P. Embrechts, C. Klüppelberg, T. Mikosch

Modelling Extremal Events

for Insurance and Finance

February 12, 1997

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

3.3.3 The Maximum Domain of Attraction of the Gumbel Distribution $\Lambda(x) = \exp\{-\exp\{-x\}\}$

Von Mises Functions

The maximum domain of attraction of the Gumbel distribution Λ covers a wide range of dfs F. Although there is no direct link with regular variation as in the maximum domains of attraction of the Fréchet and Weibull distribution, we will find extensions of regular variation which allow for a complete characterisation of $\mathrm{MDA}(\Lambda)$.

A Taylor expansion argument yields

$$1 - \Lambda(x) \sim e^{-x}, \quad x \to \infty,$$

hence $\overline{\Lambda}(x)$ decreases to zero at an exponential rate. Again the following question naturally arises:

How far away can we move from an exponential tail and still remain in $\mathrm{MDA}(\Lambda)$*?*

We will see in the present and the next section that $\mathrm{MDA}(\Lambda)$ contains dfs with very different tails, ranging from *moderately heavy* (such as the lognormal distribution) to *light* (such as the normal distribution). Also both cases $x_F < \infty$ and $x_F = \infty$ are possible. Before we give a general answer to the above question, we restrict ourselves to some absolutely continuous $F \in \mathrm{MDA}(\Lambda)$ which have a simple representation, proposed by von Mises. These distributions provide an important building block of this maximum domain of attraction, and therefore we study them in detail. We will see later (Theorem 3.3.26 and Remark 4) that one only has to consider a slight modification of the von Mises functions in order to characterise $\mathrm{MDA}(\Lambda)$ completely.

Definition 3.3.18 (Von Mises function)
Let F be a df with right endpoint $x_F \le \infty$. Suppose there exists some $z < x_F$ such that F has representation

$$\overline{F}(x) = c \exp\left\{-\int_z^x \frac{1}{a(t)}\, dt\right\}, \quad z < x < x_F, \tag{3.24}$$

where c is some positive constant, $a(\cdot)$ is a positive and absolutely continuous function (with respect to Lebesgue measure) with density a' and $\lim_{x \uparrow x_F} a'(x) = 0$.

Then F is called a von Mises function, *the function $a(\cdot)$ the* auxiliary function *of F.* □

Remark. 1) Relation (3.24) should be compared with the Karamata representation of a regularly varying function; see Theorem A3.3. Substituting into (3.24) the function $a(x) = x/\delta(x)$ such that $\delta(x) \to \alpha \in [0,\infty)$ as $x \to \infty$, (3.24) becomes a regularly varying tail with index $-\alpha$. We will see later (see Remark 2 below) that the auxiliary function of a von Mises function with $x_F = \infty$ satisfies $a(x)/x \to 0$. It immediately follows that $\overline{F}(x)$ decreases to zero much faster than any power law $x^{-\alpha}$. $\square$

We give some examples of von Mises functions.

Example 3.3.19 (Exponential distribution)

$$\overline{F}(x) = e^{-\lambda x}\,, \quad x \ge 0\,, \quad \lambda > 0\,.$$

F is a von Mises function with auxiliary function $a(x) = \lambda^{-1}$. $\square$

Example 3.3.20 (Weibull distribution)

$$\overline{F}(x) = \exp\left\{-c\,x^{\tau}\right\}\,, \quad x \ge 0\,, \quad c\,,\tau > 0\,.$$

F is a von Mises function with auxiliary function

$$a(x) = c^{-1}\tau^{-1}x^{1-\tau}\,, \quad x > 0\,.$$ $\square$

Example 3.3.21 (Erlang distribution)

$$\overline{F}(x) = e^{-\lambda x}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!}\,, \quad x \ge 0\,, \quad \lambda > 0\,, n \in \mathbb{N}.$$

F is a von Mises function with auxiliary function

$$a(x) = \sum_{k=0}^{n-1}\frac{(n-1)!}{(n-k-1)!}\,\lambda^{-(k+1)}\,x^{-k}\,, \quad x > 0\,.$$

Notice that F is the $\Gamma(n,\lambda)$ df. $\square$

Example 3.3.22 (Exponential behaviour at the finite right endpoint)
Let F be a df with finite right endpoint x_F and distribution tail

$$\overline{F}(x) = K\exp\left\{-\frac{\alpha}{x_F - x}\right\}\,, \quad x < x_F\,, \quad \alpha\,,K > 0\,.$$

F is a von Mises function with auxiliary function

$$a(x) = \frac{(x_F - x)^2}{\alpha}\,, \quad x < x_F\,.$$

For $x_F = 1$, $\alpha = 1$ and $K = e$ we obtain for example

$$\overline{F}(x) = \exp\left\{-\frac{x}{1-x}\right\}\,, \quad 0 \le x < 1\,.$$ $\square$

