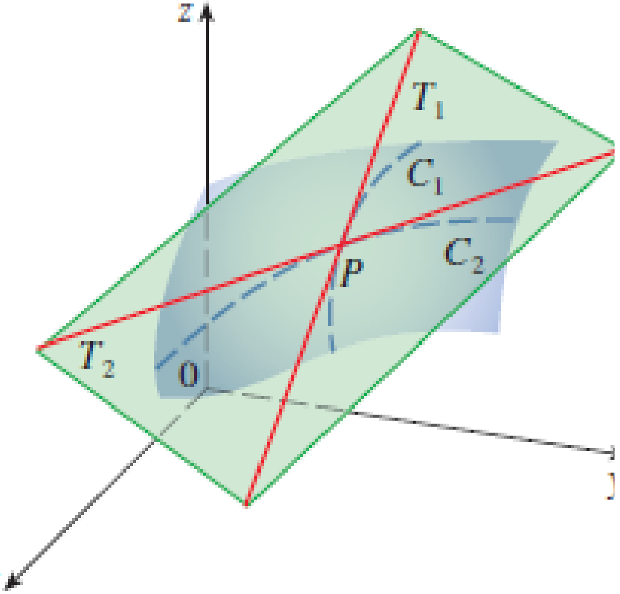
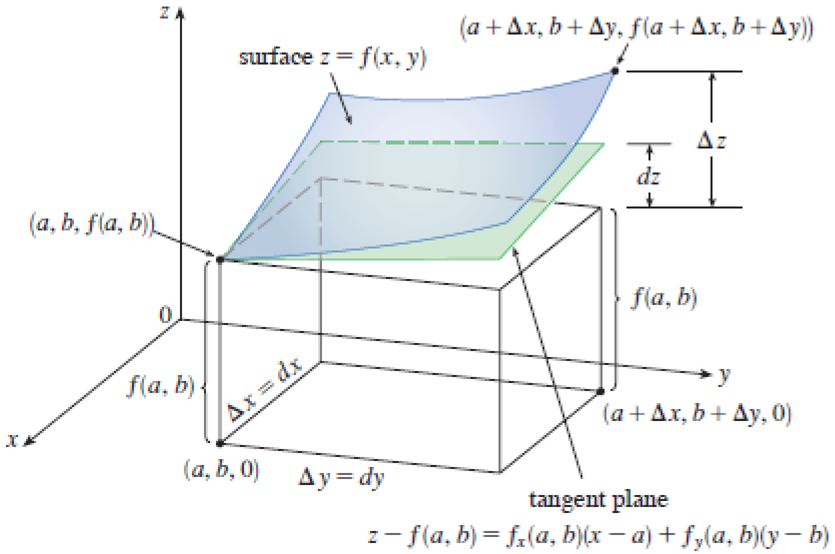
Lecture 16 – Sec. 11.4
Tangent Planes and Linear Approximations





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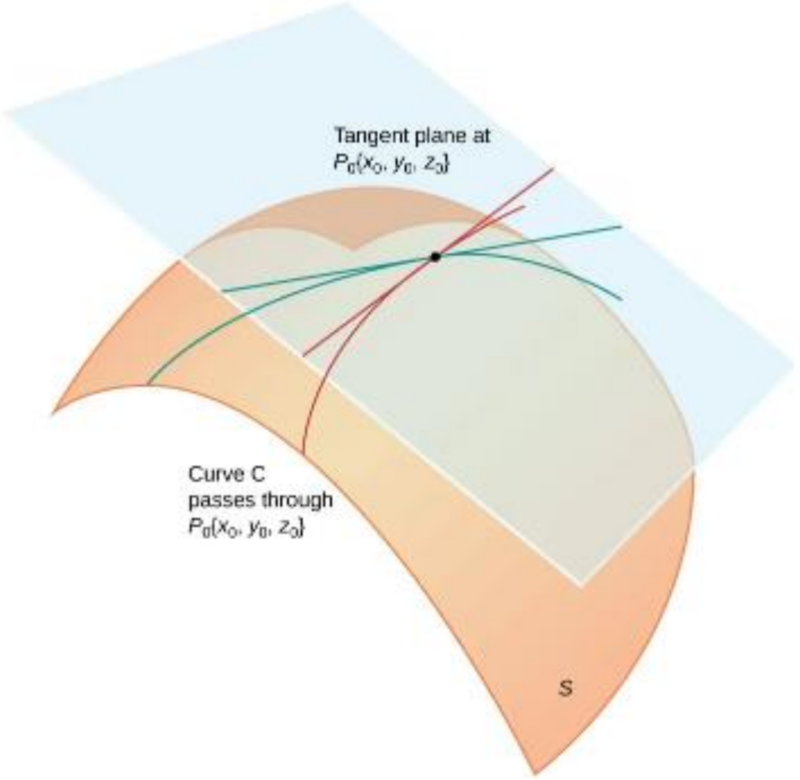
Tangent Planes

- *Determine the equation of a plane tangent to a given surface at a point.*
- *Use the tangent plane to approximate a function of two variables at a point.*
- *Explain when a function of two variables is differentiable.*
- *Use the total differential to approximate the change in a function of two variables.*

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point.

