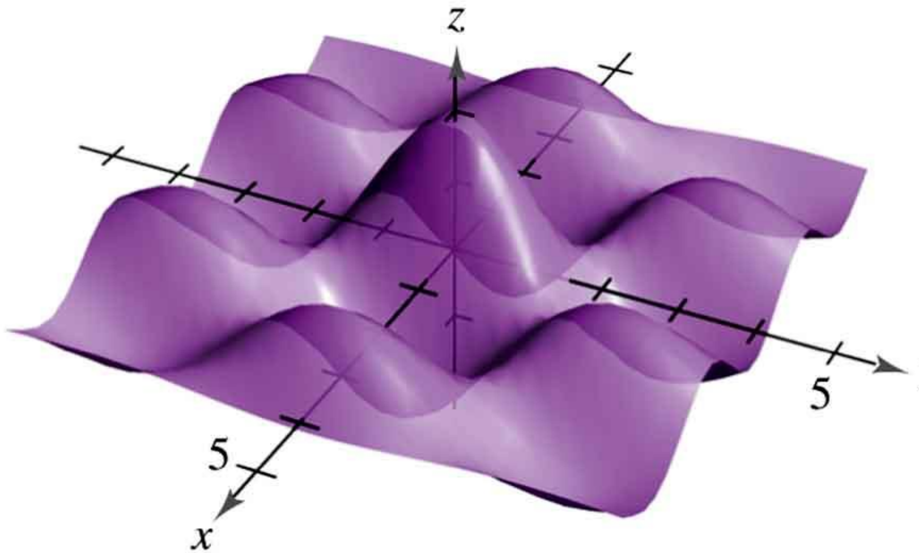
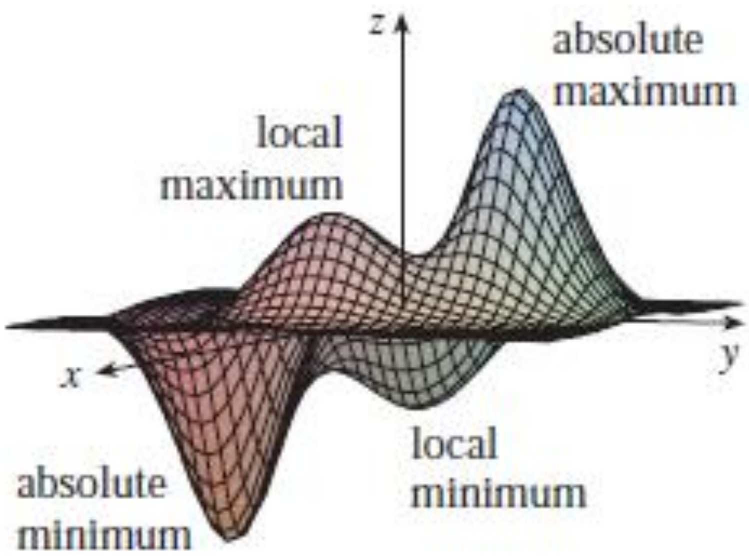
Engineering Mathematics (MA204)

Maxima and Minima





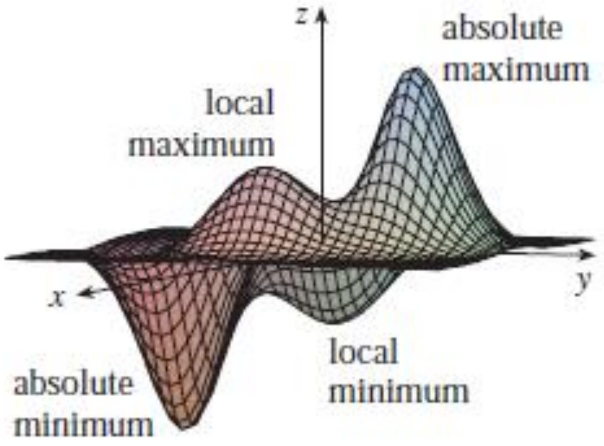
Dr M. Loan

Learning Objectives

- *Optimization*
- *Local Extrema and Derivative Theorem*
- *Critical Points and First-derivative Test and*
- *Saddle Points and Second-derivative Test*
- *Absolute Extrema*
- *Applications – Designing a Dumpster in Jahra*

Local and Absolute Extrema

Look at the hills and valleys in the graph of f(x,y).



