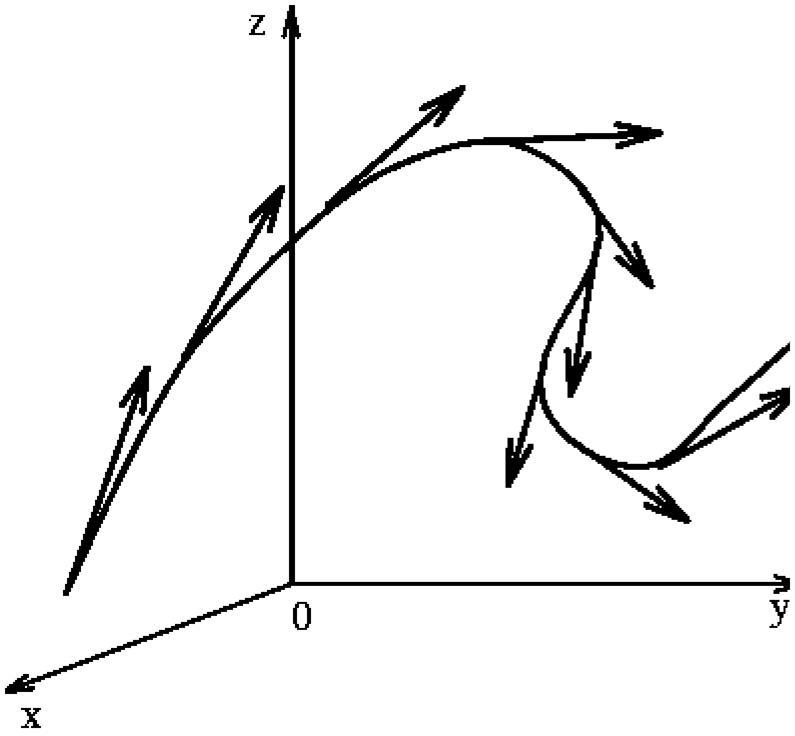
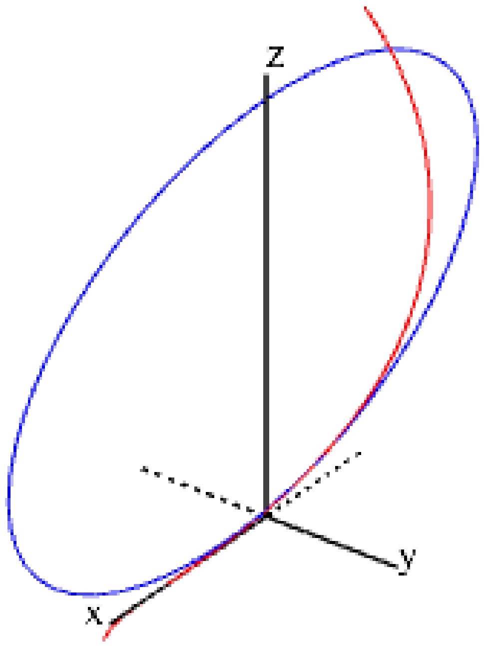
Lecture 11-II - Chapter 10 – Sec. 10.8

Arc Length & Curvature





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Learning Objectives

- *Determine the length of a particle's path in space by using the arc-length function*
- *Explain the meaning of the curvature of a curve in space and state its formula.*
- *Describe the meaning of the normal and binormal vectors of a curve in space.*

Arc Length

- *Suppose that the curve has the vector function*

$$r(t) = i\,f(t) + jg(t) + k\,h(t), \qquad a \leq t \leq b$$

We define the arc length function s by

$$s(t) = \int_a^t |r'(u)| du$$

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Thus s(t) is the length of the part of C between r(a) and r(t).

$$\frac{ds}{dt} = |r'(t)|$$

Arc Length Function

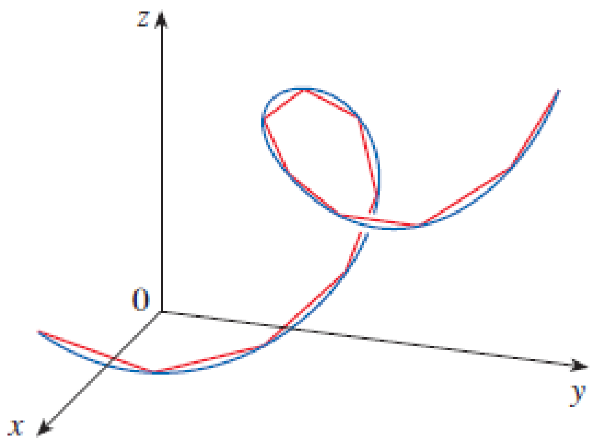
- *Suppose that the curve has the vector equation*

$$r(t) = \langle f(t), g(t), h(t) \rangle, \qquad a \le t \le b$$

or, equivalently, the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t),$$

where f, g' and h' are continuous.