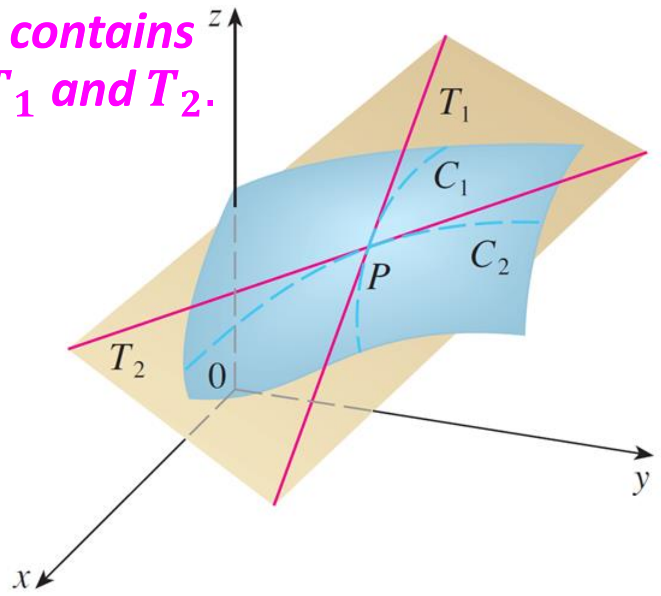
- *A tangent plane at a regular point contains all of the lines tangent to that point.*
- *A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners).*
- *Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly.*



Tangent Planes

- *Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.*
- *Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.*
- *Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.*

The tangent plane contains the tangent lines T_1 and T_2.



- *Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.*

Equation of Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

We know that any plane passing through the point P(x_0, y_0, z_0) has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ *and* $B = -B/C$, *we can write it in the form*

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

If this equation represents the tangent plane at P, then its intersection with the plane $y = y_0$ *must be the tangent line* T_1. *Setting* $y = y_0$ *gives*

$$z - z_0 = a(x - x_0), \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Equation of Tangent Planes

But we know that the slope of the tangent T_1 *is* $f_x(x_0, y_0)$. *Therefore* $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$, *we get*

$$z - z_0 = b(y - y_0)$$

which must represent the tangent line T_2, *so* $b = f_y(x_0, y_0)$.

Therefore, the equation of the tangent plane is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\boxed{z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

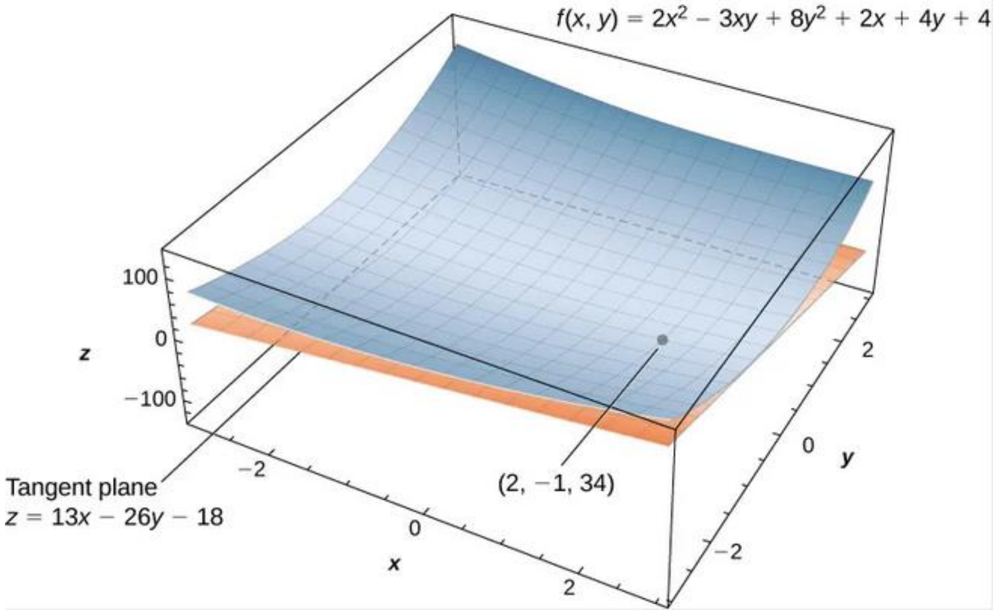
Tangent Planes

Find the equation of the tangent plane to the surface defined by the function

$$z = f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4 \quad at\ (2, -1)$$

