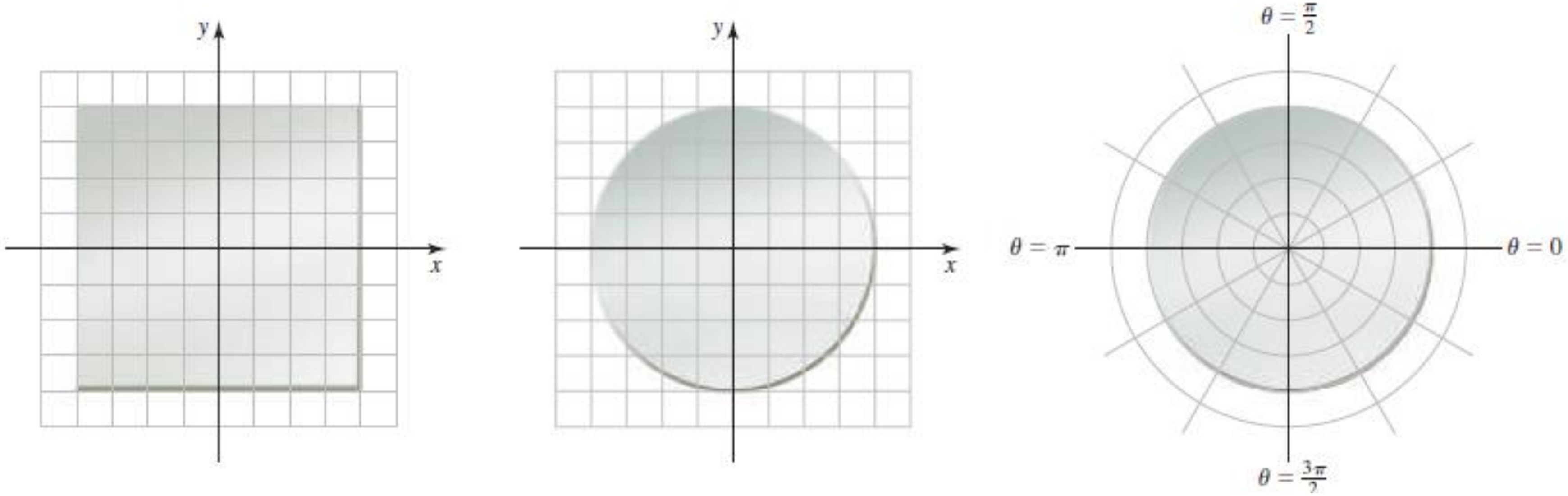
Lecture 3 (Chapter 9)

Parametric Curves and Polar Coordinates



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Learning Objectives

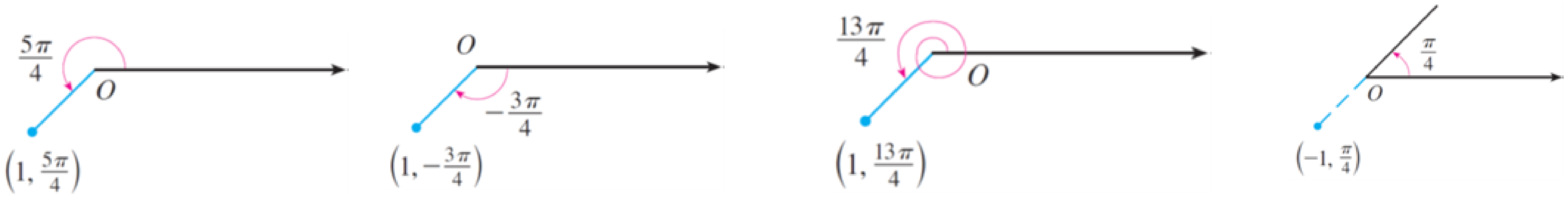
- *Locate points in a plane by using polar coordinates.*
- *Convert points between rectangular and polar coordinates.*
- *Sketch polar curves from given equations.*
- *Convert equations between rectangular and polar coordinates.*
- *Identify symmetry in polar curves and equations.*

Polar Coordinates

- *The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a one-to-one mapping from points in the plane to ordered pairs.*
- *The polar coordinate system provides an alternative method of mapping points to ordered pairs.*
- *The Cartesian coordinate system can therefore be represented as an ordered pair in the polar coordinate system.*
- *The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**.*
- *Every point in the plane can be represented in this form.*

Polar Coordinates

- *In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system, each point has many representations.*
- *For instance, the point* $(1, 5\pi/4)$ *could be written as* $(1, -3\pi/4)$ *or* $(1, 13\pi/4)$ *or* $(-1, \pi/4)$.



