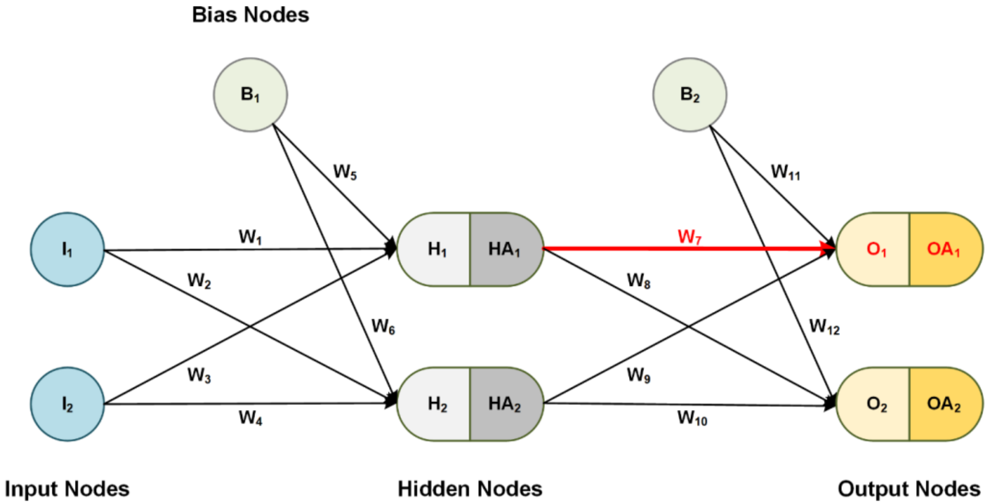
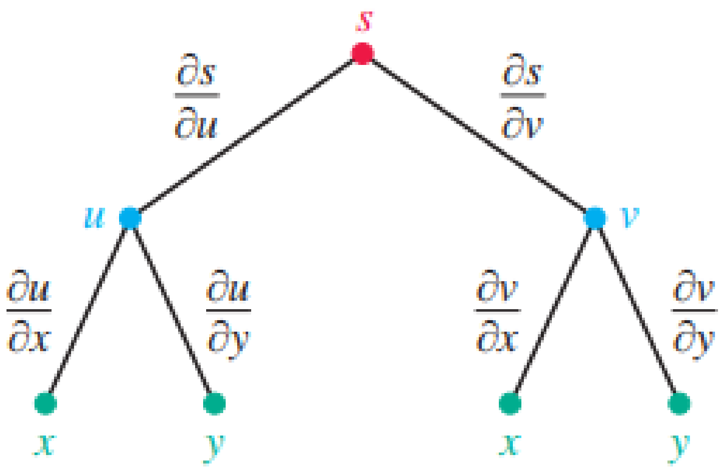
# Chapter 4 – Sec. 11.5

## Chain Rule and Implicit Partial Differentiation



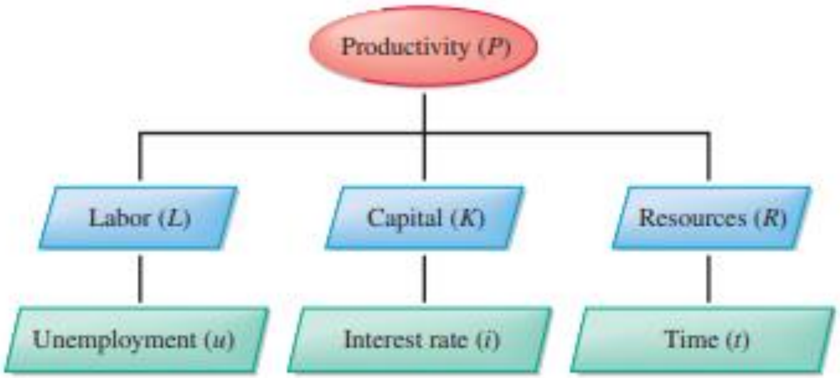

*Dr. M. Loan*

# Learning Objectives

- *Chain Rule for multivariable functions*
- *Implicit differentiation*
- *Applications*

# Chain Rule

- *A simplified production function might take the form P = (L, K, R), where L, K, and R represent the availability of labour, capital, and natural resources, respectively.*
- *However, the variables L, K, and R may be intermediate variables that depend on other variables*
- *For instance, if the unemployment rate increases by 0.1% and the interest rate decreases by 0.2%, what is the effect on productivity?  Chain Rule is the tool needed to answer such questions.*