Solution: First, we must calculate $f_x(x, y)$ and $f_y(x, y)$ with $x_0 = 2$ and $y_0 - 1$.

$$f_x(x, y) = 4x - 3y + 2,$$
$$f_x(2, -1) = 13$$
$$f_y(x, y) = -3x + 16y - 4,$$
$$f_y(2, -1) = -26$$
$$f(2, -1) = 34$$



Therefore, the equation of the tangent plane is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$z = 13x - 26y - 18$$

Example

A tangent plane to a surface does not always exist at every point on the surface.

Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution

If either $x = 0$ or $y = 0$, then $f(x,y) = 0$, so the value of the function does not change on either the x- or y-axis.

Therefore, $f_x(x,0) = f_y(0,y) = 0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

gives $z = 0$ as the equation of the tangent line.

Example

However, if we approach the origin from a different direction, we get a different story.

For example, suppose we approach the origin along the line $y = x$*. If we put* $y = x$ *into the original function, it becomes*

$$f(x,x) = \frac{x(x)}{\sqrt{x^2 + (x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}$$

When $x > 0$*, the slope of this curve is equal to* $2/\sqrt{2}$*.*
When $x < 0$*, the slope of this curve is equal to* $-2/\sqrt{2}$*.*

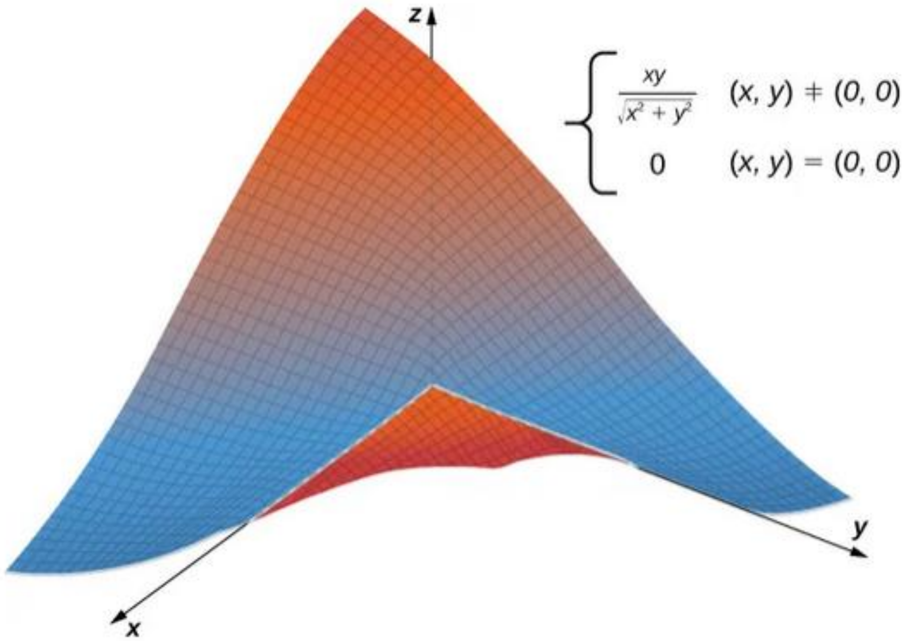
Example

This presents a problem.

- *In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane.*

- *This is clearly not the case here.*

- *This implies that this function is not differentiable at the origin.*



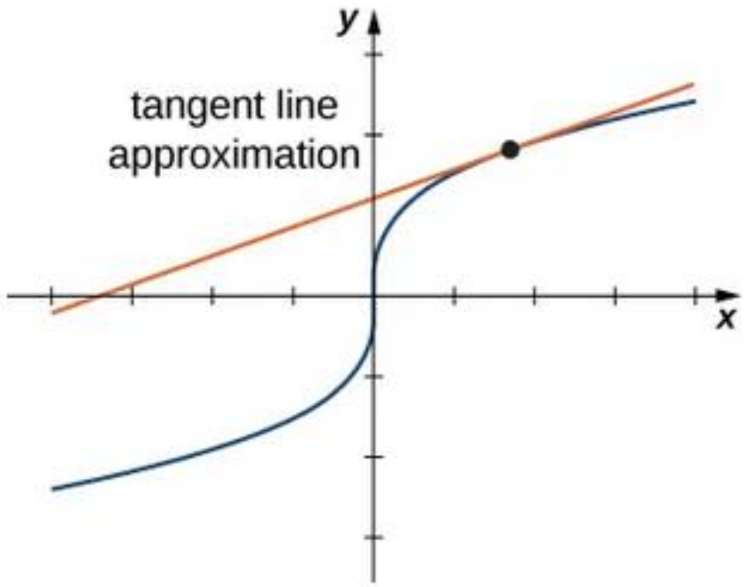
Linear Approximations

The linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a)$$

- *The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$.*

- *When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.*



Linear approximation of a function in one variable.