Example 3.3.23 (Differentiability at the right endpoint)
Let F be a df with right endpoint $x_F \le \infty$ and assume there exists some $z < x_F$ such that F is twice differentiable on (z, x_F) with positive density $f = F'$ and $F''(x) < 0$ for $z < x < x_F$. Then it is not difficult to see that F is a von Mises function with auxiliary function $a = \overline{F}/f$ if and only if

$$\lim_{x \uparrow x_F} \overline{F}(x)\, F''(x)/f^2(x) = -1\,. \tag{3.25}$$

Indeed, let $z < x < x_F$ and set $Q(x) = -\ln \overline{F}(x)$ and $a(x) = 1/Q'(x) = \overline{F}(x)/f(x) > 0$. Hence F has representation (3.24). Furthermore,

$$a'(x) = -\frac{\overline{F}(x)\, F''(x)}{f^2(x)} - 1$$

and (3.25) is equivalent to $a'(x) \to 0$ as $x \uparrow x_F$.

Condition (3.25) applies to many distributions of interest, including the normal distribution; see Example 3.3.29. □

In Remark 1 above we gained some indication that regular variation does not seem to be the right tool for describing von Mises functions. Recall the notion of *rapidly varying function* from Definition A3.11. In particular, $\overline{F} \in \mathcal{R}_{-\infty}$ means that

$$\lim_{x \to \infty} \frac{\overline{F}(xt)}{\overline{F}(x)} = \begin{cases} 0 & \text{if} \quad t > 1\,, \\ \infty & \text{if} \quad 0 < t < 1\,. \end{cases}$$

It is mentioned in Appendix A3 that some of the important results for regularly varying functions can be extended to $\mathcal{R}_{-\infty}$ in a natural way; see Theorem A3.12.

Proposition 3.3.24 (Properties of von Mises functions)
Every von Mises function F is absolutely continuous on (z, x_F) with positive densitiy f. The auxiliary function can be chosen as $a(x) = \overline{F}(x)/f(x)$. Moreover, the following properties hold.

(*a*) *If $x_F = \infty$, then $\overline{F} \in \mathcal{R}_{-\infty}$ and*

$$\lim_{x \to \infty} \frac{x f(x)}{\overline{F}(x)} = \infty\,. \tag{3.26}$$

(*b*) *If $x_F < \infty$, then $\overline{F}(x_F - x^{-1}) \in \mathcal{R}_{-\infty}$ and*

$$\lim_{x \uparrow x_F} \frac{(x_F - x) f(x)}{\overline{F}(x)} = \infty\,. \tag{3.27}$$

Remarks. 2) It follows from (3.26) that $\lim_{x\to\infty} x^{-1}a(x) = 0$, and from (3.27) that $a(x) = o(x_F - x) = o(1)$ as $x \uparrow x_F$.

3) Note that $a^{-1}(x) = f(x)/\overline{F}(x)$ is the *hazard rate* of F. □

Proof. From representation (3.24) we obtain

$$\frac{d}{dx}\left(-\ln \overline{F}(x)\right) = \frac{f(x)}{\overline{F}(x)} = \frac{1}{a(x)}\,, \qquad z < x < x_F\,.$$

(a) Since $a'(x) \to 0$ as $x \to \infty$ the Cesàro mean of a' also converges:

$$\lim_{x\to\infty} \frac{a(x)}{x} = \lim_{x\to\infty} \frac{1}{x}\int_z^x a'(t)\,dt = 0\,. \tag{3.28}$$

This implies (3.26). $\overline{F} \in \mathcal{R}_{-\infty}$ follows from an application of Theorem A3.12(b).

(b) We have

$$\begin{aligned}\lim_{x\uparrow x_F} \frac{a(x)}{x_F - x} &= \lim_{x\uparrow x_F} -\int_x^{x_F} \frac{a'(t)}{x_F - x}\,dt \\ &= \lim_{s\downarrow 0} \frac{1}{s}\int_0^s a'\left(x_F - t\right)\,dt\end{aligned}$$

by change of variables. Since $a'(x_F - t) \to 0$ as $t \downarrow 0$, the last limit tends to 0. This implies (3.27). $\overline{F}(x_F - x^{-1}) \in \mathcal{R}_{-\infty}$ follows as above. □

Now we can show that von Mises functions belong to the maximum domain of attraction of the Gumbel distribution. Moreover, the specific form of $\overline{F}$ allows to calculate the norming constants c_n from the auxiliary function.

Proposition 3.3.25 (Von Mises functions and MDA(Λ))
Suppose the df F is a von Mises function. Then $F \in \mathrm{MDA}(\Lambda)$. A possible choice of norming constants is

$$d_n = F^{\leftarrow}(1 - n^{-1}) \quad \textit{and} \quad c_n = a(d_n)\,, \tag{3.29}$$

where a is the auxiliary function of F.

