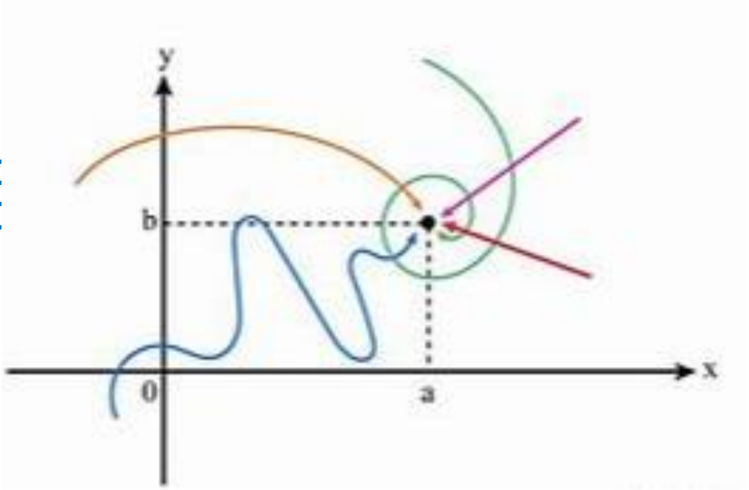
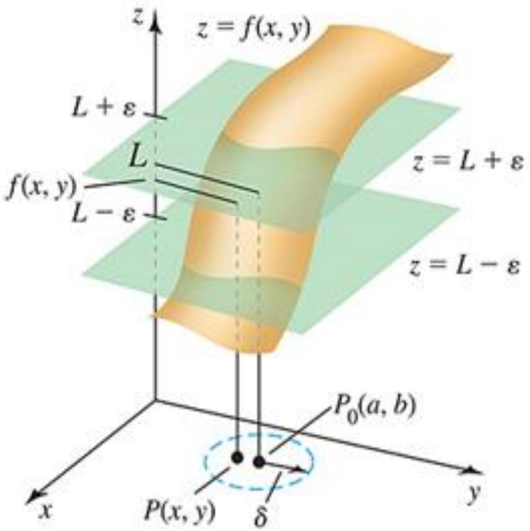
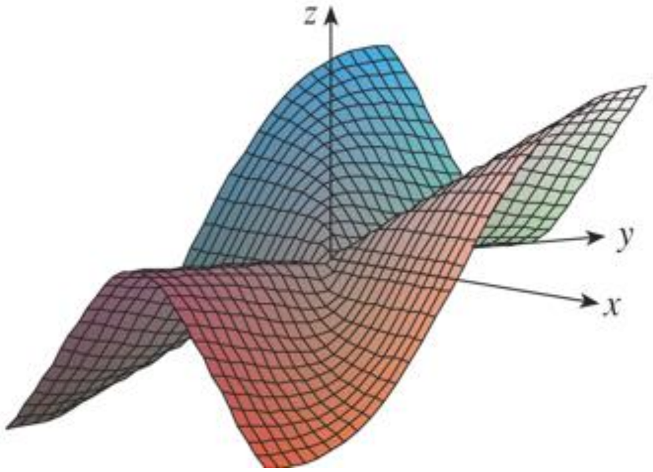
Lecture 14 - Chapter 11 – Sec. 11.2

Limits and Continuity







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Learning Objectives

- *Understanding the limit of a multivariable function*
- *Limit laws for two variable functions*
- *Limits at boundary points*
- *Limits in polar coordinates – Changing the variables*
- *Continuity of Functions of Two Variables*
- *Continuity of Composite Functions*

Limit of a function of two variables

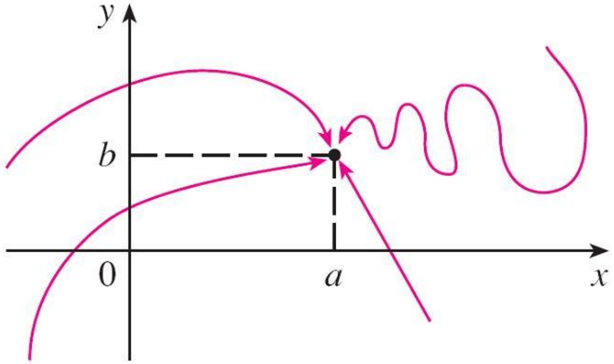
A function f of two variables has a limit L *as the point* $P(x, y)$ *approaches a fixed point* $P_0(a, b)$, *along any path that stays within the domain of f, if* $f(x, y) - L$ *can be made arbitrarily small for all P in the domain that are sufficiently close to* P_0.

