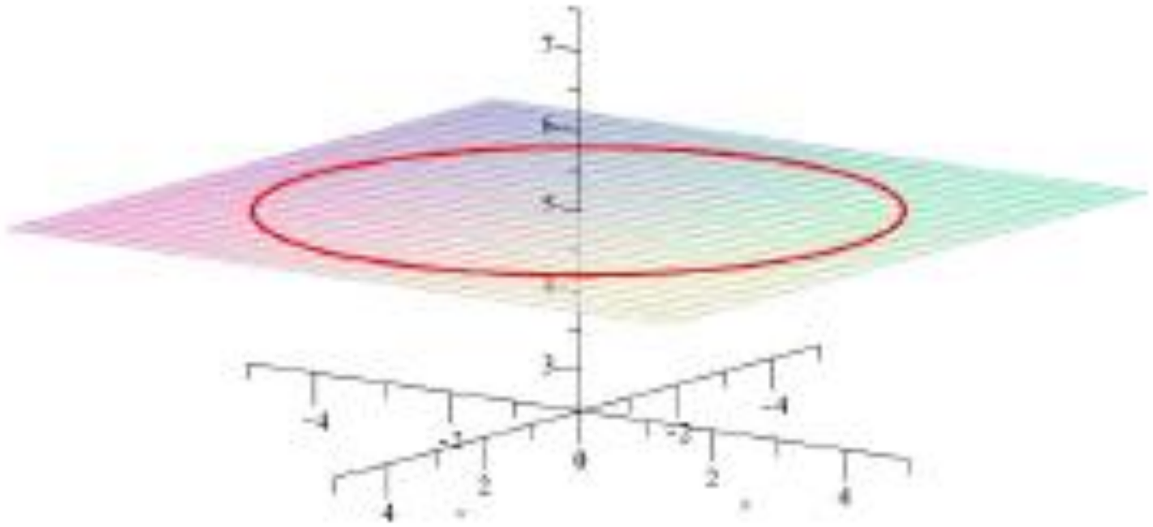
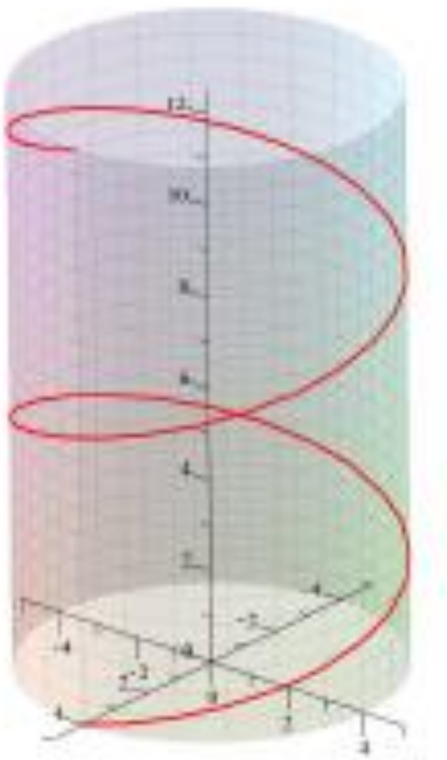
Lecture 11-I - Chapter 10 – Sec. 10.7

Vector-Valued Functions





Dr M. Loan
Department of Physics, SCUPI

Learning Objectives

- *Write the general equation of a vector-valued function in component form and unit-vector form.*
- *Recognise parametric equations for a space curve*
- *Describe the shape of a helix and write its equation.*
- *Define the limit of a vector-valued function*

Vector Functions and Space Curves

- *The functions that you have been using so far have been real-valued functions.*

- *We used vectors to represent static quantities, such as the constant force applied to the end of a wrench or the constant velocity of a boat in a current.*

- *We now study functions whose values are vectors because such functions are needed to describe the curves and surfaces in space.*

- *The calculus of vector functions can be used to solve a wealth of practical problems involving the motion of objects in space.*

Vector Functions and Space Curves

- *Imagine a projectile moving along a path in three-dimensional space. it could be an electron or a comet, a soccer ball or a rocket.*
- *If you take a snapshot of the object, its position is described by a static position vector* $\vec{r} = \langle x, y, z \rangle$.
- *However, if you want to describe the full trajectory of the object as it unfolds in time, you must represent the object's position with a vector-valued function such as*

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

  whose components change in time.
- *We aim to describe continuous motion using vector-valued functions.*

Vector-Valued Functions

A function of the form

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

may be viewed in two ways.

