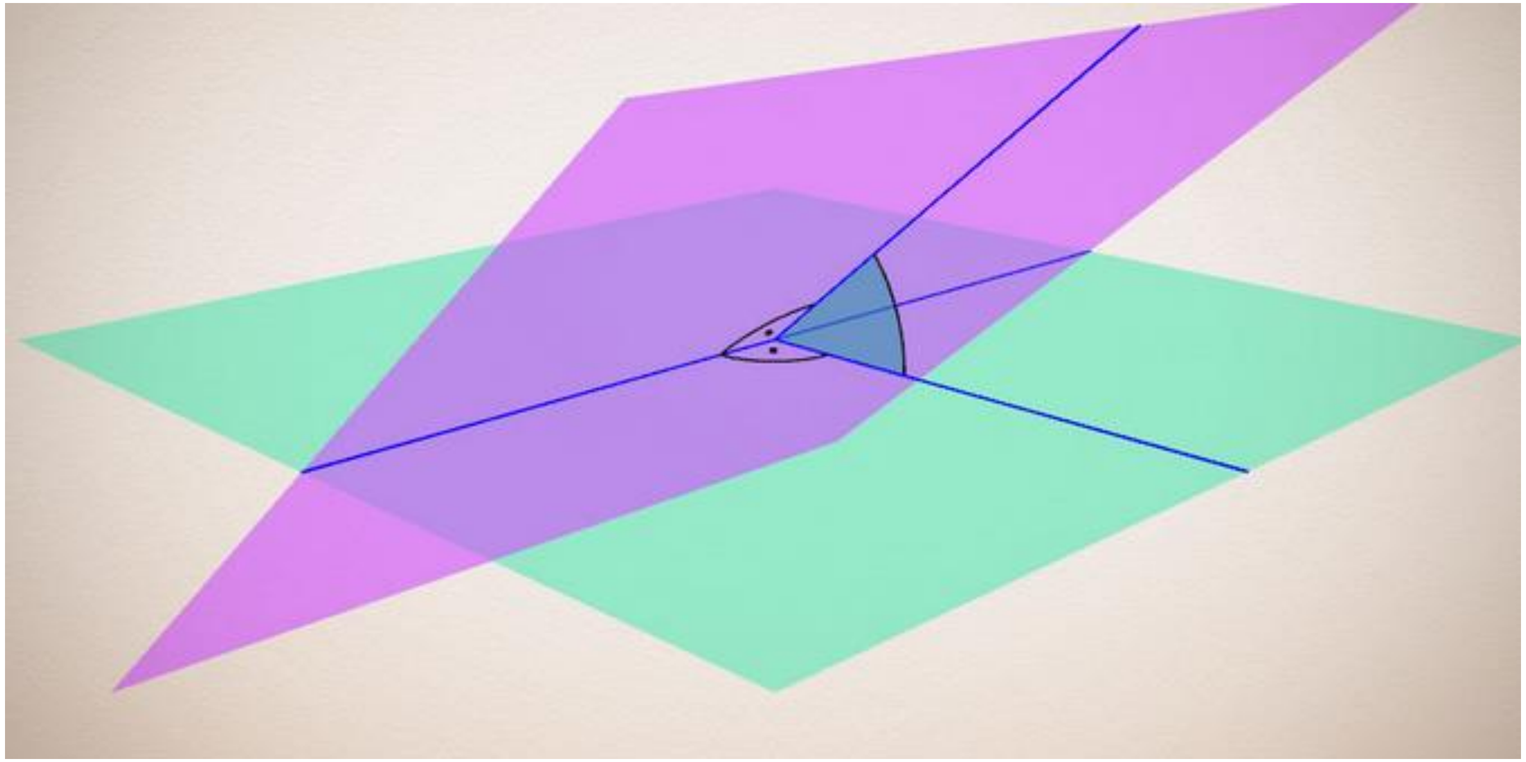
# Lecture 9 - Chapter 10 – Sec. 10.5

# Lines and Planes



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# Learning Objectives

- *Find the equation of the line represented in any one of the following ways:*
    - *Vector parameterization*
    - *Scalar parametric equations*
    - *Symmetric form*
- *Determine whether a pair of lines intersect, parallel, or skew.*
- *Find the distance from a point to a line.*
- *Find the equation of the plane*
- *Determine whether a collection of vectors is coplanar.*
- *Find the distance from a point to a plane.*