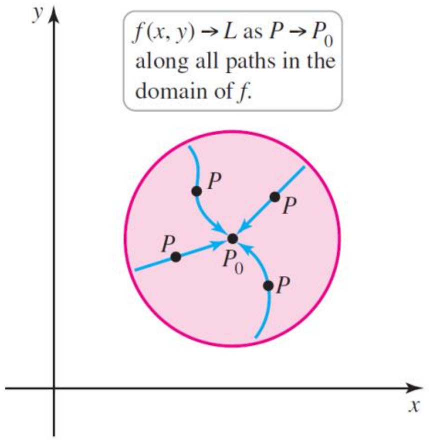
If such a limit exists, we write

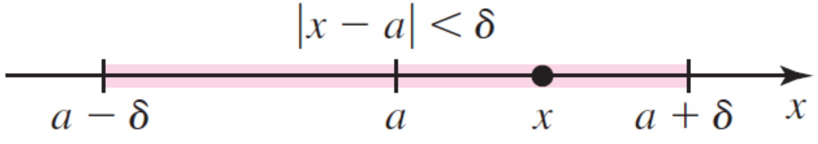
$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

Follow-up: How does evaluating the limit of a two-variable function situation differs from evaluating the limit of a one-variable function?

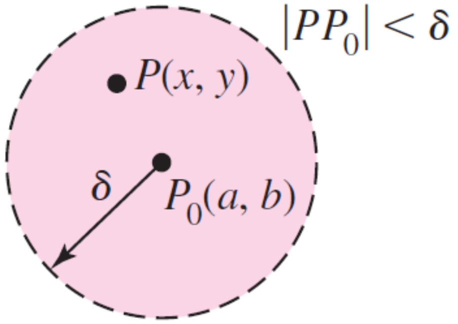
Limit of a function of two variables

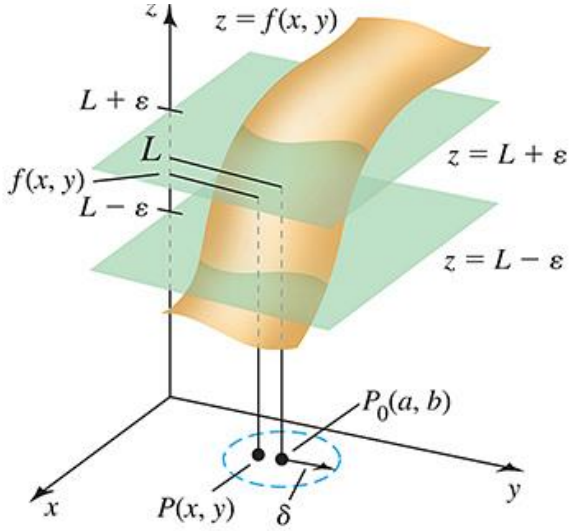






(a)





$f(x, y)$ is between $L - \varepsilon$ and $L + \varepsilon$ whenever $P(x, y)$ is within δ of P_0.

Limit of a function of two variables

- *In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).*

A more precise definition follows.

$\boxed{1}$ **Definition** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Limit of a function of two variables

THEOREM 14.1 Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant function $f(x, y) = c$: $\displaystyle\lim_{(x,y)\to(a,b)} c = c$

2. Linear function $f(x, y) = x$: $\displaystyle\lim_{(x,y)\to(a,b)} x = a$

3. Linear function $f(x, y) = y$: $\displaystyle\lim_{(x,y)\to(a,b)} y = b$

Limit of a function of two variables

THEOREM 14.2 Limit Laws for Functions of Two Variables
Let L and M be real numbers and suppose that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$. Assume c is a constant, and m and n are integers.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x, y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$
7. ***m*/*n* power** If m and n have no common factors and $n \neq 0$, then $\lim_{(x,y)\to(a,b)} [f(x, y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Limits of Polynomials and Rational Functions

The theorem 7 applied to polynomials and rational functions implies the following:

1. *To find the limit of a polynomial, we simply plug in the point.*

2. To find the limit of a rational function, we plug in the point as long as the denominator is not 0.

Follow-up: What does it mean to say that limits of polynomials may be evaluated by direct substitution?