Tangent Plane Approximations

Definition – Linear Approximation

*Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by*

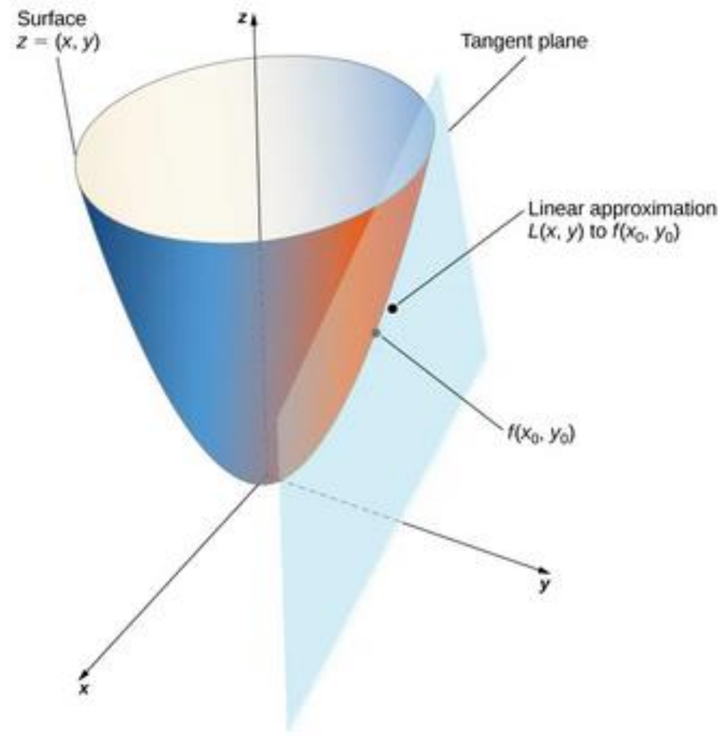
$$L(x, y) = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(y - y_0)$$

- *Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ a the point (x_0, y_0).*

Tangent Plane Approximations

- *The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (tangent plane) yields a value that is also reasonably close to exact value of $f(x, y)$.*

- *Furthermore, the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).*



Example – Q. 13. Ex. 11.4

Explain why the function

$$f(x,y) = e^{-xy}\cos y, \qquad\qquad (\pi, 0)$$

is differentiable at the given point. Then find the linearization $L(x,y)$ of the function at that point.

Solution: For the given function to be differentiable at the given point, both f_x and f_y must be continuous function.

$$f_x = -ye^{-xy}\cos y \Rightarrow f_x(\pi, 0) = 0$$

$$f_y = -e^{-xy}(\sin y + x\cos y) \Rightarrow f_y(\pi, 0) = -\pi.$$

Both f_x and f_y are continuous, so f is differentiable at $(\pi, 0)$.

The linear approximation of f at $(\pi, 0)$ is

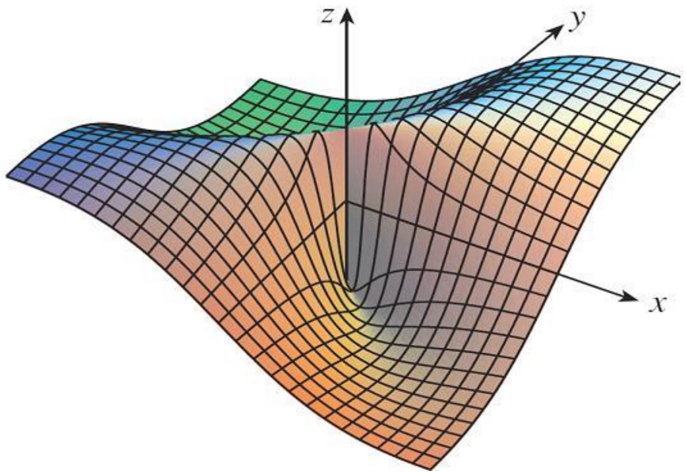
$$L(x,y) = f(\pi, 0) + (x-\pi)f_x(\pi, 0) + (y-0)f_y(\pi, 0)$$

$$L(x,y) = 1 - \pi y$$

Linear Approximations Behaving Badly!

