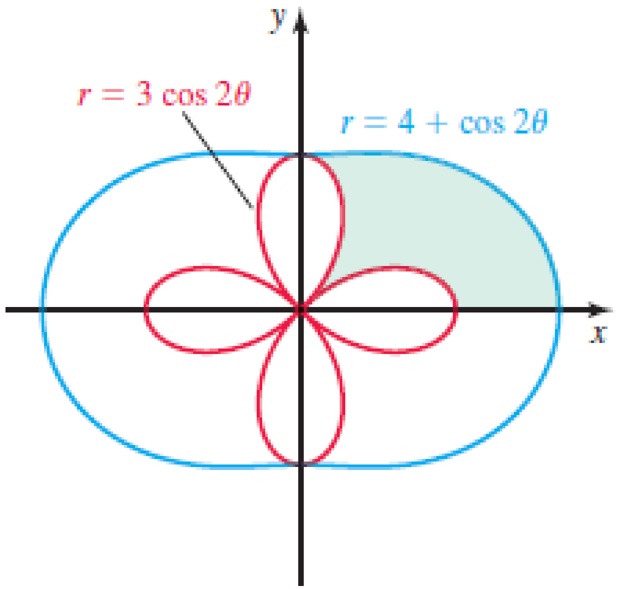
Lecture 4 (Chapter 9)

Calculus Polar Coordinates



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Learning Objectives

- *Apply the methods of calculus to polar curves.*
- *Determine the formula for the area of a region in polar coordinates.*
- *Determine the tangents and arc length of a polar curve.*

Area of Regions Bounded by Polar Curves

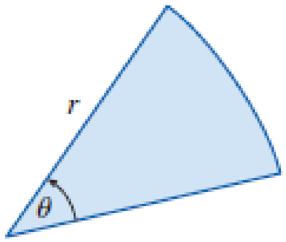
- *We have studied the formulas for the area under a curve defined in rectangular coordinates and parametrically defined curves with Riemann sum to approximate the area under a curve by using rectangles.*

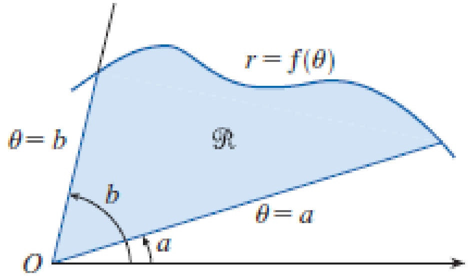
- *For polar curves, we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.*

- *The objective is to find the area of the region R bounded by the graph of* $r = f(\theta)$ *between the two rays* $\theta = \alpha$ *and* $\theta = \beta$. *We assume f is continuous and nonnegative on* $[\alpha, \beta]$.

Area of Regions Bounded by Polar Curves

- *To develop the formula for the area of a region whose boundary is given by a polar equation, we need to use the formula for the area of a sector of a circle (Fig. 1a):*





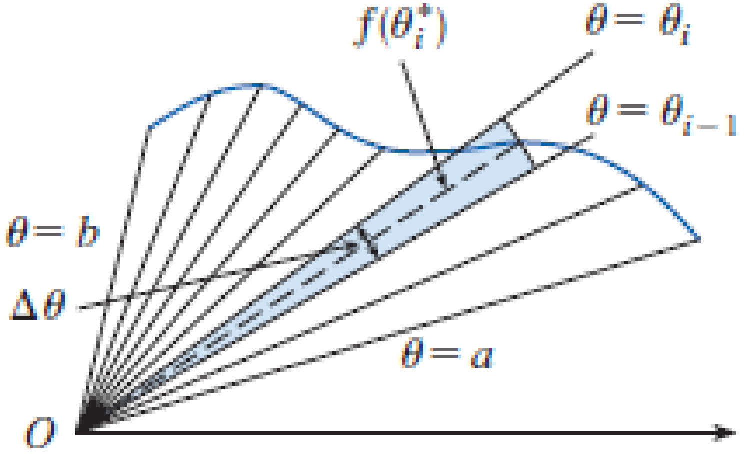
$$A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta.$$

