A Framework for Pricing and Risk Management of Loans with Embedded Options

Bernd Engelmann[1]

This version: Mai 13, 2014

ABSTRACT

A framework for the pricing and risk management of retail loans with embedded options is developed. For retail customers there is in general no market information related to their ability to pay (bond or CDS spreads) available. In this case a bank has to rely solely on statistical data to judge the credit quality of a debtor. For the pricing of a loan with embedded options like prepayment rights in this context, a model is proposed that combines a stochastic interest rate model with statistical information like a term structure of default probabilities or a one-year transition matrix and recovery rate estimations. By defining a suitable notion of risk, it is shown how the concepts of credit risk management can be transferred to this framework. It turns out that this modeling approach combines the theories of derivatives pricing and credit risk modeling in the sense that derivatives pricing theory measures the costs for hedging optional components in loans while credit risk modeling measures the risk that these hedging costs turn out to be inadequate. This risk does depend not only on the single loan's risk characteristics but also on the dependence structure and the granularity of the total loan portfolio.

JEL Classification: G12, G13
Keywords: Loan Pricing, Prepayment Options, Interest Rate Derivatives Pricing, Credit Risk Modeling, Risk Management of Loans

[1] Quantsolutions, Gärtnerweg 7, D-60322 Frankfurt, e-mail: bernd.engelmann@quantsolutions.de

A loan is probably the most traditional banking product. Moreover, it is not a simple product as loans often contain embedded options. In Germany debtors frequently have the right to pay back a certain amount of a loan, e.g. five percent of the initial notional in each year, in addition to the agreed amortization schedule. By law every debtor has the right to pay back the total notional after ten years even if the loan contract has a longer maturity, i.e. the loan contains an European style prepayment option. This right can be of considerable value because most loans have a fixed rate of interest. In Italy and the US a debtor might even have the right to pay back a loan at any time, i.e. the loan contains an American style prepayment option. In other countries prepayment options are less important. Different embedded options might play a role. For instance, floating-rate loans might have embedded caps which define an upper limit on the interest rate a debtor has to pay.

When a bank manages the risks of interest rate options, it calibrates an interest rate model to liquid instruments of the interest rate market, prices the options using the calibrated model, and uses the model sensitivities to manage the options' risks (Brigo and Mercurio 2006). This approach is not applicable here because options embedded in loans are linked to the default risk of debtors. Furthermore, debtors in the loan market do not act perfectly rational as interest rate derivatives traders would do. For these reasons, the techniques from interest rate markets cannot be directly applied to loan markets.

Therefore, we have to go beyond a term structure model by including the debtor's default risk into the model. Basically two types of models can be distinguished in the literature, structural models which describe the economic process underlying the default of a debtor explicitly and reduced form approaches which model the spreads over risk free interest rates needed for pricing risky zero bonds. A good overview on both types of models can be found in Schönbucher (2003). The pricing model proposed in this article models credit risk by rating transitions. This can be viewed as an extension of the reduced form approach as the credit quality of debtors that have not yet defaulted is represented by different states. This approach was introduced by Jarrow, Lando, and Turnbull (1997), where the default process is modeled by a Markov chain of credit ratings. The calibration of this model, however, requires that some market information - in this case bond spreads - about the defaultable entity exists. This is true only for a minority of debtors in the loan market, typically very few large companies that have bond issues outstanding.

For retail customers, the majority of debtors, no market information is available. For these customers banks have rating systems in place to evaluate their credit quality. These systems are mainly based on balance sheet information in the case of corporations and personal information in the case of private clients. This data is often combined with macroeconomic indicators. Most rating systems apply statistical models that lead to an estimation of a default probability for each rating grade or a transition matrix for the full rating system. Further, from the history of losses that have already been observed in a bank's loan portfolio, an estimation of recovery rates can be made. An overview on the statistical estimation of default probabilities and recovery rates can be found in Engelmann and Rauhmeier (2006).

