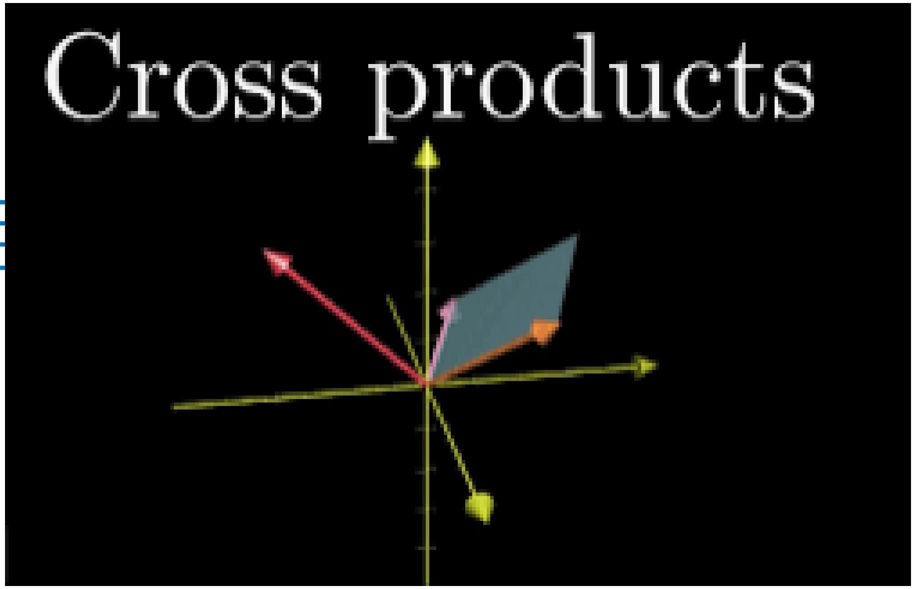
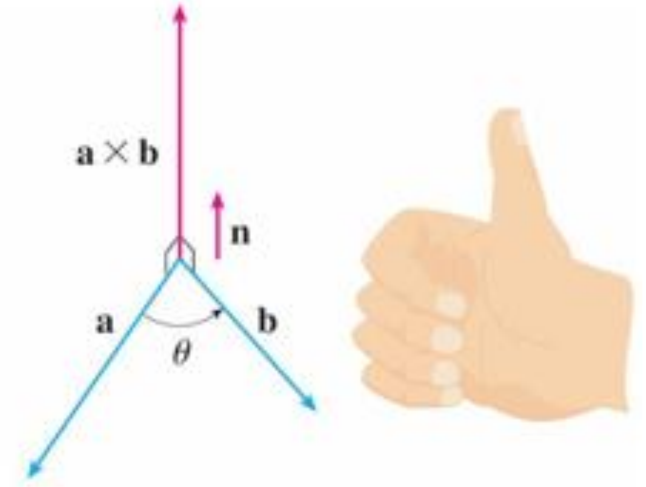
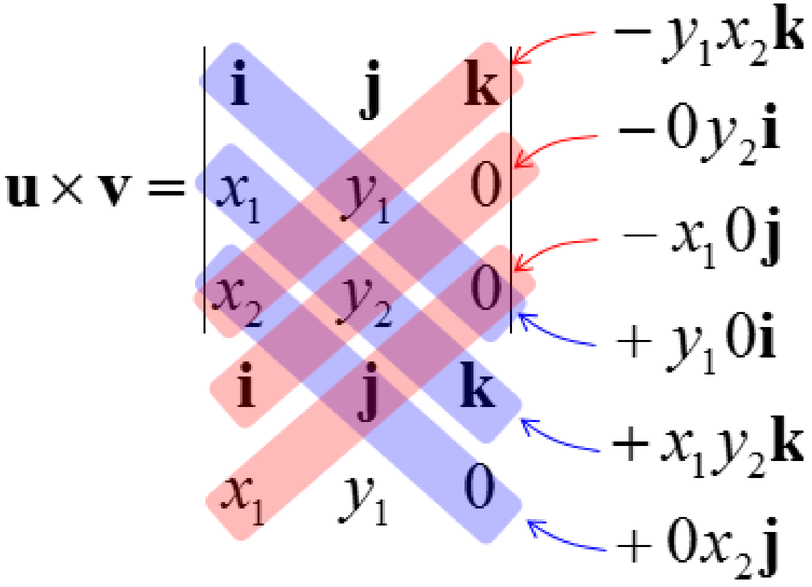
Lecture 8 - Chapter 10 – Sec. 10.4