- *In fact, since a complete counterclockwise rotation is given by an angle* 2π*, the point represented by polar coordinates* (r, θ) *is also represented by*

$$(r, \theta + 2n\pi) \quad \textit{and} \quad (-r, \theta + (2n+1)\pi)$$

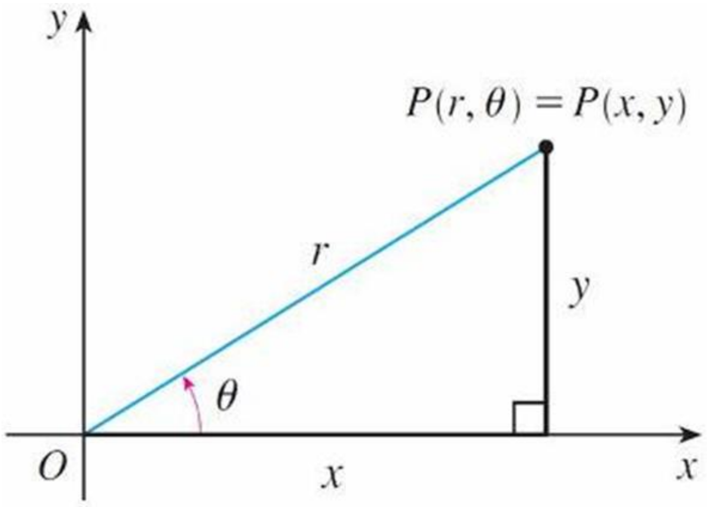
where n is an integer.

Relationship between Polar and Cartesian Coordinates

The connection between polar and Cartesian coordinates can be seen from

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad (1)$$

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = y/x \qquad (2)$$



Note that

- *Eqs. (2) do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 < \theta < 2\pi$, each value of $\tan\theta$ occurs twice.*
- *Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Eqs. (2).*
- *We must choose ϑ so that the point (r, θ) lies in the correct quadrant.*

Example

Represent the point with Cartesian coordinates (1, −1) in terms of polar coordinates.

Solution:

If we choose r to be positive, then Eqs. (2) give

$$r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point (1, −1) lies in the fourth quadrant, we can choose

$$\theta = -\pi/4 \quad \text{or} \quad \theta = 7\pi/4.$$

Thus one possible answer is $(\sqrt{2},\ -\pi/4)$*; another is* $(\sqrt{2},\ 7\pi/4)$

Polar Curves

*The **graph of a polar equation** $\boldsymbol{r = f(\theta)}$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.*

Polar Curves

(a) *Sketch the curve with polar equation*

$$r = 2\cos\theta$$

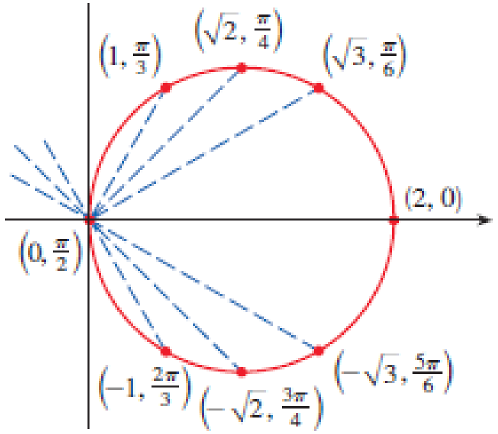
(b) *Find a Cartesian equation for this curve.*

Solution:

(a) *We find the values of r for some convenient values of* θ *and plot the corresponding points* (r, θ). *Then we join these points to sketch the curve, which appears to be a circle.*

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

We have used only values of θ *between 0 and* π, *because if we let* θ *increase beyond , we obtain the same points again.*



Polar Curves

(b) Find a Cartesian equation for this curve.