# Recap – Chain Rule for One-variable Function

- *Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function:*
- *If* $y = f(x)$ *and* $x = g(t)$, *where* $f(x)$*and* $g(t)$ *are differentiable functions, then* $y$ *is indirectly a differentiable function of* $t$ *and*

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

*For example,* $y = f(x) = 2x^3 + e^x$ *and* $x(t) = t^2 + 1$

# Chain Rule

- *For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.*

***First version: Chain Rule with one independent variable***

*deals with the case where* $z = f(x, y)$ *and each of the variables* $x$ *and* $y$ *is, in turn, a function of a variable* $t$*, i.e.,* $x = g(t)$ *and* $y = h(t)$*;*

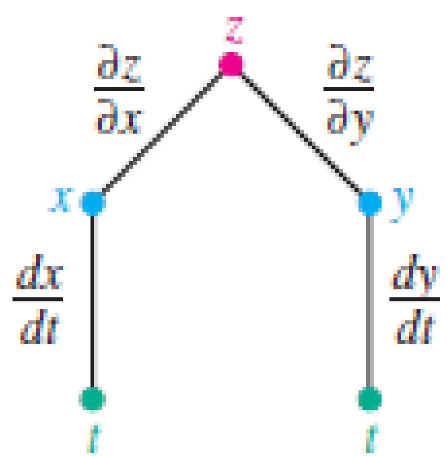
$$z = f(g(t), h(t))$$

- *This means that* $z$ *is indirectly a function of ,*$t$ *, and the Chain Rule gives a formula for differentiating* $z$ *as a function of* $t$*. We assume that* $f$ *is differentiable.*

# Chain Rule

**THE CHAIN RULE (CASE 1)** Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$




$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

# Example

*If* $z = f(x, y) = x^2y + 3xy^4$, *where* $x = \sin 2t$ *and* $y = \cos t$, *find* $dz/dt$ *when* $t = 0$.

Solution: *Since* $z$ *is a function of* $x$ *and* $y$, *which in turn, in turn, a function of a variable* $t$, *therefore the Chain Rule gives*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

*So, we need to find four derivatives: two partial derivatives* $(f_x,\ f_y)$ *and two ordinary derivatives* $(x',\ y')$

$$f_x = 2xy + 3y^4, \qquad f_y = x^2 + 12xy^3,$$

$$x' = \frac{dx}{dt} = 2\cos 2t, \qquad y' = \frac{dy}{dt} = -\sin t$$

# Example (Contd.)

*Hence*

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t),$$

*Now, at t =0,* $x(0) = 0$, $y(0) = 1$

*Therefore,*

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2) - (0+0)(0)$$

$$= 6$$

# Test Your Knowledge

*Suppose* $w$ *is a function of* $x, y,$ *and* $z$*, which are each functions of* $t$*. Explain how to find* $dw/dt$*.*

# Homework 1 – Ex. 11.5

*Use the Chain Rule to find $dz/dt$ for the following functions at $t = 2$*

*(a)*

$$z = \sqrt{1 + x^2 + y^2}, \qquad x = \ln t, \qquad y = \cos t$$

*(b)*

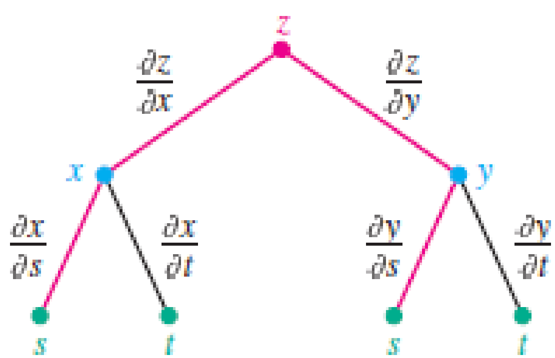
$$z = \cos 2x \sin 3y, \qquad x = \frac{t}{2}, \qquad y = t^4$$

