^{-y}cosx$$

$$u_{xx} + u_{yy} = 0$$

Therefore, $u = e^{-x}cosy + e^{-y}sinx$ *is a solution of Laplace's equation and hence describes a harmonics*

Homework 3 – Ex. 11.3

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Show that the function

$$u(x, y) = \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right)$$

is a solution of Laplace's equation.

Example – Q. 75. Ex. 11.3

You are told that there is a function f whose partial derivatives are

$$f_x(x,y) = x + 4y, \qquad \& \quad f_y(x,y) = 3x - y.$$

Should you believe it?

Solution: Using the Clairaut's Theorem, we have

$$f_x(x,y) = x + 4y$$

$$\Rightarrow f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4$$

Also,

$$f_y(x,y) = 3x - y$$

$$\Rightarrow f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3$$

Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$.
Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

Example – Q. 77. Ex. 11.3

The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ *intersects the plane* $y = 2$ *in an ellipse. Find parametric equations for the tangent line to this ellipse at the point* $(1,2,2)$.

Solution: Setting $y = 2$, *the equation of the ellipse of intersection is*

$$4x^2 + z^2 = 8.$$

By implicit differentiation, we get

$$8x + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{4x}{z}$$

So, the slope of the tangent at $x = 1, z = 2$ *is*

$$f_x(1,2) = \frac{\partial z}{\partial x}\bigg|_{1,2} = -2$$

Thus the tangent line is given by $z - 2 = -2(x - 1), y = 2$.

Taking the parameter to be $t = x - 1$, *we can write parametric equations for this lines:*

$$x = t + 1, y = 2, z = 2 - 2t$$

Wave Equation

Travelling waves (for example, water waves or electromagnetic waves) exhibit periodic motion in both time and position.

- *In one-dimension wave motion is governed by the one-dimensional wave equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

where $u(x,t)$ is the displacement of the wave surface at position x and time t, and c is the constant speed of the wave.

- *This could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.*

Example - The Diffusion Equation

The diffusion equation

$$\frac{\partial c}{\partial t} = D\,\frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x,t) = \frac{1}{\sqrt{4\pi D t}}\,e^{-x^2/(4Dt)}$$

Is a solution of the diffusion equation.

Example - The Diffusion Equation

Solution: The diffusion equation is a partial differential equation with first and second partial derivatives with respect to t and x, respectively.
So, we must calculate the first and second derivatives of the given solution to confirm it's validity.

$$c(x,t) = (4\pi Dt)^{-\frac{1}{2}} e^{-x^2/(4Dt)}$$

Differentiate with respect to t, we get

$$\frac{\partial c}{\partial t} = (4\pi Dt)^{-1/2} \left[e^{-\frac{x^2}{4Dt}} (-1)(-x^2)(4Dt)^{-2}(4D) \right] + \left[e^{-\frac{x^2}{4Dt}} \left(-\frac{1}{2} \right) (4\pi Dt)^{-3/2} (4\pi D) \right]$$

$$\frac{\partial c}{\partial t} = \frac{2\pi D}{(4\pi Dt)^{3/2}} \left(\frac{x^2}{2Dt} - 1 \right) e^{-x^2/4Dt}$$

Example - The Diffusion Equation

Partially differentiate w.r.t x

$$\frac{\partial c}{\partial x} = -\left(\frac{-2x}{4Dt}\right)\left(\frac{1}{\sqrt{4\pi Dt}}\right) e^{-\frac{x^2}{4Dt}} = -\frac{2\pi x}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2}{4Dt}}$$

Partially differentiate again w.r.t x

$$\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right) = -\frac{2x}{(4\pi Dt)^{\frac{3}{2}}}\left[x \cdot e^{-\frac{x^2}{4Dt}} \cdot \left(-\frac{2x}{4Dt}\right) + e^{-\frac{x^2}{4Dt}} \cdot 1\right]$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{2\pi}{(4\pi Dt)^{\frac{3}{2}}}\left(\frac{x^2}{2Dt} - 1\right) e^{-x^2/(4Dt)}$$