- *If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is*

- *This equation can be put into the more compact form*

$$L = \int_a^b |r'(t)| dt$$

Example – Q. 7. Ex. 10.8

Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point (6, 18, 36).

Solution: The projection of the curve C onto the xy-plane is the curve

$$x^2 = 2y \text{ or } y = \frac{1}{2}x^2, \qquad z = 0.$$

Then we can choose the parameter $x = t$

$$\Rightarrow y = \frac{1}{2}t^2$$

Since C also lies on the surface $3z = xy$, we have

$$\Rightarrow z = \frac{1}{3}(t)\left(\frac{1}{2}t^2\right) = \frac{1}{6}t^3$$

Then parametric equations for C are

$$x = t, \qquad y = (1/2)\, t^2, \qquad z = (1/6)\, t^3$$

Example – Q. 7. Ex. 10.8

The vector equation is

$$r(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3 \right\rangle$$

The origin corresponds to $t = 0$ *and the point* $(6,18,36)$ *corresponds to t=6, so*

$$L = \int_0^6 |r'(t)| dt = \int_0^6 \left\| \left\langle 1, t, \frac{1}{2}t^2 \right\rangle \right\| dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\frac{1}{2}t^2\right)^2}\; dt$$

$$L = \int_0^6 \sqrt{\left(1 + \frac{1}{2}t^2\right)^2} dt = t + \frac{1}{6}t^3 \Big|_0^6 = 42$$

Example – Q. 10. Ex 10.8

Reparametrize the curve with respect to arc length measured from the point where t=0 in the direction of increasing t.

$$r(t) = i\, e^{2t} \cos 2t + 2j + k\, e^{2t} \sin 2t$$

Solution

$$r'(t) = \; 2e^{2t}(\cos 2t \; - \sin 2t)i + 2e^{2t}(\cos 2t + \sin 2t)k$$

$$\frac{ds}{dt} = |r'(t)| = 2\sqrt{2}\, e^{2t}$$

$$s = s(t) = \int_0^t |r'(u)|du = \int_0^t 2\sqrt{2}\, e^{2t} du = \sqrt{2}\,(e^{2t} - 1)$$

$$\frac{s}{\sqrt{2}} + 1 = e^{2t} \Rightarrow t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\Rightarrow$$

Example – Q. 10. Ex 10.8

Substituting, we get

$$r\big(t(s)\big) = e^{\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)} \cos\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) i$$

$$+ 2j + e^{\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)} \sin\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) k$$

Homework 1 – 10.8

Reparametrize the curve with respect to arc length measured from the point where t=0 in the direction of increasing t.

$$r(t) = i\ 2t + j\ (1 - 3t) + k\ (5 + 4t)$$

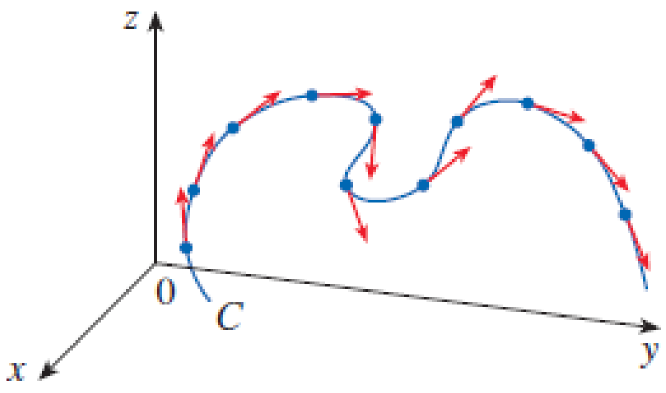
Curvature

- *The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.*
- *A parametrization $r(t)$ is called **smooth** on an interval I if $r'(t)$ is continuous and $r'(t) \neq 0$ on I.*
- *A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corner or cusp; when the tangent vector turns, it does so continuously.*

- *If C is a smooth curve defined by the vector function **r**, the unit tangent Vector $T(t)$ is given by*

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

and indicates the direction of the curve.