Proof. Representation (3.24) implies for $t \in \mathbb{R}$ and x sufficiently close to x_F that

$$\frac{\overline{F}(x + t\,a(x))}{\overline{F}(x)} = \exp\left\{-\int_x^{x+t\,a(x)} \frac{1}{a(u)}\,du\right\}\,.$$

We set $v = (u - x)/a(x)$ and obtain

$$\frac{\overline{F}(x + t\,a(x))}{\overline{F}(x)} = \exp\left\{-\int_0^t \frac{a(x)}{a(x + v\,a(x))}\,dv\right\}\,. \tag{3.30}$$

We show that the integrand converges locally uniformly to 1. For given $\varepsilon > 0$ and $x \ge x_0(\varepsilon)$,

$$|a(x+va(x)) - a(x)| = \left| \int_x^{x+va(x)} a'(s)\,ds \right| \le \varepsilon |v| a(x) \le \varepsilon |t| a(x)\,,$$

where we used $a'(x) \to 0$ as $x \uparrow x_F$. This implies for $x \ge x_0(\varepsilon)$ that

$$\left| \frac{a(x+va(x))}{a(x)} - 1 \right| \le \varepsilon |t|\,.$$

The right–hand side can be made arbitrarily small, hence

$$\lim_{x \uparrow x_F} \frac{a(x)}{a(x + v\,a(x))} = 1\,, \tag{3.31}$$

uniformly on bounded v–intervals. This together with (3.30) yields

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x + t\,a(x))}{\overline{F}(x)} = e^{-t} \tag{3.32}$$

uniformly on bounded t–intervals. Now choose the norming constants $d_n = (1/\overline{F})^{\leftarrow}(n)$ and $c_n = a(d_n)$. Then (3.32) implies

$$\lim_{n\to\infty} n\overline{F}\,(d_n + tc_n) = e^{-t} = -\ln \Lambda(t)\,, \quad t \in \mathbb{R}\,.$$

An application of Proposition 3.3.2 shows that $F \in \mathrm{MDA}(\Lambda)$. □

This result finishes our study of von Mises functions.

Characterisations of MDA(Λ)

Von Mises functions do not completely characterise the maximum domain of attraction of Λ. However, a slight modification of the defining relation (3.24) of a von Mises function yields a complete characterisation of MDA(Λ).

For a proof of the following result we refer to Resnick [530], Corollary 1.7 and Proposition 1.9.

Theorem 3.3.26 (Characterisation I of MDA(Λ))
The df F with right endpoint $x_F \le \infty$ belongs to the maximum domain of attraction of Λ if and only if there exists some $z < x_F$ such that F has representation

$$\overline{F}(x) = c(x) \exp\left\{ -\int_z^x \frac{g(t)}{a(t)}\,dt \right\}\,, \quad z < x < x_F\,, \tag{3.33}$$

where c and g are measurable functions satisfying $c(x) \to c > 0$, $g(x) \to 1$ as $x \uparrow x_F$, and $a(x)$ is a positive, absolutely continuous function (with respect to

Lebesgue measure) with density $a'(x)$ *having* $\lim_{x\uparrow x_F} a'(x) = 0$.

For F *with representation* (3.33) *we can choose*

$$d_n = F^{\leftarrow}(1 - n^{-1}) \quad \textit{and} \quad c_n = a(d_n)$$

as norming constants.

A possible choice for the function a *is*

$$a(x) = \int_x^{x_F} \frac{\overline{F}(t)}{\overline{F}(x)}\, dt\,, \qquad x < x_F\,, \tag{3.34}$$

□

Motivated by von Mises functions, we call the function a in (3.33) an *auxiliary function* for F.

Remarks. 4) Representation (3.33) is not unique, there being some trade–off possible between the functions c and g. The following representation can be employed alternatively; see Resnick [530], Proposition 1.4:

$$\overline{F}(x) = c(x) \exp\left\{-\int_z^x \frac{1}{a(t)}\, dt\right\}\,, \quad z < x < x_F\,, \tag{3.35}$$

for functions c and a with properties as in Theorem 3.3.26.