Example - Limits of Polynomials and Rational Functions

Find the limits of the following functions:

(a) $f(x,y) = x^6y + 2xy$ *as* $(x,y) \to (1,2)$

(b) $f(x,y) = \frac{x^2y}{x^4+y^2}$ *as* $(x,y) \to (1,1).$

Solution:

(a) Since the function is a polynomial in x and y, therefore one can simply plug in the points to evaluate the limit.

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} x^6y + 2xy = (1)^6(2) + 2(1)(2) = 5$$

(b) Since the denominator is non-zero for the given points, we can simply plug in the points to evaluate the limit.

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2} = 1/2$$

Example – Properties of limits

Evaluate

$$\lim_{(x,y)\to(2,3)} (3x^2y + \sqrt{xy}).$$

Solution:

$$\lim_{(x,y)\to(2,8)} (3x^2y + \sqrt{xy}) = \lim_{(x,y)\to(2,8)} 3x^2y + \lim_{(x,y)\to(2,8)} \sqrt{xy}$$

$$= 3\left[\lim_{(x,y)\to(2,8)} x\right]^2 \left[\lim_{(x,y)\to(2,8)} y\right]$$

$$+ \sqrt{\left[\lim_{(x,y)\to(2,8)} x\right]\left[\lim_{(x,y)\to(2,8)} y\right]}$$

$$= 3 \cdot 2^2 \cdot 8 + \sqrt{2 \cdot 8} = 100$$

Example - Limits of Polynomials and Rational Functions

Find the limits of the following functions:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$, *(b)* $\lim_{(x,y)\to(1,2)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Solution:

Caution - We cannot plug in the point as we get 0 in the denominator.

We try to simplify the function to see if we can avoid a vanishing denominator.

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)(x^2+xy+y^2)}{(x-y)}$$
$$= \lim_{(x,y)\to(0,0)} (x^2+xy-xy)$$
$$= 0$$

Your Turn - Pair/Share Activity

Find the limit of the following function:

(b) $\lim_{(x,y)\to(1,2)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Limit along different paths

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

Thus, if we can find two different paths of approach along which the function f (x, y) has different limits, then it follows that

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does not exist.

Example

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Solution

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

First let's approach $(0,0)$ *along the x-axis. Then* $y = 0$ *gives*

$f(x, 0) = x^2/x^2 = 1$ *for all* $x \neq 0$. *So*

$f(x, y) \to 1$ *as* $(x, y) \to (0,0)$ *along the x-axis.*

(contd.)

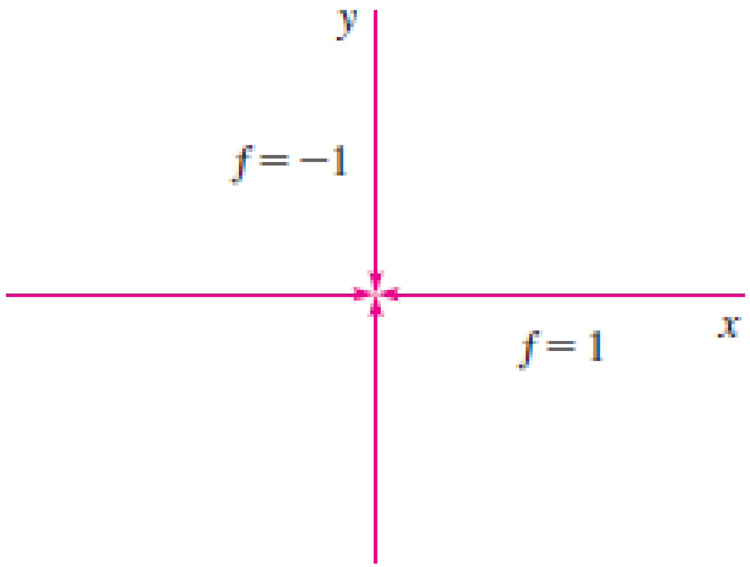
Example

We now approach $(0,0)$ *along the y-axis. Then* $x = 0$ *gives*

$f(0, y) = -y^2/y^2 = -1$ *for all* $y \neq 0$*. So*

$f(x, y) \to -1$ *as* $(x, y) \to (0,0)$ *along the y-axis.*

Since $f(x, y)$ *has two different limits along two different lines, the given limit does not exist.*