# Equation of a Line in 3-D

- *A line in the $xy$-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.*

- *A line $L$ in three-dimensional space is determined when we know a point $P(x_0, y_0, z_0)$ on $L$ and a direction for $L$, which is conveniently described by a vector $v$ parallel to the line.*

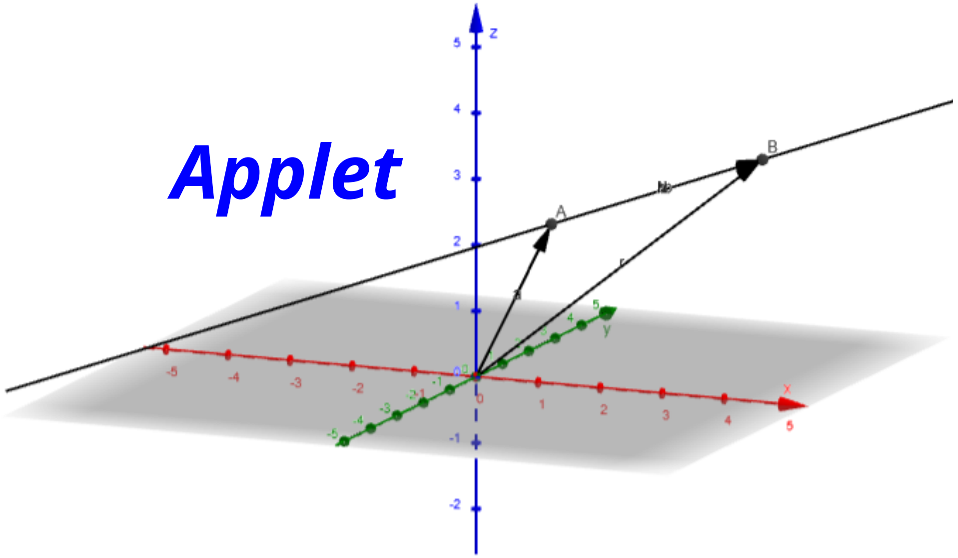
# Equation of a Line in 3-D

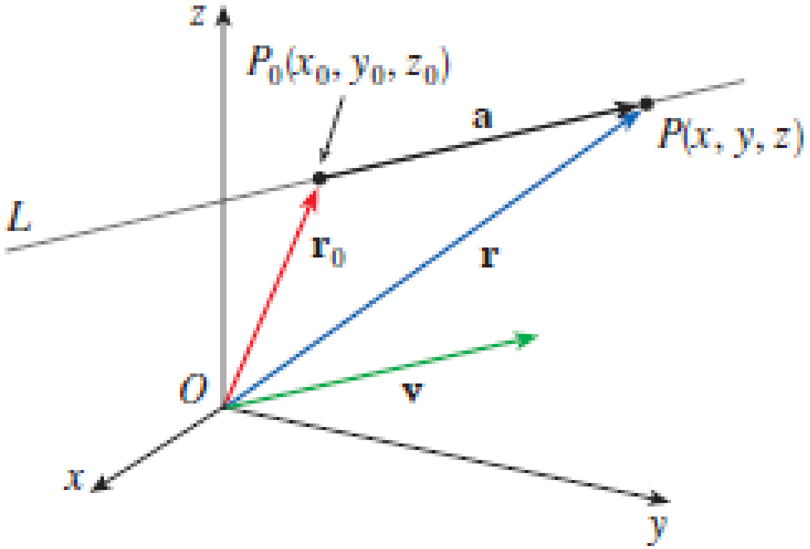
- *Let* $P(x, y, z)$ *be an arbitrary point on* $L$ *and let* $r_0$ *and* $r$ *be the position vectors of* $P_0$ *and* $P$ *(that is, they have representations* $\overrightarrow{OP_0}$ *and* $\overrightarrow{OP}$*).*