- *There are two points (a, b) where f has a* ***local maximum****, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.*
- *The larger of these two values is the* ***absolute maximum****.*
- *Likewise, f has two* ***local minima****, where $f(a, b)$ is smaller than nearby values of $f(x, y)$.*
- *The smaller of these two values is the* ***absolute minimum****.*

The goal is to locate and classify these extreme points.

Local Extrema

DEFINITIONS Local Maximum / Minimum Values

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

THEOREM 13.13 Derivatives and Local Maximum / Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

Example

- *Suppose f has a local maximum value at (a, b). The function of one variable* $g(x) = f(x, b)$*, obtained by holding y = b fixed, also has a local maximum at (a, b).*
- *By derivative Theorem,* $g'(a) = 0$*.*
- *However,* $g'(a) = f_x(a, b)$*; therefore,* $f_x(a, b) = 0$*.*
- *Similarly, the function* $h(y) = f(a, y)$*, obtained by holding* $x = a$ *fixed, has a local maximum at (a, b), which implies that* $f_y(a, b) = h'(b) = 0$*.*
- *An analogous argument is used for the local minimum case.*

Example - Local Extrema

Show that the paraboloid

$$z = f(x, y) = x^2 + y^2 - 4x + 2y + 5$$

has a local extrema at $(2, -1)$.

Solution:

If $f(x, y)$ *has a local extrema, and* $f_x(a, b)$ *and* $f_y(a, b)$ *exist, then* $f_x(a, b) = f_y(a, b) = 0$.

$$f_x(x, y) = 2x - 4, \qquad f_x(2, -1) = 0$$
$$f_y(x, y) = 2y + 2, \qquad f_y(2, -1) = 0$$

This implies that $f_x(a, b) = f_y(a, b) = 0$. *Therefore,* z *has a local minimum at* $(2, -1)$.

Critical or Stationary Points

- *The conditions $f_x(a,b) = f_y(a,b) = 0$ do not imply that f has a local extremum at (a,b).*

- *Derivative Theorem provides candidates for local extrema. We call these candidates* ***critical points*** *or* ***stationary points****.*

- *Therefore, the procedure for locating local maximum and minimum values is to find the critical points and then determine whether these candidates correspond to genuine local maximum and minimum values.*

Critical or Stationary Points

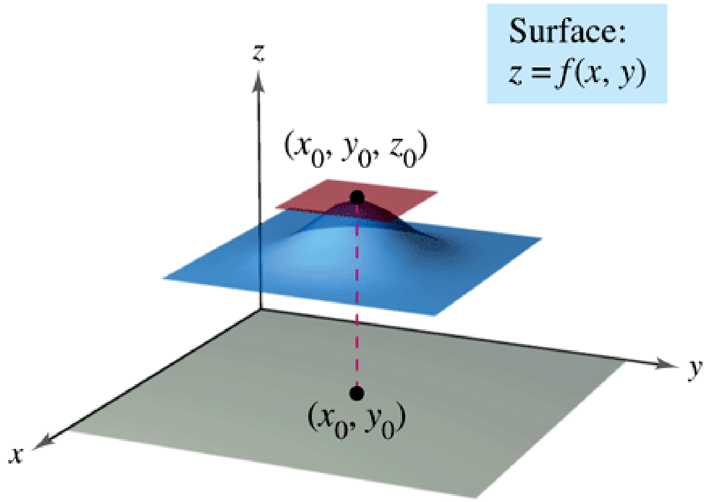
- *A point is called a* ***critical point*** *(or stationary point) of f if* f_x *and* f_y *are zero, or if one of these partial derivatives does not exist.*
- *If f has a local maximum or minimum at (a,b), then (a,b) is a critical point of f .*
- *However, not all critical points give rise to maxima or minima.*
- *At a critical point, a function could have a local maximum or a local minimum or neither.*