Solution:

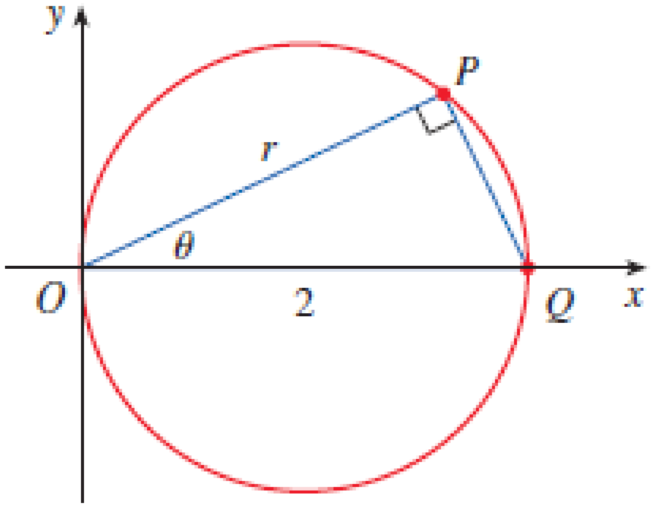
To convert the given equation to a Cartesian equation we use

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2,$$
$$\tan\theta = y/x$$

From $x = r\cos\theta$*, we have* $\cos\theta = x/r$*, so the equation* $r = 2\cos\theta$ *becomes* $r = 2x/r$*, which gives*

$$r^2 = 2x \Rightarrow x^2 + y^2 = 2x \Rightarrow x^2 + y^2 - 2x = 0$$
$$(x-1)^2 + y^2 = 1$$

which is an equation of a circle with centre (1, 0) and radius 1.



Polar Curves - Example

Sketch the curve r = 1 + sin θ.

Solution:

We first sketch the graph of $r = 1 + \sin\theta$ *in Cartesian coordinates (Fig. 1a) by shifting the sine curve up one unit.*

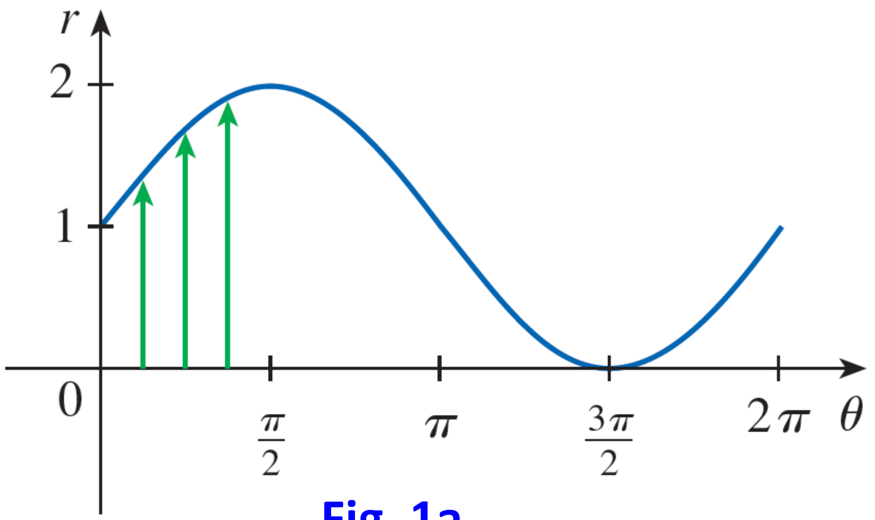


Fig. 1a

$r = 1 + sin\,\theta$ *in Cartesian Coordinates,* $0 \leq \theta \leq 2\pi$

Polar Curves - Example

Solution (Contd.)

- *This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Fig. 1(b).*
- *As θ increases from $\pi/2$ to π, r decreases from 2 to 1, so we sketch the next part of the polar curve in Fig. 1(c).*

