# CA3

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## Question 1:

Let $X$ $N_p(\mu, \Sigma)$ with $|\Sigma| > 0$. Using the spectral decomposition of the covariance matrix, prove that

$$(X-\mu)'\Sigma^{-1}(X-\mu) \quad \chi^2_p$$

1. Start with $\Sigma e_i = \lambda_i e_i$ and note that $\Sigma$ is positive definite.

$$\Sigma e_p = \lambda_p e_p \qquad (1)$$
$$\Sigma e_p = [\Sigma e_1 .... \Sigma e_p]$$
$$[\lambda_i e_1 .... \lambda_p e_p]$$
$$[\lambda_i e_1 .... \lambda_p e_p] = e_p D_p$$

$$\Sigma e_p = e_p D_p$$
$$\Sigma^{-1} = e_p \ D_p^{-1} \ e_p^{-1}$$
$$\Sigma^{-1} = e_p \ D_p^{-1} \ e_p'$$
$$\Sigma^{-1} = \sum_{i=1}^{p} \ e_i \ \lambda_i^{-1} \ e_i'(2)$$

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2. Using this, rewrite $(X-\mu)'\Sigma^{-1} \ (X-\mu)$ as the sum of p squared variables

$$
\begin{aligned}
&(X-\mu)'\Sigma^{-1}\ (X-\mu) \\
&= (X-\mu)'\ (\sum_{i=1}^{p}\ e_i\ \lambda_i^{-1}\ e_i')\ (X-\mu) \qquad (2) \\
&= \sum_{i=1}^{p}\lambda_i^{-1}\ (X-\mu)'\ e_i\ e_i'\ (X-\mu) \\
&= \sum_{i=1}^{p}\lambda_i^{-1}\ [e_i'\ (X-\mu)]^2 \\
&= \sum_{i=1}^{p}[\frac{1}{\sqrt{\lambda_i}}\ e_i'\ (X-\mu)]^2
\end{aligned}
$$

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3. Showing that this vector i.e. $[\frac{1}{\sqrt{\lambda_i}}\ e_i'\ (X-\mu)]$ is a multivariate normal with zero mean and identity covariance

$$A\ X \sim N_q(A\mu,\ A\Sigma A')$$

$$[\frac{1}{\sqrt{\lambda_i}}e_i']\ [X-\mu] \sim N[0, [(\frac{1}{\sqrt{\lambda_i}}e_i')\ \Sigma\ (\frac{1}{\sqrt{\lambda_i}}e_i')']]$$

Firstly, since X- $\mu$ is centered, we know that the <u>Mean</u> of this vector is equal to zero.

Secondly, to show that the <u>Variance</u> of this vector is equal to the identity matrix ...

$$
\begin{aligned}
&(\frac{1}{\sqrt{\lambda_i}}e_i')\ \Sigma\ (\frac{1}{\sqrt{\lambda_i}}e_i')' \\
&= \frac{1}{\sqrt{\lambda_i}}e_i'\ \Sigma\ \frac{1}{\sqrt{\lambda_i}}e_i \\
&= \frac{1}{\lambda_i}e_i'\ \Sigma\ e_i \\
&= \frac{1}{\lambda_i}e_i'\ \lambda_i\ e_i \quad (1) \\
&= I
\end{aligned}
$$

Remembering that $e_i'\ e_i = I$ because the eigen vectors are orthogonal to each other

Therefore

$$[\frac{1}{\sqrt{\lambda_i}}e_i']\ [X-\mu] = Z_i \sim N[0, I]$$

4. Finally showing Q.E.D

$$(X-\mu)'\Sigma^{-1}\ (X-\mu) \quad = \sum_{i=1}^{p} [\frac{1}{\sqrt{\lambda_i}}\ e_i'\ (X-\mu)]^2 \quad = \sum_{i=1}^{p} Z_i^2 \sim \chi_p^2$$

The sum of p independent standard normal (N(0,1) variables squared follows a chi-squared distribution with p degrees of freedom.