We have defined tangent planes for surfaces $z = f(x, y)$, *where f has continuous first partial derivatives.*

What happens if f_x *and* f_y *are not continuous?*



$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0,0) = 0$ *and* $f_y(0,0) = 0$, *but* f_x *and* f_y *are not continuous.*

Differentiability

- *The linear approximation would be* $f(x, y) = 0$*, but* $f(x, y) = 1/2$ *at all the points on the line* $y = x$.
- *So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.*
- *We know that for a function of one variable,* $y = f(x)$*, if* x *changes from* a *to* $a + \Delta x$*, we defined the increment of* y *as*

$$\Delta y = f(a + \Delta x) - f(a)$$

- *If f is differentiable at a, then*

$$\Delta y = f'(a)\Delta x = \epsilon\, \Delta x$$

where $\epsilon \to 0$ *as* $x \to 0$*,*

Differentiability

- *Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to a + Δx and y changes from b to b + Δy. Then the corresponding* ***increment*** *of z is*

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

- *Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.*

Differentiability

We define the differentiability of a function of two variables as follows.

Definition *If* $z = f(x, y)$*, then f is* ***differentiable*** *at (a, b) if* Δz *can be expressed in the form*

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \qquad (1)$$

where ε_1 *and* ε_2 *are functions of* Δx *and* Δy *such that* ε_1 *and* $\varepsilon_2 \to 0$ *as* $(\Delta x, \Delta y) \to (0, 0)$*.*

- *The above definition says that a differentiable function is one for which the linear approximation is a good approximation when (x, y) is near (a, b).*
- *In other words, the tangent plane approximates the graph of f well near the point of tangency.*

Differentiability

It's sometimes hard to use Eq. (1) directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem: If the partial derivatives f_x and f_y exist near a point (a,b) and are continuous at (a,b), then f is differentiable at (a,b)

Example

Show that $f(x,y) = xe^{xy}$ is differentiable at (1, 0) and find its linearization there. Then use it to approximate f(1.1, -0.1).

Solution: First find the partial derivatives

$$f_x(x,y) = e^{xy} + xye^{xy}, \qquad f_x(1,0) = 1$$

$$f_y(x,y) = x^2e^{xy} \qquad f_y(1,0) = 1$$

Both f_x and f_y are continuous, so f is differentiable (by Theorem).

The linearization is

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = x + y$$

The corresponding linear approximation is

$$xe^{xy} = x + y$$

$$f(1.1, -0.1) = 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{(1.1)(-0.1)} = 0.98542$

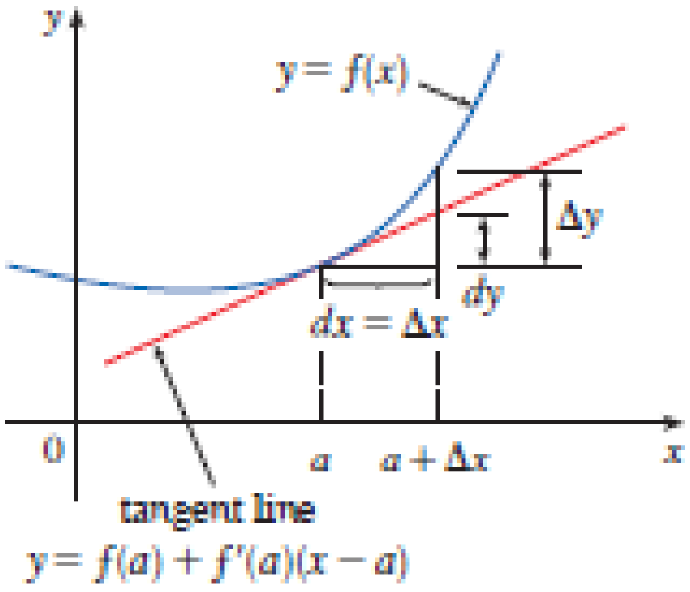
Differentials

For a function of the form $y = f(x)$*, if the independent variable changes from x to x + dx, the corresponding change* Δy *in the dependent variable is approximated by the differential*

$$dy = f'(x)dx$$

which is the change in the linear approximation.