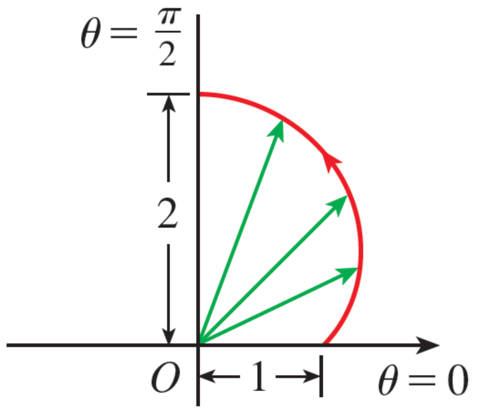


Fig. 1b

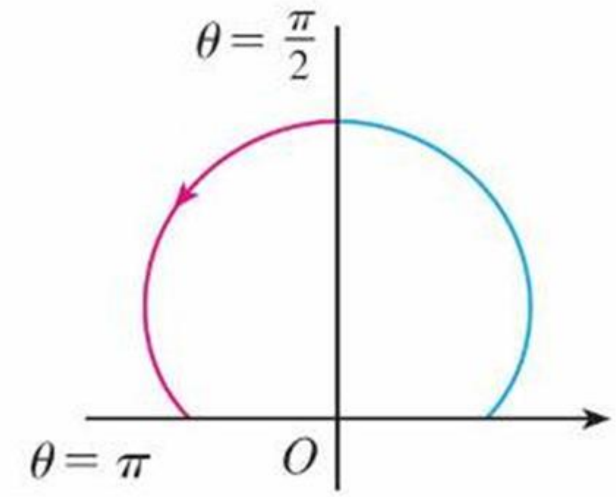


Fig. 1c

Polar Curves - Example

- *As θ increases from π to $3\pi/2$, r decreases from 0 to 1 (Fig. 1c).*
- *Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 (Fig. 1d)*
- *If we let ϑ increase beyond 2π or decrease beyond 0, we would simply retrace our path.*
- *Putting together the parts of the curve from Figure 1a–1d, we sketch the complete curve in part 1e. It is called a **cardioid** because it's shaped like a heart.*

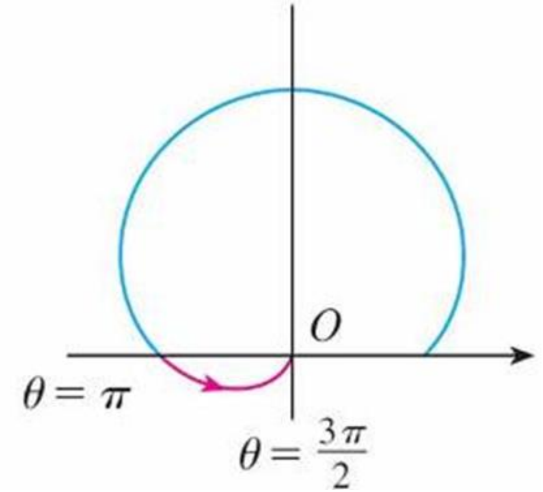


Fig. 1c

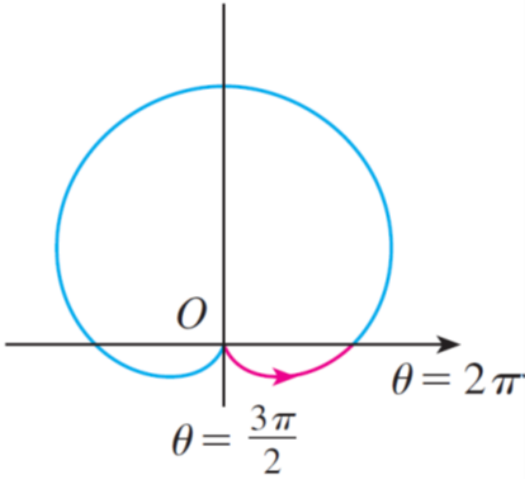


Fig. 1d

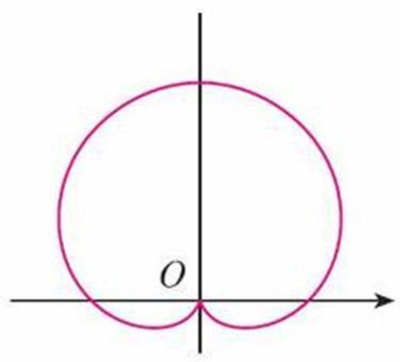


Fig. 1e

Homework 1 – 9.3

Sketch the curve with the given polar equations by first sketching the graph of r as a function of θ in Cartesian coordinates.