Your Turn - Test your understanding

If

$$f(x,y) = \frac{xy}{x^2 + y^2},$$

does $\lim_{(x,y)\to(0,0)} f(x,y)$ *exist?*

(Caution: think creatively)

Example

If $f(x,y)=\frac{xy^2}{x^2+y^4}$, *does* $\lim_{(x,y)\to(0,0)} f(x,y)$ *exist?*

Solution:

We let $(x,y)\to(0,0)$ *along any nonvertical line through the origin. Then* $y=mx$ *, where* m *is the slope, and*

$$f(x,y)=f(x,mx)=\frac{m^2x^3}{x^2+m^4x^4}=\frac{m^2x}{1+m^4x^2}$$

So $f(x,y)\to(0,0)$*, as* $(x,y)\to(0,0)$ *along* $y=mx$*.*

Thus $f(x,y)$ *has the same limiting value along every nonvertical line through the origin.*

Does not show that the given limit is 0??

Example

It does not show that the given limit is 0, for if we now let $f(x, y) \to (0,0)$ *along the parabola* $x = y^2$*, we have*

$$f(x, y) = f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2}$$

So $f(x, y) \to (0,0),$ *as* $(x, y) \to (0,0)$ *along* $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist.

Homework

Find the limit if it exists or show that limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+3y^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4+4y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$$

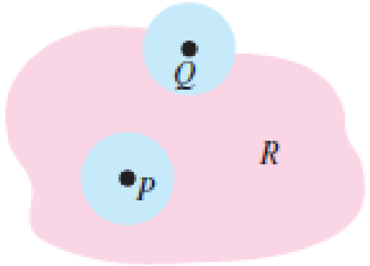
$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$

Limits at the Boundary

Interior Points *Let R be a region in* $\mathbb{R}^2$. *An* ***interior point*** *P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.*

P is an interior point: There is a disk centered at *P* that lies entirely in *R*.



Q is a boundary point: Every disk centered at *Q* contains points in *R* and points not in *R*.

A ***boundary point*** *Q of R lies on the edge of R in the sense that every disk centered at Q contains at least one point in R and at least one point not in R.*

Open and Closed Sets:

- *A region is* ***open*** *if it consists entirely of interior points.*
- *A region is* ***closed*** *if it contains all its boundary points.*

Example - Limits at the Boundary

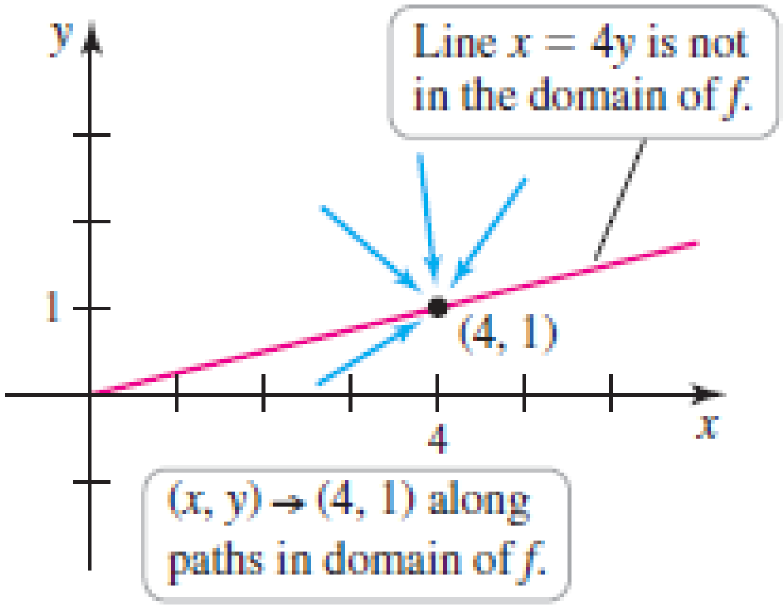
Evaluate $\lim\limits_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

Solution: Points in the domain of this function satisfy $x \geq 0$ *and* $y \geq 0$ *(because of the square roots) and* $x \neq 4y$ *(to ensure the denominator is nonzero). We see that the point (4, 1) lies on the boundary of the domain.*

$$\begin{aligned}
\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} &= \lim_{(x,y)\to(4,1)} \frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})} \\
&= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y} \\
&= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y}). \\
&= 4.
\end{aligned}$$

Example - Limits at the Boundary

Because points on the line $x = 4y$ are outside the domain of the function, we assume that $x - 4y \neq 0$.
Along all other paths to (4, 1), the function values approach 4



Homework

Evaluate the following limit

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{y} - \sqrt{x+1}}{y - x - 1}, \qquad \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$$

Consider the function f(x, y) of two variables x and y defined as

$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$

Find the limit along the following curves as (x, y) → (0, 0).

