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Extreme Values, Regular Variation, and Point Processes

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criteria are only a control on the right tail. For instance, if $F \in D(\Phi_\alpha)$ then $1 - F(x) \sim x^{-\alpha}L(x)$ as $x \to \infty$. This implies

$$\int_0^\infty x^k F(dx) < \infty \qquad \text{if } k < \alpha$$

(Exercise 1.2.2), but no control is provided over the left tail and it is possible for $\int_{-\infty}^0 |x|^k F(dx) = \infty$ for any $k > 0$. Similarly $F \in D(\Lambda)$ implies when $x_0 = \infty$ that

$$\int_0^\infty x^k F(dx) < \infty \qquad \text{for all } k > 0$$

(Exercise 1.1.1) but implies nothing about behavior of the left tail.

Thus in investigating (2.1) it is necessary to impose some condition on the left tail of F.

Proposition 2.1. *For an extreme value distribution G, suppose $F \in D(G)$.*
(i) *If $G = \Phi_\alpha$, set $a_n = (1/(1-F))^{\leftarrow}(n), b_n = 0$. If for some integer $0 < k < \alpha$*

$$\int_{-\infty}^0 |z|^k F(dx) < \infty \tag{2.3}$$

then

$$\lim_{n\to\infty} E(M_n/a_n)^k = \int_{-\infty}^\infty x^k \Phi_\alpha(dx) = \Gamma(1 - \alpha^{-1}k).$$

(ii) *If $G = \Psi_\alpha$ and F has right end x_0 set*

$$a_n = x_0 - (1/(1-F))^{\leftarrow}(n), \qquad b_n = x_0.$$

If for some integer $k > 0$

$$\int_{-\infty}^{x_0} |x|^k F(dx) < \infty \tag{2.4}$$

then

$$\lim_{n\to\infty} E((M_n - x_0)/a_n)^k = \int_{-\infty}^0 x^k \Psi_\alpha(dx) = (-1)^k \Gamma(1 + \alpha^{-1}k).$$

(iii) *If $G = \Lambda$ and F has right end x_0 with representation (1.5) set $b_n = (1/(1-F))^{\leftarrow}(n), a_n = f(b_n)$. If for some integer $k > 0$*

$$\int_{-\infty}^0 |x|^k F(dx) < \infty \tag{2.5}$$

Question 2

then

$$\lim_{n\to\infty} E((M_n - b_n)/a_n)^k = \int_{-\infty}^\infty x^k \Lambda(dx) = (-1)^k \Gamma^{(k)}(1)$$

where $\Gamma^{(k)}(1)$ is the kth derivative of the gamma function at $x = 1$.

Remarks. (i) Conditions (2.3), (2.4), and (2.5) can be weakened slightly. See Exercise 2.1.1.
(ii) For any norming constants a_n, b_n satisfying (2.1) (not just the ones specified in the statement of the proposition), we also have (2.2) satisfied provided the appropriate condition (2.3), (2.4), or (2.5) holds. See Exercise 2.1.2.

We only prove (i) and (iii) in Proposition 2.1. Part (iii) requires that our tool box be equipped with the following inequalities.

Lemma 2.2. *Suppose* $F \in D(\Lambda)$ *with representation* (1.5) *and that* a_n *and* b_n *are as specified in part* (iii) *of Proposition* 2.1.
(a) *Given* $\varepsilon > 0$, *we have for* $s > 0$ *and all sufficiently large* n

$$f(b_n)/f(a_n s + b_n) \geq (1 + \varepsilon s)^{-1} \tag{2.6}$$

and consequently if $y > 0$ *and* n *is large*

$$1 - F^n(a_n y + b_n)) \leq (1 + \varepsilon)^3 (1 + \varepsilon y)^{-\varepsilon^{-1}}. \tag{2.7}$$

(b) *Recall the meaning of* z_0 *in the representation* (1.5). *Given* ε, *pick* $z_1 \in (z_0, x_0)$ *such that* $|f'(x)| \leq \varepsilon$ *if* $x > z_1$. *Then for large* n *and* $u \in (a_n^{-1}(z_1 - b_n), 0)$ *we have*

$$f(b_n)/f(a_n u + b_n) \geq (1 + \varepsilon|u|)^{-1} \tag{2.8}$$

and consequently for large n *and* $s \in (a_n^{-1}(z_1 - b_n), 0)$

$$F^n(a_n s + b_n) \leq e^{-(1-\varepsilon)^2 (1+\varepsilon|s|)^{\varepsilon^{-1}}}. \tag{2.9}$$

Proof of Lemma 2.2. (a) For n such that $|f'(t)| \leq \varepsilon$ if $t \geq b_n$ we have for $s > 0$

$$(f(a_n s + b_n)/f(b_n)) - 1 = \int_{b_n}^{a_n s + b_n} (f'(u)/f(b_n))du$$

and recalling $a_n = f(b_n)$ this is

$$\int_0^s f'(a_n u + b_n)du \leq \varepsilon s.$$

Consequently $f(b_n)/f(a_n s + b_n) \geq (1 + \varepsilon s)^{-1}$ as asserted.
To check (2.7) note that

$$1 - F(b_n) \sim n^{-1}$$

so that for large n and $y > 0$

$$n(1 - F(a_n y + b_n) \leq (1 + \varepsilon)(1 - F(a_n y + b_n))/(1 - F(b_n))$$

and from (1.5) this is

$$(1 + \varepsilon)c(a_n y + b_n)c^{-1}(b_n)e^{-\int_{b_n}^{a_n y + b_n}(1/f(s))ds}.$$

Since $c(x) \to c > 0$ as $x \uparrow x_0$ we have the preceding

$$\leq (1 + \varepsilon)^2 e^{-\int_0^y (f(b_n)/f(a_n s + b_n))ds}$$