- *The area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$*

$$\Delta A_i = \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta \qquad -(1)$$

And so an approx. to the total area of R is

$$A = \sum_{i}^{n}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta \qquad -(2)$$



Area of Regions Bounded by Polar Curves

In the limit $n \to \infty$*, the sums in (2) are Riemann sums for the function* $g(\theta) = (1/2)[f(\theta)]^2$*, so*

$$\lim_{n\to\infty} \sum_{i}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region R is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$

or

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta, \qquad r = f(\theta)$$

Area of Regions Bounded by Polar Curves

Definition:
Let R be the region bounded by the graphs of $r = f(\theta)$ *and* $r = g(\theta)$, *between* $\theta = a$ *and* $\theta = b$, *where f and g are continuous and* $f(\theta) \geq g(\theta) \geq 0$ *on* $[a, b]$. *The area of R is*

$$A = \int_a^b \frac{1}{2}[f(\theta) - g(\theta)]^2 \, d\theta$$

Area of Regions Bounded by Polar Curves

Q. No. 17, Ex. 9.4

Find the area of the region enclosed by one loop of the curve

$$r = 1 + 2\sin\theta \quad (inner\ loop)$$

Solution

The curve passes through the pole when $r = 0$

$$\Rightarrow 1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$$

The principal solution (Use $\sin(\pi + \theta) = -\sin\theta$*, and* $\sin(2\pi - \theta) = -\sin\theta$*) to the equation are*

$$\Rightarrow \theta = 7\pi/6, \qquad and \qquad \theta = 11\pi/6$$

General Solution to $\sin x = -1/2$

To find the general solution, we use the fact that the values of $\sin x$ *repeat after interval of* 2π*. So,*

$$x = 2n\pi + \pi + \pi/6 \quad and \quad x = 2n\pi + 2\pi - \pi/6$$

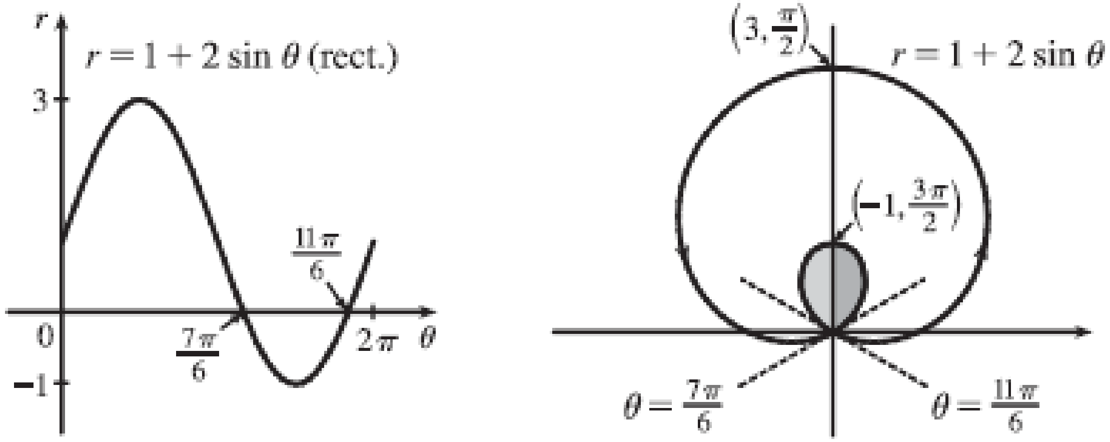
$$x = (2n+1)\pi + \frac{\pi}{6} \quad and \quad x = 2(n+1)\pi - \frac{\pi}{6}, \quad n \in Z.$$

So combination of above solutions gives us,

$$x = n\pi - (-1)^n \frac{\pi}{6}$$

Relationship between Polar and Cartesian Coordinates

The graph is a limacon, with inner loop traced between $\theta = 7\pi/6$ *and* $\theta = 11\pi/6.$