- *It is a set of three parametric equations that describe a curve in space.*
- *It is also a* ***vector-valued function****, which means that the three dependent variables* (x, y, z) *are the components of* $\vec{r}$*, and each component varies with respect to a single independent variable t (that often represents time).*

Here is the connection between these perspectives:

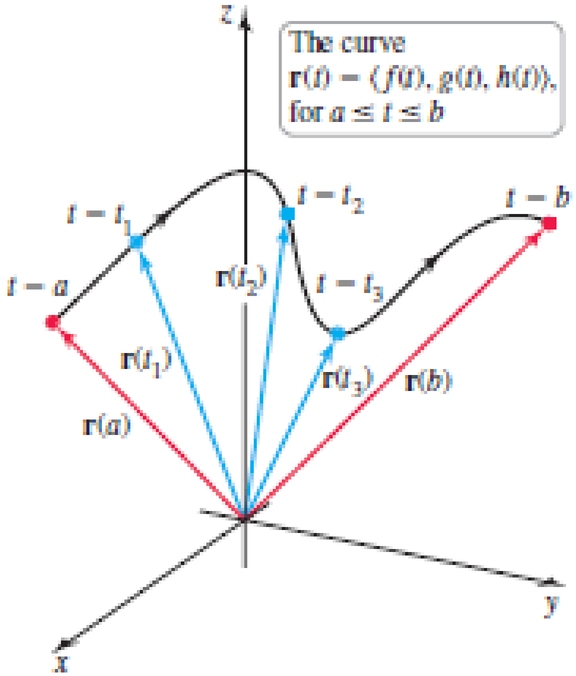
- *As t varies, a point* $(x(t), y(t), z(t))$ *on a parametric curve is also the head of the position vector* $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.
- *In other words, a vector-valued function is a set of parametric equations written in vector form. It is useful to keep both of these interpretations in mind as you work with vector-valued functions.*

Curves in Space

If $f(t), g(t)$, *and* $h(t)$ *are the components of the vector* $\vec{r}(t)$, *then f , g, and h are real-valued functions called the* ***component functions*** *of* ***r,*** *and we can write*

$$\vec{r}(t) = \langle\, f(t), g(t), h(t)\rangle = f(t)\, i + g(t) j + h(t) k$$

where f, g, and h are defined on an interval $a \leq t \leq b$. *The* ***domain*** *of* ***r*** *is the largest set of values of t on which all of f, g, and h are defined.*

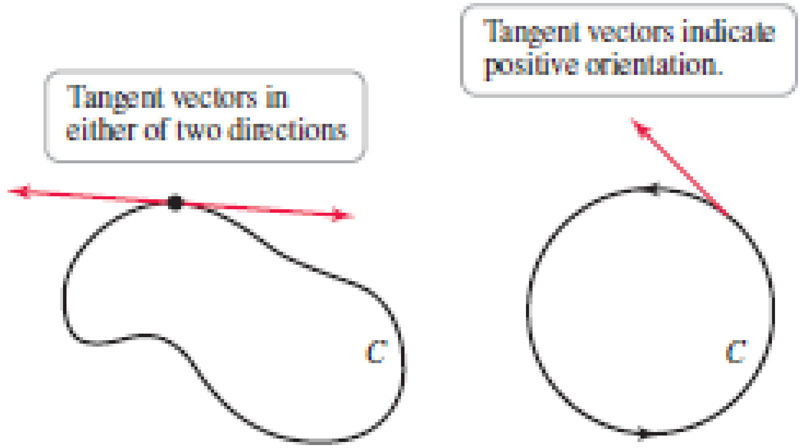


Orientation of Curves

- *If a smooth curve C is viewed only as a set of points, then at any point of C, it is possible to draw tangent vectors in two directions.*