# Chain Rule – Two Independent Variables

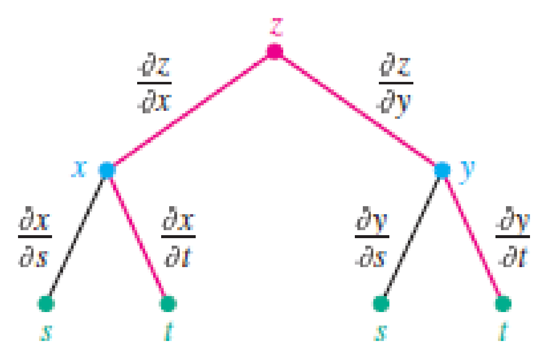
***Second version: Chain Rule with two independent variable***

*Suppose that* $z = f(x, y)$, *where* $x = g(s,t)$ *and* $y = h(s,t)$, *are differential functions of* $s$ *and* $t$, *then the Chain Rule gives*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$




$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$




$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

# Example

*Suppose* $z = u^3 + v^2$, *where* $u = xy^2$ *and* $v = x^2 \sin y$, *find* $\partial z/\partial x$ *and* $\partial z/\partial y$.

Solution: *Since* $z$ *is a function of* $u$ *and v, which in turn, a function of a variable* $x$ *and* $y$, *therefore the Chain Rule gives*

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

*and*

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

# Example

*So, we need to find six partial derivatives.*

*The set of first-partial derivatives is:*

$$f_u = \frac{\partial z}{\partial u} = 3u^2, \qquad f_v = \partial z/\partial v = 2v,$$

$$\frac{\partial u}{\partial x} = y^2 \qquad \frac{\partial v}{\partial x} = 2x\sin y$$

$$\frac{\partial u}{\partial y} = 2xy, \qquad \frac{\partial v}{\partial y} = x^2\cos y$$

*Therefore, the Chain Rules gives:*

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, = 3u^2y^2 + 4vx\sin y$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 6xyu^2 + 2vx^2\cos y$$

# Example

*We want the final solution in variables, $x$ and $y$, only.*

*Substituting the values of $u$ and $v$, we get*

$$\frac{\partial z}{\partial x} = 3x^2y^4 + 4x^3 \sin^2 y$$

$$\frac{\partial z}{\partial y} = 6x^3y^5 + 2x^4 \sin y \ \cos y$$

$$= 6x^3y^5 + x^4 \sin 2y$$

# Test Your Knowledge

*Suppose* $w$ *is a function of* $x, y,$ *and* $z$*, which are each functions of* $s$ *and* $t$*. Explain how to find* $\partial w/\partial t$*.*

# Chain Rule in Computer Engineering

## *Feed-Forward Neural Network*

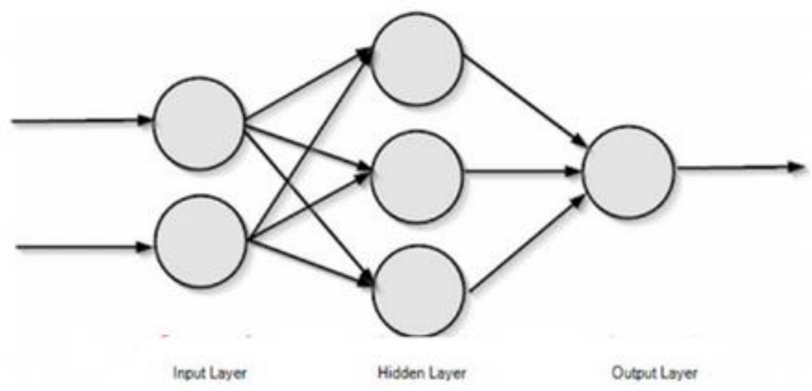
*A **feed-forward neural network** consists of a large number of simple processing units called **perceptrons** organized in multiple hidden layers*.

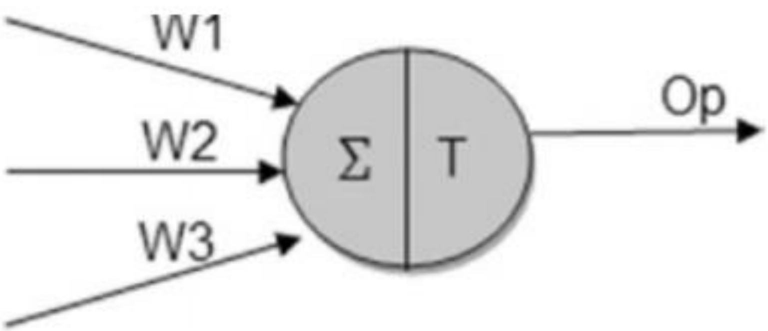
- *The **input layer** consists of neurons that accept the input values. The output from these neurons is same as the input predictors.*