The area of the region is,

$$A = 2\int_{7\pi/6}^{3\pi/2} \frac{1}{2}(1 + 2\sin\theta)^2 d\theta = \int_{7\pi/2}^{3\pi/2} [1 + 4\sin\theta + 2(1 - \cos 2\theta)]\, d\theta$$

$$A = \pi - \frac{3\sqrt{3}}{2}$$

Homework 1 – 9.4

Q. 18. Ex. 9.4 Find the area of the region enclosed by one loop of the curve

$$r = 2\cos\theta - \sec\theta$$

Example

Q. 22. Ex. 9.4: Find the area of the region that lies inside $r = 2 + \sin\theta$ and outside the second curve $r = 3\sin\theta$.

Solution:

Since the curves intersect at only one point, we will evaluate from 0 to 2π.

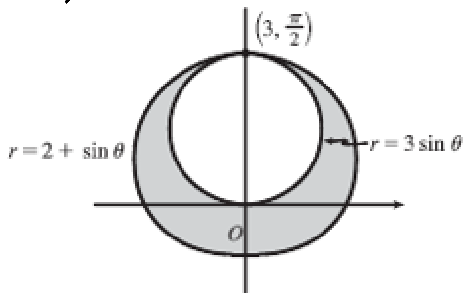
Find the area of the circle first: $A_2 = \pi r^2 = \pi(3/2)^2 = 9\pi/4$ (since $r = 3\sin\theta$ is a circle with radius $3/2$.)

To find the shaded area A, we 'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi(3/2)^2$,

$$A_1 = \int_0^{2\pi} \frac{1}{2}(2 + \sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} + 4\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta = \frac{9\pi}{2}$$

So

$$A = A_1 - \frac{9}{4}\pi = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$$