5) For a rv X the function $a(x)$ as defined in (3.34) is nothing but the *mean excess function*

$$a(x) = E(X - x \mid X > x)\,, \quad x < x_F\,;$$

see also Section 3.4 for a discussion on the use of this function. In Chapter 6 the mean excess function will turn out to be an important tool for statistical fitting of extremal event data. □

Another characterisation of MDA(Λ) was suggested in the proof of Proposition 3.3.25. There it was shown that every von Mises function satisfies (3.32), i.e. there exists a positive function $\widetilde{a}$ such that

$$\lim_{x\uparrow x_F} \frac{\overline{F}(x + t\widetilde{a}(x))}{\overline{F}(x)} = e^{-t}\,, \quad t \in \mathbb{R}\,. \tag{3.36}$$

Theorem 3.3.27 (Characterisation II of MDA(Λ))
The df F *belongs to the maximum domain of attraction of* Λ *if and only if there exists some positive function* $\widetilde{a}$ *such that* (3.36) *holds. A possible choice is* $\widetilde{a} = a$ *as given in* (3.34). □

The proof of this result is for instance to be found in de Haan [292], Theorem 2.5.1.

Now recall the notion of tail–equivalence (Definition 3.3.3). Similarly to the maximum domains of attraction of the Weibull and Fréchet distribution, tail–equivalence is an auxiliary tool to decide whether a particular distribution belongs to the maximum domain of attraction of Λ and to calculate the norming constants. In MDA(Λ) it is even more important because of the large variety of tails $\overline{F}$.

Proposition 3.3.28 (Closure property of MDA(Λ) under tail–equivalence) *Let F and G be dfs with the same right endpoint $x_F = x_G$ and assume that $F \in \text{MDA}(\Lambda)$ with norming constants $c_n > 0$ and $d_n \in \mathbb{R}$; i.e.*

$$\lim_{n\to\infty} F^n\left(c_n x + d_n\right) = \Lambda(x)\,, \quad x \in \mathbb{R}. \tag{3.37}$$

Then

$$\lim_{n\to\infty} G^n\left(c_n x + d_n\right) = \Lambda(x+b)\,, \quad x \in \mathbb{R},$$

if and only if F and G are tail–equivalent with

$$\lim_{x\uparrow x_F} \overline{F}(x)/\overline{G}(x) = e^b\,.$$

Proof of the sufficiency. For a proof of the necessity see Resnick [530], Proposition 1.19. Suppose that $\overline{F}(x) \sim c\,\overline{G}(x)$ as $x \uparrow x_F$ for some $c > 0$. By Proposition 3.3.2 the limit relation (3.37) is equivalent to

$$\lim_{n\to\infty} n\overline{F}\left(c_n x + d_n\right) = e^{-x}\,, \quad x \in \mathbb{R}.$$

For such x, $c_n x + d_n \to x_F$ and hence, by tail–equivalence,

$$n\,\overline{G}\left(c_n x + d_n\right) \sim n c^{-1}\overline{F}\left(c_n x + d_n\right) \to c^{-1}e^{-x}\,, \quad x \in \mathbb{R}.$$

Therefore by Proposition 3.3.2,

$$\lim_{n\to\infty} G^n\left(c_n x + d_n\right) = \exp\left\{-e^{-(x+\ln c)}\right\} = \Lambda(x+\ln c)\,, \quad x \in \mathbb{R}.$$

Now set $\ln c = b$. $\square$

The results of this section yield a further complete characterisation of MDA(Λ).

$$\boxed{\begin{array}{l}\text{MDA}(\Lambda)\text{ consists of von Mises functions}\\ \text{and their tail–equivalent dfs.}\end{array}}$$

This statement and the examples discussed throughout this section show that MDA(Λ) consists of a large variety of distributions whose tails can be very different. Tails may range from moderately heavy (lognormal, heavy–tailed Weibull) to very light (exponential, dfs with support bounded to the right). Because of this, MDA(Λ) is perhaps the most interesting among all maximum domains of attraction. As a natural consequence of the variety of tails in MDA(Λ), the norming constants also vary considerably. Whereas in MDA(Φ_α) and MDA(Ψ_α) the norming constants are calculated by straightforward application of regular variation theory, more advanced results are needed for MDA(Λ). A complete theory has been developed by de Haan involving certain subclasses of $\mathcal{R}_{-\infty}$ and $\mathcal{R}_0$; see de Haan [292] or Bingham et al. [72], Chapter 3. Various examples below will illustrate the usefulness of results like Proposition 3.3.28.

Example 3.3.29 (Normal distribution)
See also Figure 3.3.30. Denote by Φ the df and by φ the density of the standard normal distribution. We first show that Φ is a von Mises function and check condition (3.25). An application of l'Hospital's rule to $\overline{\Phi}(x)/(x^{-1}\varphi(x))$ yields *Mill's ratio*, $\overline{\Phi}(x) \sim \varphi(x)/x$. Furthermore $\varphi'(x) = -x\varphi(x) < 0$ and

step 2

$$\lim_{x\to\infty} \frac{\overline{\Phi}(x)\,\varphi'(x)}{\varphi^2(x)} = -1\,.$$