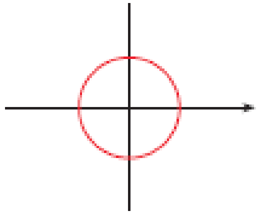
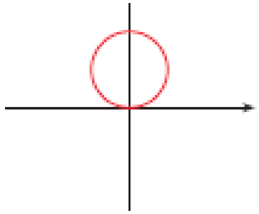
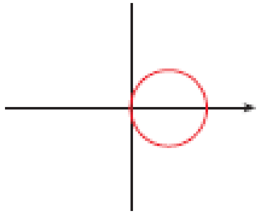
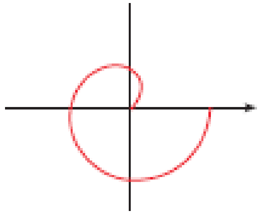
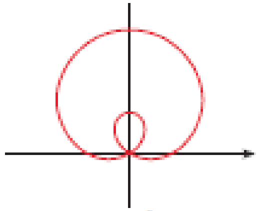
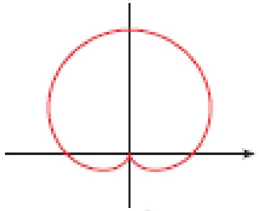
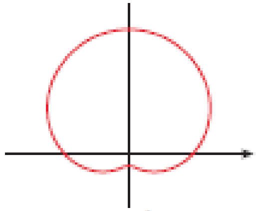
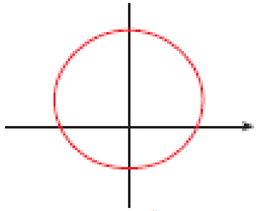
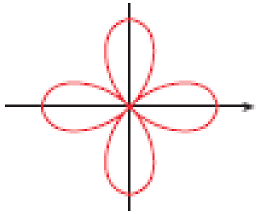
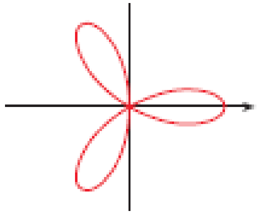
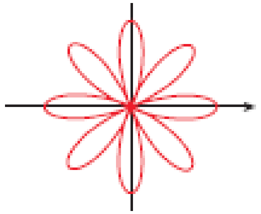
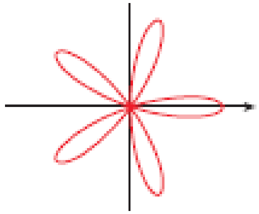
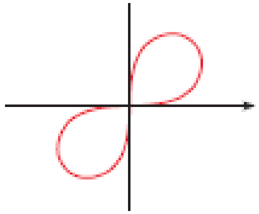
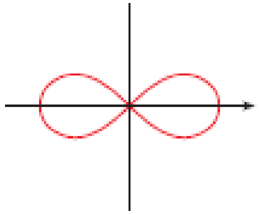
Q. 31. Ex. 9.3

$$r = 1 - 2\sin 4\theta$$

Q. 40. Ex. 9.3

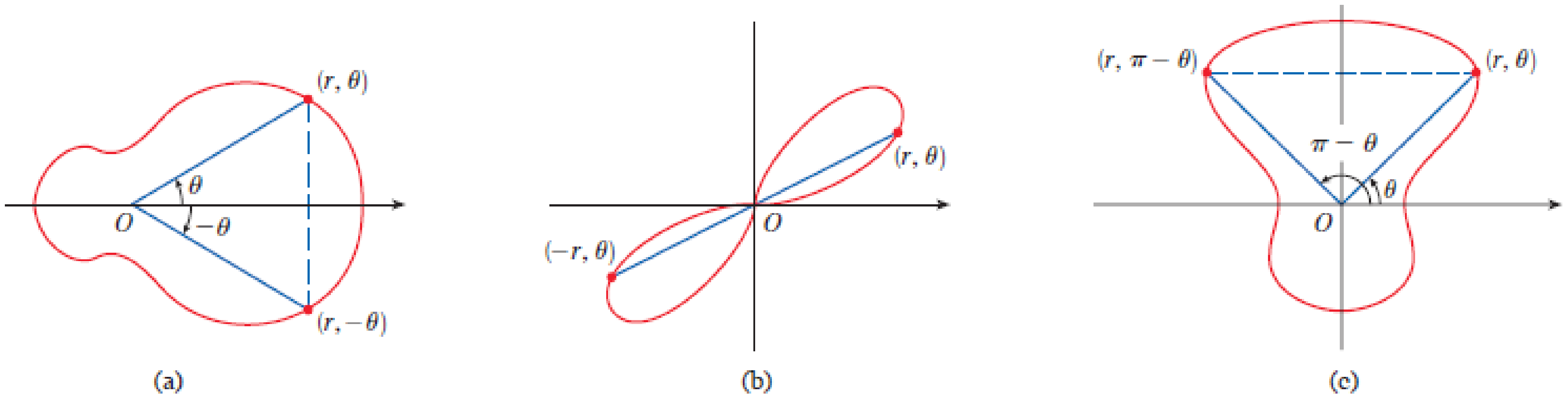
$$r = 3 + 4\cos\theta$$

Common Polar Curves

Circles and Spiral	 $r = a$ circle	 $r = a \sin \theta$ circle	 $r = a \cos \theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geq 2b$ convex limaçon
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a \cos 2\theta$ four-leaved rose	 $r = a \cos 3\theta$ three-leaved rose	 $r = a \cos 4\theta$ eight-leaved rose	 $r = a \cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2 \sin 2\theta$ lemniscate	 $r^2 = a^2 \cos 2\theta$ lemniscate		

Symmetry in Polar Equations

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)
(c) If the equation is unchanged when θ is replaced by $\theta - \pi$, the curve is symmetric about the vertical line $\theta = \pi/2$.



Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation

$$r = 3\sin 2\theta$$

and create a graph.

Solution:

Suppose the point (r, θ) *is on the graph of* $r = 3\sin 2\theta$.

(i) To test for symmetry about polar axis, first try replacing θ *with* $-\theta$. *This gives*

$$r = 3\sin\big(2(-\theta)\big) = -3\sin(2\theta)$$

Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r *by* $-r$ *and* θ *with* $\pi - \theta$ *yields*

$$-r = 3\sin\big(2(\pi - \theta)\big) = 3\sin(-2\theta) = -3\sin 2\theta$$

$$r = 3\sin 2\theta$$

This demonstrates that the graph is symmetric with respect to the polar axis.

Using Symmetry to Graph a Polar Equation

Solution (Contd.):

(ii) To test for symmetry with respect to the pole, first replace r *by* $-r$ *which yields*

$$-r = 3\sin 2\theta \Rightarrow r = -3\sin 2\theta$$

Therefore the equation does not pass the test for this symmetry.

However, returning to the original equation and replacing θ *by* $\theta + \pi$ *gives*

$$r = 3\sin\big(2(\theta + \pi)\big) = 3\sin(2\theta + 2\pi) = 3\sin 2\theta$$

Since this agrees with the original equation, the graph is symmetric about the pole.

(ii) To test for symmetry with respect to the vertical line, $\theta = \pi/2$, *first replace both* r *by* $-r$ *and* θ *by* $-\theta$