Homework 2 – 9.4

Q. 21. Ex. 9.4: Find the area of the region that lies inside

$$r = 3\cos\theta$$

and outside the second curve

$$r = 1 + \cos\theta.$$

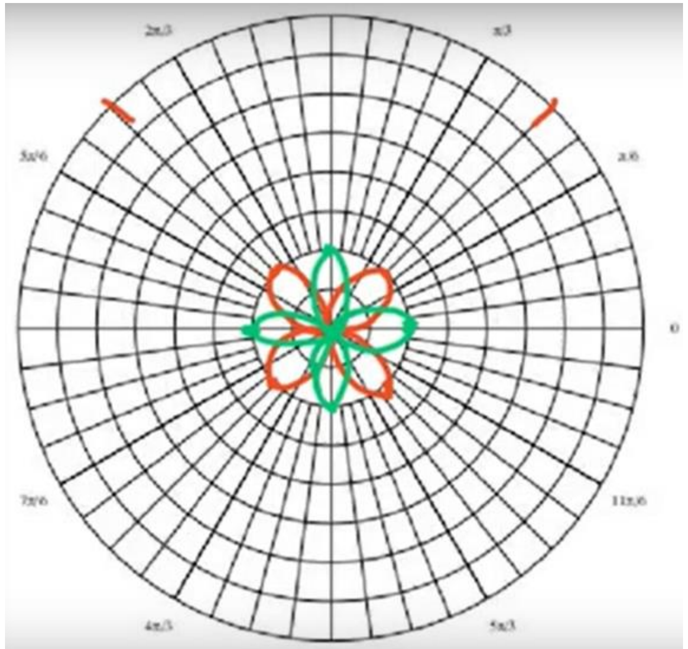
Example

Q. 25. Ex. 9.4: Find the area of the region that lies inside both curves.

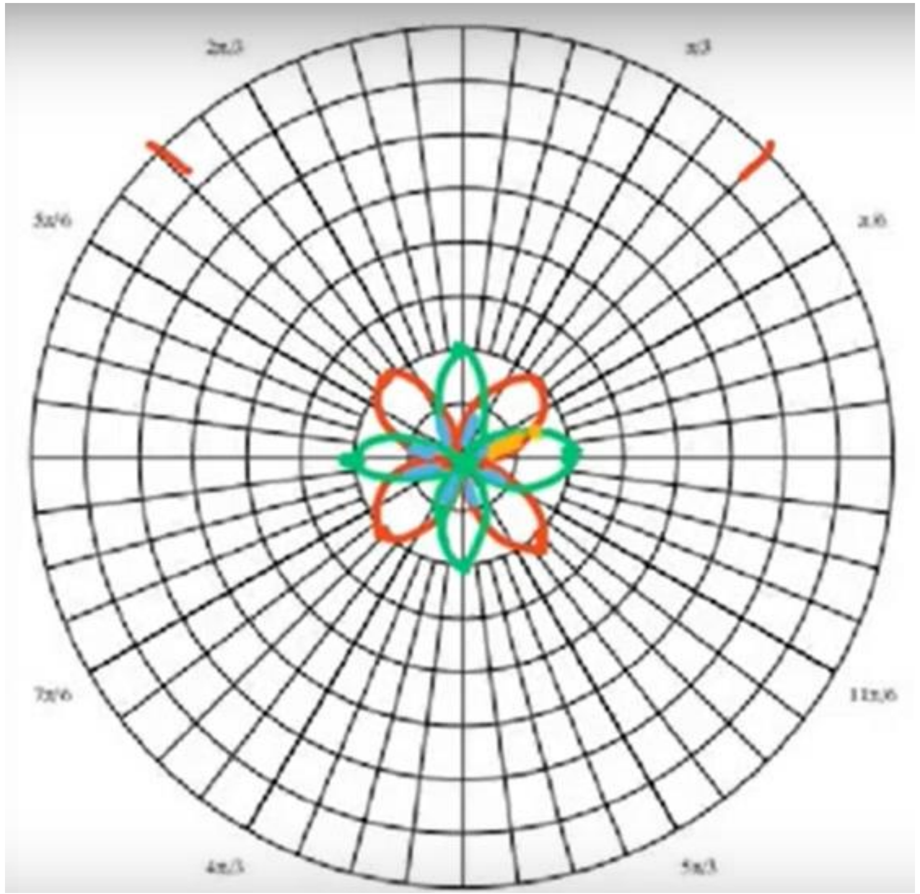
$$r = \sin 2\theta, \qquad r = \cos 2\theta.$$

Solution: *To find the limits on θ, we have*

$$\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow \theta = \frac{\pi}{8}.$$