- *On the other hand, a parameterized curve described by the function $r(t)$, where $a \leq t \leq b$, has a natural direction, or* ***orientation****.*

- *The positive orientation is the direction in which the curve is generated as the parameter increases from a to b.*



Unparameterized curve
(a)

Parameterized curve
(b)

Limits and Continuity for Vector-valued Functions

DEFINITION

*A vector-valued function **r** approaches the limit **L** as t approaches a, written*

$$\lim_{t \to a} \vec{r}(t) = L$$

provided

$$\lim_{t \to a} |\vec{r}(t) - L| = 0.$$

Suppose $\lim_{t \to a} f(t) = L_1$, $\lim_{t \to a} g(t) = L_2$, *and* $\lim_{t \to a} h(t) = L_3$, *then*

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle = \langle L_1, L_2, L_3 \rangle$$

Example – Q. 5. Ex. 10.7

Evaluate the following limits.

$$\lim_{t\to\infty}\left\langle te^{-t}, \quad \frac{t^3+t}{2t^3-1}, \quad t\sin 1/t\right\rangle$$

Solution:

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0$$

$$\lim_{t\to\infty}\frac{t^3+t}{2t^3-1} = \lim_{t\to\infty}\frac{1+(1/t^2)}{2-(1/t^3)} = \frac{1+0}{2-0} = \frac{1}{2},$$ *By l' Hospital's Rule*

$$\lim_{t\to\infty} t\sin\frac{1}{t} = \lim_{t\to\infty}\frac{\sin(1/t)}{1/t} = \lim_{t\to\infty}\frac{\cos(1/t)(-1/t^2)}{-1/t^2}$$

$$= \lim_{t\to\infty}\cos\frac{1}{t} = \cos 0 = 1$$ *By l' Hospital's Rule*

Thus, $$\lim_{t\to\infty}\left\langle te^{-t}, \frac{t^3+t}{2t^3-1}, t\sin\frac{1}{t}\right\rangle = \langle 0, \tfrac{1}{2}, 1\rangle.$$

Example - Q. 12. Roller coaster curve

Graph the curve $\vec{r}(t) = i\cos t + j\sin(t) + 0.4k\sin(2t) \qquad 0 \le t \le 2\pi$.

Solution: We begin by setting z = 0 to determine the projection of the curve in the xy-plane. We begin by setting z = 0 to determine the projection of the curve in the xy-plane. Without the z-component, the resulting function

$$r(t) = i\cos\theta + j\sin\theta$$

describes a circle of radius 1 in the xy-plane.

The z-component of the function varies between -0.4 and 0.4 with a period of π *units.*

Therefore, on the interval $0 \le t \le 2\pi$*, the z-coordinates of points on the curve oscillate twice between -0.4 and 0.4, while the x- and y-coordinates describe a circle.*

The result is a curve that circles the z-axis once in the counterclockwise direction with two peaks and two valleys.

Example - Roller coaster curve

Writing the vector function in parametric form, we have

$$x = \cos t, \qquad y = \sin t, \qquad z = 0.4 \sin 2t$$

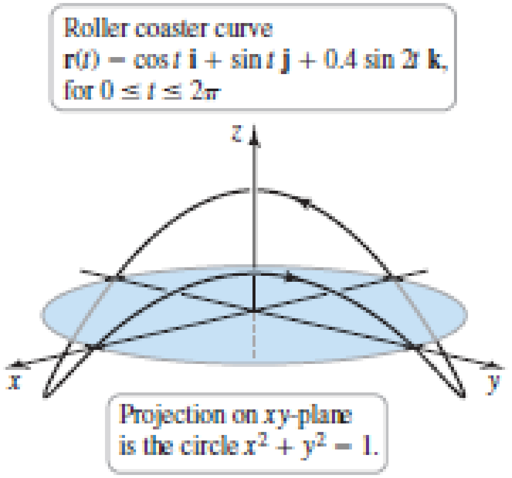
Noting that $x^2 + y^2 = \cos^2 t + \sin^2 = 1$*, we conclude that the curve lies on the cylinder* $x^2 + y^2 = 1$.