(a) the x-axis *(b) the y-axis* *(c) the line* $y = x$

(d) the line $y = -x$ *(e) the parabola* $y = x^2$.

Limits with Change of Coordinates

Consider the function f(x, y) of two variables x and y defined as

$$f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

as $(x, y) \to (0,0)$.

Solution:

A quick inspection reveals that simple plugging of rectangular points results in 0/0 form.

Consider coordinate transformation – from rectangular coordinates to polar coordinates.

$$x = r cos\theta, \qquad y = r sin\theta$$

$$\Rightarrow r^2 = x^2 + y^2$$

Limits with Change of Coordinates

As $(x, y) \to 0,\ r \to 0^+$.

Therefore, the given limit can be written as

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0^+} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2}$$

$$= \lim_{r\to 0^+} r(\cos^3\theta + \sin^3\theta)$$

$$= 0$$

Squeeze Theorem

- *In calculus, the **squeeze theorem** (also known as the **sandwich theorem**) is a theorem regarding the limit of a function that is trapped between two other functions.*
- *The squeeze theorem is used in calculus and mathematical analysis, typically to confirm the limit of a function via comparison with two other functions whose limits are known.*

Theorem: Let I be an interval containing the point a. Let g, f, and h be functions defined on I, except possibly at a itself.

Suppose that for every x in I not equal to a, we have $g(x) \leq f(x) \leq h(x)$ and also suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L\,, \qquad \textit{then} \qquad \lim_{x\to a} f(x) = L$$

Squeeze Theorem

Use the Squeeze Theorem to find the limit

$$\lim_{(x,y \to 0,0)} xy \sin \frac{1}{x^2+y^2}$$

Solution:

Since $-1 \le \sin\theta \le 1$, *therefore*

$$-1 \le \sin \frac{1}{x^2+y^2} \le 1$$

$$\Rightarrow \quad -xy \le xy \sin\left(\frac{1}{x^2+y^2}\right) \le xy \quad for\ every\ xy > 0.$$

If $xy < 0$, *we have*

$$-xy \ge xy \sin\left(\frac{1}{x^2+y^2}\right) \ge xy$$

Squeeze Theorem

In either case,

$$\lim_{(x,y\to 0,0)} xy = 0, \qquad \textit{and} \qquad \lim_{(x,y\to 0,0)} (-xy) = 0,$$

Thus,

$$\lim_{(x,y\to 0,0)} xy \sin\left(\frac{1}{x^2+y^2}\right) = 0$$ *by the squeeze Theorem.*

Continuity of Multivariable Functions

Definition: A function $f(x, y)$ is said to be continuous at point (a, b) if

1. (a, b) *is in the domain of* f.
2. $\lim_{(x,y)\to(a.b)} f(x, y)$ *exists.*
3. $\lim_{(x,y)\to(a.b)} f(x, y) = f(a, b)$.

Note: All these three conditions must hold for a function to be continuous.

- *A function of two (or more) variables is continuous at a point, provided its limit equals its value at that point (which implies the limit and the value both exist).*
- *The definition of continuity applies at boundary points of the domain of f provided the limits in the definition are taken along paths that lie in the domain.*

Continuity of Multivariable Functions

- *A function f is continuous on a set D if it is continuous at every point in D.*
- *This means that a surface that is the graph of a continuous function has no hole or break.*
- *If a function is not continuous at a point (a,b), the function is said to be discontinuous at (a,b).*
- *If a function is continuous at a point, to find the limit of the function at the point it is enough to plug-in the point*

Continuity of Multivariable Functions

- *Using the properties of limits, you can see that sums, differences, products, quotients of continuous functions are continuous on their domains.*

The following results are true for multivariable functions:

1. The sum, difference and product of continuous functions is a continuous function.

2. The quotient of two continuous functions is continuous as long as the denominator is not 0.

3. Polynomial functions are continuous.

4. Rational functions are continuous in their domain.

5. If $f(x; y)$ is continuous and $g(x)$ is defined and continuous on the range of f, then $g(f(x; y))$ is also continuous

Polynomial Function and Continuity

- *Limit of a polynomial exits everywhere in its domain, so polynomials are continuous everywhere in their domain*

Example: Discuss the continuity of

$$f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y.$$

is $f(x,y)$ continuous at (0; 0)?