$\theta \rightarrow 0 \; to \; \pi/8$



Example

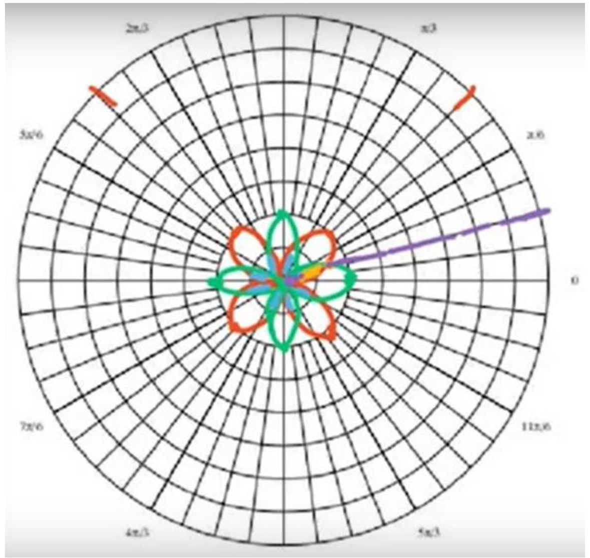
The area of the polar region is

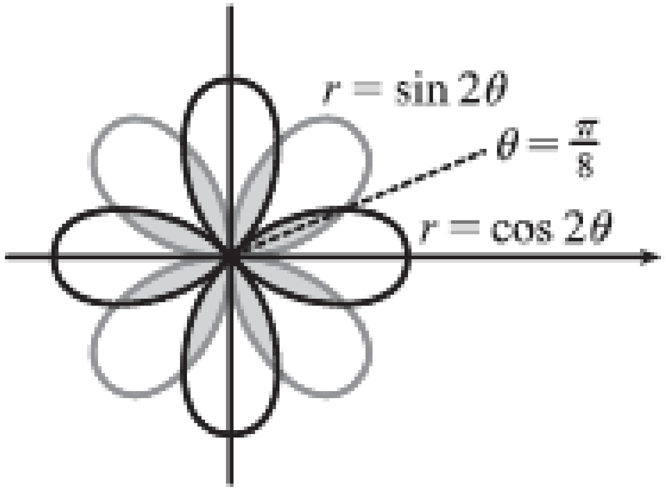
$$A = \int_{a}^{b} \frac{1}{2} r^2 d\theta$$

$$A = 8 \times 2 \int_{0}^{\pi/8} \frac{1}{2} \sin^2 2\theta \, d\theta$$

$$A = 8 \int_{0}^{8} \frac{1}{2} (1 - \cos 4\theta) d\theta$$

$$A = \frac{\pi}{2} - 1$$





Homework 3 – 9.4

Find the area of the region that lies inside both curves

$$r = \sqrt{3}\cos\theta, \qquad r = \sin\theta$$

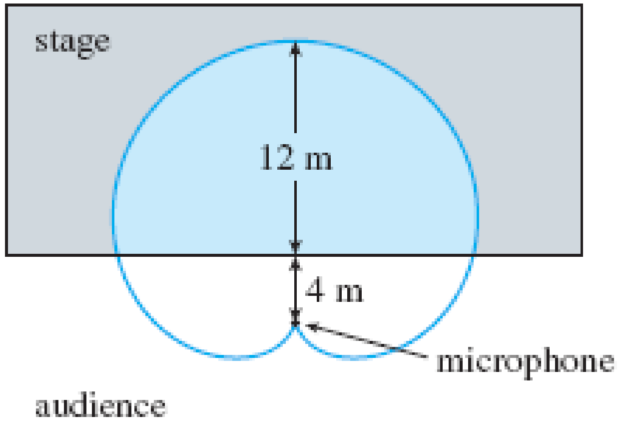
Application

Q. 28. Ex. 9.4: When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience.

Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid

$$r = 8 + 8\sin\theta$$

where is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



Homework 6 – 9.4

Solution: Method 1.

The cardioid equation yields if the microphone is placed at the pole.

The stage's front is visible 4m from the pole. This can be expressed in terms of the cartesian equation $y = 4$, which translates into a polar equation as

$$r \sin\theta = 4 \Rightarrow r = 4/\sin\theta$$

To find the portion of the area that is on the stage, we must subtract the area of the front of the stage from the area of the cardioid.

To identify the integration's bounds, one must first find the intersections

The line intersects the curve $r = 8 + 8\sin\theta$ when

$$8 + 8\sin\theta = 4/\sin\theta$$

$$2\sin^2\theta + 2\sin\theta - 1 = 0 \;\Rightarrow\; \sin\theta = \frac{-1 \pm \sqrt{3}}{2}$$

$$\Rightarrow \theta = 21.5^o \qquad or\ 0.3747\ radians$$

Homework 6 – 9.4

Using symmetry, we may just obtain the opposite angle.

However, the method is to use the second limit $\pi/2$ and double the result, taking advantage of symmetry as well.