Vector or Cross Product







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Learning Objectives

- *Evaluate the cross product of two vectors.*
- *Interpret the cross product geometrically.*
- *Define scalar triple product.*
- *Use the scalar triple product to find the volume of a parallelepiped.*

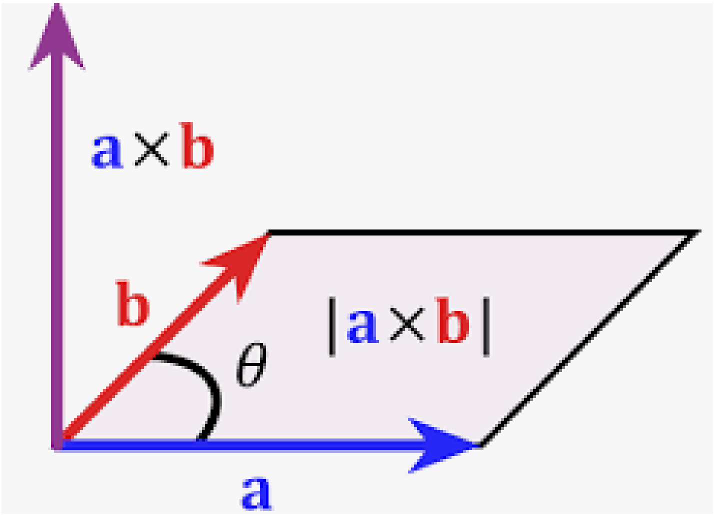
Cross Product

DEFINITION* - *Cross Product

*Given two nonzero vectors **u** and **v** in $\mathbb{R}^3$, the **cross product** $\boldsymbol{u} \times \boldsymbol{v}$ is a vector with magnitude*

$$\boldsymbol{u} \times \boldsymbol{v} = |\,\boldsymbol{u}\,||\,\boldsymbol{v}\,|\,\sin\theta\;\;\hat{n}$$