Thus $\Phi \in$ MDA(Λ) by Example 3.3.23 and Proposition 3.3.25. We now calculate the norming constants. Use Mill's ratio again:

$$\overline{\Phi}(x) \sim \frac{\varphi(x)}{x} = \frac{1}{\sqrt{2\pi}\,x}\, e^{-x^2/2}\,, \quad x \to \infty\,, \tag{3.38}$$

and interpret the right–hand side as the tail of some df G. Then by Proposition 3.3.28, Φ and G have the same norming constants c_n and d_n. According to (3.29), $d_n = G^{\leftarrow}(1-n^{-1})$. Hence look for a solution of $-\ln \overline{G}(d_n) = \ln n$; i.e.

$$\frac{1}{2}\,d_n^2 + \ln d_n + \frac{1}{2}\ln 2\pi = \ln n\,. \tag{3.39}$$

Then a Taylor expansion in (3.39) yields

Upper bound

$$d_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln 4\pi}{2(2\ln n)^{1/2}} + o\left((\ln n)^{-1/2}\right)$$

as a possible choice for d_n. Since we can take $a(x) = \overline{\Phi}(x)/\varphi(x)$ we have that $a(x) \sim x^{-1}$ and therefore

$$c_n = a\,(d_n) \sim (2\ln n)^{-1/2}\,.$$

As the c_n are unique up to asymptotic equivalence, we choose

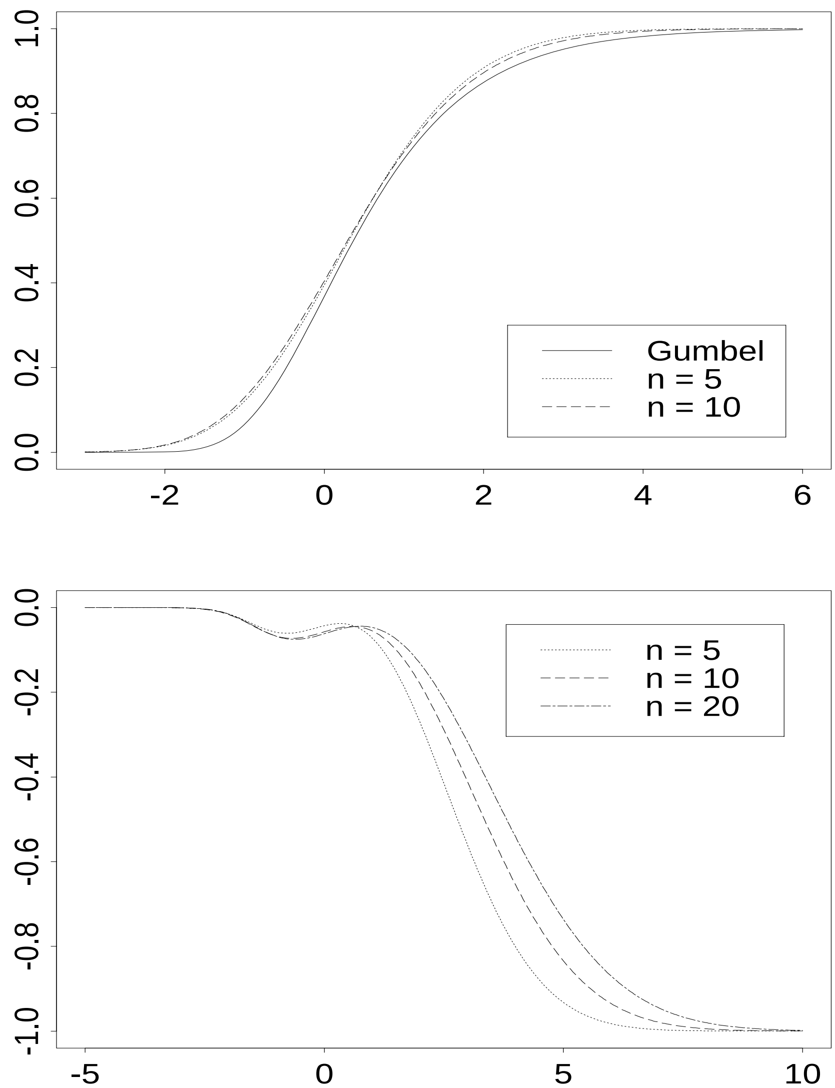

Figure 3.3.30 *Dfs of the normalised maxima of n standard normal rvs and the Gumbel df (top). In the bottom figure the relative error of this approximation for the tail is illustrated. The rate of convergence appears to be very slow.*

$$c_n = (2 \ln n)^{-1/2} .$$

We conclude that

$$\sqrt{2 \ln n} \left(M_n - \sqrt{2 \ln n} + \frac{\ln \ln n + \ln 4\pi}{2(2 \ln n)^{1/2}} \right) \xrightarrow{d} \Lambda . \tag{3.40}$$

Note that $c_n \to 0$, i.e. the distribution of M_n becomes less spread around d_n as n increases. □

Similarly, it can be proved that the gamma distribution also belongs to MDA(Λ). The norming constants are given in Table 3.4.4.