Solution:

f (x; y) is a polynomial function, therefore it is continuous on $\mathbb{R}^2$.

In particular, it is continuous at (0; 0).

Continuity of Rational Functions

Where is the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

continuous?

Solution: f is the quotient of two continuous functions therefore it is continuous as long as its denominator is not 0.

The function f is discontinuous at (0, 0) because it is not defined there.

Therefore, it is continuous on its domain, which is the set

$$D = \{(x,y)|(x,y) \neq 0\}$$

Test your knowledge

At what points of $\mathbb{R}^2$ is a rational function of two variables continuous?

Your Turn

Where is

$$f(x,y) = \frac{1}{x^2 - y}$$

continuous?

Solved Example

.

Find where $\tan^{-1}\frac{xy^2}{x+y}$ *is continuous?*

Solution: *Here, we have the composition of two functions. We know that inverse tan is continuous on its domain, that is on R. Therefore,* $\tan^{-1}\frac{xy^2}{x+y}$ *will be continuous where*

$\frac{xy^2}{x+y}$ is continuous.

Since $\frac{xy^2}{x+y}$ *is the quotient of two polynomial functions, therefore it will be continuous as long as its denominator is not 0, that is as long as* $y \neq -x$.

It follows that the given function is continuous on

$$\{(x, y) \in \mathbb{R}^2 : y \neq -x\}$$

Test your understanding

.

Find where

$$g(x, y) = \ln(x^2 + y^2 - 1)$$

$$f(x, y) = \frac{x^2 + y^2}{x(y^2 - 1)}$$

$$h(x, y) = \frac{2}{x(y^2 + 1)}$$

are continuous?

Continuity of Piecewise Functions

Where is

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \ \ (x,y) = (0,0) \end{cases}$$

continuous?

Solution:

Away from (0; 0), f is a rational function always well-defined. So, it is continuous.

We still need to investigate continuity at (0; 0)

Continuity of Piecewise Functions

To investigate the continuity at (0,0), we evaluate the limit of the function

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

First let's approach $(0,0)$ *along the x-axis. Then* $y = 0$ *gives* $f(x,0) = x^2/x^2 = 1$ *for all* $x \neq 0$. *So*

$$f(x,y) \to 1 \text{ as } (x,y) \to (0,0) \text{ along the x-axis.}$$

We now approach $(0,0)$ *along the y-axis. Then* $x = 0$ *gives* $f(0,y) = -y^2/y^2 = -1$ *for all* $y \neq 0$. *So*

$$f(x,y) \to -1 \text{ as } (x,y) \to (0,0) \text{ along the y-axis.}$$

Since limit does not exist, f(x,y) is continuous everywhere expect at (0,0).

Test your understanding

1. Where is

$$f(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2} & if \quad (x,y)\neq(0,0)\\ 0 & if \quad (x,y)=(0,0)\end{cases}$$