For loans without embedded options the pricing and risk management are based on statistically estimated default probabilities and recovery rates. Based on this information all risk and cost components of a loan are calculated to determine the interest margin a bank charges to a debtor. The expected loss of a loan is one of these components and a bank has to ensure that the interest margin is high enough to cover expected losses on portfolio level. Further, to back a loan portfolio against the risk of unexpected losses a bank computes economic capital. In this context, risk is usually defined as uncertainty about the portfolio loss at a future point in time (typically in one year) and a credit risk model is used to compute the loss distribution. Popular modeling frameworks are based on Gupton, Finger, and Bhatia (1997), CSFB (1997), or Wilson (1997a) and Wilson (1997b). These frameworks have been extended and refined in recent years by various authors. A good example of a very rich and still numerically tractable approach belonging to the class of asset value models is Castagna, Mercurio, and Mosconi (2009). Once the loss distribution is computed, economic capital is defined by a risk measure like value-at-risk or expected shortfall. Although value-at-risk is still more popular in banking practice, expected shortfall has the superior properties as analyzed in articles by Artzner, Dalbaen, Eber, and Heath (1999), Acerbi and Tasche (2002), and Tasche (2002). Once the economic capital is determined it has to be allocated to each credit exposure to measure the main drivers of credit risk in the portfolio. Reasonable approaches are explained in Kalkbrener (2005), Kalkbrener, Lotter, and Overbeck (2004), and Kurth and Tasche (2003). The capital allocated to each loan has to earn a return which is also a part of the interest margin a debtor pays. Finally, the interest margin has to cover all other bank-internal costs.

In this article, credit risk modeling is combined with derivatives pricing theory to build a generic framework for the pricing and risk management of retail loan portfolios. We assume that for each loan the rating grade of a debtor is known and that either a term structure of default probabilities or a one-year transition matrix from the bank's rating system and a (possibly time-dependent) recovery rate have been estimated. From the interest rate market, we know the prices of zero bonds for all maturities which are bootstrapped from deposits, futures, and swaps, and the implied volatilities of European swaptions quoted in the market. Based on this data, we construct a pricing model that can value loans including embedded options. After the introduction of the pricing model we show how to combine the concepts from the credit and derivatives modeling worlds to risk manage loan portfolios. Regarding embedded options, we focus on the right to prepay the loan because this is the most important type of embedded option in the loan market.

There already exists a lot of literature on prepayment options in the context of mortgages and mortgage-backed securities (MBS), see e.g. Kau and Keenan (1995). In these articles interest rates are driven by a term structure model and the optimality of prepayment is either derived endogenously from the interest rate level and some additional conditions like transaction costs as in Stanton (1995) or, alternatively, prepayment is modeled explicitly by a prepayment process as, e.g., in Kolbe and Zagst (2008). In the latter case, the parameters of the prepayment process are determined in a calibration process from observed market prices of MBS. This kind of modeling is not applicable in our context as we have assumed the absence of market prices. In the mortgage literature, default is typically captured by an explicit modeling of the house price as in Ciochetti, Deng, Gao, and Yao (2002) where debtors strategically default when house prices fall and the drivers of default are analyzed empirically. This modeling approach is applicable to mortgages only, not to other types of loans. Furthermore, it is related to mortgages where the real estate is the only available collateral which is typical for the US loan market but not for loan markets in other countries.

In comparison to this literature the model in this article is more generic and applicable to any type of loan. Default is modeled in a reduced form approach by explicitly modeling the credit rating of a debtor. A further advantage of this model is that it is based almost entirely on risk parameters that are already available in banks' risk management systems and, therefore, hardly any model calibration is necessary. In modeling prepayment, optimality conditions are derived endogenously from the future level of interest rates and the debtor's rating at prepayment times. Other embedded options like caps and floors of loans with floating interest rates or combinations of caps with prepayment rights can be included easily into this model. Finally, if a bank decides to hedge some of its exposure to embedded options with market instruments like European swaptions the model can be used for calculating the appropriate hedge ratios. However, as the model is not complete these hedges will not be perfect. It will be shown how techniques from credit risk modeling can be used to quantify the risk of losses when hedging embedded options in loans on portfolio level. We finally remark that a similar approach of modeling prepayment endogenously as in this article was suggested by Aguais, Forest, and Rosen (2000). However, they were not very detailed on model calibration nor did they work out a link between derivatives pricing and credit modeling.

The article is structured as follows. Section 1 starts with a short review of a pricing framework for loans without embedded options based on RAROC (risk-adjusted return of capital). After that, the pricing model will be generalized and the valuation of prepayment rights is explained in detail. In the extended model, risk-neutral probabilities from the calibration of a short rate model to traded instruments are combined with real-world probabilities from statistical information about debtors. We will show that this approach can be viewed as a combination of credit risk modeling with derivatives pricing under a suitable notion of risk. This will be done in Section 2. Furthermore, it will be outlined how the proposed pricing model can be generalized to more advanced modeling approaches. In Section 3 we will illustrate the model with numerical examples. The final section concludes.

1. Pricing Model

We divide this section into four parts. In the first part we briefly review the basic concepts of loan pricing. The second part deals with modeling rating transitions, the third part explains the term structure model, while in the fourth parts both models are combined and the application of the model to loan pricing is explained in detail.

