Applied Numa Assignment 2

Group 13
maaxen@kth.se
hfag@kth.se

November 8, 2022

Problem 1

We are considering a slow moving fluid in a pipe of length L. The fluid have a velocity v and is heated by a coil. The temperature distribution $T(z)$ is determined by the following equations:

$$-\frac{d^2T}{dz^2} + v\frac{dT}{dz} = Q(z) \tag{1}$$

$$Q(z) = \begin{cases} 0, & 0 \leq \text{z} < \text{a}, \\ Q_0 \sin\left(\frac{(z-a)\pi}{b-a}\right), & \text{a} \leq \text{z} \leq \text{b}, \\ 0, & \text{b} < \text{z} \leq \text{L}. \end{cases} \tag{2}$$

The boundary conditions ($z = 0$ and $z = L$) are given by the following equations:

$$T(0) = T_0 \tag{3}$$

$$-\frac{dT}{dz}(L) = \alpha(v)(T(L) - T_{out}), \quad \alpha(v) = \sqrt{\frac{v^2}{4} + \alpha_0^2} - \frac{v}{2} \tag{4}$$

The following values used are: $L = 1$, $a = 0.2$, $b = 0.3$, $Q_0 = 4000$, $\alpha_0 = 100$, $T_{out} = 20$, $T_0 = 50$.

a) Different step-size - *h*

The boundary value problem was solved using the finite difference method. The method can be divided in to five steps:

1. Discretise: $z_j = jh$, h is a constant step-size
2. Approximate derivatives with (second order) differences:

$$-\frac{\kappa_{j+\frac{1}{2}}u(x_{j+1}) - (\kappa_{j+\frac{1}{2}} + \kappa_{j-\frac{1}{2}})u(x_j) + \kappa_{j-\frac{1}{2}}u(x_{j-1})}{h^2} + p_j\frac{u(x_{j+1}) - u(x_{j-1})}{2h} + q_j u(x_j) = f_j + O(h^2)$$

   where $\kappa_j = 1$, $q_j = 0$, $p_j = 10$ and $f_j = Q(z_j)$ for all j.

3. Define the approximation:

$$\underbrace{\left(-\frac{\kappa_{j-\frac{1}{2}}}{h^2}-\frac{p_j}{2h}\right)}_{a_j}u_{j-1}+\underbrace{\left(\frac{\kappa_{j-\frac{1}{2}}+\kappa_{j+\frac{1}{2}}}{h^2}+q_j\right)}_{b_j}u_j+\underbrace{\left(-\frac{\kappa_{j+\frac{1}{2}}}{h^2}+\frac{p_j}{2h}\right)}_{c_j}u_{j+1}=f_j.$$

4. Apply boundary conditions: $z = 0$ and $z = L$.
5. Formulate as matrix equation: $A\boldsymbol{u} = \boldsymbol{f}$

The last value is approximated by the derivative at $x = L$ from equation (4) using ghost point motivation:

$$-\frac{u_{n+1}-u_{n-1}}{2h}=\alpha(v)(T(L)-T_{out}) \tag{5}$$

$$T(L)=u_n$$

Rewrite for u_{n+1}:

$$u_{n+1}=-2h\alpha(v)(u_n-T_{out})+u_{n-1} \tag{6}$$

The approximation for $x = L$ can be written as followed:

$$au_{n-1}+bu_n+cun+1=f_n \tag{7}$$

Use equation (6) and move over the constants:

$$(a+c)\cdot u_{n-1}+(b-c2h\alpha(v))\cdot u_n=f_n-c2h\alpha(v)\cdot T_{out} \tag{8}$$

The system can then be written on matrix-form:

$$\begin{pmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a+c & b-c2h\alpha(v) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}=\begin{pmatrix} f_1-a\cdot T_0 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n-c2h\alpha(v)\cdot T_{out} \end{pmatrix}$$

The solution *T(z)* was plotted for h = 0.05, 0.025, 0.0125 and 0.00625 with v = 10.

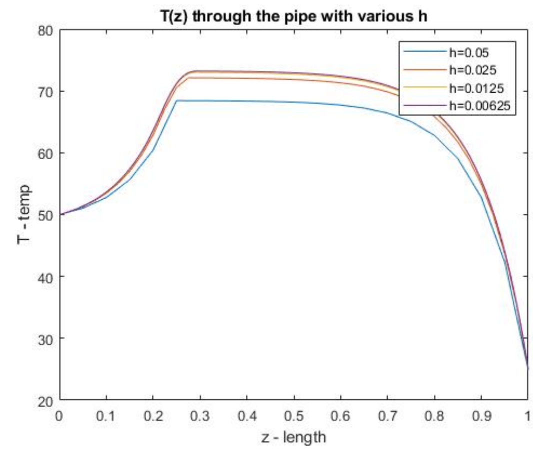

Figure 1: Curves of solution using various step-sizes.