$$-r = 3\sin(-2\theta)$$

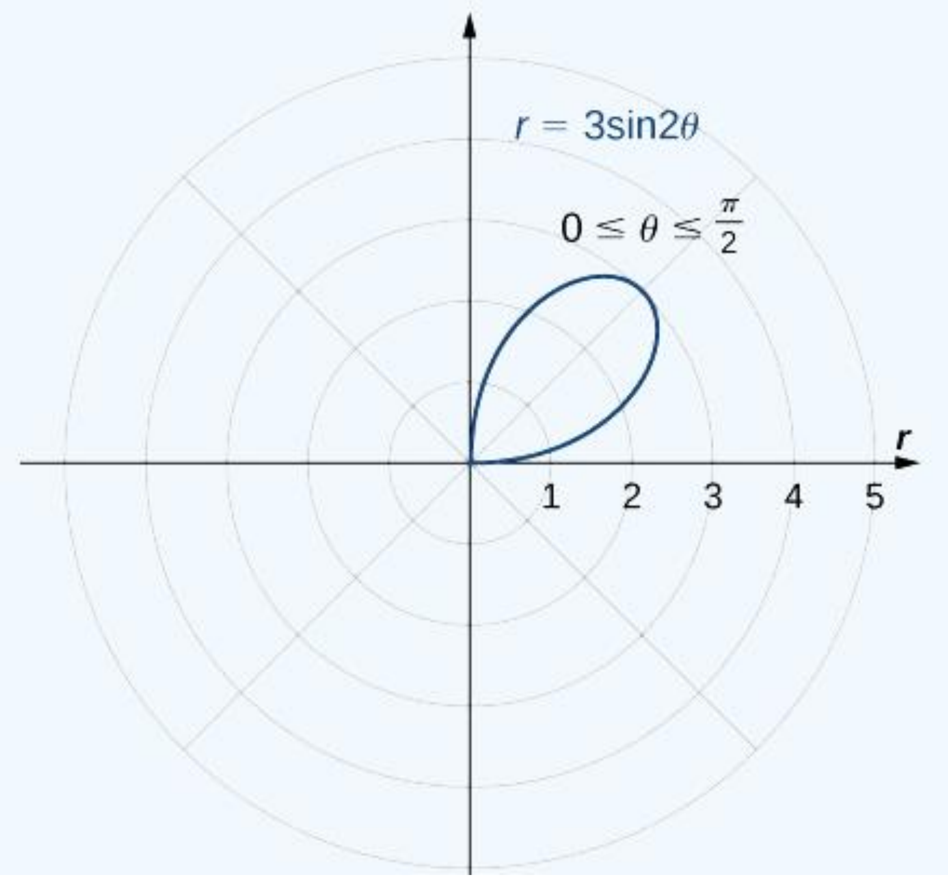
$$r = 3\sin(2\theta)$$

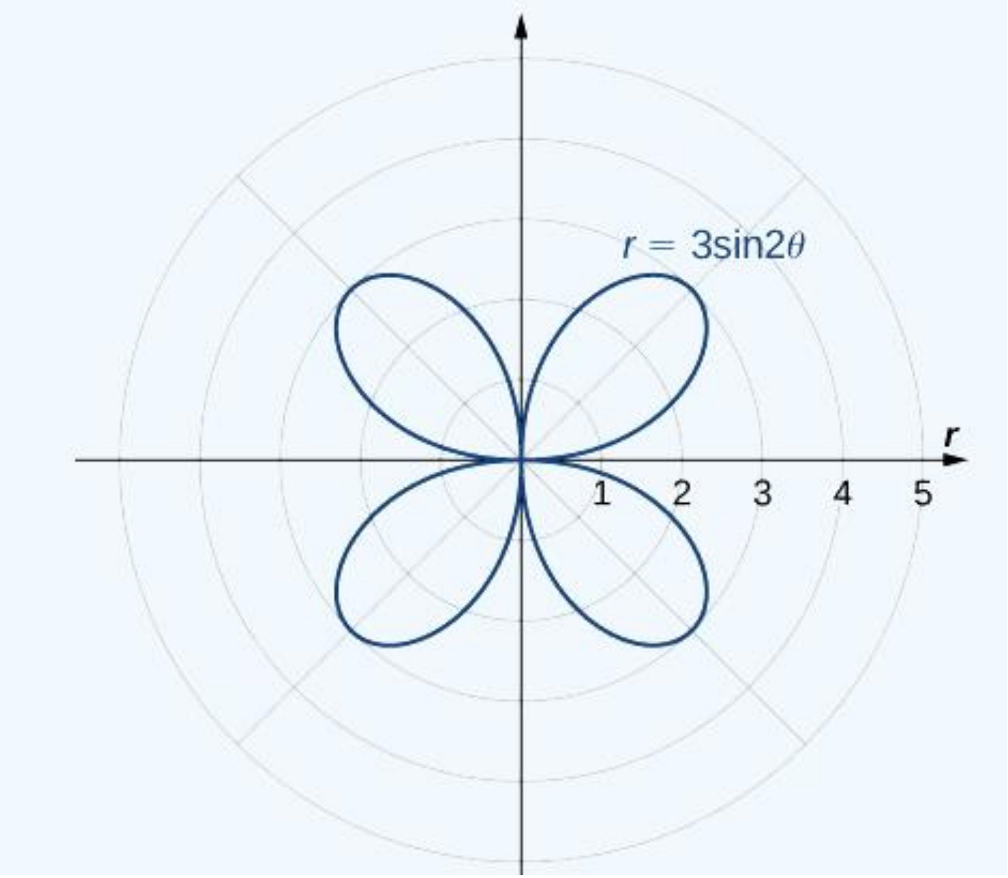
Therefore the graph is symmetric about the vertical line $\theta = \pi/2$.

Using Symmetry to Graph a Polar Equation

To graph the function, tabulate the values of θ from 0 to $\pi/2$ and reflect the resulting graph.

$\boldsymbol{\theta}$	$\boldsymbol{r}$
0	0
$\pi/6$	2.6
$\pi/4$	3
$\pi/3$	2.6
$\pi/2$	0





Homework 2 – 9.3

(i) Determine the symmetry of the graph determined by the equation

$$r = 2\cos 3\theta$$

and create a graph.

(ii) Determine the symmetry of the graph determined by the equation

$$r^2 = 9\cos\theta$$

and create a graph.