Therefore, $\Delta y \approx dy$*, with the approximation improving as* dx *approaches 0.*



Differentials

For functions of the form $z = f(x, y)$, we start with the linear approximation to the surface

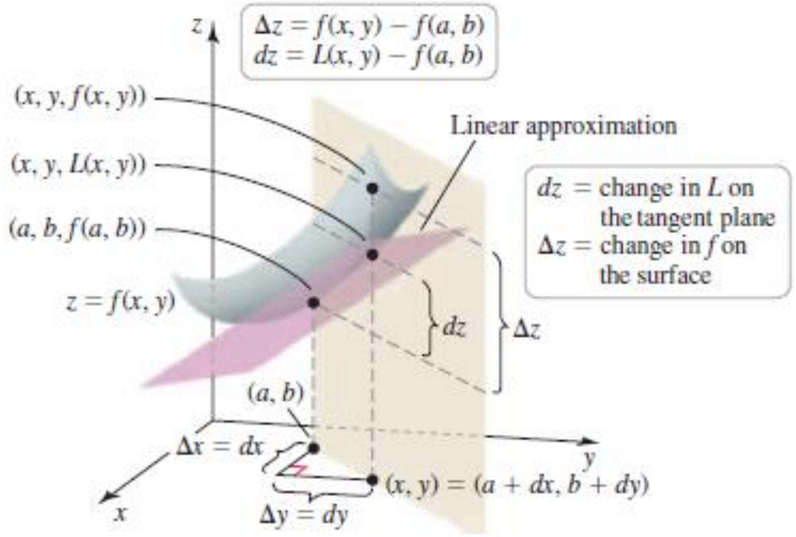
$$f(x, y) \approx L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

The exact change in the function between the points (a, b) and (x, y) is

$$\Delta z = f(x, y) - f(a, b).$$

Replacing $f(x, y)$ with its linear approximation, the change Δz is approximated by

$$\Delta z \approx \underbrace{L(x, y) - f(a, b)}_{dz} = f_x(a, b)\underbrace{(x - a)}_{dx} + f_y(a, b)\underbrace{(y - b)}_{dy}.$$

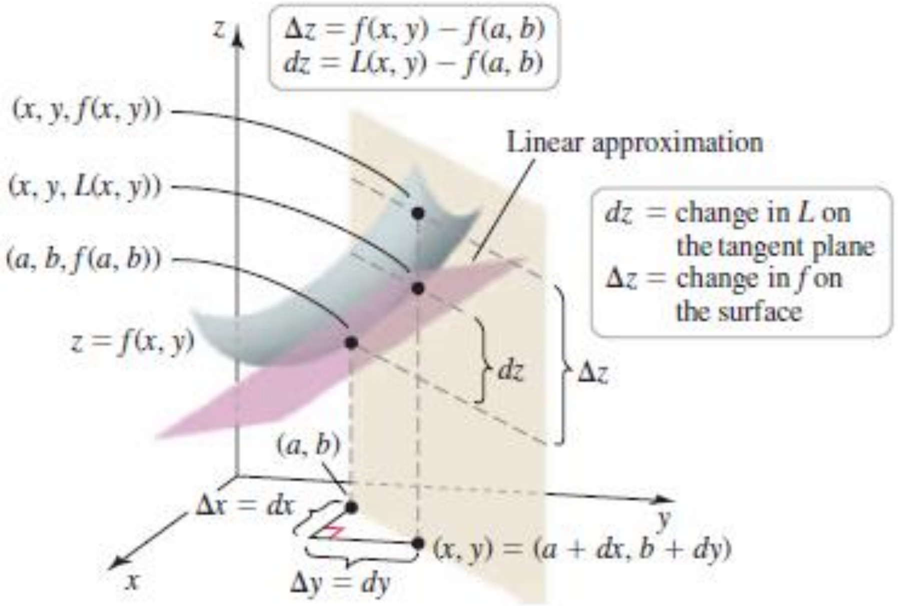


Differentials

The change in the x-coordinate is $dx - x - a$ *and the change in the y-coordinate is* $dy = y - b$.

As before, we let the differential dz *denote the change in the linear approximation. Therefore, the approximate change in the z-coordinate is*

$$\Delta z \approx dz = \underbrace{f_x(a, b)\,dx}_{\text{change in } z \text{ due to change in } x} + \underbrace{f_y(a, b)\,dy}_{\text{change in } z \text{ due to change in } y}.$$