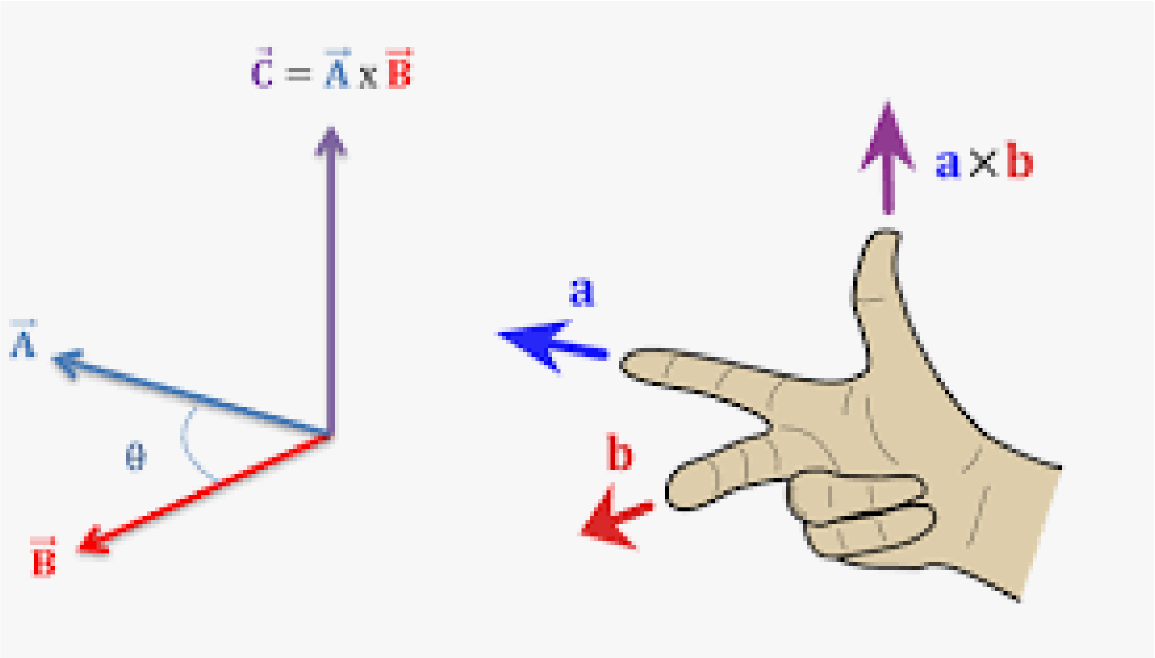
*where θ is the angle between **u** and **v** and the unit vector $\hat{n}$ gives the direction which is perpendicular to the plane containing $\boldsymbol{u}$ and $\boldsymbol{v}$.*



Cross Product

The direction of $\boldsymbol{u} \times \boldsymbol{v}$ *is given by the* ***right-hand rule****:*

- *Stretch your right hand so that the index finger of the right hand is in the direction of the first vector and the middle finger is in the direction of the second vector. Then, the thumb of the right hand indicates the direction or unit vector* $\hat{n}$.



Cross Product of Coordinate Unit Vectors

The cross product of the coordinate unit vectors is given by,

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = -(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}) = \widehat{\boldsymbol{k}},$$
$$\hat{\jmath} \times \widehat{\boldsymbol{k}} = -\left(\widehat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}\right) = \hat{\boldsymbol{\imath}}$$
$$\widehat{k} \times \hat{\boldsymbol{\imath}} = -\left(\hat{\boldsymbol{\imath}} \times \widehat{\boldsymbol{k}}\right) = \hat{\boldsymbol{\jmath}}$$

and

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} \times \widehat{\boldsymbol{k}} = \mathbf{0}$$

Cross Product in Component Form

Let

$$u = \hat{\imath}\, u_1 + \hat{\jmath}\, u_2 + \hat{k}\, u_3 \qquad \textit{and}$$
$$v = \hat{\imath}\, v_1 + \hat{\jmath}\, v_2 + \hat{k}\, v_3$$

then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$$u \times v = (u_2 v_3 - v_2 u_3)\ \hat{\imath} - (u_1 v_3 - v_1 u_3)\ \hat{\jmath} + (u_1 v_2 - v_1 u_2)\ \hat{k}$$
$$u \times v = \langle (u_2 v_3 - v_2 u_3), \quad (u_1 v_3 - v_1 u_3), \quad (u_1 v_2 - v_1 u_2) \rangle$$

Orthogonality of Vector Product

Theorem

The vector ***a*** *and* ***b*** *is orthogonal to both* ***a*** *and* ***b****.*

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$$

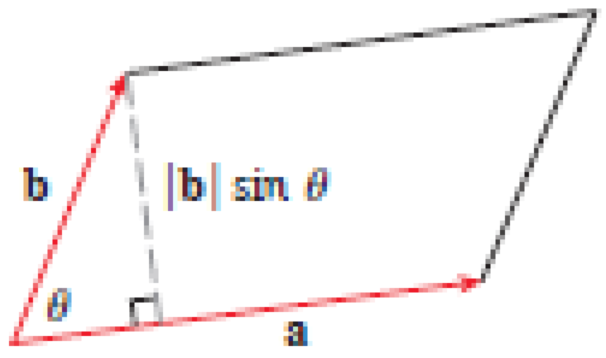
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b} = 0$$

Corollary

Corollary Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.