First-Derivative Test for Local Extrema

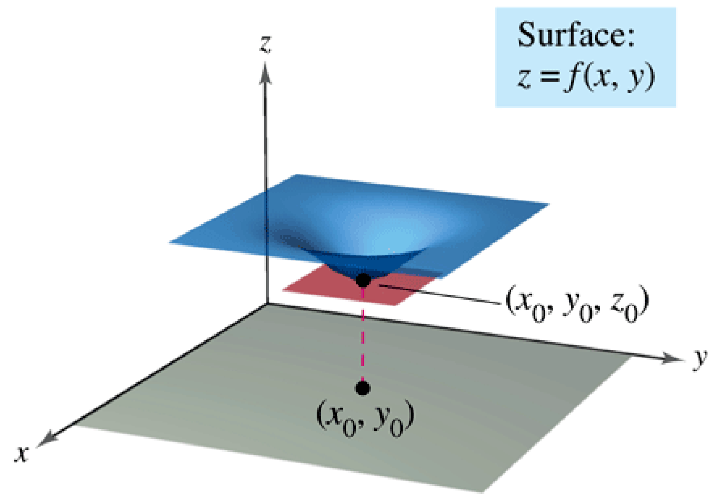
DEFINITION Critical Point

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. one (or both) of f_x or f_y does not exist at (a, b).



Relative maximum



Relative minimum

Example

Find the critical points and the local extrema for the following functions:

(a) $f(x, y) = -x^2 - y^2 + 6x + 8y - 20$

(b) $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

Solution

(a) $$f(x,y) = -x^2 - y^2 + 6x + 8y - 20$$

Step 1: Find the partial derivative

$$f_x(x,y) = -2x + 6$$
$$f_y(x,y) = -2y + 8$$

Step 2: Find the critical points (by setting the pds equal to 0)

$$-2x + 6 = 0 \Rightarrow x = 3$$
$$-2y + 8 = 0 \Rightarrow y = 4$$

Therefore (3,4) is a critical point.

Step 3: For f(3,4) to be a local minimum or a local maximum, check $f(x,y) \geq$ *or* $\leq$ $f(a,b)$.

$$f(x,y) = -(x-3)^2 - (y-4)^2 + 5$$

Solution

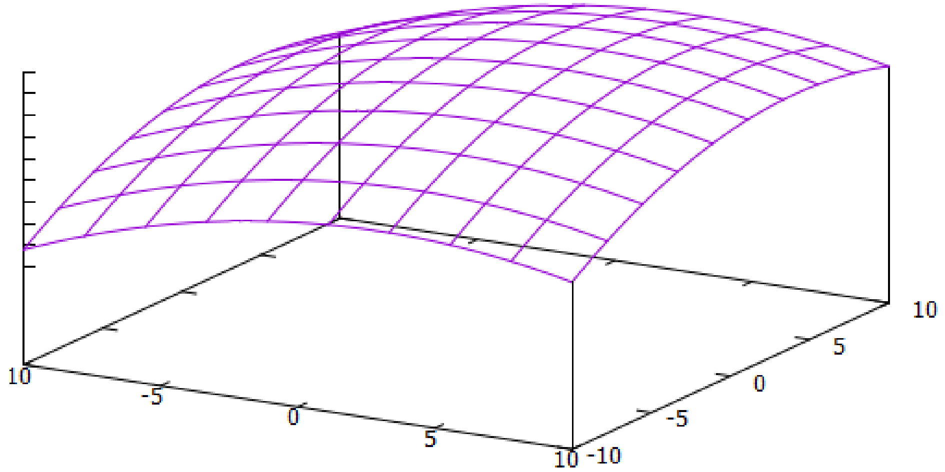
$$f(x,y) = -(x-3)^2 - (y-4)^2 + 5$$

$$f(3{,}4) = 5, \qquad f(3{,}3) = +4, \qquad f(4{,}4) = +4$$

$$f(x,y) \le f(3{,}4)$$

Therefore, (3,4) is a local maximum point.

The maximum value of the function at the critical point is 5.



Solution (Contd.)