- *The **output layer** is the final layer of a neural network that returns the result back to the user environment. It also signals the previous layers on how they have performed in learning the information and accordingly improved their functions.*

# Chain Rule in Computer Engineering

- ***Hidden layers*** *are in between input and output layers. These are the central computation layers that have the functions that map the input to the output of a node.*



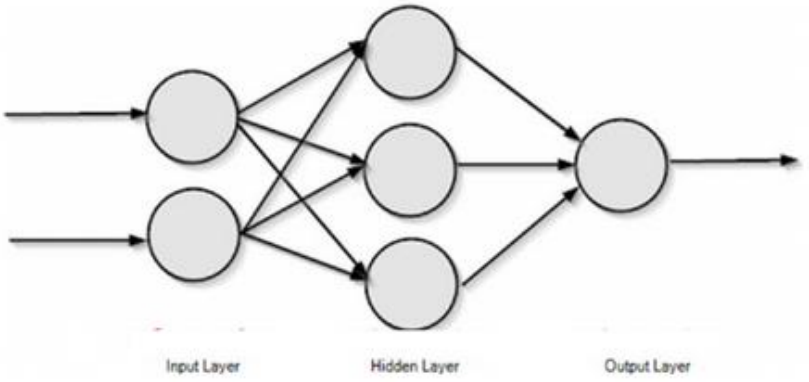

- *A perceptron is the basic processing unit of an artificial neural network.*

- *A perceptron takes several inputs and produces an output.*

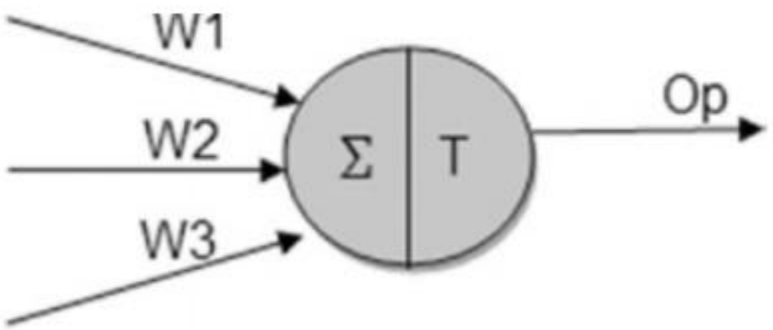
# Chain Rule in Computer Engineering

## *Feed-Forward Neural Networks - General Architecture*

- *A perceptron is the basic processing unit of an artificial neural network.*

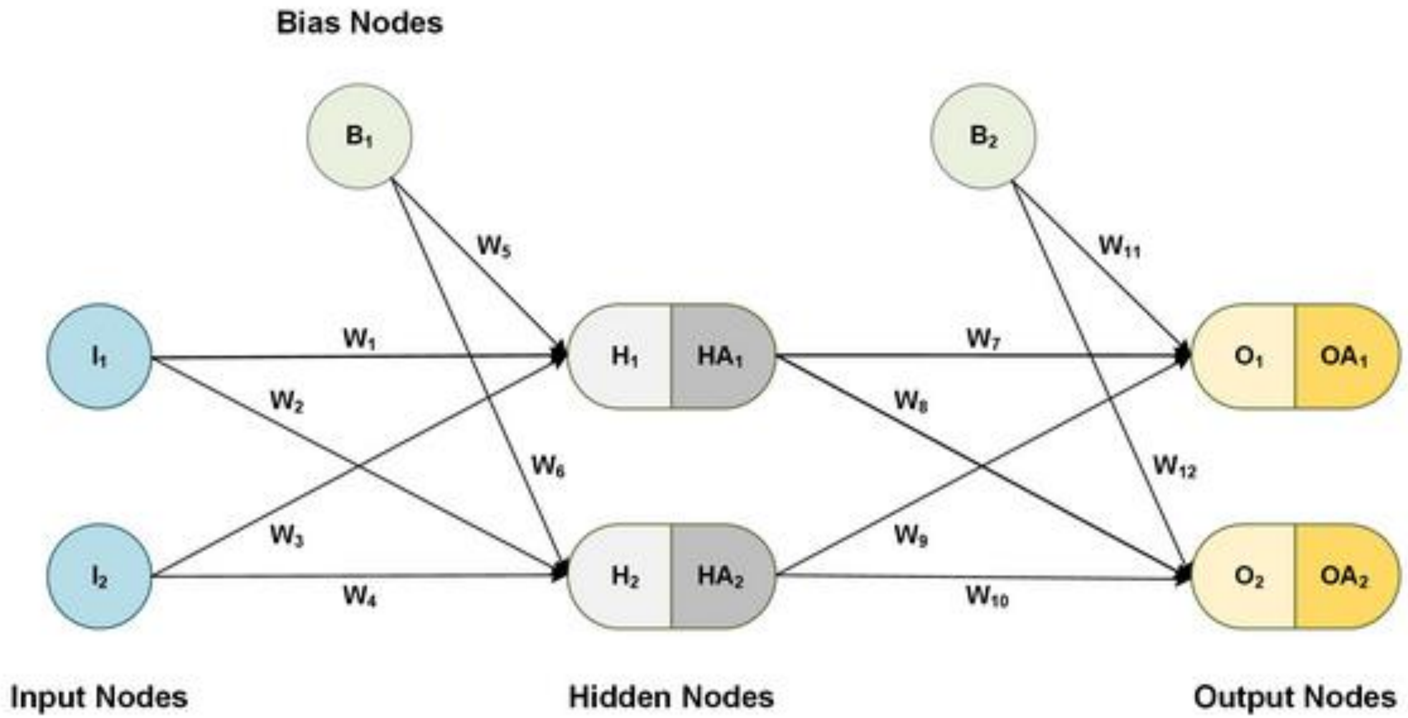



- *A perceptron takes several inputs and produces an output.*



# Applications – Circuit Theory

*Chain rule for weights between input and hidden layer*




$$\frac{\partial e}{\partial W_1} = \underbrace{\left(\frac{\partial e}{\partial OA_1}\frac{\partial OA_1}{\partial O_1}\frac{\partial O_1}{\partial HA_1}\frac{\partial HA_1}{\partial H_1}\frac{\partial H_1}{\partial