Another useful trick to calculate the norming constants is via monotone transformations. If g is an increasing function and $\widetilde{x} = g(x)$, then obviously

$$\widetilde{M}_n = \max \left(\widetilde{X}_1, \ldots, \widetilde{X}_n \right) = g\left(M_n\right) .$$

If $X \in$ MDA(Λ) with

$$\lim_{n \to \infty} P\left(M_n \le c_n x + d_n\right) = \Lambda(x) , \quad x \in \mathbb{R},$$

then

$$\lim_{n \to \infty} P\left(\widetilde{M}_n \le g\left(c_n x + d_n\right)\right) = \Lambda(x) , \quad x \in \mathbb{R}.$$

In some cases, g may be expanded in a Taylor series about d_n and just linear terms suffice to give the limit law for $\widetilde{M}_n$, with changed constants $\tilde{c}_n = c_n g'(d_n)$ and $\tilde{d}_n = g(d_n)$. We apply this method to the lognormal distribution.

Example 3.3.31 (Lognormal distribution)
Let X be a standard normal rv and $g(x) = e^{\mu + \sigma x}$, $\mu \in \mathbb{R}$, $\sigma > 0$. Then

$$\widetilde{X} = g(X) = e^{\mu + \sigma X}$$

defines a lognormal rv. Since $X \in$ MDA(Λ) we obtain

$$\lim_{n \to \infty} P\left(\widetilde{M}_n \le e^{\mu + \sigma (c_n x + d_n)}\right) = \Lambda(x) , \quad x \in \mathbb{R},$$

where c_n, d_n are the norming constants of the standard normal distribution as calculated in Example 3.3.29. This implies

$$\lim_{n \to \infty} P\left(e^{-\mu - \sigma d_n} \ \widetilde{M}_n \le 1 + \sigma c_n x + o\left(c_n\right)\right) = \Lambda(x) , \quad x \in \mathbb{R}.$$

Since $c_n \to 0$ it follows that

$$\frac{e^{-\mu - \sigma d_n}}{\sigma c_n} \left(\widetilde{M}_n - e^{\mu + \sigma d_n} \right) \xrightarrow{d} \Lambda ,$$

so that $\widetilde{X} \in$ MDA(Λ) with norming constants

$$\widetilde{c}_n = \sigma c_n e^{\mu + \sigma d_n} , \qquad \widetilde{d}_n = e^{\mu + \sigma d_n} .$$

Explicit expressions for the norming constants of the lognormal distribution can be found in Table 3.4.4. □

Further Properties of Distributions in MDA(Λ)

In the remainder of this section we collect some further useful facts about distributions in MDA(Λ).

Corollary 3.3.32 (Existence of moments)
Assume that the rv X has df $F \in \mathrm{MDA}(\Lambda)$ with infinite right endpoint. Then $\overline{F} \in \mathcal{R}_{-\infty}$. In particular, $E(X^+)^\alpha < \infty$ for every $\alpha > 0$, where $X^+ = \max(0, X)$.

Proof. Every $F \in \mathrm{MDA}(\Lambda)$ is tail–equivalent to a von Mises function. If $x_F = \infty$, the latter have rapidly varying tails; see Proposition 3.3.24(a), which also implies the statement about the moments; see Theorem A3.12(a). □

In Section 3.3.2 we showed that the maximum domains of attraction of Ψ_α and Φ_α are linked in a natural way. Now we show that MDA(Φ_α) can be embedded in MDA(Λ).

Example 3.3.33 (Embedding MDA(Φ_α) in MDA(Λ))
Let X have df $F \in \mathrm{MDA}(\Phi_\alpha)$ with norming constants c_n. Define

$$X^* = \ln(1 \vee X)$$

with df F^*. By Proposition 3.3.2 and Theorem 3.3.7, $F \in \mathrm{MDA}(\Phi_\alpha)$ if and only if

$$\lim_{n\to\infty} n\overline{F}(c_n x) = \lim_{n\to\infty} \frac{\overline{F}\,(c_n x)}{\overline{F}\,(c_n)} = x^{-\alpha}\,, \quad x > 0\,.$$

This implies that

$$\lim_{n\to\infty} \frac{\overline{F^*}\left(\alpha^{-1}x + \ln c_n\right)}{\overline{F^*}\,(\ln c_n)} = \lim_{n\to\infty} \frac{\overline{F}\left(c_n \exp\left\{\alpha^{-1}x\right\}\right)}{\overline{F}\,(c_n)} = e^{-x}\,, \quad x \in \mathbb{R}\,.$$