(b) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Step 1: Find the partial derivative

$$f_x(x,y) = 2x - 2$$
$$f_y(x,y) = 2y - 6$$

Step 2: Find the critical points (by setting the pds equal to 0)

$$2x - 2 = 0 \Rightarrow x = 1$$
$$2y - 6 = 0 \Rightarrow y = 3$$

Therefore (1,3) is a critical point.

Step 3: For f(1,3) to be a local minimum or a local maximum, check $f(x,y) \geq$ *or* $\leq$ $f(a,b)$.

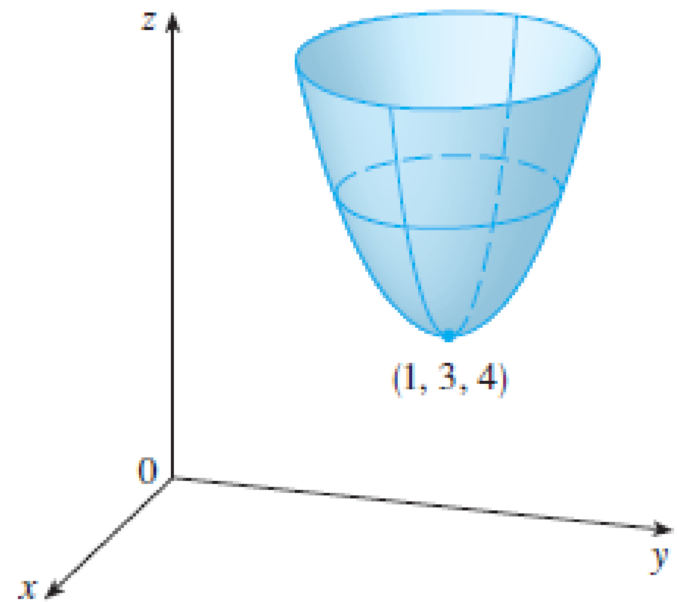
$$f(x,y) = (x-1)^2 + (y-3)^2 + 4$$

Solution

Since $(x-1)^2 \geq 0$ *and* $(y-3)^2 \geq 0$ *, we have* $f(x,y) \geq 4$ *for all values of* x *and* y*.*

Therefore $f(1,3)$ *is a local minimum.*

This can be confirmed geometrically from the graph of which is the elliptic paraboloid with vertex $(1,3,4)$.



Example – No Extrema

Find the extreme values of $f(x, y) = y^2 - x^2$.

Solution:

Since $f_x = -2x$ *and* $f_y = 2y$, *the only critical point is* $(0,0)$.

Is $(0,0)$ *a local extrema then???*

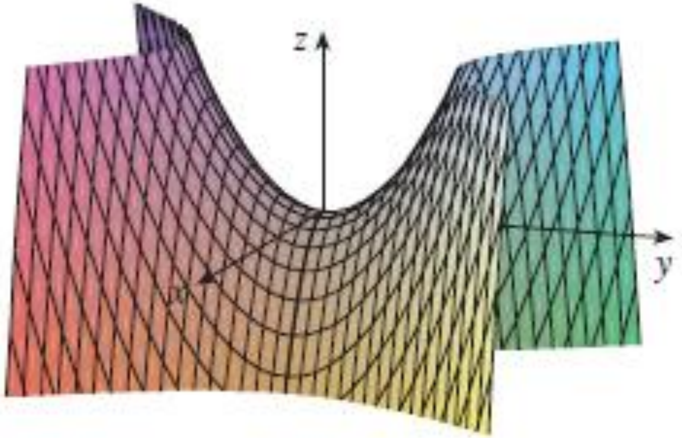
Notice that for points on the x-axis we have $y = 0$, *so* $f(x, y) = -x^2 < 0$ *(if* $x \neq 0$*).*

However, for points on the y-axis we have $x = 0$, *so* $f(x, y) = y^2 > 0$ *(if* $y \neq 0$*).*

Example – No Extrema

Thus every disk with centre contains points where $f(x, y)$ *takes positive values as well as points where* $f(x, y)$ *takes negative values.*

Therefore $f(0,0) = 0$ *can't be an extreme value for f , so f has no extreme value.*



Note: *You can see that* $f(0,0)$ *is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis. Near the origin the graph has the shape of a saddle and so is called a* ***saddle point*** *of f.*