Now remove the inner area enclosed by the line from the cardioid using the polar area formula.

$$A = \int_a^b \frac{1}{2}[\left(f(\theta)\right)^2 - (g(\theta)^2]d\theta$$

$$A = \int_{0.3747}^{\pi/2} \frac{1}{2}\left[(8 + 8\sin\theta)^2 - \left(\frac{4}{\sin\theta}\right)^2\right]d\theta$$

$$A = \int_{0.3747}^{\pi/2} \frac{1}{2}[6 + 8\sin\theta - 2\cos 2\theta - \csc^2\theta]d\theta$$

$$A = 204.16\, m^2$$

Application

Solution – Method 2:

The objective is to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4$

$$\Rightarrow r\sin\theta = 4 \Rightarrow r = 4/\sin\theta$$

The line intersects the curve $r = 8 + 8\sin\theta$ *when*

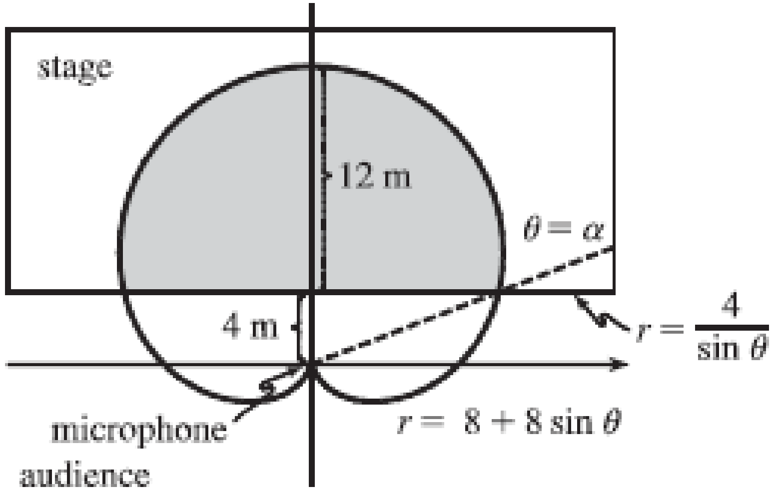
$$8 + 8\sin\theta = 4/\sin\theta$$

$$2\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\sin\theta = \frac{-1 \pm \sqrt{3}}{2}$$

$$\Rightarrow \theta = 21.5^o \qquad (denoted\ by\ \alpha\ in\ the\ Fig.)$$

(The other value is less than -1)



Application

Solution:

$$A = 2\int_{\alpha}^{\pi/2} \frac{1}{2}(8 + 8\sin\theta)^2 d\theta - 2\int_{\alpha}^{\pi/2} \frac{1}{2}(4\csc\theta)^2 d\theta$$

$$= 16[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta]_{\alpha}^{\pi/2}$$

$$= 48\,\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

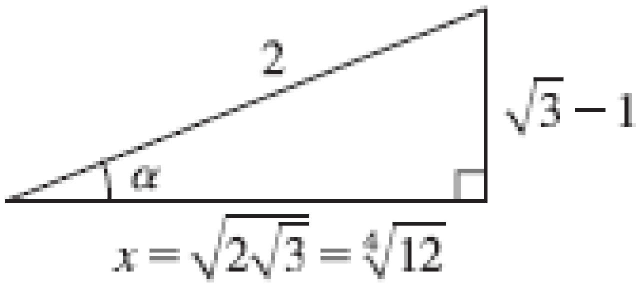
From the Fig., we have

$$x^2 + \left(\sqrt{3} - 1\right)^2 = 2^2 \Rightarrow x = \sqrt[4]{12}$$

$\cos\alpha = \sqrt[4]{12}/2, \quad \sin\alpha = (\sqrt{3} - 1)/2, \;\; \cot\alpha = \sqrt[4]{12}\,/(\sqrt{3} - 1).$