Curvature

Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \implies \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

$But\ dsdt = |r'(t)|$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

For the special case of a plane curve with $y = f(x)$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Example – Q. 21. Ex. 10.8

Find the curvature of

$$r(t) = \langle t, t^2, t^3 \rangle$$

at the point $(1,1,1)$

Solution: The curvature is given by $\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^2}$

The point (1,1,1) corresponds to t=1. We must calculate the derivatives first

$$r'(t) = \langle 1,2t,3t^2 \rangle \Rightarrow r'(1) = \langle 1,2,3 \rangle, \qquad |r'(1)| = \sqrt{1+4+9} = \sqrt{14}$$

$$r''(t) = \langle 0,2,6t \rangle \Rightarrow r''(1) = \langle 0,2,6 \rangle$$

$$|r'(1) \times r''(1)| = |\langle 1,2,3 \rangle \times \langle 0,2,6 \rangle| = \sqrt{76}$$

Therefore,

$$\kappa = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^2} = \frac{1}{7}\sqrt{\frac{19}{14}}$$

Homework 2 – Ex. 10.8

Find the curvature of

$$r(t) = \langle e^t \cos t, e^t \sin t, t \rangle$$

at the point $(1,0,0)$

Homework 3 – Ex. 10.8

At what point does the curve

$$y = e^x$$

has maximum curvature ? What happens to the curvature as $x \to \infty$.
(Hint: Use

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{3/2}}$$

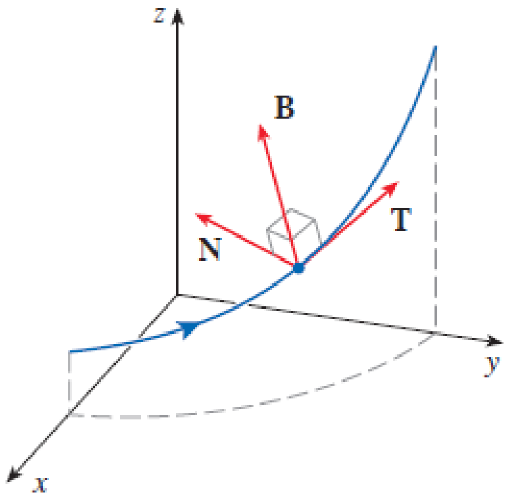
and then apply the optimization method)

The Normal and Binormal Vectors

- *At a given point on a smooth space curve r(t), there are many vectors that are orthogonal to the unit tangent vector T(t).*
- *We single out one by observing that, because* $|T(t)| = 1$ *for all t, we have*

$$T(t) \cdot T'(t) = 0$$

so $T'(t)$ *is orthogonal to* $T(t)$*.*



- *Note that, typically,* $T'(t)$ *is itself not a unit vector. But at any point where* $\kappa \neq 0$ *we can define the* ***principal unit normal vector N(t)*** *(or simply* ***unit normal****) as*

$$N(t) = T'(t)/|T'(t)|$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point.

The Normal and Binormal Vectors

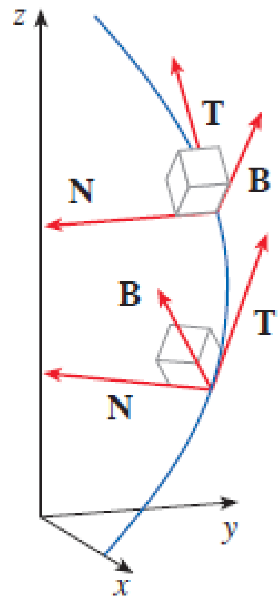
The vector

$$B(t) = T(t) \times N(t)$$

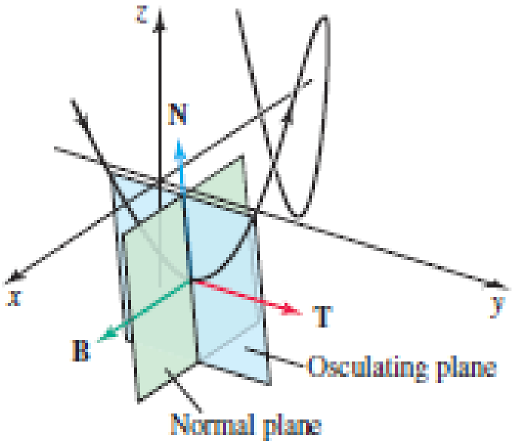
*is called the **binormal vector.***

*It is perpendicular to both **T** and **N** and is also a unit vector.*

- *In general, the vectors **T**, **N**, and **B**, starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB** frame, that moves along the curve as t varies.*