- *If* $\vec{a}$ *is the vector with representation* $\overrightarrow{P_0P}$*, then the Triangle Law for vector addition gives*

$$\vec{r} = \vec{r}_0 + \vec{a}$$

***Applet***



# Equation of a Line

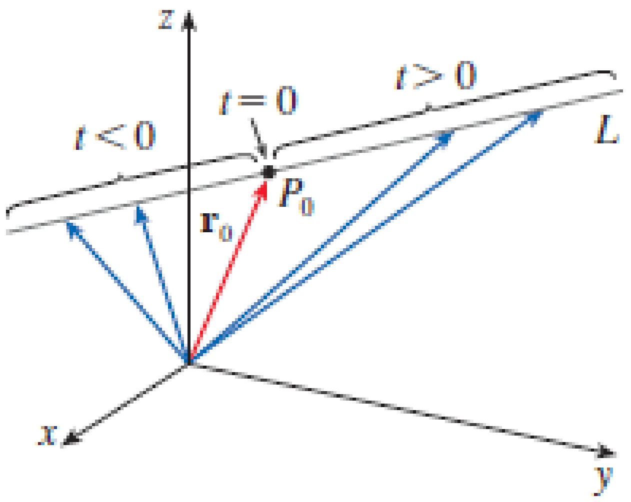
*Since $\vec{a}$ and $\vec{v}$ are parallel vectors, there is a scalar $t$ such that $\vec{a} = t\vec{v}$. Thus*

$$\vec{r} = \vec{r}_0 + t\,\vec{v}$$

*which is a **vector equation** of $L$.*

- *Each value of the **parameter** t gives the position vector **r** of a point on L. In other words, as t varies, the line is traced out by the tip of the vector **r**.*

- *The positive values of t correspond to points on L that lie on one side of $P_0$, whereas negative values of t correspond to points that lie on the other side of $P_0$.*



# Parametric Equations of the Line

- *If the vector **v** that gives the direction of the line L is written in component form as*

$$v = \langle a, b, c \rangle$$

*then we have* $tv = \langle ta, tb, tc \rangle$

- *Also, we can write* $r = \langle x, y, z \rangle$ *and* $r_0 = \langle x_0, y_0, z_0 \rangle$*, so the vector equation becomes*

$$\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

- *Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:*

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

- *These equations are called **parametric equations** of the line L through the point* $P(x_0, y_0, z_0)$ *and parallel to the vector* $v = \langle a, b, c \rangle$*.*
- *Each value of the parameter t gives a point* $(x, y, z)$ *on L.*

# Summary

**Equation of a Line**

A vector equation of the line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

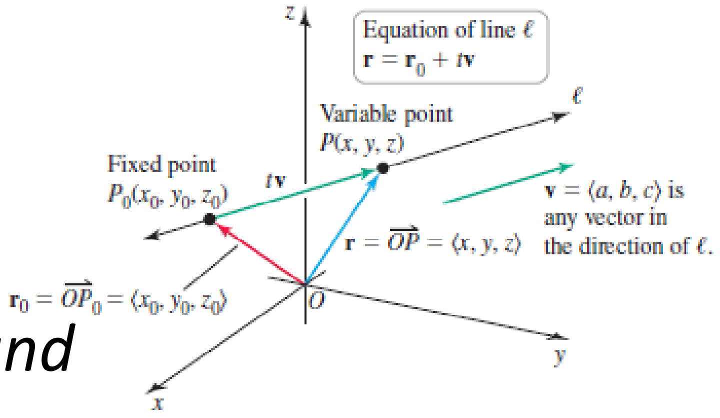
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \quad \text{for} \quad -\infty < t < \infty.$$

Equivalently, the corresponding parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty.$$

*Eliminating parameter t yields:*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$




*These equations are called* ***symmetric equations*** *of L, and a, b, c are called* ***direction numbers*** *of L.*

# Test Your Understanding

*True or False and why?*

*(a) Two lines parallel to a third line are parallel.*

*(b) Two lines perpendicular to a third line are parallel*

# Example – Q. 8-9, Ex. 10.5

*Find parametric equations and symmetric equations for the line*

*(a) that passes through the point (2, 1, 0) and perpendicular to both* $i + j$ *and* $j + k$.

*(b) That passes through (1,-1,1) and parallel to the line* $x + 2 = (1/2)\, y = z - 2$.