W_1}\right)}_{(1)} + \underbrace{\left(\frac{\partial e}{\partial OA_2}\frac{\partial OA_2}{\partial O_2}\frac{\partial O_2}{\partial HA_1}\frac{\partial HA_1}{\partial H_1}\frac{\partial H_1}{\partial W_1}\right)}_{(2)}$$

# Applications – Circuit Theory

*The voltage $V$ in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance is slowly increasing as the resistor heats up.*

*Use Ohm's Law, V= IR , to find how the current is changing at the moment when $R = 400\ \Omega$, $I = 0.08\ A$, $dV/dt = -0.01 V/s$, and $dR/dt = 0.03\ \Omega/s$.*

# Solution

*Solution: Since both the current and the resistance change with time, therefore, the Chain Rule gives*

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

*Now* $\partial V/\partial I = R$ *and* $\partial V/\partial R = I$*, therefore*

$$\frac{dV}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}$$

*Using* $R = 400\ \Omega$, $I = 0.08\ A, dV/dt = -0.01V/s$, *and* $dR/dt = 0.03\ \Omega/s$, *we get*

$$-0.01 = 400\frac{dI}{dt} + 0.08 \times 0.03$$

$$\frac{dI}{dt} = -0.000031\ A/s$$

# Application - Changing Cylinder

*The volume of a right circular cylinder with radius r and height h is* $V = \pi r^2 h$.

*(a) Assume r and h are functions of t. Find* $V'(t)$.

*(b) Suppose* $r = e^t$ *and* $h = e^{-2t}$, *for* $t \geq 0$. *Use part (a) to find* $V'(t)$.