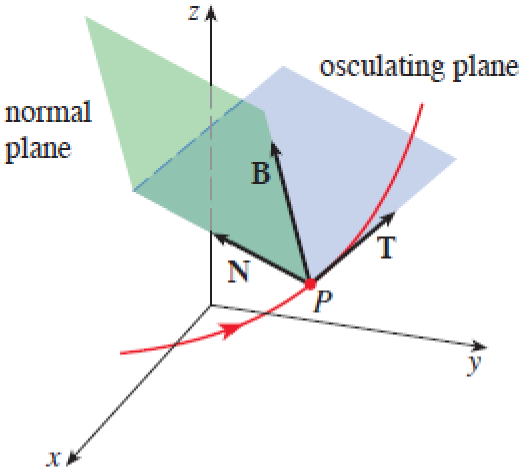


- *The plane determined by N and B at a point on a curve is called the normal plane. The plane determined by T and n is called the osculating plane of C.*
- *This **TNB** frame plays an important role in its applications to the motion of spacecraft.*



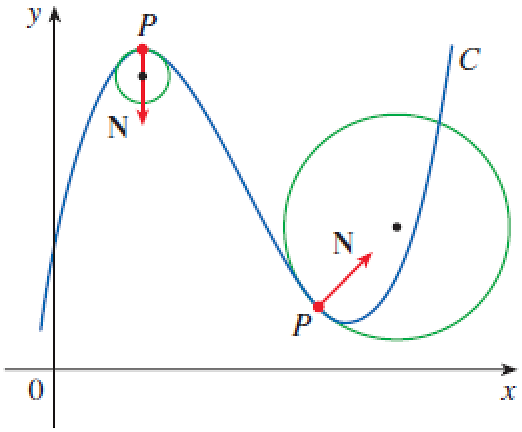
Osculating Circle

- *The **circle of curvature**, or the **osculating circle**, of C at P is the circle in the osculating plane that passes through P with radius $1/\kappa$ and centre a distance $1/\kappa$ from P along the vector **N**. The centre of the circle is called the **centre of curvature** of C at P.*



- *We can think of the circle of curvature as the circle that best describes how C behaves near P— it shares the same tangent, normal, and curvature at P.*

- *Figure illustrates two circles of curvature for a plane curve.*



Example - Q. 40. Ex. 10.8

Find the vectors T, N and B at the given point

$$\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, \qquad (1,0,0).$$

Solution: The point (1,0,0) corresponds to t=0.

$$r'(t) = \langle -\sin t, \cos t, -\tan t \rangle$$

$$|r'(t)| = |\sec t| = \sec t \qquad (\sec t > 0)$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle, \qquad T(0) = \langle 0, 1, 0 \rangle$$

$$T'(t) = \langle -\sin^2 t - \cos^2 \mathrm{t}, 2\sin t \cos t, -\cos t \rangle$$

Therefore,

$$N(0) = \frac{T'(0)}{|T'(0)|} = \frac{1}{\sqrt{2}} \langle -1,0,1 \rangle$$

$$B(0) = T(0) \times N(0) = \langle 0,1,0 \rangle \times \frac{1}{\sqrt{2}} \langle -1,0,1 \rangle = \frac{1}{\sqrt{2}} \langle -1,0,1 \rangle$$

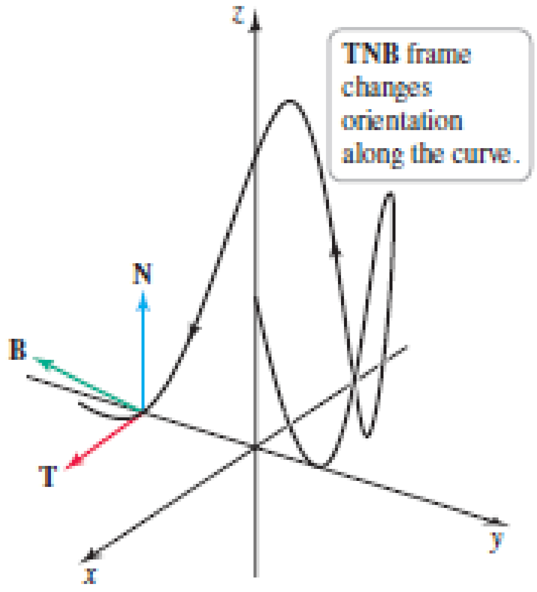
Homework 4 – Ex. 10.8

Find the equations of the normal plane and osculating plane of the curve

$$x = t, y = t^2, z = t^3; \quad at \quad (1,1,1,)$$

The Binormal Vector and Torsion

- *We have seen that the curvature function and the principal unit normal vector tell us how quickly and in what direction a curve turns.*
- *For curves in two dimensions, these quantities give a fairly complete description of motion along the curve. However, in three dimensions, a curve has more "room" in which to change its course, and another descriptive function is often useful.*