Hence $F^* \in \mathrm{MDA}(\Lambda)$ with norming constants $c_n^* = \alpha^{-1}$ and $d_n^* = \ln c_n$. As auxiliary function one can take

$$a^*(x) = \int_x^\infty \frac{\overline{F^*}(y)}{\overline{F^*}(x)}\,dy$$ □

Example 3.3.34 (Closure of MDA(Λ) under logarithmic transformations)
Let X have df $F \in \mathrm{MDA}(\Lambda)$ with $x_F = \infty$ and norming constants c_n, d_n, chosen according to Theorem 3.3.26. Define X^* and F^* as above. We intend to show that $F^* \in \mathrm{MDA}(\Lambda)$ with norming constants $d_n^* = \ln d_n$ and $c_n^* = c_n/d_n$. Since $a'(x) \to 0$, (3.28) holds, and since $d_n = F^{\leftarrow}(1 - n^{-1}) \to \infty$, it follows that

according as $\epsilon \in (0, \epsilon_0)$ *or* $\epsilon \in (-\epsilon_0, 0)$ *implies that*

$$\limsup_{n\to\infty} c_n^{-1} (M_n - d_n) = 1 \quad \text{a.s.}$$

(*b*) *Assume that the sequences* $u_n(\epsilon) = c_n(1+\epsilon) + d_n$, $n \in \mathbb{N}$, *are non–decreasing and satisfy* (3.57), (3.58) *for every* $\epsilon \in (-\epsilon_0, \epsilon_0)$. *Then the relation*

$$\sum_{n=1}^{\infty} \overline{F}(u_n(\epsilon)) \exp\left\{-n\overline{F}(u_n(\epsilon))\right\} < \infty \quad \text{or} \quad = \infty$$

according as $\epsilon \in (-\epsilon_0, 0)$ *or* $\epsilon \in (0, \epsilon_0)$ *implies that*

$$\liminf_{n\to\infty} c_n^{-1} (M_n - d_n) = 1 \quad \text{a.s.} \qquad \square$$

We continue with several examples in order to illustrate the different options for the a.s. behaviour of the maxima M_n. Throughout we will use the following notation

$$\ln_0 x = x\,, \quad \ln_1 x = \max(0, \ln x)\,, \quad \ln_k x = \max(0, \ln_{k-1} x)\,, \quad k \geq 2\,,$$

i.e. $\ln_k x$ is the *kth iterated logarithm of* x.

Question 2 (referenced in main reference)

Example 3.5.4 (Normal distribution, continuation of Example 3.3.29)
Assume that $F = \Phi$ is the standard normal distribution. Then

$$\overline{\Phi}(x) \sim \frac{1}{\sqrt{2\pi}x} \exp\{-x^2/2\}\,. \tag{3.64}$$

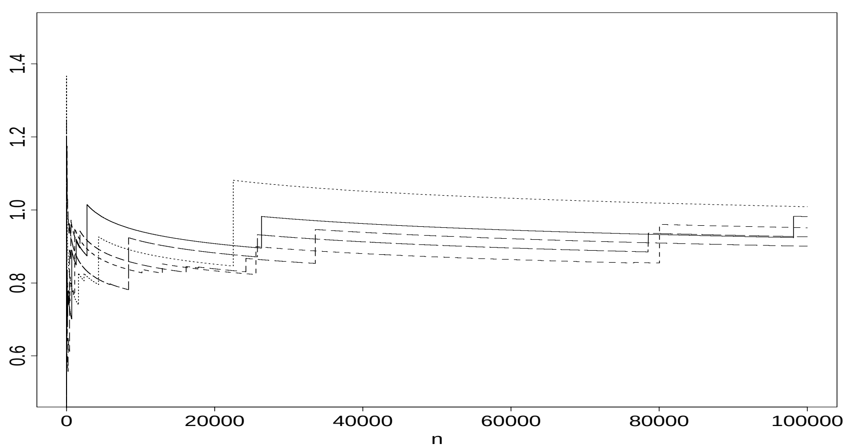

Figure 3.5.5 *Five sample paths of* $(M_n/\sqrt{2\ln n})$ *for* 100 000 *realisations of iid standard normal rvs. The rate of a.s. convergence to* 1 *appears to be very slow.*

From (3.40) we conclude

$$\frac{M_n}{\sqrt{2\ln n}} \xrightarrow{P} 1\,, \quad n \to \infty\,. \tag{3.65}$$

We are interested in a.s. refinements of this result.
Choose

$$u_n(\epsilon) = \sqrt{2\ln\left(\frac{(\ln_0 n \cdots \ln_r n)\ln_r^{\epsilon} n}{\sqrt{\ln n}}\right)}\,, \quad r \geq 0\,.$$