Properties of the Vector Product

Properties of the Cross Product If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Example – Q. 8, Ex. 10.4

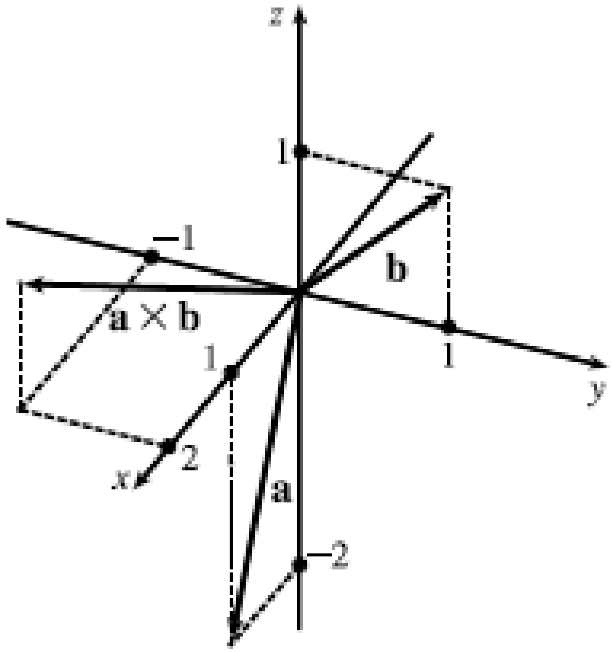
If $a = i - 2k$ and $b = j + k$, find $a \times b$. Sketch a, b, and $a \times b$ as vectors starting at the origin.

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$

$$= 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$



Example – Q. 19, Ex. 10.4

Find two unit vectors orthogonal to both

$$\langle 3,2,1\rangle \quad and\ \langle -1,1,0\rangle$$

Solution: The cross product of two vectors is orthogonal to both vectors. So we calculate

$$\langle 3,2,1\rangle \times \langle -1,1,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{k}$$

$$= -\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$$

So, two unit vectors orthogonal are

$$\pm \frac{\langle -1,-1,+5\rangle}{\sqrt{(-1)^2 + (-1)^2 + 5^2}}$$

$$\pm \frac{1}{\sqrt{27}} \langle -1,-1,+5\rangle$$

Example – Q. 27, Ex. 10.4

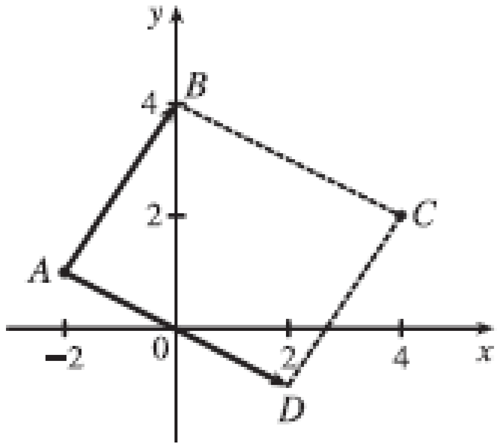
Find the area of the parallelogram with vertices

$$A(-2,1), B(0,4), C(4,2) \textit{ and } \ D(2,-1)$$

Solution: *By plotting the vertices, we can see that the parallelogram is determined by the vectors*

$$\overrightarrow{AB} = \langle 2, 3 \rangle, \overrightarrow{AD} = \langle 4, -2 \rangle$$

We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors.