*(c) Does the volume of the cylinder in part (b) increase or decrease as t increases?*

# Application - Changing Cylinder

*Solution:*

*(a) By the chain rule,*

$$V'(t) = 2\,\pi r(t)h(t)r'(t) + \pi[r(t)]^2 h'(t)$$

*(b) Substituting* $r = e^t$ *and* $h = e^{-2t}$ *gives*

$$V'(t) = 2\pi e^t e^{-2t} e^t + \pi e^{2t}(-2e^{-2t})$$
$$= 0.$$

*(c) Because* $V'(t) = 0$*, the volume remains constant.*

# Homework 2- Application – Doppler Effect

*If a sound signal with frequency $f_s$ is produced by a source traveling along a line with speed $v_s$and an observer is traveling with speed $v_0$ along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is*

$$f_0 = \left(\frac{c + v_0}{c - v_0}\right) f_s$$

*where $c$ is the speed of sound, about $332\ ms$.*

*Suppose that, at a particular moment, you are in a train traveling at $34\ ms$ and accelerating at $1.2\ ms^2$. A train is approaching you from the opposite direction on the other track at 40 m/s , accelerating at $1.4ms^2$ , and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?*

# Homework 3

*(a) Use a tree diagram to write the required Chain Rule formula for the following:*

$$w = f(x, y, z),$$

*Where* $x = g(t), \quad y = h(t)$*, and* $z = p(t)$*. Find* $dw/dt$.

*(b) Use a tree diagram to write the required Chain Rule formula for the following:*

$$w = f(x, y, z),$$

*Where* $x = g(t, s), \quad y = h(t, s)$*, and* $z = p(t, s)$*. Find* $\partial w/\partial t$ *and* $\partial w/\partial s$.

# Implicit Differentiation

- *Using the Chain Rule for partial derivatives, the technique of implicit differentiation can be put in a larger perspective.*

- *If x and y are related through an implicit relationship, such as*

$$\sin xy + \pi y^2$$

*then $dy/dx$ is computed using* ***implicit differentiation***

*Follow-up: How can the Chain Rule be used to give a more complete description of the process of implicit differentiation?*

# Implicit Differentiation

*We suppose that an equation of the form*

$$F(x, y) = 0$$

*defines* $y$ *implicitly as a differentiable function of* $x$ *, that is,* $y = f(x)$*, where* $F\big(x, f(x)\big) = 0$ *for all* $x$ *in the domain of* $f$*.*

- *If* $F$ *is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation* $F(x, y)$ *with respect to* $x$*. Since both* $x$ *and* $y$ *are functions of* $x$*, we obtain*

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

# Example – Implicit Differentiation

*Find $dy/dx$ when*

$$F(x,y) = \sin xy + \pi y^2 - x = 0$$

*Solution:*

$$F_x = \frac{\partial F}{\partial x} = y\cos xy \ - 1$$

$$F_y = \frac{\partial F}{\partial x} = x\cos xy + 2\pi y$$

*The implicit differentiation theorem gives*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{ycosxy - 1}{xcos\ xy + 2\pi y}$$

$$\frac{dy}{dx} = \frac{1 - ycosxy}{xcos\ xy + 2\pi y}$$

# Example – Implicit Differentiation

*Use implicit differentiation to find $dy/dx$ when*

$$\cos(x-y) = xe^y$$

*Solution:*

$$F(x,y) = \cos(x-y) - xe^y = 0$$

$$F_x = \frac{\partial F}{\partial x} = -\sin(x-y) \ - e^y$$

$$F_y = \frac{\partial F}{\partial x} = -\sin(x-y)(-1) - xe^y = \sin(x-y) - xe^y$$

*The implicit differentiation theorem gives*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(x-y) \ - e^y}{\sin(x-y) - xe^y}$$

$$\frac{dy}{dx} = \frac{\sin(x-y) \ - e^y}{\sin(x-y) - xe^y}$$

# Homework 4

*Use implicit differentiation to find $dy/dx$ when*

$$(i) \qquad \sqrt{xy} = 1 + x^2y + ye^{x^2}$$

$$(ii) \qquad y\ln(x^2 + y^2 + 4) = \sin x \cos x$$

# Example

*Show that if $f$ is homogenous of degree n, then*

$$(a) \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\,f(x,y)$$

$$(b) \qquad x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y)$$

*Solution:*

*A function f is called homogeneous of degree n if it satisfies the equation*

$$f(tx,ty) = t^n f(x,y)$$

*For all t, where n is a positive integer and f has continuous second-order partial derivatives.*

# Solution

*(a) Differentiating both sides of* $f(tx,ty) = t^n f(x,y)$ *with respect to t, we get*

$$\frac{\partial}{\partial t} f(tx,ty) = \frac{\partial}{\partial t}[t^n f(x,y)]$$

$$\frac{\partial}{\partial(tx)} f(tx,ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial}{\partial(ty)} f(tx,ty) \cdot \frac{\partial(ty)}{\partial t} = nt^{n-1} f(x,y)$$