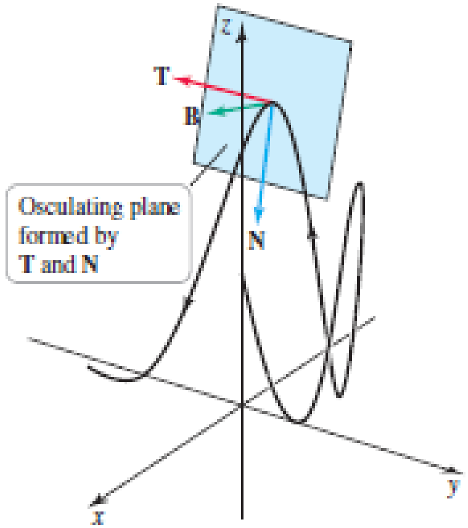
- *Figure shows a smooth parameterized curve C with its unit tangent vector **T** and its principal unit normal vector **N**.*
- *These two vectors determine a plane called the osculating plane.*

The Binormal Vector and Torsion

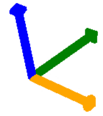
*The question we now ask is, How quickly does the curve C move out of the plane determined by **T** and **N**?*

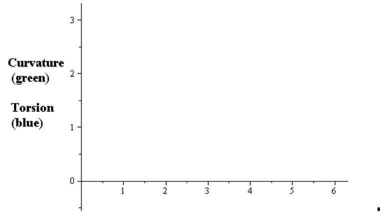
- *To answer this question, we begin by defining the unit binormal vector **B** = **T** × **N**.*
- *By the definition of the cross product, **B** is orthogonal to **T** and **N**. Because **T** and **N** are unit vectors, **B** is also a unit vector.*
- *Notice that **T**, **N**, and **B** form a right-handed coordinate system (like the xyz-coordinate system) that changes its orientation as we move along the curve.*

- *This coordinate system is often called the **TNB frame** (also called the Frenet-Serret frame)*



Torus knot with tangent vector (brown), normal vector (green) and binormal vector (blue)

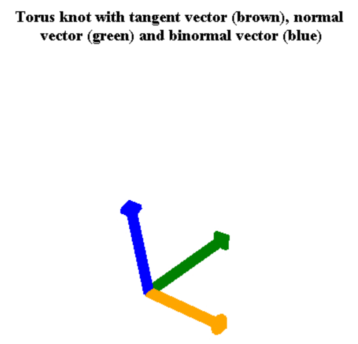


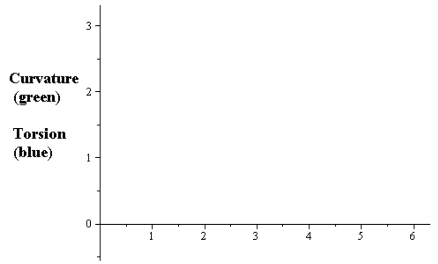


Torsion

The torsion τ(s) ***measures the turnaround of the binormal vector****. The larger the torsion is, the faster the binormal vector rotates around the axis given by the tangent vector.*

- *A plane curve with non-vanishing curvature has zero torsion at all points. Conversely, if the torsion of a regular curve with non-vanishing curvature is identically zero, then this curve belongs to a fixed plane.*
- *The curvature and the torsion of a helix are constant. Conversely, any space curve whose curvature and torsion are both constant and non-zero is a helix.*
- *The torsion is positive for a right-handed helix and is negative for a left-handed one.*





Torsion

*The rate at which the curve C twists out of the plane determined by **T** and **N** is the rate at which **B** changes as we move along C, which is dB/ds.*

DEFINITION - *Unit Binormal Vector and Torsion*

*Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors **T** and **N**, respectively. Then at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is*

$$B = T \times N$$

*and the **torsion** is*

$$\tau = -\frac{dB}{ds} \cdot N$$

Summary

Summary of the formulas for unit tangent, unit normal and binormal vectors, curvature and torsion

SUMMARY Formulas for Curves in Space

Position function: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\mathbf{v} = \mathbf{r}'$