Your Turn

Find the critical points of

$$f(x,y) = xy(x-2)(y+3)$$

Second-Derivative Test

- *We need to be able to determine whether or not a function has an extreme value at a critical point.*
- *Enter the Second Derivative Test for functions*

THEOREM **Second Derivative Test**
Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

Example

Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

Solution: Step 1- Locate the critical points

$$f_x = 4x^3 - 4y, \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain

$$x^3 = y, \qquad and\ y^3 = x$$

To solve these equations we substitute $y = x^3$ *from the first equation into the second to obtain*

$$x^9 - x = 0 \Rightarrow x(x^2-1)(x^2+1)(x^4+1) = 0$$

so there are three real roots: $0,1,-1$. *The three critical points are* ,$(0,0$, $(1,1)$, *and* $(-1,-1)$.

Solution

Step 2- Find the second derivatives

$$f_{xx} = 12x^2, \qquad f_{yy} = 12y^2, \qquad f_{xy} = -4$$

Step 3 – Calculate $D(x, y)$

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

Step 4 – Analyse $D(x, y)$ at the critical points.

$$D(0,0) = -16 < 0, \qquad (0,0)\ a\ saddle\ point$$

$$D(1,1) = 128 > 0, \qquad and \quad f_{xx}(1,1) = 12 > 0$$

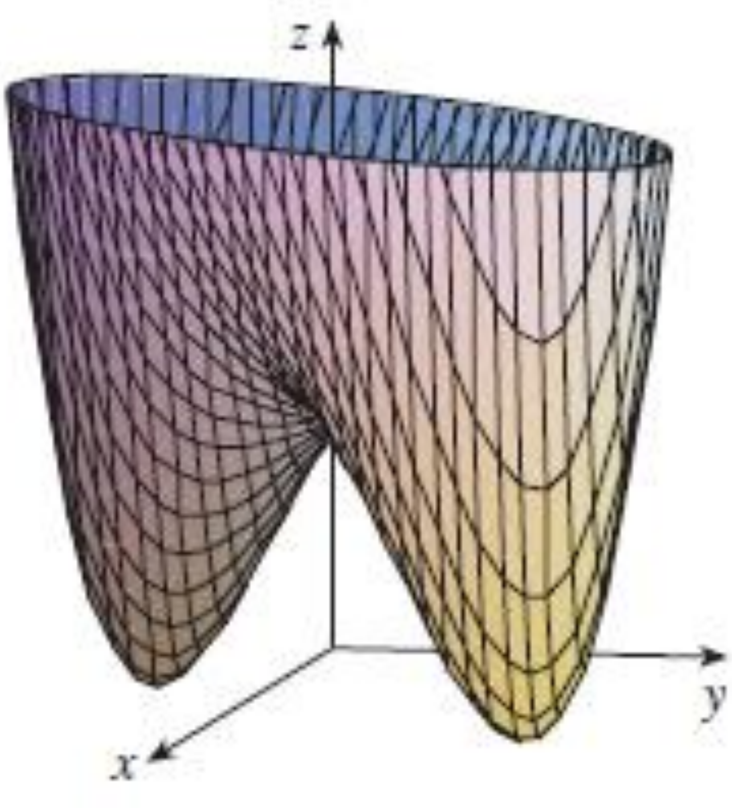
So, $f(1,1) = -1$ is a local minimum.

$$D(-1,-1) = 128 > 0, \qquad and \quad f_{xx}(-1,-1) = 12 > 0$$

So, $f(-1,-1) = -1$ is a local minimum

Solution (Contd.)

The graph of $f(x, y) = x^4 + y^4 - 4xy + 1$



Example

Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^2 - 4xy + y^3 - 4y$$

Solution: Step 1- Locate the critical points

$$f_x = 2x - 4y, \qquad f_y = 4x + 3y^2 - 4$$

Setting these partial derivatives equal to 0, we obtain

$$x = 2y, \qquad \text{and } 4x + 3y^2 - 4 = 0$$

To solve these equations we substitute $x = 2y$ *from the first equation into the second to obtain*

$$3y^2 + 8y - 4 = 0 \Rightarrow y = 2, 2/3$$

The two critical points are, $(4,2)$, *and* $(4/3, 2/3)$.