An application of Theorem 3.5.1 together with (3.64) yields

$$P\left(M_n > u_n(\epsilon) \quad \text{i.o.}\right) = 0 \quad \text{or} \quad = 1$$

according as $\epsilon > 0$ or $\epsilon < 0$ for small $|\epsilon|$ and hence, by Corollary 3.5.3,

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2\ln n}} = 1 \quad \text{a.s.} \tag{3.66}$$

This result can further be refined. For example, notice that

$$\begin{aligned}
&P\left(M_n > \sqrt{2\ln\left(\frac{n\ln^{1+\epsilon} n}{\sqrt{\ln n}}\right)} \quad \text{i.o.}\right)\\
&= \quad P\left(M_n - \sqrt{2\ln\left(\frac{n}{\sqrt{\ln n}}\right)}\right.\\
&\qquad\qquad \left. > \sqrt{2\ln\left(\frac{n\ln^{1+\epsilon} n}{\sqrt{\ln n}}\right)} - \sqrt{2\ln\left(\frac{n}{\sqrt{\ln n}}\right)} \quad \text{i.o.}\right)\\
&= \quad 0 \quad \text{or} \quad = 1
\end{aligned}$$

according as $\epsilon > 0$ or $\epsilon \leq 0$. By the mean value theorem, for small $|\epsilon|$ and certain $\theta_n \in (0,1)$,

$$\begin{aligned}
&P\left(M_n - \sqrt{2\ln\left(\frac{n}{\sqrt{\ln n}}\right)} > \frac{1}{2}\,\frac{2(1+\epsilon)\ln_2 n}{\left(2\ln(n/\sqrt{\ln n}) + \theta_n 2(1+\epsilon)\ln_2 n\right)^{1/2}} \quad \text{i.o.}\right)\\
&\quad = 0 \quad \text{or} \quad = 1
\end{aligned}$$

according as $\epsilon > 0$ or $\epsilon \leq 0$. In view of Corollary 3.5.3 this just means that

$$\limsup_{n\to\infty} \frac{\sqrt{2\ln n}}{\ln_2 n}\left(M_n - \sqrt{2\ln\left(\frac{n}{\sqrt{\ln n}}\right)}\right) = 1 \quad \text{a.s.}$$

By the same arguments,

$$\limsup_{n\to\infty} \frac{\sqrt{2\ln n}}{\ln_{r+1} n}\left(M_n - \sqrt{2\ln\left(\frac{\ln_0 n\cdots\ln_{r-1} n}{\sqrt{\ln n}}\right)}\right) = 1 \quad \text{a.s.}, \quad r\geq 1\,.$$

Now choose

$$u_n'(\epsilon) = \sqrt{2\ln\left(\frac{n}{\sqrt{4\pi\ln n}\ln((\ln_1 n\cdots\ln_r n)\ln_r^{\epsilon} n)}\right)}\,, \quad r\geq 1\,.$$

An application of Theorem 3.5.2 yields that

$$P\left(M_n \leq u_n'(\epsilon) \quad \text{i.o.}\right) = 0 \quad \text{or} \quad = 1$$

according as $\epsilon > 0$ or $\epsilon < 0$ for small $|\epsilon|$. In particular, we may conclude that

$$\liminf_{n\to\infty} \frac{M_n}{\sqrt{2\ln n}} = 1 \quad \text{a.s.} \tag{3.67}$$

which, together with (3.66), yields the a.s. analogue to (3.65):

$$\lim_{n\to\infty} \frac{M_n}{\sqrt{2\ln n}} = 1 \quad \text{a.s.}$$

Refinements of relation (3.67) are possible in the same way as for lim sup. □

Example 3.5.6 (Exponential tails)
Let X be a rv with tail

$$\overline{F}(x) \sim Ke^{-ax}\,, \quad x\to\infty\,,$$

for some $a\,,K > 0$. From Example 3.2.7 we conclude that

$$\frac{M_n}{\ln n} \stackrel{P}{\to} \frac{1}{a}\,, \quad n\to\infty\,. \tag{3.68}$$

We are interested in a.s. refinements of this result.
Choose

$$u_n(\epsilon) = \frac{1}{a}\,\ln\left(K\left(\ln_0 n\ln_1 n\cdots\ln_r n\right)\ln_r^{\epsilon}\right)\,, \quad r\geq 0\,.$$

Then, for large n and small $|\epsilon|$,

$$\overline{F}\left(u_n(\epsilon)\right) \sim \frac{1}{(\ln_0 n\cdots\ln_r n)\ln_r^{\epsilon} n}\,.$$

An application of Theorem 3.5.1 yields that

$$P\left(M_n > u_n(\epsilon) \quad \text{i.o.}\right) = 0 \quad \text{or} \quad = 1$$

according as $\epsilon > 0$ or $\epsilon < 0$ for small $|\epsilon|$ and hence