## Question 2

$$\begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{bmatrix} \sim N_5(\begin{bmatrix} 5 \\ 0 \\ -2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 & 3 & -1 & 0 & 5 \\ 3 & 12 & 2 & 2 & -2 \\ -1 & 2 & 9 & 0 & 1 \\ 0 & 2 & 0 & 8 & 2 \\ 5 & -2 & 1 & 2 & 10 \end{bmatrix})$$

Now define $X1 = \begin{bmatrix} x1 \\ x2 \\ x4 \end{bmatrix}$ and $X2 = \begin{bmatrix} x3 \\ x5 \end{bmatrix}$. Find the conditional (joint) distribution of $X1|X2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

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1. First change the matrix around

$$\left[\begin{array}{c} x1 \\ x2 \\ x4 \\ \hline x3 \\ x5 \end{array}\right] \sim N_5(\left[\begin{array}{c} 5 \\ 0 \\ 6 \\ \hline -2 \\ 2 \end{array}\right], \left[\begin{array}{ccc|cc} 8 & 3 & 0 & -1 & 5 \\ 3 & 12 & 2 & 2 & -2 \\ 0 & 2 & 8 & 0 & 2 \\ \hline -1 & 2 & 0 & 9 & 1 \\ 5 & -2 & 2 & 1 & 10 \end{array}\right])$$

$$X1|X2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N(E[X1|X2], cov(X1, x2))$$

$$\mu_1 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

$$\mu_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Sigma_{11} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$

$$\Sigma_{12} = \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix}$$

$$\Sigma_{21} = \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\Sigma_{22} = \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}$$

$$\Sigma_{21} = \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

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2. Calculate the $E[X1|X2]$

$$E[X1|X2] = \mu_1 + \Sigma_{12}\ \Sigma_{22}^{-1}(X_2 -\ \mu_2)$$

$$E[X1|X2] = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}^{-1} \left( \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$$

$$E[X1|X2] = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ 2 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{10}{89} & \frac{-1}{89} \\ \frac{-1}{89} & \frac{90}{89} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E[X1|X2] = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{-15}{89} \\ \frac{22}{89} \\ \frac{-2}{89} \end{bmatrix}$$

$$E[X1|X2] = \begin{bmatrix} \frac{430}{89} \\ \frac{22}{89} \\ \frac{532}{89} \end{bmatrix}$$

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[1] "R output of the same answer"

      [,1]
[1,] 4.831
[2,] 0.247
[3,] 5.978
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3. Calculate $cov[X1, X2]$

$$Cov(X1, X2) = \Sigma_{11} - \Sigma 12 \ \Sigma_{22}^{-1} \ \Sigma_{21}$$

$$Cov(X1, X2) = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 1 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 1 \end{bmatrix}$$

$$Cov(X1, X2) = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} - \begin{bmatrix} -1 & 5 \\ 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{10}{89} & \frac{-1}{89} \\ \frac{-1}{89} & \frac{90}{89} \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 5 & -2 & 1 \end{bmatrix}$$

$$Cov(X1, X2) = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 12 & 2 \\ 0 & 2 & 8 \end{bmatrix} - \begin{bmatrix} \frac{245}{89} & \frac{-122}{89} & \frac{92}{89} \\ \frac{-122}{89} & \frac{84}{89} & \frac{-40}{89} \\ \frac{92}{89} & \frac{-40}{89} & \frac{36}{89} \end{bmatrix}$$

$$Cov(X1, X2) = \begin{bmatrix} \frac{467}{89} & \frac{389}{89} & \frac{-92}{89} \\ \frac{389}{89} & \frac{984}{89} & \frac{218}{89} \\ \frac{-92}{89} & \frac{218}{89} & \frac{676}{89} \end{bmatrix}$$

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[1] "R output of the same answer"

       [,1]   [,2]   [,3]
[1,]  5.247  4.371 -1.034
[2,]  4.371 11.056  2.449
[3,] -1.034  2.449  7.596
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Therefore

$$X1|X2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N(E[X1|X2], cov(X1, x2))$$

$$X1|X2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \sim N(\begin{bmatrix} \frac{430}{89} \\ \frac{22}{89} \\ \frac{532}{89} \end{bmatrix}, \begin{bmatrix} \frac{467}{89} & \frac{389}{89} & \frac{-92}{89} \\ \frac{389}{89} & \frac{984}{89} & \frac{218}{89} \\ \frac{-92}{89} & \frac{218}{89} & \frac{676}{89} \end{bmatrix})$$