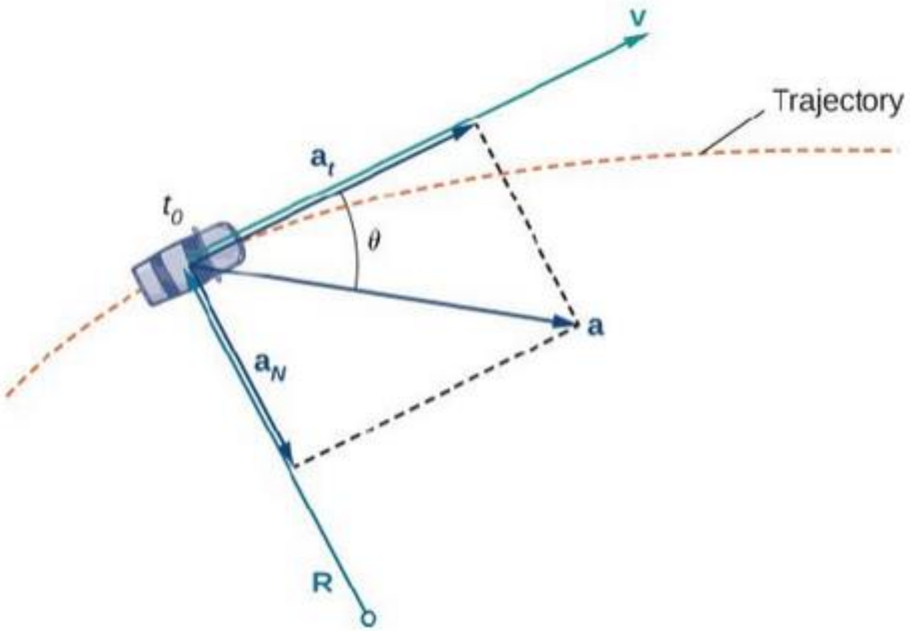
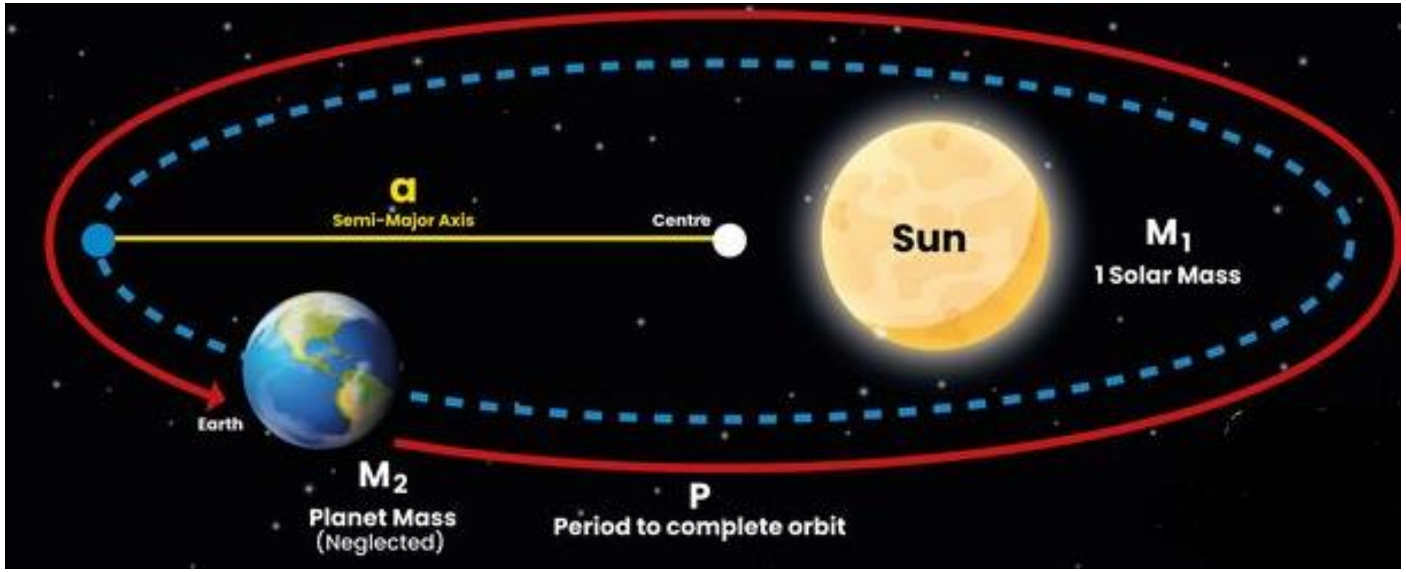
Lecture 12 - Chapter 10 – Sec. 10.8
Applications