The curves converges towards ≈ 25.
The following loglog-plot demonstrates the error for of the solution in at the end of the pipe (z = L) while decreasing the step-size h. By comparing it with h^2 we can conclude that the method is second order accurate.

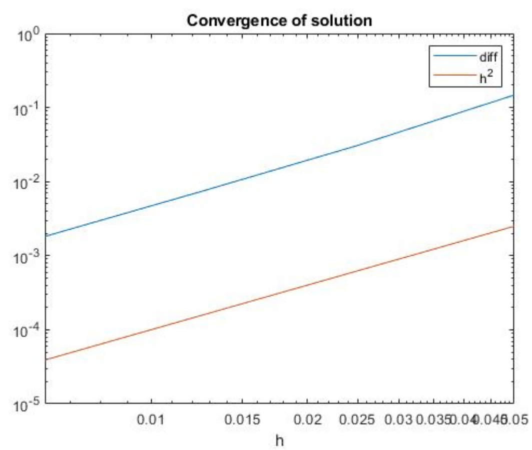

Figure 2: loglog-plot of error as function of h at end of pipe and h^2.

b) Different velocity - v

Now we are solving the same problem using different values for the fluids velocity, v. The values used are $v = 1,\ 10,\ 30,\ 100$. The curves for the solution are demonstrated in the plot below.

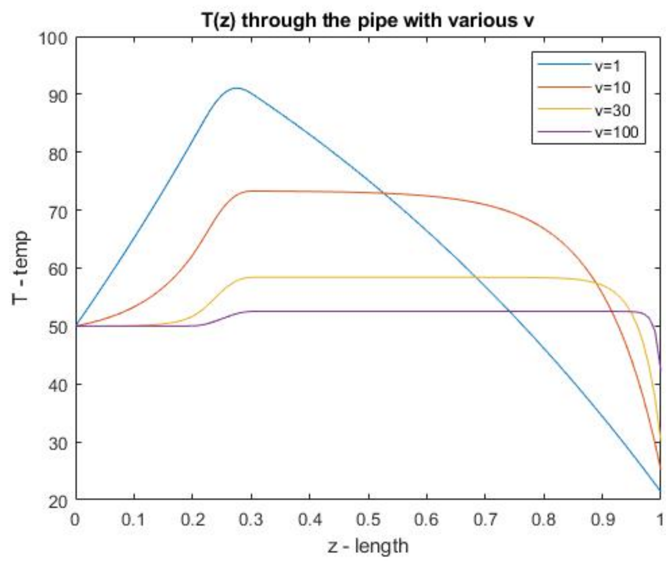

Figure 3: Curves of solution using various velocities, h = 0.01.

The curve, $v = 100$, became unstable when using $h > 0.02$ with this high of a velocity it seems reasonable that it requires smaller step-size to converge (after this value they all converged). The plot shows that the fluid was heated up to a much higher temperature when flowing with a smaller velocity. This result matches with our expectation since with a smaller velocity, the fluid have more time close to the coil and can therefore reach a higher temperature before it then decreases with a higher amount per distance unit. And likewise when the fluid flows with a high velocity it does not have as much time to be heated but consist in its initial temperature for a longer distance.

Problem 2

In this problem we have a 2D problem with dimensions, $\Omega = [0 < x < 5, 0 < y < 2]$, with boundary conditions,

$$\begin{aligned} &T(0,y) = 40, \quad 0 < y < 2 \\ &T(5,y) = 400, \quad 0 < y < 2 \\ &\tfrac{\partial T}{\partial y}(x,0) = 0, \quad 0 < x < 5 \\ &\tfrac{\partial T}{\partial y}(x,2) = 0, \quad 0 < x < 5 \end{aligned}$$

An external source, $f(x,y)$ heats the block. We encounter the following elliptic problem to model the temperature $T(x,y)$,

$$-\Delta T = -\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = f, \quad (x,y) \in \Omega$$

a)

We are asked in this section to solve the problem with $f = 100$ and $h = 0.1$ and then to visualise it using the mesh function in Matlab.