*Differentiate again with respect to t*

$$x\left[\frac{\partial^2}{\partial(tx)^2} f(tx,ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty)\partial(tx)} f(tx,ty) \cdot \frac{\partial(ty)}{\partial t}\right]$$
$$+ y\left[\frac{\partial^2}{\partial(tx)\partial(ty)} f(tx,ty) \cdot \frac{\partial(tx)}{\partial t} + \frac{\partial^2}{\partial(ty)^2} f(tx,ty) \cdot \frac{\partial(ty)}{\partial t}\right]$$
$$= n(n-1)t^{n-2} f(x,y)$$

# Solution

*(b) Setting t=1, we get*

$$x\left[\frac{\partial^2}{\partial x^2} f(x,y)\cdot\frac{\partial(x)}{\partial t}+\frac{\partial^2}{\partial y\partial x} f(x,y)\cdot\frac{\partial y}{\partial t}\right]$$
$$+\,y\left[\frac{\partial^2}{\partial x\partial y} f(x,y)\cdot\frac{\partial x}{\partial t}+\frac{\partial^2}{\partial y^2} f(x,y)\cdot\frac{\partial y}{\partial t}\right]=n(n-1)\,f(x,y)$$

*Using the Clairaut's Theorem* $f_{yx}=f_{xy}$, *we get*

$$x^2 f_{xx}+2xyf_{xy}+y^2 fyy=n(n-1)f(x,y)$$

# Example – Q. 37. Ex. 11.5

*Assume that the given function*

$$z = f(x, y), where \quad x = r\cos\theta, \quad y = r\sin\theta$$

*show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

*Solution: By the Chain Rule,*

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

# Example – Q. 37. Ex. 11.5

$$\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} r^2 \sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2\theta$$

*Thus,*

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\sin^2\theta + \cos^2\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

# Homework 5

*Assume that the given function*

$$z = f(x, y), where \qquad x = r\cos\theta, \qquad y = r\sin\theta$$

*show that*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2 z}{\partial \theta^2}\right) + \frac{1}{r}\left(\frac{\partial z}{\partial r}\right)$$

# Example – Q. 48. Ex. 11.5

*The derivative $dy/dx$ of a function defined implicitly by an equation $F(x, y) = 0$, provided that $F$ is differentiable and $F_y \neq 0$. Prove that if F has continuous second derivatives, then a formula for the second derivative of $y$ is*

$$\frac{d^2y}{dx^2} = -\frac{FxxF_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}$$

*Solution: Given the function defined implicitly by $F(x, y) = 0$, where F is differentiable and $F_y \neq 0$,*

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

*Let* $$G(x, y) = -\frac{F_x}{F_y} = \frac{dy}{dx}$$

# Example – Q. 48. Ex. 11.5

*Differentiating both side with respect to* $x$

$$\frac{d^2y}{dx^2} = \frac{\partial G}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$$

*where*

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right) = -\frac{F_y F_{xx} - F_x F_{yx}}{F_y^2}$$

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right) = -\frac{F_y F_{xy} - F_x F_{yy}}{F_y^2}$$

*Thus*

$$\frac{d^2y}{dx^2} = \left(-\frac{F_y F_{xx} - F_x F_{yx}}{F_y^2}\right)(1) + \left(-\frac{F_y F_{xy} - F_x F_{yy}}{F_y^2}\right)\left(-\frac{F_x}{F_y}\right)$$

# Example – Q. 48. Ex. 11.5

$$\frac{d^2y}{dx^2} = \left(-\frac{F_yF_{xx} - F_xF_{yx}}{F_y^2}\right) + \left(\frac{F_yF_{xy} - F_xF_{yy}}{F_y^2}\right)\left(\frac{F_x}{F_y}\right)$$

$$= -\frac{F_{xx}F_y^2 - F_{yx}F_xF_y - F_{xy}F_yF_x + F_{yy}F_x^2}{F_y^3}$$

*But F has continuous second derivative, so by Clairaut's Theorem* $F_{yx} = F_{xy}$ *and*

$$\frac{d^2y}{dx^2} = \frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}$$