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Learning Objectives

- *Describe the velocity and acceleration vectors of a particle moving in space.*
- *Explain the tangential and normal components of acceleration.*
- *Kepler's Laws of Planetary Motion*

Tangential and Normal Components of Acceleration

- *We have seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature.*

- *All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves.*

- *Our starting point is using vector-valued functions to represent the position of an object as a function of time.*

- *For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle travelling along a helix moves on a curve in three dimensions.*

Acceleration Vector

Let r(t) be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

Position $$r(t) = i\,x(t) + j\,y(t) + k\,z(t)$$

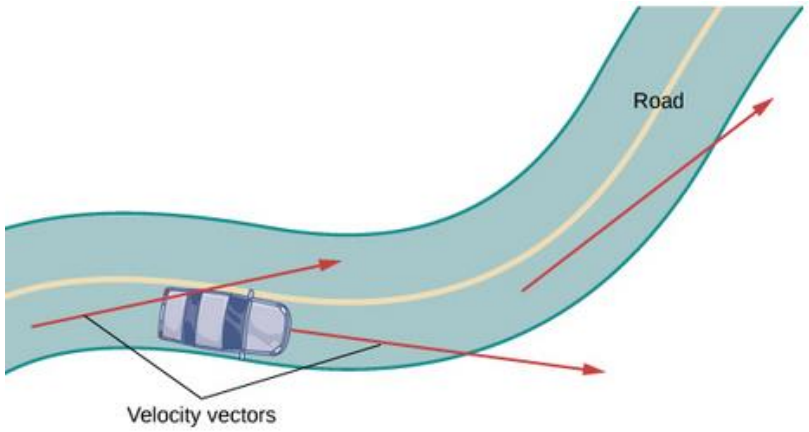
Velocity $$v(t) = r'(t) = i\,x'(t) + j\,y'(t) + k\,z'(t)$$

Speed $$|v(t)| = |r'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}$$

Acceleration $$a(t) = v'(t) = r''(t) = i\,x''(t) + j\,y''(t) + k\,z''(t)$$

Example – Q. 7. Ex. 10.8

- *To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road.*
- *The speed at which you are travelling when you run off the road, coupled with the direction, gives a vector representing your velocity.*
- *The fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your direction is constantly changing to keep you on the road.*
- *As you turn to the left or right, your acceleration vector also points to the left or right.*
- *This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies*