Tangents to Polar Curves

To find the tangent to the polar curve $r = f(\theta)$, *regard* θ *as a parameter and write its parametric equations as*

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta$$

The slope to the cuve is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

We locate the horizontal tangents by finding the point where $dy/d\theta = 0$ *(provided that* $dy/d\theta \neq 0$*).*

We locate the vertical tangents by finding the point where $dx/d\theta = 0$ *(provided that* $dy/d\theta \neq 0$*).*

If we are looking for tangent lines at the pole, then $r = 0$, *and*

$$dy/dx = \tan\theta \qquad if\ dr/d\theta \neq 0.$$

Tangents to Polar Curves

Q. No. 52. Ex. 9.3:
Find the points where the tangent line is horizontal or vertical on the curve

$$r = e^{\theta}$$

Solution:

$$r = e^{\theta} \Rightarrow x = r\cos\theta = e^{\theta}\cos\theta, \qquad y = e^{\theta}\sin\theta$$

$$\Rightarrow \frac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta = e^{\theta}(\sin\theta + \cos\theta)$$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta = e^{\theta}(\cos\theta - \sin\theta)$$

We locate the horizontal tangents by finding the point where $dy/d\theta = 0$

$$e^{\theta}(\sin\theta + \cos\theta) = 0 \Rightarrow \tan\theta = -1 \Rightarrow \theta = -\frac{\pi}{4} + n\pi$$

The horizontal tangents are at

$$\left(e^{\pi(n-1/4)}, \qquad \pi(n-1/4)\right)$$

Tangents to Polar Curves

Solution (Contd.):

We locate the vertical tangents by finding the point where $dx/d\theta = 0$

$$e^{\theta}(\cos\theta - \sin\theta) = 0$$
$$\Rightarrow \tan\theta = 1$$
$$\theta = \frac{\pi}{4} + n\pi, \qquad (n \text{ is an integer})$$

The vertical tangents are at

$$\left(e^{\pi(n+1/4)}, \qquad \pi(n+1/4)\right)$$

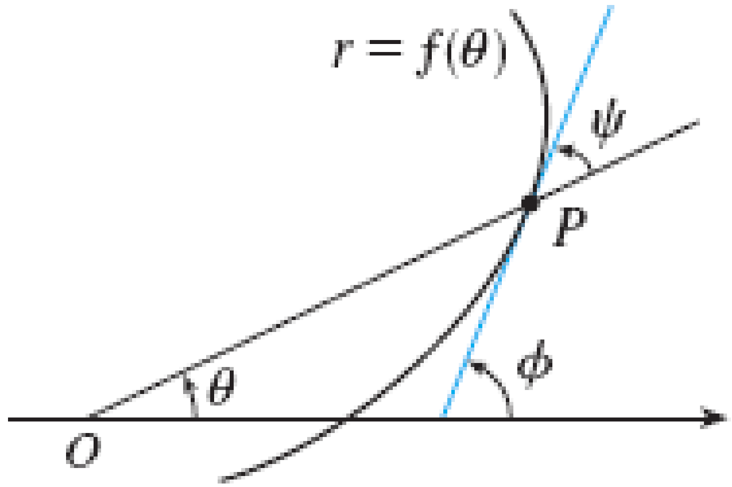
Tangents to Polar Curves

Q. No. 67. Ex. 9.3

Let P be any point (except the origin) on the curve $r = f(\theta)$*. If* ψ *is the angle between the tangent line at P and the radial line OP, show that*

$$\tan\psi = \frac{r}{dr/d\theta}$$

[Hint: Observe that $\psi = \phi - \theta$*].*



Tangents to Polar Curves

Solution:

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$

$$\tan\psi = \frac{(dy/dx) - \tan\theta}{1 + (dy/dx)\tan\theta} = \frac{(dy/d\theta)/(dx/d\theta) - \tan\theta}{1 + (dy/d\theta)/(dx/d\theta)\tan\theta}$$

$$\tan\psi = \frac{(dy/d\theta) - (dx/d\theta)\tan\theta}{(dx/d\theta) + (dy/d\theta)\tan\theta}$$

$$\tan\psi = \left(\frac{[(dr/d\theta)\sin\theta + r\cos\theta] - \tan\theta[(dr/d\theta)\cos\theta - r\sin\theta]}{[(dr/d\theta)\cos\theta - r\sin\theta] + \tan\theta[(dr/d\theta)\sin\theta + r\cos\theta]}\right)$$

$$\tan\psi = \frac{r\cos\theta + r.\sin^2\theta/\cos\theta}{(dr/d\theta)\cos\theta + (dr/d\theta).\sin^2\theta/\cos\theta}$$

$$\tan\psi = \frac{r}{dr/d\theta}$$