Solution

Step 2- Find the second derivatives

$$f_{xx} = 2, \qquad f_{yy} = 6y, \qquad f_{xy} = -4$$

Step 3 – Calculate $D(x,y)$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

Step 4 – Analyse $D(x,y)$ at the critical points.

$$D(4,2) = 8 > 0, \qquad and \quad f_{xx}(4,2) = 2 > 0$$

So, $f(4,2) = 0$ *is a local minimum.*

$$D(4/3, 2/3) = -8 < 0$$

So, $f(4/3,\ 2/3)$ *is a saddle point*.

Pair/Share Activity

Find the critical points and determine the local extrema or a saddle point for the following function.

$$f(x, y) = -x^2 - y^2 + 6x + 8y - 20$$

Homework

Find the critical points of f(x,y) and determine whether f(x,y) at critical point(s) is a local maximum, a local minimum or a saddle point.

(a) $f(x, y) = x^2 + y^2 + 2x - 6y - 14$

(b) $f(x, y) = xy + 2x - 3y - 2$

(c) $f(x, y) = 2x^2 - xy + y^2 - x - 5y + 8$

(d) $f(x, y) = 2x^4 + y^2 - 12xy$

Absolute Maxima and Minima

EXTREME VALUE THEOREM FOR FUNCTIONS OF TWO VARIABLES If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem EVT, we note that, if f has an extreme value at (x_1, y_1), then (x_1, y_1), is either a critical point of or a boundary point of D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example – Absolute Extrema

Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 - 2xy + 2y$$

on the rectangle $D = \{(x, y) | \ 0 \leq x \leq 3, \ 0 \leq y \leq 2\}$

Solution:

Since is a polynomial, it is continuous on the closed, bounded rectangle D , so there is both an absolute maximum and an absolute minimum.

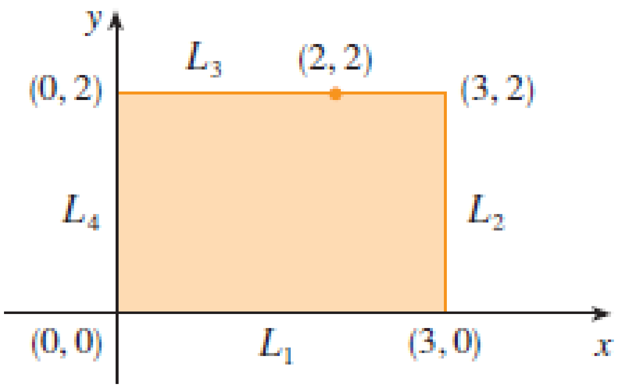
Step 1: Find the critical points.

$$f_x = 2x - 2y, \qquad f_y = -2x + 2$$

The only critical point is $(1,1)$ *and the value of the function f(1,1) = 1.*

Example – Absolute Extrema

Step 2: Look for the values at the boundary D which consists of four segments L_1, L_2, L_3, *and* L_4.



On L_1 *we have* $y = 0$, *and*

$$f(x, 0) = x^2, \qquad 0 \le x \le 3$$

This is an increasing function of x, *so its maximum value is* $f(3,0) = 9$ *and minimum value* $f(0,0) = 0$.

On L_2 *we have* $x = 3$, *and*

$$f(3, y) = 9 - 4y, \qquad 0 \le y \le 2$$

This is an decreasing function of y, *so its maximum value is* $f(3,0) = 9$ *and minimum value* $f(3,2) = 1$

Example – Absolute Extrema

On L_3 *we have* $y = 2$, *and*

$$f(x,2) = x^2 - 4x + 4, \qquad 0 \le x \le 3$$

The maximum value is $f(0,2) = 4$ *and minimum value* $f(2,2) = 0$

On L_4 *we have* $x = 0$, *and*

$$f(0,y) = 2y, \qquad 0 \le y \le 2$$

The maximum value is $f(0,2) = 4$ *and minimum value* $f(0,0) = 0$.