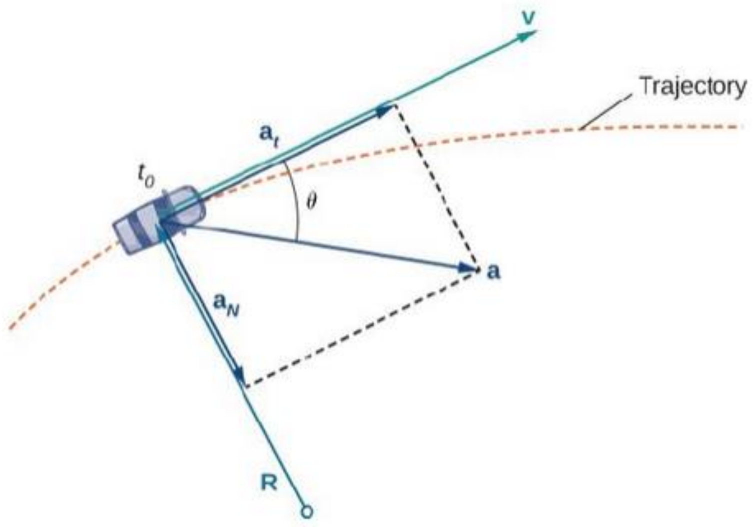


Components of Acceleration Vector

- *We can combine some of the concepts of Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space*

- *Recall that the unit tangent vector* T *and the unit normal vector N form an osculating plane at any point P on the curve defined by a vector-valued function* $r(t)$.

The acceleration vector $a(t)$ *lies in the osculating plane and can be written as a linear combination of the unit tangent and unit normal vectors.*



Components of Acceleration Vector

Since $v(t) = r'(t)$ *and unit tangent vector is given by* $T(t) = r'(t)/|r'(t)| = r'(t)/v_s$ *therefore*

$$v(t) = T(t)v_s \qquad (v_s \rightarrow speed)$$

If we differentiate both sides of this equation with respect to t, we get

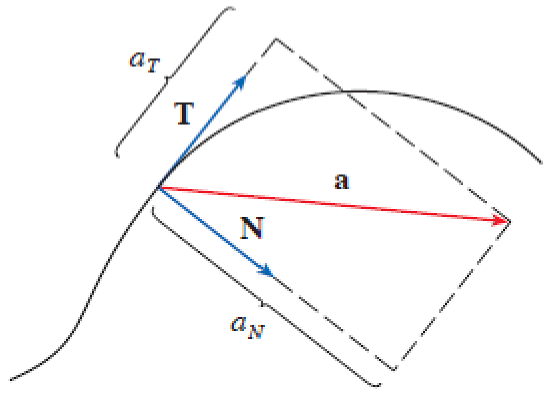
$$a(t) = v'(t) = T'(t)v_s + T(t){v_s}'$$

Using the expression for the curvature $\kappa = |T'|/|r'| = |T'|/v_s \Rightarrow |T'| = \kappa v_s$

The unit normal vector is defined as $N = T'/|T'|$, *so*

$$T' = |T'|N = \kappa v_s N$$

Therefore $a(t) = v_s' T + \kappa v_s^2 N \qquad (1)$



Writing a_T *and* a_N *for the tangential and normal components of acceleration, we have*

$$a_T = v_s' = r'(t) \cdot r''(t)/|r'(t)| \quad and$$

$$a_N = \kappa v_s^2 = |r'(t) \times r''(t)|/|r'(t)|$$

Components of Acceleration Vector

- *Let's look at what Eq. (1) says. The first thing to notice is that the binormal vector* ***B*** *is absent.*
- *No matter how an object moves through space, its acceleration always lies in the plane of* ***T*** *and* ***N*** *(the osculating plane). (Recall that* ***T*** *gives the direction of motion and* ***N*** *points in the direction the curve is turning.)*
- *Next we notice that the tangential component of acceleration is* v_s'*, and the normal component of acceleration is* κv_s^2.

- *This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature, so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against the car door. High speed around the turn has the same effect; in fact, if you double your speed,* a_N *is increased by a factor of 4.*

Example – Q. 31. Ex 10.9

Find the tangential and normal components of the acceleration vector

$$r(t) = i\cos t + j\sin t + k\,t$$

Solution: The velocity vector is given by

$$r'(t) = v(t) = -i\sin t + j\cos t + k, \qquad v_s = |r'(t)| = \sqrt{2}$$

The acceleration is given by

$$a(t) = r''(t) = -i\cos t\ - j\sin t, \qquad r'(t) \times r''(t) = i\sin t - j\cos t + k$$