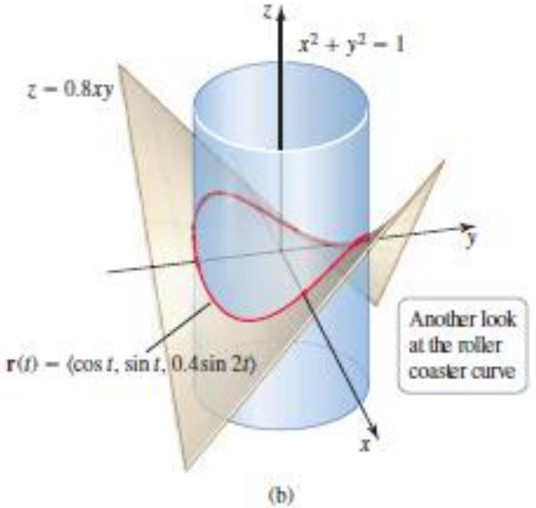
In this case, we can also eliminate the parameter by writing

$$z = 0.4 \sin 2t = 0.4 \times 2 \times \cos t \sin t = 0.8xy$$

which implies that the curve also lies on the hyperbolic paraboloid $z = 0.8xy$.

In fact, the roller coaster curve is the curve in which the surfaces $x^2 + y^2 = 1$ *and* $z = 0.8xy$ *intersect.*





Lines as vector-valued functions – Q. 16

Find a vector function for the line that passes through the points P(2, -1, 4) and Q(3, 0, 6).

Solution: *The parametric equation of a line parallel to the vector* $\vec{v} = \langle a, b, c\rangle$ *and passing through* $P_0(x_0, y_0, z_0)$ *are*

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

The vector $v = \overrightarrow{PQ} = \langle 3, -2, 0 - (-6), 6 - 4\rangle = \langle 1,1,2\rangle$ *is parallel to the line, and we let* $P_0 = P(2, -1,4)$.

Therefore, parametric equations for the line are

$$x = 2 + t, \qquad y = -1 + t, \qquad z = 4 + 2t$$

And the corresponding vector function for the line is

$$\vec{r}(t) = \langle 2 + t, -1 + t, 4 + 2t\rangle$$

With a domain of all real numbers. As t increases, the line is generated in the direction of $\overrightarrow{PQ}$.

Example – Ex. 10.7

Find three different surfaces that contain the curve

$$r(t) = i\, t^2 + j \ln t + k\, 1/t$$

Solution:

The component functions are

$$x = t^2, \qquad y = \ln t, \qquad z = 1/t$$

The domain of r is $(0, \infty)$, *so* $x = t^2 \Rightarrow t = \sqrt{x} \Rightarrow y = \ln\sqrt{x}$

Thus, one surface containing the curve is the cylinder $y = \ln\sqrt{x} = \frac{1}{2}\ln x$.

Also, $z = 1/t \Rightarrow t = 1/z \Rightarrow y = \ln(1/z)$, *so the curve also lies on the cylinder* $y = \ln(1/z)$ *or* $y = -\ln z$.

Note that the surface $y = \ln(xz)$ *also contains the curve, since*

$$\ln(xz) = \ln(t^2 . 1/t) = \ln t = y$$

Example – Q. 29. Ex. 10.7

Find a vector function that represents the curve of the intersection of the cone

$$z = \sqrt{x^2 + y^2}$$

And the plane $z = 1 + y$.