We solved this in Matlab using a central difference scheme for the partial derivatives, that is,

$$\begin{aligned} T_{xx}(x_k, y_l) &= \tfrac{T(x_k+\Delta x, y_l) - 2T(x_k,y_l) + T(x_k - \Delta x, y_l)}{\Delta x^2} + O(\Delta x^2), \\ T_{yy}(x_k, y_l) &= \tfrac{T(x_k, y_l+\Delta y) - 2T(x_k,y_l) + T(x_k, y_l - \Delta y)}{\Delta x^2} + O(\Delta x^2). \end{aligned}$$

And using $\Delta x = \Delta y$ we gain, after discretisation and approximation,

$$f(x_k, y_l) = -\frac{T_{k+1,l} + T_{k-1,l} + T_{k,l-1} + T_{k,l+1} - 4T_{k,l}}{\Delta x^2}$$

or

$$f(x_k, y_l) = \frac{-1}{\Delta x^2}(T_{k+1,l} + T_{k-1,l} + T_{k,l-1} + T_{k,l+1} - 4T_{k,l}).$$

Using the previously mentioned boundary conditions we shall determine the values of the boundaries, namely, $T_{i,0}, T_{0,M+1}, T_{0,k}, T_{N+1,k}$.

Using the Boundary conditions we have that, no matter where we are on the boundary of the left or right side of the rectangle, the temperature is the same. That is in other words,

$$\begin{aligned} T_{0,l} &= 40 \\ T_{N+1,l} &= 400. \end{aligned}$$

for the boundaries. And here again as in the earlier part by ghost point motivation, that is

$$\frac{T_{k,l+1} - T_{k,l-1}}{2\Delta x} = 0$$

and we can establish the relations on the boundaries as follows,

$$\begin{aligned} &T_{k,0} = T_{k,2}, \\ &T_{k,M+1} = T_{k,M-1}. \end{aligned}$$

Then inserting this back into our boundary condition we can establish the following,

$$\begin{aligned}
&\tfrac{-1}{\Delta x^2}(T_{2,l} + T_{0,l} + T_{1,l-1} + T_{1,l+1} - 4T_{1,l}) = f_{1,l} + \tfrac{1}{\Delta x^2}40, \\
&\tfrac{-1}{\Delta x^2}(T_{N+1,l} + T_{N-1,l} + T_{N,l-1} + T_{N,l+1} - 4T_{N,l}) = f_{N,l} + \tfrac{1}{\Delta x^2}400, \\
&\tfrac{-1}{\Delta x^2}(T_{k+1,1} + T_{k-1,1} + 2T_{k,l+1} - 4T_{k,l}) = f_{k,1}, \\
&\tfrac{-1}{\Delta x^2}(T_{k+1,l} + T_{k-1,l} + 2T_{k,l-1} - 4T_{k,l}) = f_{k,M}.
\end{aligned}$$

The initial solution to the homogeneous Poisson equation when using the central difference stencil , has the following solution

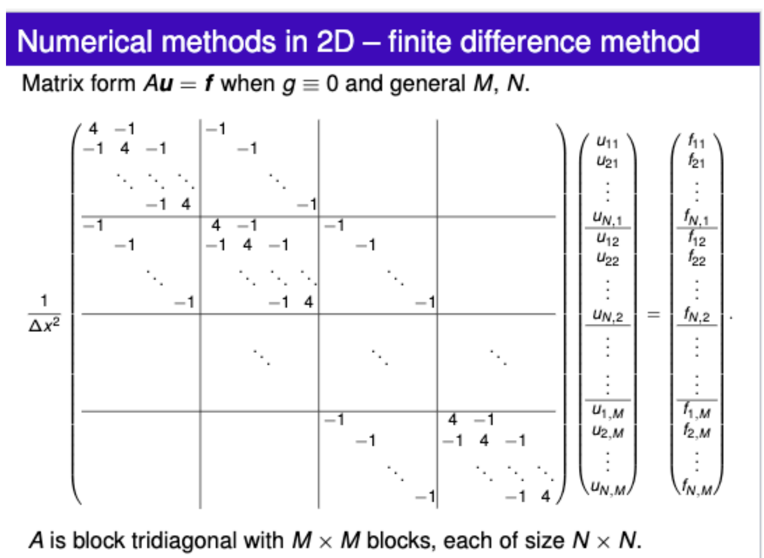

Figure 4: Stencil

When adjusted for boundary conditions we can thus form the following scheme, (the twos here is only for one block)