Then

$$a_T = \frac{r'(t)\cdot r''(t)}{|r'(t)|} = \frac{\sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$$

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = 1$$

Homework 1 – 10.9

Find the tangential and normal components of the acceleration vector

$$r(t) = i\,(3\mathrm{t} - \mathrm{t}^3) + j\,3t^2$$

Example – Q. 24. Ex. 10.9

A batter hits a baseball 1 m above the ground toward the centre field fence, which is 4 m high and 120 m from home plate. The ball leaves the bat with speed 35 m/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

Solution:

$$v_0 = 35\, ms, \qquad angle\ of\ elecation\ \ \alpha = 50^o$$

If the origin is placed at home plate, then

$$r(0) = 1j$$

Then the position vector is given by $r(t) = r(0) + v_0 t - 1/2\ gt^2$, *where* $D = r(0) = 1\ j$ *and* $v_0 = i\ v_0 \cos\alpha + j \sin\alpha$, *so*

$$r(t) = i\ (vo \cos\alpha)t + j\ (v_0 \sin\alpha\ - \frac{1}{2} gt^2 + 1)$$

Example – Q. 24. Ex. 10.9

Thus, parametric equations for the trajectory of the ball are

$$x = (v_0 \cos \alpha)t, \qquad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 + 1$$

The ball reaches the fence when

$$x = 120 \Rightarrow (v_0 \cos \alpha)t = 120 \Rightarrow t \approx 5.33\ s$$

At this time, the height of the ball is

$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 + 1 \approx 4.7\ m$$

Since the fence is 4 m high, the ball clears the fence.

Homework 2 – Ex. 10.9

A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)

Example – Q. 28. Ex. 10.9

A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land, and with what speed?

Solution:

Place the ball at the origin and consider j to be pointing in the northward direction with i pointing east and k pointing upward.

Force = mass × acceleration ⇒ acceleration = force/mass, so the wind applies a constant acceleration of $4/0.8 = 5m/s^2$ in the easterly direction.

Combined with the acceleration due to gravity, the acceleration acting on the ball is $a(t) = 5\,i\ - 9.8\,k$.

Then

$$v(t) = \int a(t)dt = i\,5t - k\,9.8t + C, \qquad C \text{ is a constant vector}.$$

Example – Q. 28. Ex. 10.9

We know $v(0) = C = -30\cos 30^o\ j + 30\sin 30^o\ k = -15\sqrt{3}\, j + 15\, k$

$$\Rightarrow C = -15\,\sqrt{3}\, j + 15\, k$$

The velocity is given by

$$5t\, i\ - 15\sqrt{3}\, j + (15 - 9.8t)k$$

The position vector $r(t)$ *is given by*

$$r(t) = \int v(t)dt = 2.5t^2\, i\ - 15\sqrt{3}\, t\, j + (15t - 4.9t^2)k + D$$

Using the initial condition $r(0) = 0$. *This gives* $D = 0$. *Thus*

$$r(t) = 2.5t^2\, i\ - 15\sqrt{3}\, t\, j + (15t - 4.9t^2)k$$

The ball lands when $15t - 4.9t^2 \Rightarrow t = 0, \quad t = 3.0612\ s$, *so the ball lands at approximately*

$$r(3.0612) = 23.43\, i - 79.53\, j$$

Example – Q. 28. Ex. 10.9

This is 82.9 m away in the direction $S16.4^o E$.

It's speed is approximately

$$|v(3.0612)| \approx |15.3061\, i\ - 15\sqrt{3}\, j - 15\, k|$$
$$\approx 33.68\, m/s$$

Homework 3 – Ex. 10.8

A rocket burning its onboard fuel while moving through space has velocity $v(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity v_e relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$\frac{mdv}{dt} = \frac{dm}{dt} v_e$$

(a) Show that

$$v(t) = v(0) - \ln \frac{m(0)}{m(t)} v_e$$

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?