Solution: *Solve both the equations for z*

$$\sqrt{x^2 + y^2} = 1 + y \Rightarrow y = \frac{1}{2}(x^2 - 1).$$

We can form parametric equations for the curves C of intersection by choosing a parameter $x = t$, *then*

$$y = \frac{1}{2}(t^1 - 1), \qquad and \quad z = 1 + y = \frac{1}{2}(t^2 + 1).$$

Thus a vector function representing C is

$$r(t) = it + j\frac{1}{2}(t^2 - 1) + k\frac{1}{2}(t^2 + 1)$$

Calculus of Vector-Valued Functions

The Derivative and Tangent Vector

If $r(t) = \langle f(t), g(t), h(t)\rangle = i\, f(t) + jg(t) + kh(t)$, *where* f, g, *and* h *are differential functions, then*

$$r'(t) = \left\langle f'^{(t)}, g'^{(t)}, h'^{(t)} \right\rangle = i\, f'(t) + jg'(t) + kh'(t)$$

Proof: $dr/dt = r'(t) = \lim\limits_{\Delta t \to 0} [r(t + \Delta t) - r(t)]/\Delta\mathrm{t}$

$$r'(t) = \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$r'(t) = \langle f'(t), \quad g'(t), \quad h'(t) \rangle$$

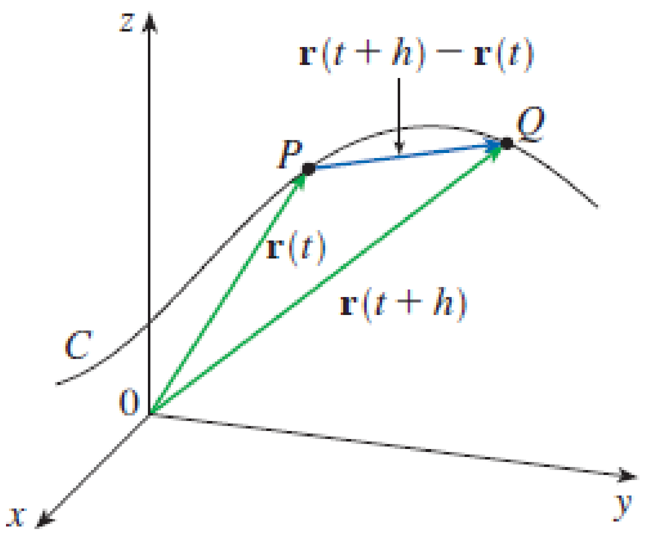
Tangent Vector

DEFINITION Derivative and Tangent Vector

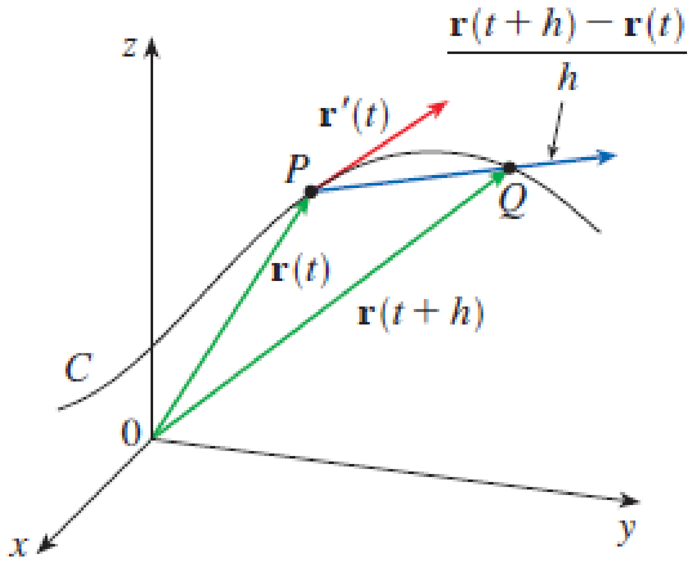
Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$

Calculus of Vector-Valued Functions

Thus vector $r'(t)$ *is called the* ***tangent vector*** *to the curve defined by* $r(t)$ *at a point P, provided that* $r'(t)$ *exists and* $r'(t) \neq 0$.

