UNIVERSITY OF CAPE TOWN
Department of Statistical Sciences Honours

Tim Gebbie : Portfolio Theory
Assignment 1 : MV Backtesting and Out-of-Sample Performance

1 PART I : Introduction to Strategy Backtesting

A *backtest* is a historical simulation of a quantitative investment strategy [1]. The objective of a backtest is to compute the series of returns in terms of the temporal sequence of profits and losses that the strategy would have generated had it been used to make investment decisions of the time period investigated.

Performance measures such as the *Sharpe Ratio* and the *Information Ratio* are often used to quantify the performance of a backtest.

Simulated Performance and Measured Performance are unlikely to be the same. It is very useful to differentiate between insample performance (IS) and out-of-sample performance (OOS). In sample performance considers the simulation over the data used to design or estimate the model. In the more modern statistical learning approach this is teh *training data-set* used in the learning period. The out-of-sample performance is the simulated performance over a selection of data from the period not used in the strategy design or estimation - this is the *testing data-set*. A *backtest* is only consistent when the training set performance is consistent with the testing set performance. In fact there should be three periods of data: 1.) a training set, 2.) a validation set (used to tune the strategy parameters to avoid over-fitting) and 3.) a testing set. Here we will only be concerned with the training set (for the in-sample period) and the testing set (for the out-of-sample period).

The problem of *overfitting* is the situation when the strategy targets specific observations in the training period rather than generic structures or properties. The validation set can be useful in managing overfitting by tuning models in terms of the *bias-variance* payoff.

Here we will be writing an R-script to generate out-of-sample performance of Sharpe Ratio maximising portfolios for the Bond Index (ALBI) and Industrial Indices (based on the Industrial ICB classification). We will compare this to in-sample estimation of the same portfolio using data from the complete data-set. We will also compute the *minimum backtest length*. The minimum length of data that an investment manager should use given the number of simulation trials or realisations used.

It is strongly recommended that that MikTeX and the associated TeXworks editor is used to create the required assignment PDF for submission. This can be integrated into your R markdown file for PART II.

Only a single PDF file will be accepted for the online assignment submission.

If multiple files are submitted only the first file (in alphabetic named ordering) will be graded. All code is to be provide in the appendices of the R markdown file.

```
a <- c(1,2,3)
b <- data.frame(this=a,that=c(3,4,5))
```

Question 1 : Sample Error when Estimating the Sharpe Ratio

The Sharpe Ratio is a statistic the evaluates a strategies past performance based on historical performance computed from the excess returns $\mu = E[\mathbf{R}] - R_f$ and variance $\sigma^2 = \mathrm{Cov}(\mathbf{R})$. If we assume that the excess returns are IID following a Normal distribution then the Sharpe Ratio can be computed as:

$$\mathrm{SR} = \frac{\mu}{\sigma}\sqrt{q} \tag{1}$$

where q is the number of returns per year. This is usually expressed as an annualised quantity. Most models are built on the IID Normal assumption which is why the Sharpe Ratio is so popular. This assumption is questionable. Nevertheless we will use the IID Normal assumption in this assignment.

The true value of SR is unknown because μ and σ are unknown. We need to estimate the Sharpe Ratio: $\hat{\mathrm{SR}}$ from measured data $\tilde{\mathbf{R}}$; we need to estimate the mean and variance: $\hat{\mu}$ and $\hat{\sigma}$.

Show that the distribution of the estimated annualised Sharpe Ratio $\hat{SR}$ converges asymptotically to:

$$\hat{\mathrm{SR}} \xrightarrow[y\to\infty]{a} \mathbb{N}\left(\mathrm{SR}, \frac{1+\frac{\mathrm{SR}^2}{2q}}{y}\right) \tag{2}$$

where y is the number data points (here years) used to estimate the statistic.