Acceleration: $\mathbf{a} = \mathbf{v}'$

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa|\mathbf{v}|^2 = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \dfrac{d^2s}{dt^2} = \dfrac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}$

Unit binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \dfrac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

Homework 5 – Ex. 10.8

(a) Show that dB/ds *is perpendicular to* ***B****.*
(b) Show that dB/ds *is perpendicular to* ***T****.*
(c) Deduce from parts (a) and (b) that dB/ds *is parallel to* ***N****.*

Example – Ex. 10.7

Find the torsion of the helix

$$r(t) = \langle \cos t, \sin t, t \rangle.$$

Solution: We first compute the ingredients needed for the torsion.

$$r'(t) = \langle -\sin t, \cos t, 1 \rangle, \qquad |r'(t)| = \sqrt{2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle, \qquad |T'(t)| = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos t, \ -\sin t, 0 \rangle$$

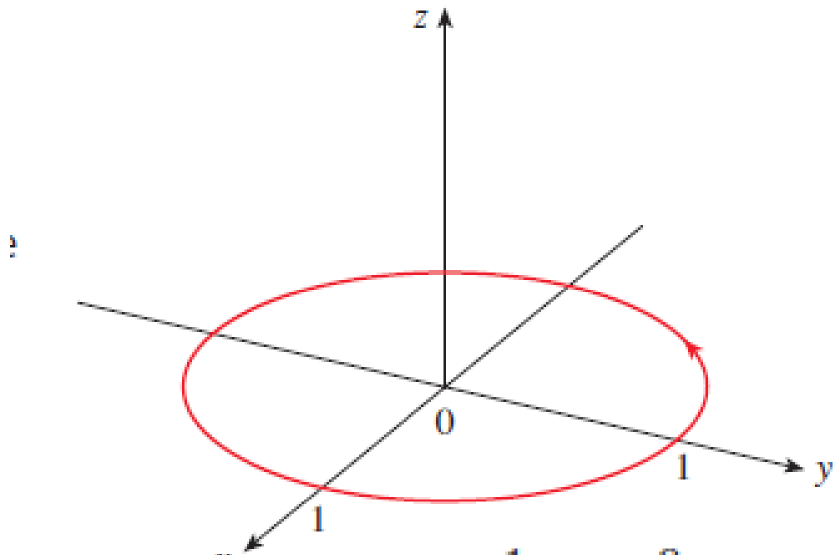
$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

Example – Ex. 10.7

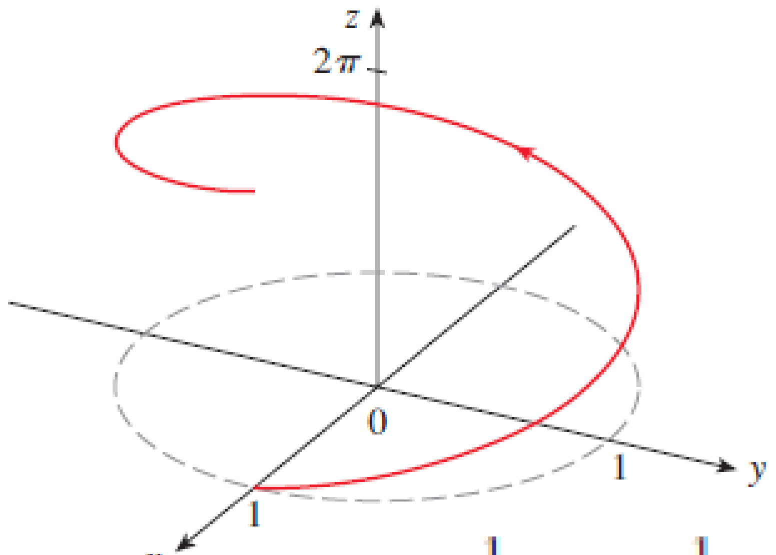
$$B(t) = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle, \qquad B'(t) = \frac{1}{\sqrt{2}}\langle \cos t, \sin t, 0\rangle$$

Using the formula

$$\tau(t) = -\frac{B'(t)\cdot N(t)}{|r'(t)|} = \frac{1}{2}\langle \cos t, \sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle = \frac{1}{2}$$



$\kappa = 1, \tau = 0$



$\kappa = \frac{1}{2}, \tau = \frac{1}{2}$

Homework 6 – Ex. 10.8

Find the torsion at the given value of t.

$$r(t) = \langle \sin t, 3t, \cos t \rangle, \qquad\qquad t = \pi/2$$