The ***tangent line*** *to C at P is defined to be the line through P parallel to the tangent vector* $r'(t)$.

A ***unit tangent vector*** *is defined as*

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Differentiation Rules

Derivative Rules

Let **u** and **v** be differentiable vector-valued functions, and let f be a differentiable scalar-valued function, all at a point t. Let **c** be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule
4. $\dfrac{d}{dt}\big(\mathbf{u}(f(t))\big) = \mathbf{u}'(f(t))f'(t)$ Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

Example – Q. 53. Ex. 10.7

Find a vector equation for the tangent line to the curve of the intersection of the cylinders

$$x^2 + y^2 = 25 \quad \text{and} \quad y^2 + z^2 = 20$$

at the point (3, 4, -2).

Solution: First, parametrize the curve C of intersection.

The projection of C onto the xy-plane is contained in the circle $x^2 + y^2 = 25, z = 0$, so we can write

$$x = 5\cos t, \qquad y = 5\sin t$$

C also lies on the cylinder $y^2 + z^2 = 20$ and $x \geq 0$ near the point (3,4,2), so we can write $z = \sqrt{20 - y^2} = \sqrt{20 - 25\sin^2 t}$.

A vector equation then for C is

$$r(t) = \langle 5\cos t, 5\sin t, \sqrt{20 - 25\sin^2 t} \rangle$$

Example – Q. 53. Ex. 10.7

The tangent vector is

$$r'(t) = \langle -5\sin t, 5\cos t, \frac{1}{2}(20 - 25\sin^2 t)(-50\sin t cos\ t)\rangle\,.$$

The point (3,4,2) corresponds to $t = \cos^{-1} 3/5$*, so the tangent vector there is*

$$r'(\cos^{-1} 3/5) = \langle -5\left(\frac{4}{5}\right), 5\left(\frac{3}{5}\right), \frac{1}{2}\left(20 - 25\left(\frac{4}{5}\right)^2\right)^{-12}\left(-50\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\right)\rangle$$

$$= \langle -4, 3, -6\ \rangle$$

The tangent line is parallel to this vector and passes through (3,4,2), so a vector equation for the line

$$r(t) = i(3 - 4t) + j\ (4 + 3t) + k(2 - 6t)$$

Example – Q. 57. Ex. 10.7

The curves $r_1(t) = \langle t, t^2, t^3 \rangle$ and $r_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

Solution:

The angle of intersection of the two curves is the angle between two tangent vectors to the curves at the point of intersection.

$$r_1'(t) = \langle 1,2t, 3t^2 \rangle$$

At t=0 and (0,0,0),

$$r_1'(t) = \langle 1,0,0 \rangle$$

is a tangent vector to r_1 at $(0,0,0)$.

Similarly,

$$r_2'(t) = \langle \cos t \,, 2 \cos 2t \,, 1 \rangle$$

At t=0 and (0,0,0),

$$r_2'(t) = \langle 1,2,1 \rangle$$

is a tangent vector to r_2 at $(0,0,0)$.

Example – Q. 57. Ex. 10.7

If θ is the angle between these two tangent vectors, then

$$\cos\theta = \frac{\langle l_1, m_1, n_1\rangle \cdot \langle l_2, m_2, n_2\rangle}{\sqrt{l_1^2 + m_1^2 + n_1^2} \times \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\cos\theta = \frac{\langle 1,0,0\rangle \cdot \langle 1,2,1\rangle}{\sqrt{1}\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^0$$

Homework 1 - Ex. 10.7

If two objects travel through space along two different curves, it is often important to know whether they will collide, for example, will a missile hit its target? Will two aircrafts collide? The curves might intersect, but we need to know whether the objects are in the same position at the same time.