*Solution:*

*(a) We need the direction vector of the line. Since it is perpendicular the lines,* $i + j$ *and* $j + k$*, therefore*

$$v = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i - j + k$$

*With* $P_0(2,1,0)$*, the parametric equations are* $x = 2 + t,\quad y = 1 - t,\qquad z = t.$

*The symmetric equations are*

$$x - 2 = -(y - 1) = z$$

# Example – Q. 8-9, Ex. 10.5

*(b) The line has direction vector* $v = (1,2,1)$

*The line passing through (1,-1,1) will have the same direction vector since it is parallel to the given line. Therefore parametric equations are*

$$x = 1 + t, \qquad y = -1 + 2t, \qquad z = 1 + t$$

*and the symmetric equations are*

$$x - 1 = \frac{y - 1}{2} = z - 1$$

- *Lines are parallel if the direction vectors are a scalar multiple of each other.*
- *Lines are perpendicular if the dot product of their direction vectors is zero.*

# Example – Q. 13, Ex. 10.5

*(a) Find parametric equations for the line through (5,1,0) that is perpendicular to the plane $2x - y + z = 1$.*

*(b) In what points does this line intersect the coordinate planes?*

*Solution*

*(a) The vector normal to the plane is* $n = \langle 2, -1, 1 \rangle$

*Since the line is to be perpendicular to the plane, n is also a direction vector for the line. Thus parametric equations of the line are*

$$x = 5 + 2t, \qquad y = 1 - t, \qquad z = t$$

*(b) On the xy-plane* $z = 0$*, so* $z = t = 0 \Rightarrow t = 0$ *in the parametric equations of the line, and therefore* $x = 5, y = 1$

*giving the point of intersection* $(5,1,0)$.

*Similarly, for the yz- and zx-planes give the intersection points* $(0, 7/2, -5/2)$ *and* $(7, 0, 1)$*, respectively.*

# Skew Lines

- *Lines that do not intersect and are not parallel (and therefore do not lie in the same plane) are called skew lines.*

*Show that the lines $L_1$ and $L_2$ with parametric equations*

$$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

*are skew lines.*

*Solution: The vectors corresponding to lines are* $\langle 1,3,-1 \rangle$ *and* $\langle 2,1,4 \rangle$

*Since the vectors are not parallel (there components are not proportional), hence corresponding lines are not parallel.*

*If $L_1$ and $L_2$ had a point of intersection, then there would be values of t and s such that* $1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s$

*There are no values of t and s that satisfy the above three eqns., so $L_1$ and $L_2$ do not intersect. The lined are skew lines.*

# Homework 1 – 10.5

*Determine whether the lines $L_1$ and $L_2$ are parallel, skew, or intersecting. If they intersect, find the point of intersection*

$$L_1 = \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$$

$$L_2 = \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

# Equation of Plane

- *Unlike a line in space, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the direction of the plane.*

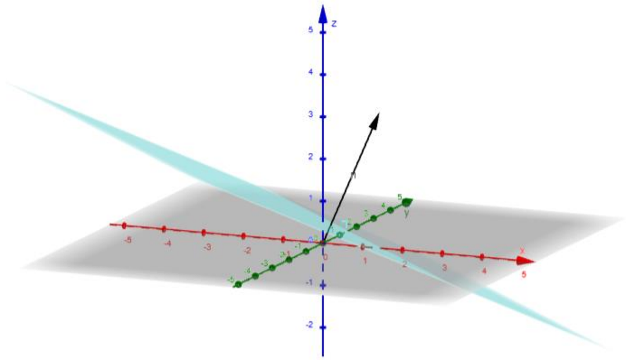
- *However, a vector perpendicular to the plane does completely specify its direction.*

- *Thus, a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector n that is orthogonal to the plane.*

- *This orthogonal vector is called n is called a normal vector.*

# Equation of Plane

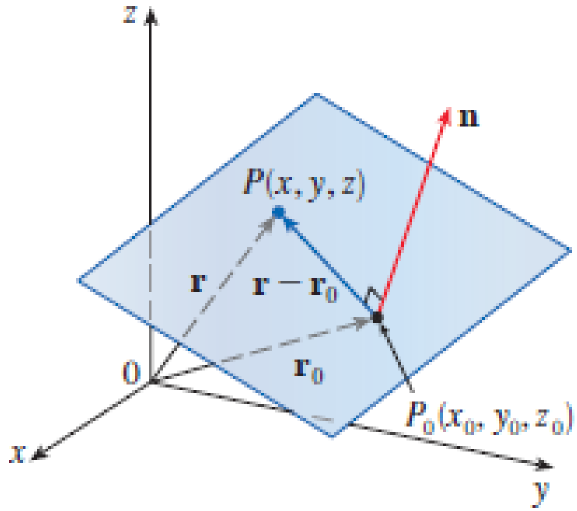
- *Let $P(x, y, z)$ be an arbitrary point in the plane, and let $r_0$ and $r$ be the position vector of $P_0$ and $P$. Thus the vector $r - r_0$ is represented by $PP_0$.*
- *The normal vector n is orthogonal to every vector in the given plane. In particular, it is perpendicular to $r - r_0$ and so we have*