1.1. Pricing Loans without Embedded Options

To explain the basic principles of loan pricing, we start with a simple loan that does not contain any options. The framework presented here is based on ideas that were already introduced in articles by Aguais and Forest (2000), Aguais, Forest, Krishnamoorthy, and Mueller (1998), and Aguais and Santomero (1998), where also some details on the institutional background of lending practice can be found. We use a bullet loan for illustration, i.e. a loan without amortization payments during its lifetime. For loans with amortization payments, like annuity loans or installment loans, the extension of the formulas we derive in the sequel is straightforward.

The notional of the bullet loan is denoted with N. The interest rate periods are defined by the times $T_0,\ T_1,\ldots,T_m$ where $[T_i,\ T_{i+1}]$ is an interest rate period, $\tau_i = T_{i+1} - T_i$ is the year fraction of the period, and we assume that interest is paid in T_{i+1}. In the case of a fixed-rate loan, the interest payment in each period is $y\,\tau_i\,N$, where y is the fixed interest rate. In the case of a floating-rate loan, the interest rate payment is $(L_i + s)\,\tau_i\,N$, where we have assumed that the floating rate L_i, e.g. a Libor rate, is fixed at the beginning of each interest rate period i and s is a spread that is constant throughout the loan's lifetime. We will use the notation z_i for the interest rate in period i with

$$z_i = \begin{cases} y, & \text{if the loan's interest rate is fixed,} \\ f_i + s, & \text{if the loan's interest rate is floating,} \end{cases} \tag{1}$$

where f_i is the forward rate corresponding to the floating rate L_i. At T_m the notional N is paid back. In the case of default, the lender receives a payment of $R\,N$ at the time of default, where R is the estimated recovery rate. We will assume a time-independent recovery rate.

The value V of a loan is defined as the expected discounted value of all future cash flows. This leads to

$$\begin{aligned} V(t) &= N \sum_{t<T_i} z_i\,\tau_i\,\delta(T_i)\,v(T_i) + N\,\delta(T_m)\,v(T_m) \\ &+ N\,R \int_t^{T_m} \delta(u)\,d(1 - v(u)), \end{aligned} \tag{2}$$

where $\delta(u)$ is the discount factor corresponding to maturity u and $v(u)$ is the probability that the debtor will survive until time u. The first part on the right hand side of (2) is the expected discounted value of the interest rate payments, the second part the expected discounted value of the reimbursement of the notional, and the third part the expected discounted value of the liquidation proceeds of the collateral in case of default. Note, that in the above pricing formula implicitly the independence of interest rate fluctuations and the default event was assumed. The integral in (2) can be approximated by the trapezoid rule which leads to

$$V(t) = N \sum_{t<T_i} z_i\,\tau_i\,\delta(T_i)\,v(T_i) + N\,\delta(T_m)\,v(T_m)$$

$$+N\,R\sum_{t<T_i}\delta\left(\frac{T_{i-1}+T_i}{2}\right)\,(v(T_{i-1})-v(T_i)). \tag{3}$$

The above valuation formula requires two discount curves. The first is the market swap discount curve which will be denoted with $\delta_M(t)$ and is needed for calculating forward rates as

$$f_i=\frac{1}{\tau_i}\left(\frac{\delta_M(T_{i-1})}{\delta_M(T_i)}-1\right). \tag{4}$$

The second discount curve is the bank's funding curve which is used for discounting a loan's cash flows. This discount curve is provided by a bank's treasury. It will become immediately clear why this curve is a sensible choice for discounting a loan's cash flows.

In the next step, it will be shown how the valuation formula (3) is used for calculating the risk-adjusted interest rate of a loan. The starting point of this calculation is the determination of the market base rate y_M for a fixed rate loan. This is the interest rate a bank should charge when the loan is risk free, i.e. all survival probabilities are equal to one, and the bank does not have any costs. To ensure that a bank's assets equal its liabilities in this situation, the condition

$$N\sum_{t<T_i}y\,\tau_i\,\delta_M(T_i)+N\,\delta_M(T_m)=N \tag{5}$$

has to be fulfilled. Solving this condition for y gives the market base rate which is equal to the swap rate corresponding to the loan's maturity

$$y_M=\frac{1-\delta_M(T_m)}{\sum_{t<T_i}\tau_i\,\delta_M(T_i)}. \tag{6}$$

This allows us to rewrite the fixed rate y as $y=y_M+s$ where s has an analogous meaning as for the floating rate loan. It is the spread over market rates to cover all loss risks and costs of a loan.