Suppose the trajectories of two particles are given by the vector functions

$$r_1(t) = \langle t^2, 7t - 12, t^2 \rangle, \qquad r_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. *Do the particles collide?*

Integrals of a Vector-Valued Function

DEFINITION Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function, and let $\mathbf{R}(t) = F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + H(t)\,\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of **r** is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where **C** is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle\,dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

DEFINITION Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are integrable on the interval $[a, b]$. The **definite integral** of **r** on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Example – Q. 66. Ex. 10.7

Find $r(t)$ *if* $r'(t) = it + j\,e^t + k\,te^t$ $\quad$ *and* $r(0) = i + j + k$

Solution:

$$r(t) = \int (it + j\,e^t + k\,te^t)dt$$

$$r(t) = i\frac{1}{2}t^2 + j\,e^t + k(te^t - e^t) + C$$

Use the initial condition $r(0) = i + j + k$ *to evaluate the integration constant C.*

$$r(0) = i + j + k = j - k + C \Rightarrow i + 2k$$

Therefore,

$$r(t) = i\frac{1}{2}t^2 + j\,e^t + k(te^t - e^t) + i + 2k$$

$$r(t) = i(\frac{1}{2}t^2 + 1) + j\,e^t + k(te^t - e^t + 2)$$

Homework 2 – Ex. 10.7

Evaluate the integral

$$\int \left(ite^{2t} + j\frac{t}{1-t} + k\frac{1}{\sqrt{1-t^2}} \right) dt$$

Example – Q. 68. Ex. 10.7

Two particles travel along the space curves

$$r_1(t) = \langle t, t^2, t^3 \rangle, \qquad r_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle.$$

Do the particles collide? Do their paths intersect?

Solution:

The particles collide provided $r_1(t) = r_2(t)$

$$\Rightarrow \quad \langle t, t^2, t^3 \rangle = \langle 1+2t, 1+6t, 1+14t \rangle$$

$$\Rightarrow \quad t = 1+2t, \qquad t^2 = 1+6t, \qquad t^3 = 1+14t$$

This first equation gives $t = -1$, *but this does not satisfy the other equations, so the particles do not collide.*

For the paths to intersect, we need to find a value for t and a value for s where

$$r_1(t) = r_2(s)$$

$$\Rightarrow \quad t = 1+2s, \qquad t^2 = 1+6s, \qquad t^3 = 1+14s$$

Example – Q. 68. Ex. 10.7

Substituting first equation into second gives,

$$s = 0 \Rightarrow t = 1 \textit{ and } s = 1/2 \Rightarrow t = 2$$

Both pairs of values satisfy the third equation.

Thus the paths intersect twice, at the point $(1,1,1)$ *when s=0 and t=1, and at* $(2,4,8)$ *when s=1/2 and t=2.*

Example – Q. 80. Ex. 10.7

Find an expression for

$$\frac{d}{dt}[u(t)\cdot(v(t)\times w(t))].$$

Solution:

$$\frac{d}{dt}(\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)])=\mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]+\mathbf{u}(t)\cdot\frac{d}{dt}[\mathbf{v}(t)\times\mathbf{w}(t)]$$

$$=\mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]+\mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)+\mathbf{v}(t)\times\mathbf{w}'(t)]$$

$$=\mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]+\mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)]+\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}'(t)]$$

$$=\mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]-\mathbf{v}'(t)\cdot[\mathbf{u}(t)\times\mathbf{w}(t)]+\mathbf{w}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]$$

Homework 3 – Ex. 10.7

If $r(t) = u(t) \times v(t)$, *where*

$$u(2) = \langle 1, 2, -1 \rangle, \qquad u'(2) = \langle 3, 0, 4 \rangle, \qquad v(t) = \langle t, t^2, t^3 \rangle$$

find $r'(2)$.

Homework 4 – Ex. 10.7

If a curve has the property that the position vector $r(t)$ *is always perpendicular to the tangent vector* $r'(t)$*, show that the curve lies on the sphere with centre the origin.*

Homework 5 – Ex. 10.7

If

$$u(t) = r(t) \cdot [r'(t) \times r''(t)]$$

show that

$$u'(t) = r(t) \cdot [r'(t) \times r'''(t)]$$