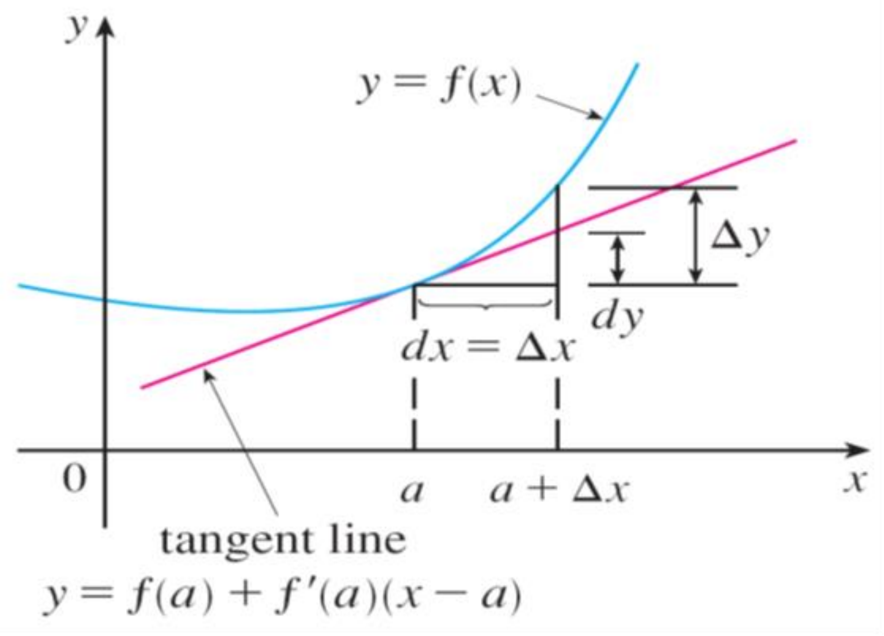


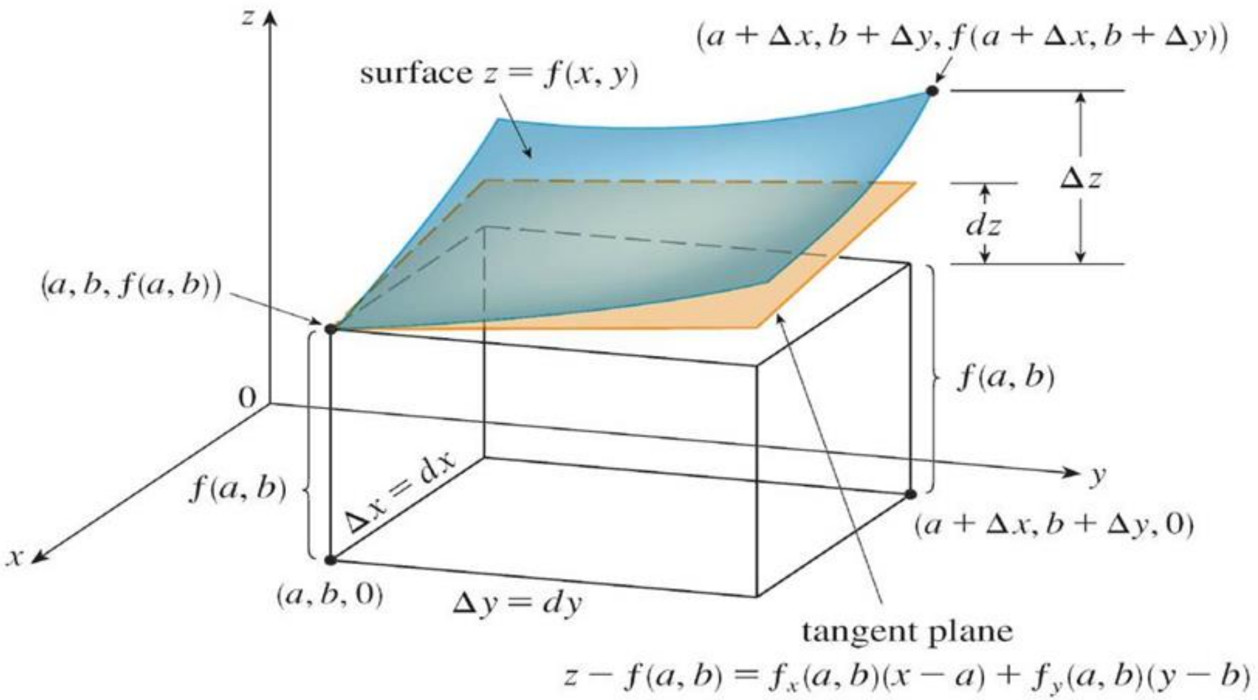
This expression says that if we move the independent variables from (a, b) to (x, y) = (a + dx, b + dy), the corresponding change in the dependent variable z has two contributions—one due to the change in x and one due to the change in y. If dx and dy are small in magnitude, then so is z.

Differentials

The geometric interpretation of the differential dz and the increment Δz:

dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface z = f (x, y) when (x, y) changes from (a, b) to (a + Δx, b + Δy).