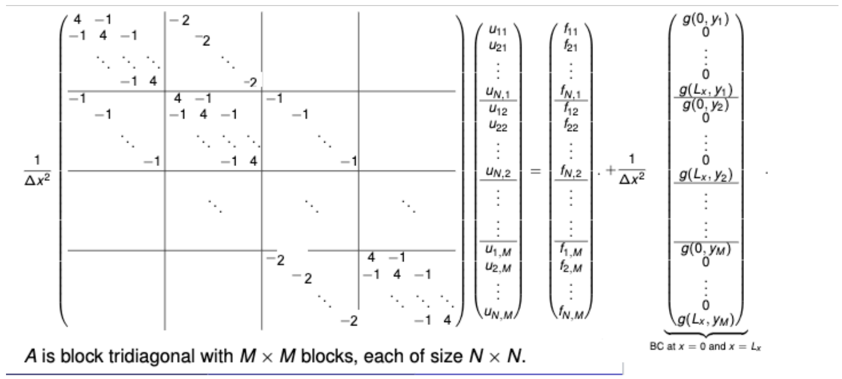

Figure 5: Stencil

where u is our T which we want to solve for and g is the function with our boundary values, that is,

$$T(0, y) = 40, \quad T(5, y) = 400.$$

This system is then solved for T with backslash and then our temperature vector T was attained. The recorded temperature at the point $(x, y) = (3, 1)$ with $h = 0.1, N = 49, M = 21$ was 556 degrees.

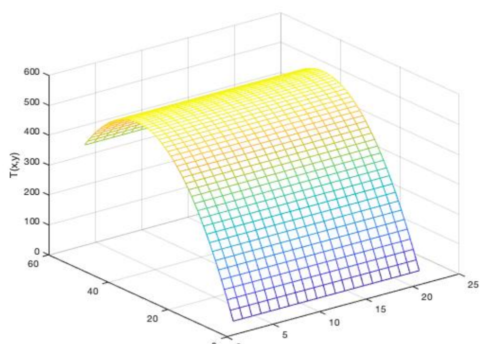

Figure 6: Plotted solution f=100

b)

As by the formulation of the problem we suspect to have a second order solution or lower (as we solve the problem exactly with our method). Starting by rewriting the equation as ,

$$T(x, y) = X(x)Y(y),$$

we shall motivate here that we will regard $Y(y)$ as a constant. Seeing from the plot in the solution that all the values on the y-axis is the same for each respective x. Additionally we also have the Boundary conditions, to support this.

$$\frac{\partial T}{\partial y}(x, 0) = 0, \frac{\partial T}{\partial y}(x, 2) = 0$$

Let $X(x) = ax^2 + bx + c$ and $Y(y) = d$ then rewriting the expression for T,

$$T(x, y) = d(ax^2 + bx + c) = dax^2 + dbx + dc$$

letting the multiplications be new constant, denoted by a_d, b_d, c_d we can solve for all three with the boundary conditions and Possion equation,

$$T(0, y) = 40 = c_d, \quad , T(5, y) = 400 = (a_d 5^2 + b_d 5 + c_d)$$

and

$$-(T_{xx} + T_{yy}) = -(2a_d) = 100$$

and we arrive at the final expression of $T(x, y)$

$$T(x, y) = a_d x^2 + b_d x + c_d = -50x^2 + 322x + 40$$

and evaluated at $x = 3$ is 556 which is indeed the same as with our findings in the solution above. The numerical scheme does indeed find the exact solution as the solution is a second degree polynomial and our method of solving the problem is of second order. Additionally this is supported by the visual plot that we found of the solution, as it indeed takes the characteristic appearance of a second degree polynomial.

c)

In this section we adjust the f such that instead of being a mere constant as before it is now a function dependant on x and y.

$$f = 6000\exp(-5(x-1)^2 - 10(y-1.5)^2).$$

The reported T values for the various h were as follows,

h	T(x,y)
0.1	781.7898
0.05	782.2751
0.025	782.3972

And the requested plots from the problem is listed below.