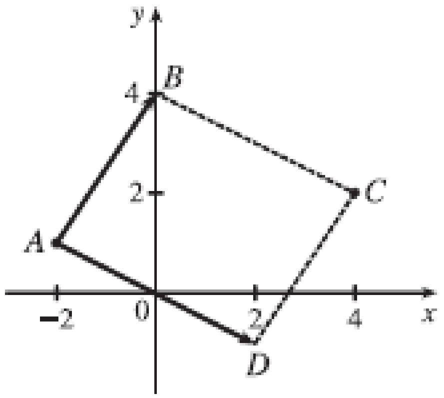


In order to compute the cross product, we consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as the three dimensional vectors,

$$\overrightarrow{AB} = \langle 2, 3, 0 \rangle, \overrightarrow{AD} = \langle 4, -2, \ 0 \rangle$$

Example – Q. 27, Ex. 10.4

Then the area of parallelogram ABCD is



$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} \right\|$$

$$= \left|(0)\,\mathbf{i} - (0)\,\mathbf{j} + (-4 - 12)\,\mathbf{k}\right|$$

$$= \left|-16\,\mathbf{k}\right| = 16$$

Example – Q. 28, Ex. 10.4

Find the vector not with the determinant, but by using the properties of cross product.

$$(\hat{j} - \hat{k}) \times (\hat{k} - \hat{\imath})$$

Solution

Using the property of the cross product

$$(a + b) \times c = a \times c + b \times c$$

we have

$$(\hat{j} - \hat{k}) \times (\hat{k} - \hat{\imath}) = (\hat{j} - \hat{k}) \times \hat{k} + (\hat{j} - \hat{k}) \times (-\hat{\imath})$$

$$= \hat{j} \times \hat{k} + \left(-\hat{k}\right) \times \hat{k} + \hat{j} \times (-\hat{\imath}) + (-\hat{k}) \times (-\hat{\imath})$$

$$= \hat{\imath} + 0 + \hat{k} + \hat{j}$$

$$= \hat{\imath} + \hat{j} + \hat{k}$$

Example – Q. 29, Ex. 10.4

(a) Find the nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of the triangle PQR.

$$P(1,0,1), \qquad Q(-2,1,3), \qquad R(4,2,5)$$

Solution: Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane.

$$\overrightarrow{PQ} = \langle -3, 1, 2 \rangle, \qquad \overrightarrow{PR} = \langle 3, 2, 4 \rangle$$

So

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(4) - (2)(2), (2)(3) - (-3)(4), (-3)(2) - (1)(3) \rangle$$

$$= \langle 0, 18, -9 \rangle$$

Therefore, $= \langle 0, 18, -9 \rangle$ (or any nonzero scalar multiple thereof, e.g., $\langle 0, 2, -1 \rangle$) is orthogonal to the plane through P, Q and R.

Example – Q. 29, Ex. 10.4

(b) The area of the triangle PQR is equal to half of the area of the parallelogram determined by the three points.
The area of the ||gram is

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \langle 0, 18, -9 \rangle$$
$$= \sqrt{0 + 324 + 81}$$
$$= \sqrt{405} = 9\sqrt{5}$$

So, the area of the triangle is

$$\frac{9}{2}\sqrt{5}$$

Triple Product

The scalar triple product (also called the mixed product, box product, or triple scalar product) is defined as the dot product of one of the vectors with the cross product of the other two.

Given three nonzero vectors a, b, and c, the product $a \cdot (b \times c)$ *is called the scalar triple product.*