Therefore,

$$A = 204.16\, m^2$$

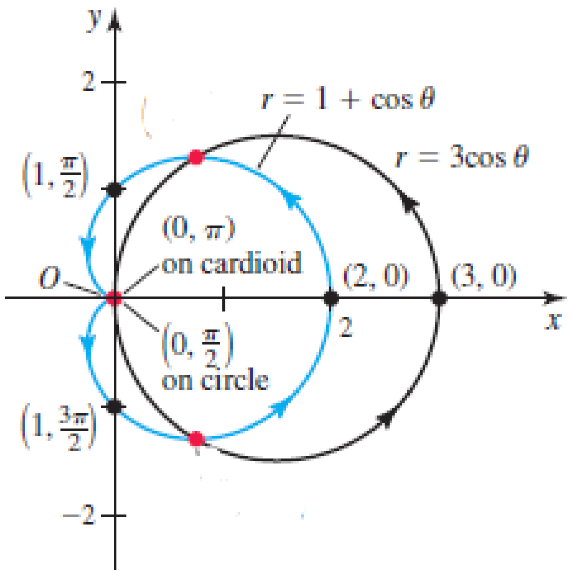


Points of intersection

Find the points of intersection of the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

Solution:

The fact that a point has multiple representations in polar coordinates may lead to subtle difficulties in finding intersection points.



We first proceed algebraically. Equating the two expressions for r and solving for θ, we have

$$3\cos\theta = 1 + \cos\theta \Rightarrow \theta = \pm\pi/3$$

Points of intersection

- *Therefore, two intersection points are* $= (3/2, \pi/3)$ *and* $(3/2, -\pi/3)$. *Without examining graphs of the curves, we might be tempted to stop here.*
- *Yet the figure shows another intersection point at the origin O that has not been detected.*
- *To find this intersection point, we must investigate the way in which the two curves are generated. As* θ *increases from 0 to* 2π*, the cardioid is generated counterclockwise, beginning at (2, 0). The cardioid passes through O when* $\theta = \pi$.
- *As* θ *increases from 0 to* π*, the circle is generated counterclockwise, beginning at (3, 0). The circle passes through O when* $\theta = \pi/2$.

Points of intersection

- *Therefore, the intersection point O is* $(0, \pi)$ *on the cardioid (and these coordinates do not satisfy the equation of the circle), while O is* $(0, \pi/2)$ *on the circle (and these coordinates do not satisfy the equation of the cardioid).*

- *There is no foolproof rule for detecting such "hidden" intersection points. Care must be used.*

Homework 4 – 9.4

Q. 30. Ex. 9.4: Find the points of intersection of the curves

$$r = \cos 3\theta, \qquad r = \sin 3\theta$$

Arc Length of a Polar Curve

- *Given the polar equation $r = f(\theta)$, what is the length of the corresponding curve for $\alpha \leq \theta \leq \beta$ (assuming the curve does not retrace itself on this interval)?*
- *The key idea is to express the polar equation as a set of parametric equations in Cartesian coordinates and then use the arc length formula for parametric equations derived in Section 9.1.*

Letting θ play the role of a parameter and using $r = f(\theta)$, parametric equations for the polar curve are

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta$$

where $\alpha \leq \theta \leq \beta$.

Arc Length of a Polar Curve

The arc length, in terms of θ, *is*

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \quad d\theta$$

where

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

The resulting arc length integral is

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} \quad d\theta}$$

Arc Length of a Polar Curve

Q. 34. Ex. 9.4: Find the exact length of the polar curve

$$r = e^{2\theta}, \qquad 0 \le \theta \le 2\pi$$

Solution:

$$r = f(\theta) = e^{2\theta}, \qquad f'(\theta) = 2e^{2\theta}$$

Therefore

$$L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} \quad d\theta$$

gives

$$L = \int_{0}^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} \quad d\theta$$

$$L = \int_{0}^{2\pi} \sqrt{5e^{4\theta}} \quad d\theta = \frac{\sqrt{5}}{2}(e^{4\pi} - 1).$$