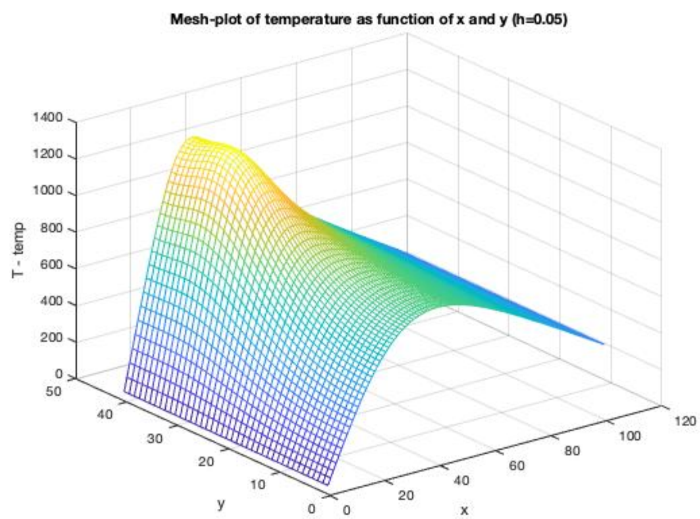

Figure 7: Mesh plot of solution with h=0.05

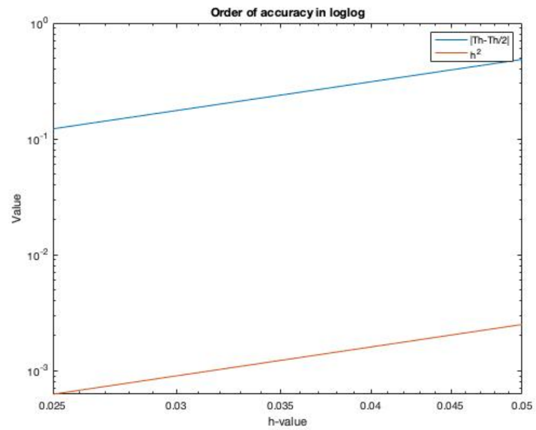

Figure 8: Order of accuracy

The order of accuracy of the method is two as expected from the properties of the central difference scheme. And yes we think its safe to say that the various results are as expected. The Meshed plot, ImageSc and Contour plot seems to all align.

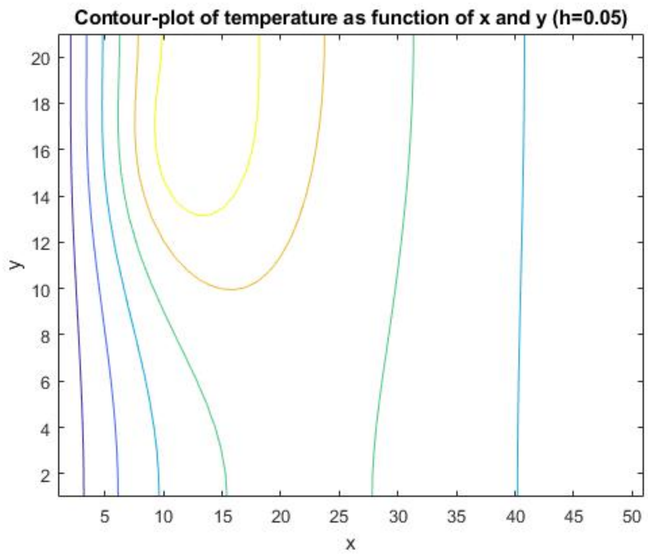

Figure 9: Contour plot

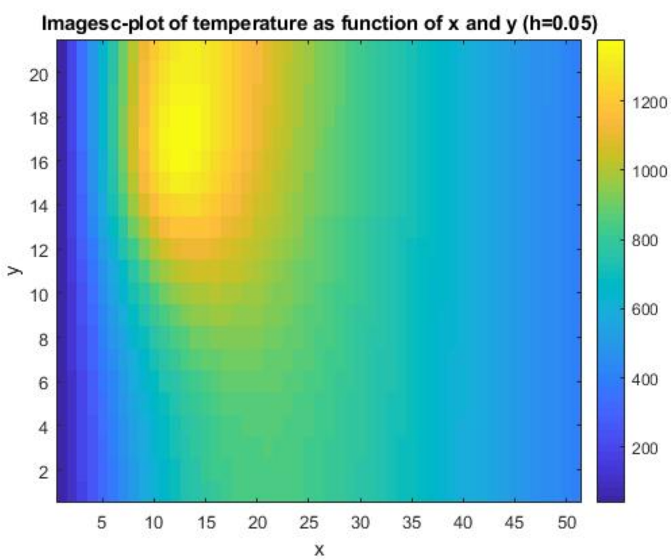

Figure 10: ImageSc plot with temperature

Problem 3

a)

After the first meshing (with Element size: Extra coarse) with 32 triangles, a temperature at point $(3, 1)$ was registered to be 782.445. Then the Element size: Extra Fine was used, it had 2536 triangles and registered a temperature of 782.438 at the point $(3, 1)$ and after that Element size: Extremely fine, with 10040 triangles and a temperature at 782.438 at point $(3, 1)$. Increasing Element size after this point on would not increase the accuracy of the temperature (by more than 0.001) thus we conclude part a. As a last point the plot here is indeed in line with our findings with the Matlab code.
For clarification in this part and in the coming part the following f is used in solving $-\Delta T = f$,