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

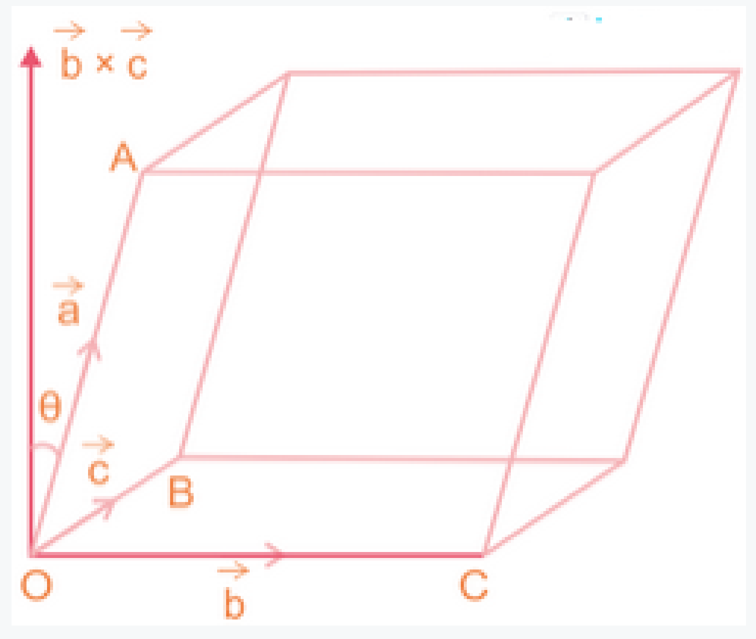
Triple Product

Geometric interpretation

Geometrically, the scalar triple product

$$a \cdot (b \times c)$$

is the (signed) volume of the parallelepiped defined by the three vectors given.



*The scalar triple product is unchanged under a circular shift of its three operands (**a**, **b**, **c**):*

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Example – Q. 36, Ex. 10.4

Find the volume of the parallelepiped determined by the vectors

$$P(3,0,1), \quad Q(1,2,5), \quad R(5,1,-1), \quad S(0,4,2)$$

Solution:

$$\mathbf{a} = \overrightarrow{PQ} = \langle -4, 2, 4 \rangle, \ \mathbf{b} = \overrightarrow{PR} = \langle 2, 1, -2 \rangle \text{ and } \mathbf{c} = \overrightarrow{PS} = \langle -3, 4, 1 \rangle.$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

$$= -4 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}$$

$$= -36 + 8 + 44 = 16$$

Therefore, the volume of the parallelepiped is 16 cubic units.

Homework 1– Q. 35, Ex. 10.4

Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS

$P(-2, 1, 0)$, $Q(2. 3, 2)$, $R(1, 4, -1)$, $S(3,6,1)$.

Homework 2 – Q. 46, Ex. 10.4

Distance from a Point to a Plane: *Let P be a point not on the plane that passes through the points Q, R, and S.*

(a) Show that the distance d from P to the plane is

$$d = \frac{|\, a \cdot (b \times c)\, |}{|\, a \times b\, |}$$

where $a = \overrightarrow{QR}$, $b = \overrightarrow{QS}$, *and* $c = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point P(2,1,4) to the plane through the points Q(1, 0, 0), R(0, 2, 0), and S(0,0,3).

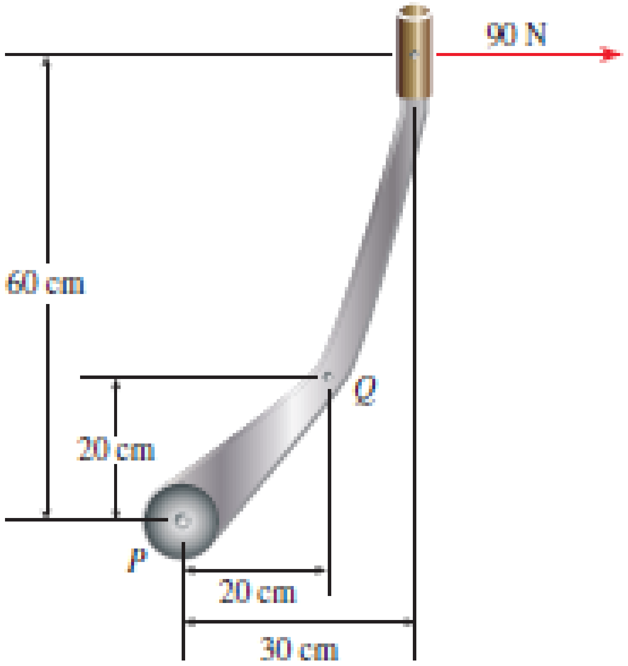
Applications - Robotics and Computer Graphics