$$n \cdot (r - r_0) = 0$$

$$n \cdot r = n \cdot r_0$$

*Either of the above equations is called the vector equation of the plane.*




$$\langle a, b, c \rangle \cdot \left( x_{x_0}, y - y_0, z - z_0 \right) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

*Scalar equation of a plane with normal vector n.*

# Example – Q. 30, Ex. 10.5

*Find the equation of the plane that passes through the points* $(0,-2,5)$ *and* $(-1,3,1)$ *and is perpendicular to the plane* $2z = 5x + 4y$.

Solution: *The points (0, -2, 5) and (-1, 3,1) lie in the desired plane, so the vector*

$$v_1 = \langle -1-0, 3-(-2), 1-5 \rangle = \langle -1, 5, -4 \rangle$$

*connecting them is parallel to the plane.*

*The desired plane is perpendicular to the plane* $2z = 5x + 4y$ *or* $5x + 4y - 2z = 0$.

*For perpendicular planes, a normal vector for one plane is parallel to the other plane, so* $v_2 = \langle 5, 4, -2 \rangle$ *is also parallel to the desired plane.*

*The normal vector to the desired plane is*

$$n = v_1 \times v_2 = \langle -10+16, -20-2, -4-25 \rangle = \langle 6, -22, -29 \rangle$$

*Thus, the equation of the plane passing through* $(x_0, y_0, z_0) = (0, -2, 5)$ *and normal vector* $\langle 6, -22, -29 \rangle$ *is*

$$6(x-0) - 22(y+2) - 29(z-5) = 0 \qquad \text{or } 6x - 22y - 29z = 0$$

# Homework 2 – 10.5

*Find the equation of the plane that contains the line*

$$x = 3 + 2t, \qquad y = t, \qquad z = 8 - t$$

*and is parallel to the plane*

$$2x + 4y + 8z = 17.$$

# Example – Q. 32, Ex. 10.4

*Find the equation of the plane that passes through the line of intersection of the planes* $x - z = 1$ *and* $y + 2z = 3$ *and is perpendicular to the plane* $x + y - 2z = 1$.

*Solution:*

*The normal vector of the two planes are* $n_1 = \langle 1, 0, -1 \rangle$ *and* $n_2 = \langle 0,1,2 \rangle$

*Setting* $z = 0$, *it is easy to see that* $(1,3,0)$ *is a point on the line of intersection of*

$$x - z = 1 \quad and \; y + 2z = 3$$

*This direction of this line is* $v_1 = n_1 \times n_2 = \langle 1, -2,1 \rangle$

*A second vector parallel to the desired plane is* $v_2 = \langle 1,1, -2 \rangle$, *since it is perpendicular to* $x + y - 2z = 1$. *Therefore, a normal to the plane in question is*

$$n = v_1 \times v_2 = \langle 4 - 1,1 + 2,1 + 2 \rangle = \langle 3,3,3 \rangle = 3\langle 1,1,1 \rangle$$

*Thus the eqn. of the plane passing through (1,3,0) and normal vector* $3\langle 1,1,1 \rangle$ *is*

$$(x - 1) + (y - 3) + z = 0 \quad or \;\; x + y + z = 4$$

# Homework 3 – 10.5

*(a). Determine whether the planes*

$$x = 4y - 2z, \qquad 8y = 1 + 2x + 4z$$

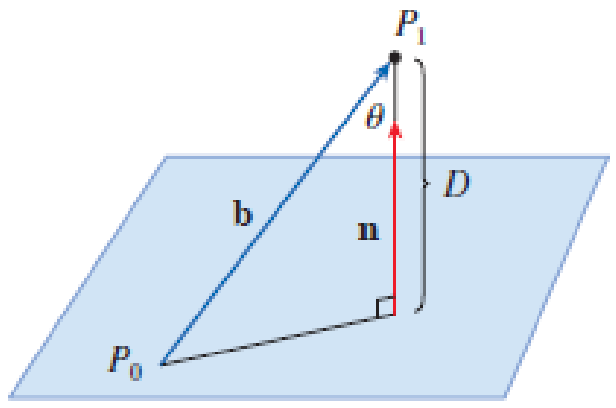
*are parallel, perpendicular or neither. If neither, find the angle between them.*