$$f = 6000 \exp(-5(x-1)^2 - 10(y-1.5)^2).$$

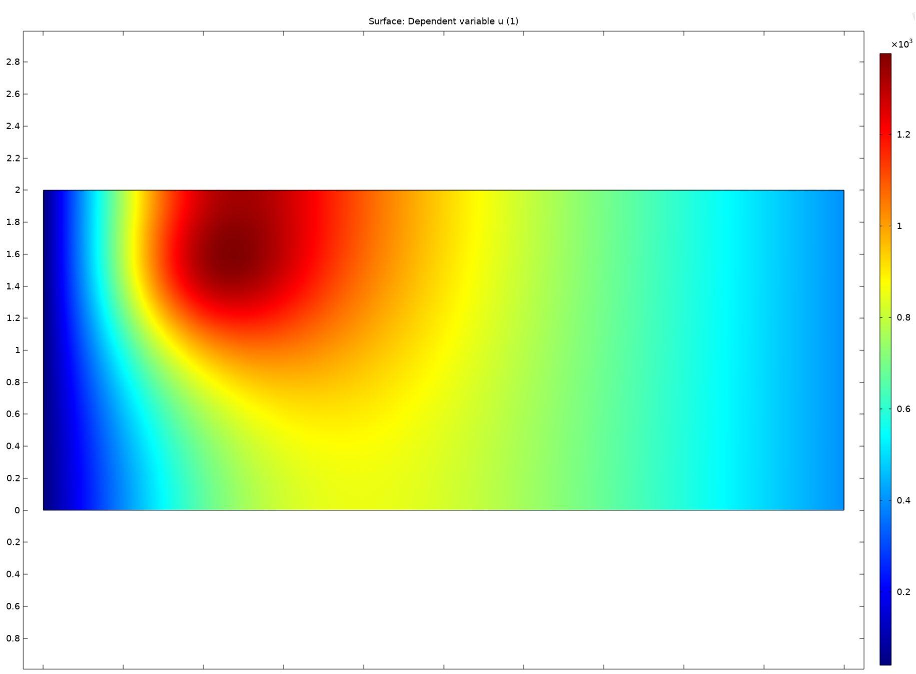

Figure 11: Comsol solution

b)

In this part we change our boundary condition at $x = 5$, namely,

$$T(5, y) = 40,$$

to the new Neumann condition,

$$\frac{\partial T}{\partial x}(5, y) = 0.$$

Below listed is the plot for the new boundary condition. The registered temperature at the boundary $x = 5$ was roughly 1356.06 degrees.

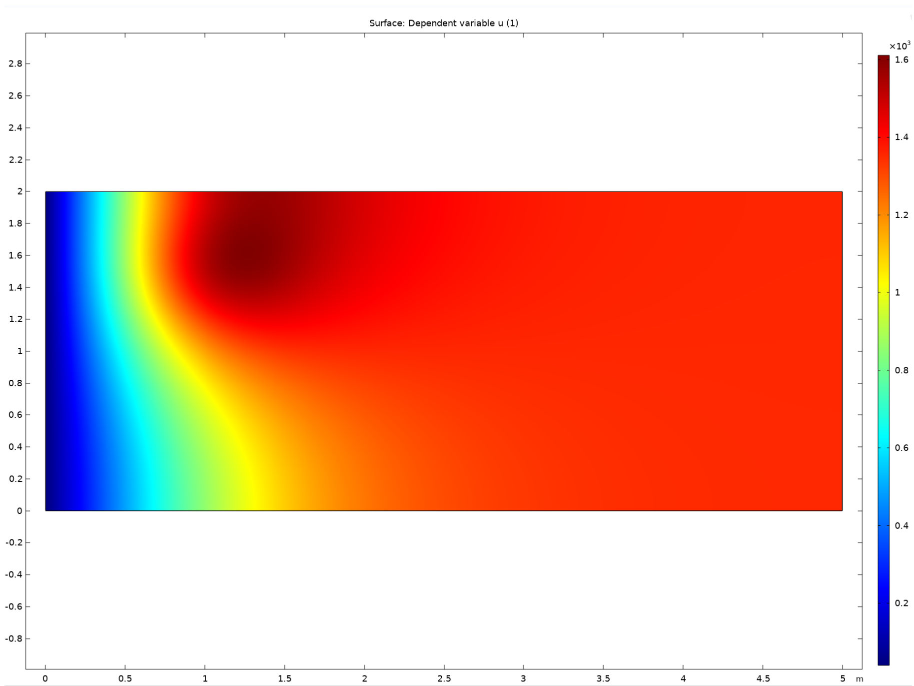

Figure 12: Comsol solution with Neumann boundary condition

c)

In this part we drill a circle at the point $(3, 1)$ with radius 0.5 to try to cool down the temperature at the boundary of $x = 5$. The boundary of said circle fufills the following condition

$$\frac{\partial T}{\partial n} = T_0 - T,$$

where $\frac{\partial}{\partial n} = \hat{n}\nabla$ is the normal derivative and $T_0 = 20$ is the liquid temperature given in the assignment. Then the average temperature at the boundary $x = 5$ was registered to be 179.8 degrees.