- *The cross-product of two vectors is used to calculate the torque required to move a robotic arm. The torque vector is the cross product of the joint angle vector and the joint axis vector.*

- *Similarly, you can also use cross-products in computer graphics to calculate the surface normal vector of a 3D object. The surface normal vector is the cross-product of two vectors that lie on the surface of the object.*

Applications - Robotics and Computer Graphics

A horizontal force of 90 N is applied to the handle of a gearshift lever of a robot as shown in the figure. Find the magnitude of the torque about the pivot point P. (b) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.



Solution: Let r be the position vector from point P to the handle. Then

$$|\vec{r}| = \sqrt{30^2 + 60^2} = 10\sqrt{45}\ cm = \sqrt{45}/10\ m$$

Moving F so that both vectors start from the same point makes an angle (θ) between r and F

$$\tan\theta = 60/30 \Rightarrow \quad \theta = 63.4^o$$

Therefore, the torque τ is given by

$$|\tau| = |r||F|\sin\theta = \left(\sqrt{45}/10\right)(90)\sin 63.4^o \approx 54\ Nm$$

(b) Let r_Q be the position vector from P to Q. then

$$|\vec{r}_Q| = \sqrt{20^2 + 20^2} = 20\sqrt{2}\ cm = \sqrt{2}/5\ m$$

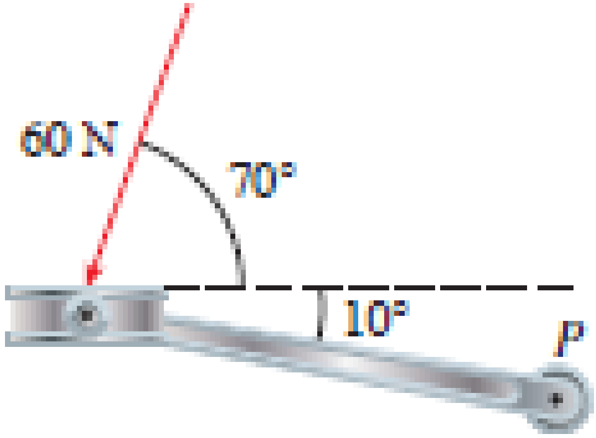
And the angle that force vector makes with $|\vec{r}_Q|$ is $\theta = 45^o$.

Therefore,

$$|\tau| = |r||F|\sin\theta = \left(\sqrt{2}/5\right)(90)\sin 45^o \approx 18\ Nm$$

Homework 3 – Q. 39, Ex. 10.4

A bicycle pedal is pushed by a foot with a 60-N force as shown in the figure. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



Homework 4** – Q. 39, Ex. 10.4

If v_1, v_2*, and* v_3 *are nonplanner vectors, let*

$$k_1 = \frac{v_2 \times v_3}{v_1 \cdot (v_2 \times v_3)}, \qquad k_2 = \frac{v_3 \times v_1}{v_1 \cdot (v_2 \times v_3)}, \qquad k_3 = \frac{v_1 \times v_2}{v_1 \cdot (v_2 \times v_3)}$$

These vectors occur in the study of crystallography.

Vectors of the form

$$n_1 v_1 + n_2 v_2 + n_3 v_3$$

where each n_i *is an integer, form a lattice for a crystal. Vectors written similarly in terms of* k_1, k_2*, and* k_3 *form the reciprocal lattice.)*

(a) Show that k_i *is perpendicular to* v_j *if* $i \neq j$*.*

(b) Show that $k_i \cdot v_i = 1$ *for* $i = 1, 2, 3$*.*

(c) Show that

$$k_1 \cdot (k_2 \times k_3) = \frac{1}{v_1 \cdot (v_2 \times v_3)}$$