[10 marks]

Question 2 : The Maximum of the Sample

Motivate the following proposition for large N:

Theorem 1.1 (1)**.** *Given a sample of N IID Normal random variables X_n: $n = 1, 2, \ldots, N$ where Z is the CDF for the standard normal distribution. The expected maximum of the sample is:*

$$\mathbb{E}[\max_N] := \mathbb{E}[\max\{X_n\}]. \tag{3}$$

The expected maximum can be approximated as:

$$\mathbb{E}[\max_N] \approx (1-\gamma)Z^{-1}(1-\frac{1}{N}) + \gamma Z^{-1}(1-\frac{1}{N}e^{-1}) \tag{4}$$

for some constant γ.

[10 marks]

Question 3 : Minimum Backtest Length

Derive and discuss the minimum backtest length T_{min} [1]:

Theorem 1.2. *The minimum backtest length T_{min} needed to avoid selecting a strategy with an in-sample Sharpe Ratio as the average $\mathbb{E}[\max_N]$ among N independent strategies with an out-of-sample Sharpe Ratio of zero is:*

$$T_{min} < \frac{2\ln(N)}{\mathbb{E}[\mathbb{E}[\max_N]]^2} \tag{5}$$

The shows that the minimum backtest length grows as the analyst tries more independent configurations of the model so as to keep the Sharpe Ratio at a given level. The key point here: the analyst that does not report the number of simulations used to select a particular strategy configuration makes it very difficult to assess the overall risk of strategy overfitting.

[10 marks]

2 PART II : Backtest Performance of the Tangency Portfolio

For the test data compute Sharpe Ratio maximising portfolios (the Tangency Portfolios) from rolling windows of past data. Divide your historical sample of data into two separate data-sets. You will be writing a script file to carry out two numerical experiments for fully invested no-short selling tangency portfolios.

2.1 Experiment 1 : *In-Sample and Out-Of-Sample Sharpe Ratios*

Compute the portfolio Sharpe Ratio for the test set period and compare this to that from the training set period. The data is monthly sampled. Write a R script file using the data provided to compute the tangency portfolio (the Sharpe Ratio maximing portfolio) for the first period, the training set. Implement the portfolio as a Buy-and-Hold (BH) strategy using the data in the second period, the test set. The PDF R markdown report must at least provide a table that compares the in-sample and out-of-sample portfolio means, variances and Sharpe Ratios, and a table with the Buy-and-Hold (BH) portfolio weights. Additional statistical sophistication will lead to extra credits.

[20 marks]

2.2 Experiment 2 : *Out-Of-Sample Backtesting using a Rolling Window*

Compute the sequence of Sharpe Ratio maximising portfolios using rolling monthly incremented data windows. Use the estimated portfolio controls to generate a time-series of strategy performance (for the Tangency Portfolios). Compare the in-sample Sharpe Ratios and the out-of-sample Buy-and-Hold strategy Sharpe Ratios with those of the above strategy. Once again we consider monthly sampled. For each month from the end of the first data-set we consider a window of fixed length. This window is rolled incrementally across the data to incrementally estimate, month by month, the next SR maximising portfolio weights. Implement these weights to compute the portfolio returns for the next period (here a month later). The returns are computed out-of-sample a month after each control vector is estimate using a numerical scheme (as provided in lectures). Store this out-of-sample portfolio performance, increment the window forward. The PDF R markdown report must at least have well labelled and discussed time-series plots of the out-of-sample performance of the incrementally rebalanced strategy from experiement 2 compared to the out-of-sample time-series performance of the Buy-and-Hold strategy from experiment 1, and a table of the various portfolio means, variances, Sharpe Ratios. Additional statistial sophistication will lead to extra credits.

[20 marks]

References

[1] Bailey, D. H., Borwein, J. M., Lopez de Prado, M., Zhu, Q-J, (2014) Pesudo-Mathematics and Financial Charlatanism: The Effects of Backtest Overfitting on Out-of-Sample Performance, Notices of the AMS, 61,5, 458-471

[2] Lee, W. Theory and Methodology of Tactical Asset Allocation, Fabrozzi Associates, 2000