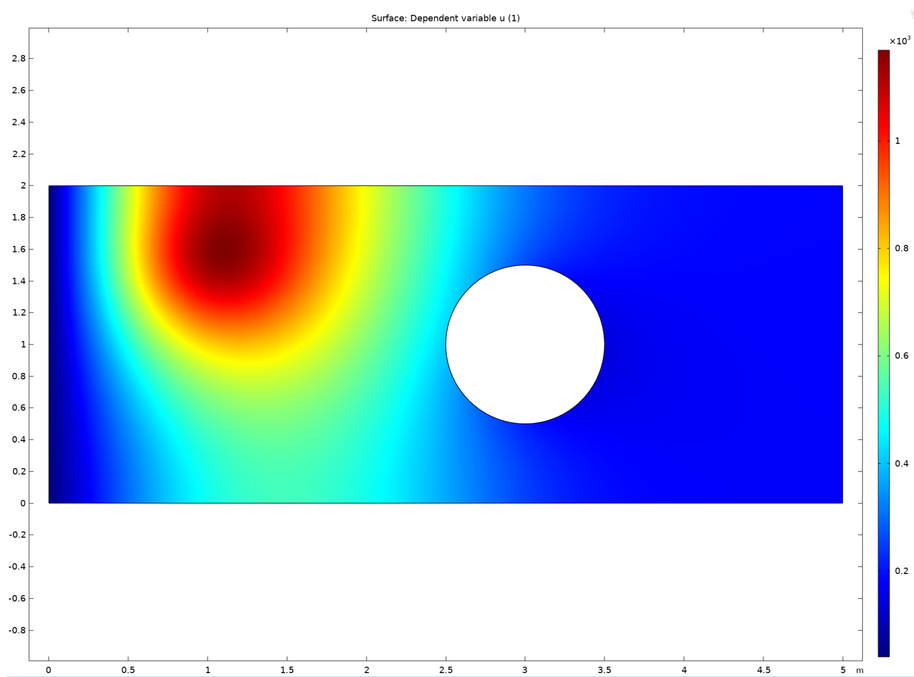

Figure 13: Comsol solution with hole

d)

In this subsection we replace the previous circle by 4 smaller holes placed at $(3 \pm 0.25, 1 \pm 0.25)$ all with a radius of 0.2 needing to satisfy the same boundary condition whilst keeping the boundary conditions the exact same as in subsection c. The recorded temperature at the boundary was roughly 147.9, giving us a decreased temperature of approximately 31.9 degrees from the earlier solution.

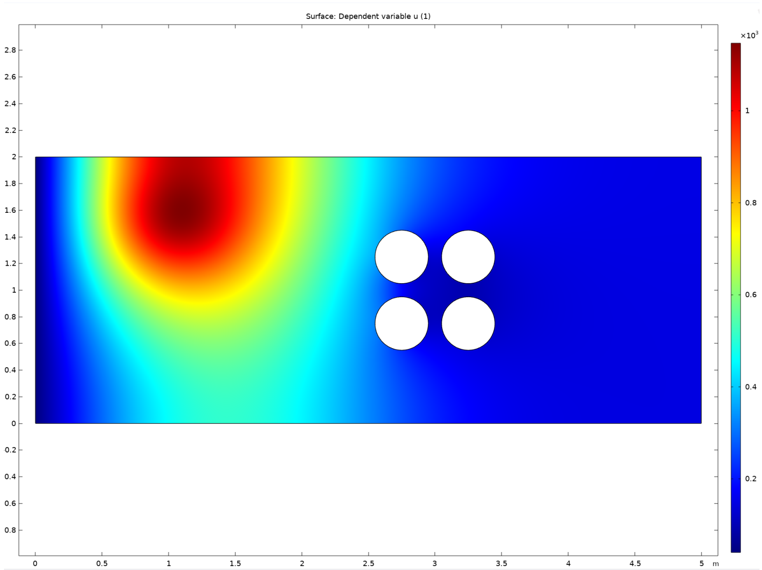

Figure 14: Comsol solution with 4 smaller holes

As per request the added grid plot is seen below. Note the frequency (and smaller size) of the triangles surrounding the 4 holes in contrast to the rest of the grid. This is to be certain the boundary condition,

$$\frac{\partial T}{\partial n} = T_0 - T,$$

of the four holes is satisfied and thus the grid must be finer in this area.