Thus on the boundary, max. value of f is 9 and minimum 0.

Step 4: Compare with value of f at critical point:

Absolute max of f on D: $f(3,0) = 9$,

Absolute min. of f on D: $f(0,0) = f(2,2) = 0$.

Example – Absolute Extrema

On L_3 we have $y = 2$, and

$$f(x, 2) = x^2 - 4x + 4, \qquad 0 \leq x \leq 3$$

The maximum value is $f(0,2) = 4$ and minimum value $f(2,2) = 0$

On L_4 we have $x = 0$, and

$$f(0, y) = 2y, \qquad 0 \leq y \leq 2$$

The maximum value is $f(0,2) = 4$ and minimum value $f(0,0) = 0$.

Thus on the boundary, max. value of f is 8 and minimum 0.

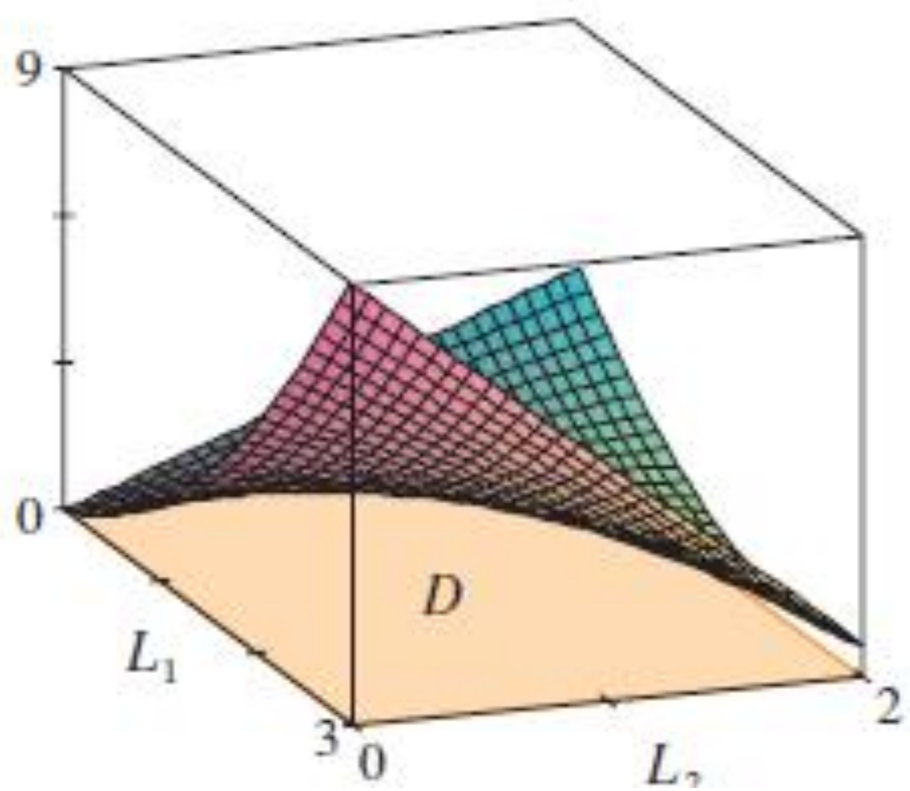
Step 4: Compare with value of f at critical point:

Absolute max of f on D: $f(3,0) = 9$,

Absolute min. of f on D: $f(0,0) = f(2,2) = 0$.

Example – Absolute Extrema

Graph of the function



Homework

Find the absolute maximum and minimum values of the following functions on the given set D.

(a)

$$f(x, y) = 4 + 2x^2 + y^2,$$
$$D = \{(x, y) | -1 \le x \le 1, -1 \le y \le 1\}$$

(b)

$$f(x, y) = 6 - x^2 - 4y^2,$$
$$D = \{(x, y) | -2 \le x \le 2, -1 \le y \le 1\}$$

Homework

Find the points on the cone

$$z^2 = x^2 + y^2$$

that are closest to the point $(4,2,0)$.

Solution: *Let d be the distance from the point (4,2,0) to any point* (x, y, z) *on the cone, so*

$$d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

where $z^2 = x^2 + y^2$*, and we minimize*

$$d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x, y)$$

Then

$$f_x = 4x - 8, \qquad f_y = 4y - 4$$

The critical point is (2,1).