Homework 5 – 9.4

Find the are length of the polar curve

$$r = \frac{\sqrt{2}}{1 + \cos\theta}, \qquad 0 \le \theta \le \pi/2.$$

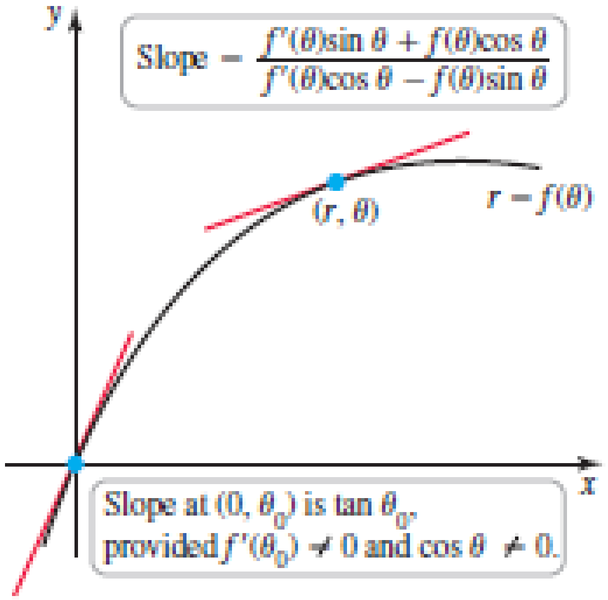
Slope of a Tangent Line

THEOREM

Let f be a differentiable function at θ. *The slope of the line tangent to the curve* $r = f(\theta)$ *at the point* $(f(\theta), \theta)$ *is*

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided the denominator is nonzero at the point. At angles θ_0 *for which* $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, *and* $\cos\theta_0 \neq 0$, *the tangent line is* $\theta = \theta_0$ *with slope* $\tan\theta_0$.



Slope of a Tangent Line

Using the method for finding the slope of a parametric curve and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$.

Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then r = 0 and the above Eq. simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad if\,\frac{dr}{d\theta} \neq 0.$$

Slope of a Tangent Line

For the cardioid

$$r = 1 - \sin\theta$$

(a) find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution:

With $r = 1 - sin\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

Slope of a Tangent Line

(a) The slope of the tangent line when $\theta = \pi/3$.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)[1 + 2\sin(\pi/3)]}{[1 + \sin(\pi/3)][1 - 2\sin(\pi/3)]} = \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})}$$

$$= \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$$

(b) The points on the cardioid where the tangent line is horizontal or vertical.

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when} \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when} \quad \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Slope of a Tangent Line

Therefore there are horizontal tangents at the points $(2, \pi/2)$, $(12, 7\pi/6)$, $(1/2, 11\pi/6)$ *and vertical tangents at* $(3/2, \pi/6)$ *and* $(3/2, 5\pi/6)$.

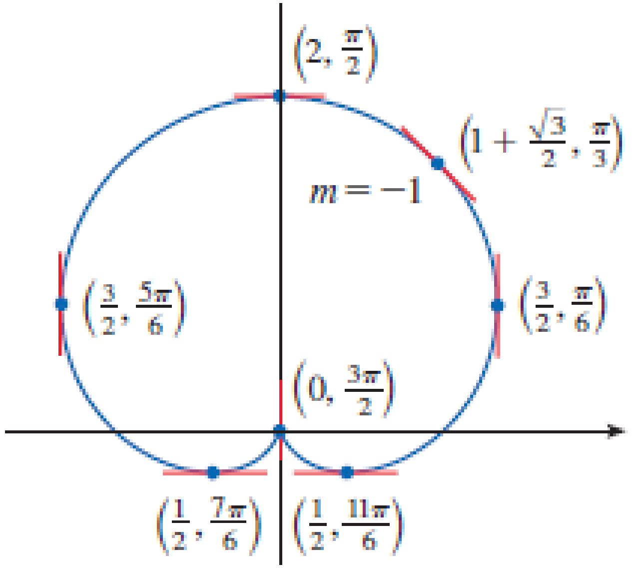
When $\theta = 3\pi/2$, *both* $dy/d\theta$ *and* $dx/d\theta$ *are 0, so we must be careful.*

Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry, $$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$



Thus there is a vertical tangent line at the pole.

Homework 6 – 9.4

Find the slope of the tangent line to the polar curve

$$r = \sin\theta + 2\cos\theta$$

at $\theta = \pi/2$.