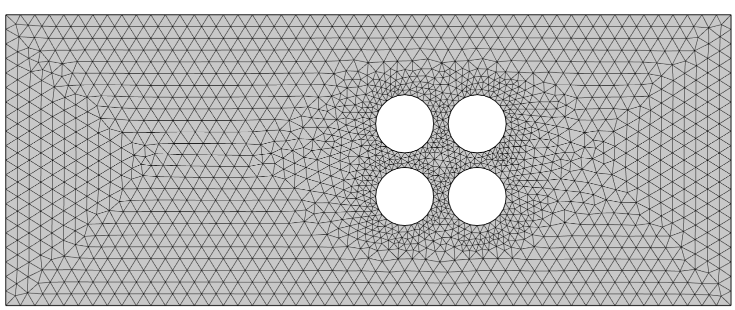

Figure 15: Comsol Mesh grid

e)

Staying in tradition we simulated similar holes as in the previous question, only this time making various grids with more and more holes. We initially created an evenly-spaced grid in the square $(3 \pm 0.5, 1 \pm 0.5)$ with 4x4 holes with radius 0.12 as in the picture but gained a boundary degree of approximately 180. Lastly we created instead a 20x20 grid, also evenly-spaced in the same square, $(3 \pm 0.5, 1 \pm 0.5)$ with radius 0.024 to then gain a boundary temperature of roughly 69.4. (Here we increased the Element size till it was correct by one decimal to affirm it was indeed correct). Additionally the right most side, $(5, y)$ had a temperature of roughly 49.9.

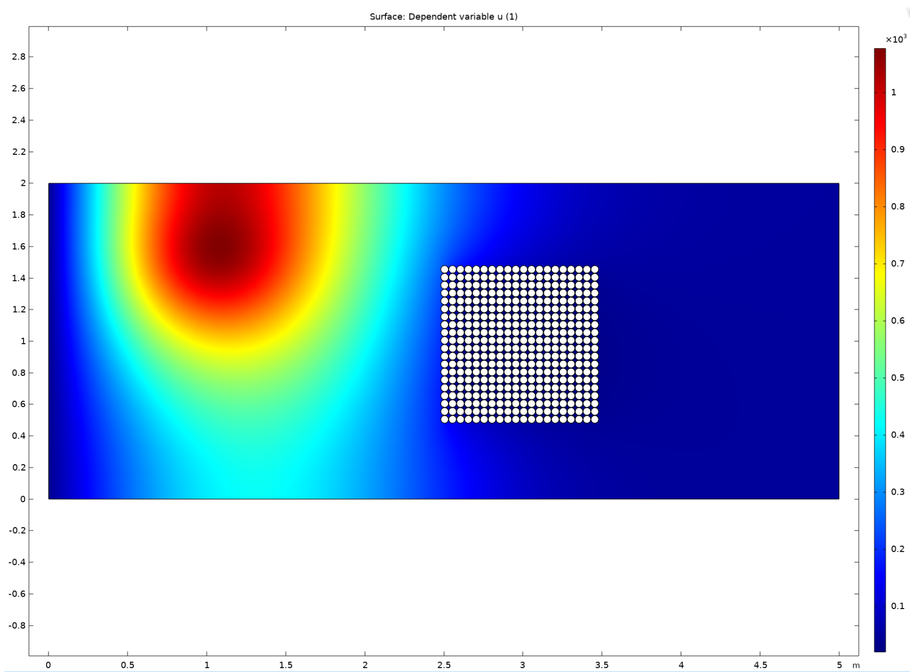

Figure 16: Solution with 20 holes

The idea behind said construction is just from the earlier exercises, we are obviously interested in maximizing the length on which our boundary lies such that we are utilizing the condition,

$$\frac{\partial T}{\partial n} = T_0 - T,$$

to its maximum.

There probably exits better such formulation which increases the area on which we are utilizing the condition and thus make it more cool. Additionally one could make more and more fine holes, but seeing as this is quite demanding (computationally) perhaps it is not recommended.
From a purely physical standpoint it is relevant, seeing one has a desired cooling on a specific point or area, and also wanting the right side to be more cool. As for feasibility the tiny gap between the holes might be less probable, here perhaps its desired to have larger gaps between the holes seeing as the material might crack or break. As long as the length units specified is not to small, one should be able to manufacture a similar configuration using fine drilling tools (e.g. laser cnc-machine).
Looking at only the meshed grid at and around the 400 holes it is easily seen that it is computationally heavy, (we could add that it took some few seconds to actually compute it). And we also attempted in doing it on a 50x50 grid of holes, which did not work due to the time-complexity and increasing time of the problem.