is continuous

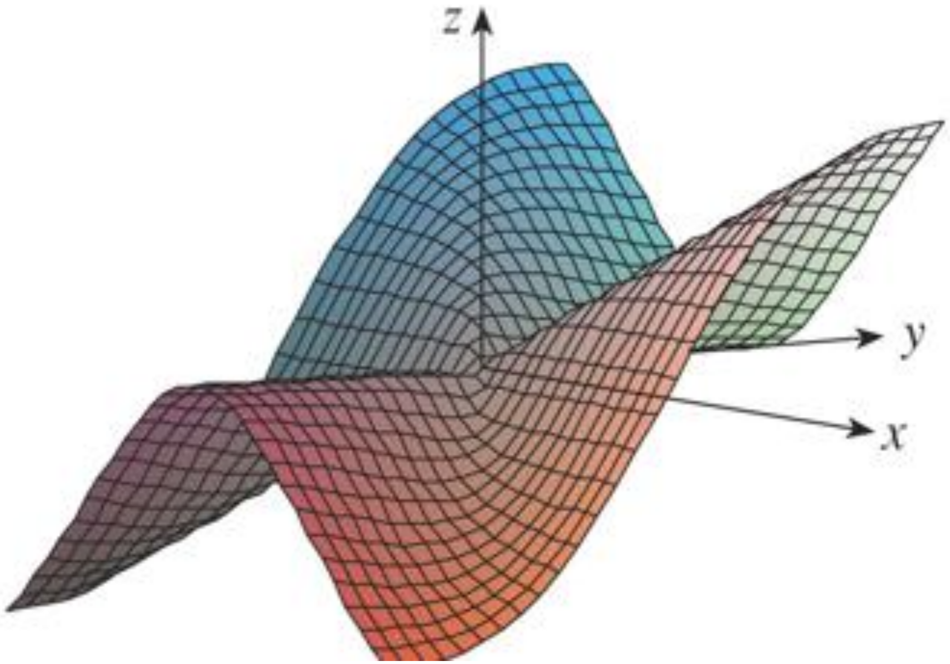
2. Discuss the continuity of

$$f(x,y)=\begin{cases}\dfrac{y^4-2x^2}{x^2+y^4} & if \quad (x,y)\neq(0,0)\\ 0 & if \quad (x,y)=(0,0)\end{cases}$$

Continuity

This figure shows the graph of the continuous function in

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$



COMPOSITE FUNCTIONS

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that, if f is a continuous function of two variables and g is a continuous function of a single variable defined on the range of f, then

The composite function $h = g \circ f$ *defined by*

$$h(x, y) = g(f(x, y)$$

is also a continuous function.

COMPOSITE FUNCTIONS

Determine the points at which the following functions are continuous.

(a) $h(x, y) = \ln(x^2 + y^2 + 42)$, *(b)* $h(x, y) = e^{x/y}$

Solution: The function is the composition of $g(f(x, y))$, *where*

$g(u) = \ln u$ *and* $u = f(x, y) = x^2 + y^2 + 4$

As a polynomial, f is continuous for all (x, y) *in* $\mathbb{R}^2$.

The function g is continuous for $u > 0$.

Because $u = x^2 + y^2 + 4 > 0$ *for all* (x, y), *it follows that h is continuous at all points of* $\mathbb{R}^2$.

COMPOSITE FUNCTIONS

Where is the function

$$h(x, y) = \arctan\left(\frac{y}{x}\right)$$

continuous?

Solution:

The function $f(x, y) = y/x$ *is a rational function and therefore continuous except on the line* $x = 0$.

The function $g(t) = \tan^{-1} t$ *is continuous everywhere. So, the composite function*

$$g\big(f(x, y)\big) = \arctan\left(\frac{y}{x}\right) = h(x, y)$$

is continuous except where x = 0.

In-Class Activity

Determine the points at which the following functions are continuous.

(a), $$f(x,y) = \begin{cases} sin\frac{(x^2+y^2)}{x^2+y^2}, & if\ (x,y) \neq 0 \\ 1 & if\ (x,y) = (0,0) \end{cases}$$

Homework

Determine the points at which the following functions are continuous.

(a), $f(x,y) = e^{x^2-y^2}$

(b) $g(x,y) = \cos(x^2 - y)$

(c) $h(x,y) = xysin\left(\frac{y}{x}\right)$

(d) $f(x,y) = \begin{cases} \frac{1-cos(x^2+y^2)}{x^2+y^2}, & if\ (x,y) \neq 0 \\ 0 & if\ (x,y) = (0,0) \end{cases}$

Squeeze Theorem

- *In calculus, the **squeeze theorem** (also known as the **sandwich theorem**) is a theorem regarding the limit of a function that is trapped between two other functions.*
- *The squeeze theorem is used in calculus and mathematical analysis, typically to confirm the limit of a function via comparison with two other functions whose limits are known.*

Theorem: Let I be an interval containing the point a. Let g, f, and h be functions defined on I, except possibly at a itself.

Suppose that for every x in I not equal to a, we have $g(x) \leq f(x) \leq h(x)$ and also suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L\,, \qquad then \qquad \lim_{x\to a} f(x) = L$$