Differentials

DEFINITION **The differential dz**

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + dx, y + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)\, dx + f_y(x, y)\, dy.$$

Example – Q. 24. Ex. 11.4

Find the differential of the function

$$w = xy\, e^{xz}$$

Solution

The differential of the given function is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Now

$$\frac{\partial w}{\partial x} = ye^{xz} + xyze^{ez}, \qquad \frac{\partial w}{\partial y} = xe^{xz}, \qquad \frac{\partial w}{\partial z} = x^2ye^{xz}$$

Therefore,

$$dw = (ye^{xz} + xyze^{ez})dx + xe^{xz}dy + x^2ye^{xz}dz$$

Homework 1 – 11.4

If

$$z = x^2 - xy + 3y^2$$

And (x,y) change from (3,-1) to (2.96, -0.95), compare the values of Δz *and* dz.

Applications – Example: Q. 32, Ex. 11.4

The wind-chill index is modelled by the function

$$W = 13.12 + 0.6215\,T - 11.37\,v^{0.16} - 0.3965Tv^{0.16}$$

where T is the actual temperature (in °C) and v is the wind speed (in km/h). The wind speed is measured as 26 km/h, with a possible error of ± 2 km/h, and the actual temperature is measured as -11°C, with a possible error of ± 1°C. Use differentials to estimate the maximum error in the calculated value of W due to the measurement errors in T and v.

Applications – Example: Q. 32, Ex. 11.4

Solution: The differential of W is

$$dW = \frac{\partial W}{\partial T} dT + \frac{\partial W}{\partial v} dv$$

$$= (0.6215 + 0.3965 v^{0.16}) dT + [-11.37(0.16) v^{-0.84} + 0.3965\&(0.16) v^{-0.84}] dv$$

$$= (0.6215 + 0.3965 v^{0.16}) dT + (-1.8192 + 0.06344T) v^{-0.84} dv$$

Here we have $|\Delta T| \leq 1,\ \ |\Delta v| \leq 2$, *so we take* $dT = 1$ *and* $dv = 2$ *with T=-11 and v=26. The maximum error in the calculated value of W is about*

$$dW = (0.6215 + 0.3965(26)^{0.16})(1) + \big(-1.8192 + 0.06344(-11)\big)(26)^{-0.84}(2)$$
$$\approx 0.96$$

Application - Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within ε cm.

(a) Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

(b) What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm.

Solution:

(a) *If the dimensions of the box are x, y, and z, its volume is V = xyz and so*

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\ dx + xz\ dy + xy\ dz$$

Application - Example

We are given that

$$|\Delta x| \leq \epsilon, \qquad |\Delta y| \leq \epsilon \quad and \quad |\Delta z| \leq \epsilon$$

To estimate the largest error in the volume, we therefore use dx = ε, dy = ε, and dz = ε together with x = 75, y = 60, and z = 40:

$$\Delta V \approx dV = (60)(40)\epsilon + (75)(40)\epsilon + (75)(6)\epsilon$$
$$= 9900\,\epsilon$$

Thus the maximum error in the calculated volume is about 9900 times larger than the error in each measurement taken.

Application - Example

(b) If the largest error in each measurement is ε = 0.2 cm, then

$$dV = 9900(0.2) = 1980$$

So an error of only 0.2 cm in measuring each dimension could lead to an error of approximately $1980\ cm^3$ *in the calculated volume.*

This may seem like a large error, but you can verify that it's only about 1% of the volume of the box.

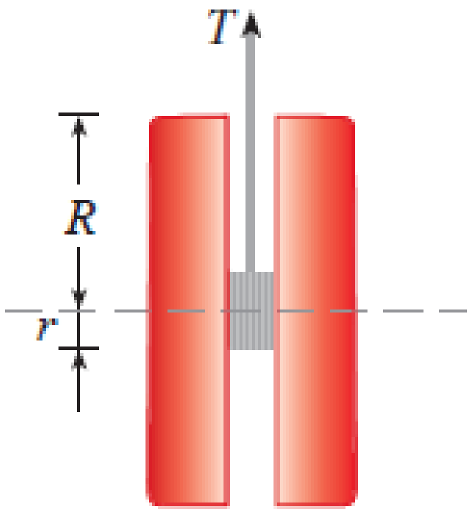
Homework 2 – 11.4

The tension T in the string of the yo-yo in the figure is

$$T = \frac{mgr}{2r^2 + R^2}$$

where m is the mass of the yo-yo and g is acceleration due to gravity.

Use differentials to estimate the change in the tension if R is increased from 3 cm to 3.1 cm and r is increased from 0.7 cm to 0.8 cm. Does the tension increase or decrease?



Homework 3 – 11.4

The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV - 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins.

Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.