*(b). Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.*

# Distances - Point the Plane

*In order to find a formula for the distance D from a point* $P(x_1, y_1, z_1)$ *to the plane* $ax + by + cz + d = 0$, *we let* $P(x_0, y_0, z_0)$ *be any point in the given plane and* ***b*** *be the vector corresponding to* $\overrightarrow{P_0P_1}$. *Then*

$$b = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$




*The distance D from* $P_1$ *to the plane is equal to the absolute value Of the scalar projection of b onto the normal vector* $n = \langle a, b, c \rangle$

$$D = |comp_n b| = \frac{|n \cdot b|}{|n|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

# Example – Q. 49, Ex. 10.5

*Since* $P_0$ *lies in the plane, its coordinates satisfy the equation of the plane and so we have* $ax_0 + by_0 + cz_0 + d = 0$*. Thus we have the following formula*

The distance $D$ from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

*The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane.*

# Example – Q. 56, Ex. 10.5

*Find the distance between the given parallel planes*

$$2x - 3y + z = 4, \qquad 4x - 6y + 2z = 3$$

*Solution:*

*Put* $y = z = 0$ *in the equation of the first plane to obtain the point* $(2, 0, 0)$ *on the plane.*

*Because the planes are parallel, the distance D between them is the distance from* $(2,0,0)$ *to the second plane.*

$$D = \frac{|4(2) - 6(0) + 2(0) - 3|}{\sqrt{4 + (-6)^2 + (2)^2}}$$

$$= \frac{5}{\sqrt{56}}$$

# Homework 4 – 10.5

*(a) Find distance from the point (1, -2, 4) to the plane*

$$3x + 2y + 6z = 5$$

*(b) Find the distance between the given parallel planes*

$$6z = 4y - 2x, \qquad 9z = 1 - 3x + 6y$$

# Example – Q. 56, Ex. 10.5

*Find the distance between the skew lines with parametric equations*

$$x = 1 + t, \qquad y = 1 + 6t, \qquad z = 2t \ \ and$$
$$x = 1 + 2s, \qquad y = 5 + 15s, \qquad z = -2 + 6s.$$

*Solution: From the equations of the lines, we can write their direction vectors*

$$v_1 = \langle 1, 6,2 \rangle, \qquad and \ \ v_2 = \langle 2, 15, 6 \rangle$$

*Then*

$$n = v_1 \times v_2 = \langle 6, -2, 3 \rangle$$

*is perpendicular to both line.*

*Pick any point on each of the lines,* $(1, 1, 0)$ *and* $(1,5, -2)$, *and form the vector b connecting the two points*

$$b = \langle 0, 4, -2 \rangle$$

# Example – Q. 56, Ex. 10.5

*Then, the distance between the two skew lines is the absolute value of the scalar projection of b along n, that is*

$$D = \frac{|n \cdot \ b|}{|n|}$$

$$= \frac{1}{\sqrt{36+4+9}}|0-8-6|$$

$$= \frac{14}{7}$$

$$= 2$$

# Summary – Lines & Planes

- *Equation of a line passing through a point and having direction vector (a,b,c)*

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}=t \qquad (t \to parameter)$$

- *Angle between two lines using direction ratios*

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\times\sqrt{a_2^2+b_2^2+c_2^2}}$$

- *Perpendicularity and Parallelism*

  *Two lines with direction cosines* $l_1, m_1, n_1$ *and* $l_2, m_2, n_2$ *are*

  *- Perpendicular if* $l_1l_2+m_1m_2+n_1n_2=0$

  *- Parallel if*

$$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$$

# Homework 5 – 10.5

*Let* $L_1$ *be the line through the origin and the point* $(2, 0, -1)$*. Let* $L_2$ *be the line through the points* $(1, -1, 1)$ *and* $(4,1,3)$*.*

*Find the distance between* $L_1$ *and* $L_2$*.*