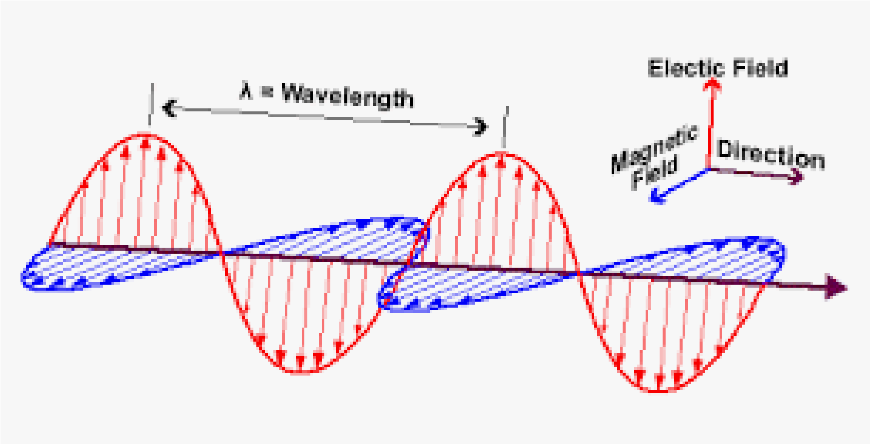
$$D\frac{\partial^2 c}{\partial x^2} = \frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}}\left(\frac{x^2}{2Dt} - 1\right) e^{-x^2/(4Dt)} = \frac{\partial c}{\partial t}$$

Homework 4 - Ex. 11.3

The wave equations of an electromagnetic wave are given by

$$\frac{\partial^2 E(x,t)}{\partial t^2} = c^2 \frac{\partial^2 E(x,t)}{\partial x^2}.$$

$$\frac{\partial^2 B(x,t)}{\partial t^2} = c^2 \frac{\partial^2 B(x,t)}{\partial x^2}.$$



Show that

$$u(x,t) = Acos(x + ct) + Bsin(x - ct)$$

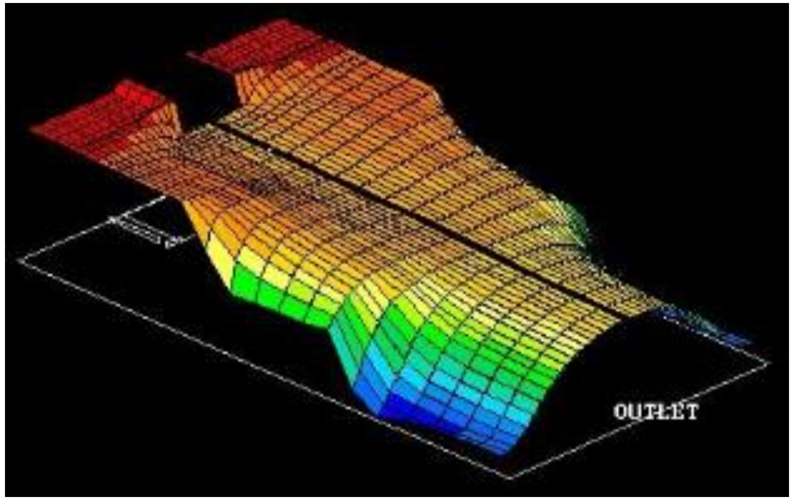
Is the solution of above equations.

Heat Equation

The flow of heat along a thin conducting bar is governed by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where u is a measure of the temperature at a location x on the bar at time t and the positive constant k is related to the conductivity of the material.



Homework 5 – Ex 11.3

Show that the following functions satisfy the heat equation with k = 1.

$$u(x,t) = e^{-t}(2sinx + 3cosx)$$

$$u(x,t) = Ae^{-a^2t}cosax, \qquad for\ any\ real\ a\ \&\ A$$

$$u = \sin(x - at) + \ln(x + at)$$