Homework

An absolute minimum exists (since there is a minimum distance from cone to the point) which must occur at a critical point, so the points on the closest to (4,2,0) are

$$(2,1,\pm\sqrt{5})$$

Homework

Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

Solution: The volume of the box is

$$V = xyz$$

Since one vertex is in the plane $x + 2y + 3z = 6$

$$\Rightarrow z = \frac{1}{3}(6 - x - 2y),$$

The volume is given by

$$V(x, y) = \frac{1}{3}(6xy - x^2y - 2xy^2)$$

Homework

To optimize V, we have

$$V_x = \frac{1}{3}(6 - 2x - 2y), \qquad V_y = \frac{1}{3}(6 - x - 4y)$$

The critical point is $(2,1)$, which geometrically must give a maximum.

Thus the volume of the largest such box is

$$V = 2(1)\left(\frac{2}{3}\right) = 4/3$$

Homework

Find the dimensions of the rectangular box with largest volume if the total surface area is given as $64\ cm^2$.

Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.

Example

A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of $10\ units/m^2$ *per day, the north and south walls at a rate of* $8\ units/m^2$ *per day, the floor at a rate of* $1\ units/m^2$ *per day, and the roof at a rate of* $5\ units/m^2$ *per day. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly* $4000\ m^3$ *.*

(a) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)

(b) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed

Example

Let x be the length of the north and south walls, y the length of the east and west wall, and z the height of the building. The heat loss is given by

$$h(x,y,z) = 10(2yz) + 8(2xz) + 1(xy) + 5(xy)$$

$$h(x,y,z) = 6xy + 16xz + 20yz.$$

The volume is $4000\ m^3 \Rightarrow xyz = 4000 \Rightarrow z = 4000/xy$.

Therefore,

$$h(x,y) = 6xy + 80{,}000/x + 64{,}000/y$$

(a) Since

$$z = 4000xy \geq 4, \qquad xy \leq 1000 \Rightarrow y \leq 1000x$$

Also, $x \geq 30$ *and* $y \geq 30$, *so the domain of h is*

$$D = \{(x,y) | x \geq 30, \qquad 30 \leq y \leq 1000/x\}$$

Example

$$h(x,y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1}$$

$$\Rightarrow h_x = 6y - 80{,}000x^2, \qquad h_y = 6x - 64{,}000y^{-2}$$

The critical point is

$$x = 10\sqrt[3]{\frac{50}{3}}, \qquad y = \frac{80}{\sqrt[3]{60}} \Rightarrow (25.54, 20.43)$$

which is not in D.

Next we check the boundary of D.

On $L_1 \colon y = 30,\ h(x,30) = 180x + 80{,}000/x + 64{,}000/3$

$$30 \le x \le 100/3$$

Since $h'(x,30) = 180 - 80{,}000/x^2 > 0$ *for* $30 \le x \le 100/3$, $h(30,x)$ *is an increasing function with minimum* $h(30,30) = 10{,}200$ *and maximum* $h(100/3,30) \approx 10{,}587$.

Example

On L_2: $y = 1000/x, h(x, 1000/x) = 6000 + 64x + 80{,}000/x, \quad 30 \le x \le 100/3.$

Since

$h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ *for* $30 \le x \le 100/3$

$h(x, 1000/x)$ *is a decreasing function with minimum* $h(100/3, 30) \approx 19{,}533$ *and maximum* $h(30, 100/3) \approx 10{,}587$.

On L_3: $x = 30,\ h(30, y) = 180y + 64{,}000/y + 8000/3$

$$30 \le y \le 100/3$$

$h'(30, y) = 180 - 64{,}000/y^2 > 0$ *for* $30 \le y \le 100/3$

$h(30, y)$ *is an increasing function of y with minimum* $h(30,30) = 10{,}200$ *and maximum* $h(30,1003) \approx 10{,}587$.

Homework

(a) If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

(b) Find an equation of the plane that passes through the point (1, 2, 3) and cuts off the smallest volume in the first octant