The first cost component we consider is the margin s_f for funding costs. To ensure that a bank's assets equal its liabilities when funding costs are included into consideration, the condition

$$N\sum_{t<T_i}z_i\,\tau_i\,\delta(T_i)+N\,\delta(T_m)=N \tag{7}$$

has to be met for each loan. This leads to an expression for the funding cost spread s_f

$$s_f=\begin{cases}\frac{1-\delta(T_m)}{\sum_{t<T_i}\tau_i\,\delta(T_i)}-y_M, & \text{if the loan's interest rate is fixed,}\\ \frac{1-\sum_{t<T_i}f_i\,\tau_i\,\delta(T_i)-\delta(T_m)}{\sum_{t<T_i}\tau_i\,\delta(T_i)}, & \text{if the loan's interest rate is floating.}\end{cases} \tag{8}$$

In reality loan portfolios are not risk free and liabilities can be matched by assets only on average. To achieve this, an additional component s_{EL} of s which is needed to cover expected losses is computed from the condition $V(t) = N$, which means that expected assets equal a bank's liabilities. In this calculation survival probabilities are included. In the case of a fixed-rate loan the expected loss margin s_{EL} is given as

$$s_{EL} = \frac{1 - \delta(T_m)\, v(T_m) - R\, \sum_{t<T_i} \delta\left(\frac{T_{i-1}+T_i}{2}\right)\, (v(T_{i-1}) - v(T_i))}{\sum_{t<T_i} \tau_i\, \delta(T_i)\, v(T_i)} - s_f - y_M, \tag{9}$$

for the floating-rate loan, we find

$$s_{EL} = \frac{1 - \delta(T_m)\, v(T_m) - \sum_{t<T_i} f_i\, \tau_i\, \delta(T_i)\, v(T_i)}{\sum_{t<T_i} \tau_i\, \delta(T_i)\, v(T_i)} - \frac{R\, \sum_{t<T_i} \delta\left(\frac{T_{i-1}+T_i}{2}\right)\, (v(T_{i-1}) - v(T_i))}{\sum_{t<T_i} \tau_i\, \delta(T_i)\, v(T_i)} - s_f. \tag{10}$$

Expected losses are only estimations of losses a bank has to bear on average. In a bad year these losses might be exceeded considerably. To avoid bankruptcy in these years a bank has to hold an equity capital buffer E, the economic capital, for each loan as a protection against unexpected losses which reflects the risk contribution of this loan to portfolio risk. This equity capital has to earn a return. A bank defines a target return w on its equity capital, e.g. $w = 10\%$. This leads to the margin component for unexpected losses s_{UL}

$$s_{UL} = \frac{w\, E}{N}. \tag{11}$$

Finally, all other internal costs like maintenance costs for the loan have to be summarized in a cost margin s_C. The interest rate s that has to be paid by a debtor to cover all cost components is therefore

$$s = s_f + s_{EL} + s_{UL} + s_C. \tag{12}$$

We remark that this pricing approach is related to the RAROC concept. The target return w can be computed as $w = (s - s_f - s_{EL} - s_C)/(E/N)$, i.e. as the loan's return reduced by the loan's risk costs divided by its economic capital. Therefore, this target return is equivalent to the bank's target RAROC.

In this pricing approach we have implicitly assumed that interest rate movements and the default event are independent. Further, we have assumed independence of the realized recovery rate from interest rate movements and the default event. These assumptions will be maintained throughout this article. However, we have to introduce stochastic models for the rating grade and the level of interest rates to value embedded options which will be done in the next two subsections.

We finish this section with a comment on the modeling assumption of independence between recovery rates and default probabilities. The recent financial crisis and empirical work by Frye (2000) and Frye (2003) have shown that this assumption can be heavily violated in reality. As long as we are concerned with pricing, i.e. with computing expectations as in (3), this is not a big limitation. In (3) we are using the expression $R \cdot (v(T_{i-1}) - v(T_i))$ which can be viewed as an expected recovery value corresponding to a default in period i. The most popular (but, of course, not all) approaches to model dependence of defaults and recoveries lead to the same expectations as the independence assumption. This is in particular true for those relying on the asset-value framework, see for instance Barco (2007). In this case (3) will be unchanged when the independence assumptions is relaxed. However, when it comes to risk measurement which will be discussed later in this article, the independence assumption might be too restrictive. Instead of making the model more complicated in this case with all the consequences for parameterization complexity and computational workload, one could instead keep the model simple and use more conservative estimates of recovery rates, e.g. instead of using historical averages one could use a lower quantile of a historical distribution, to mitigate model inefficiencies. This way of thinking is reflected in the concept of a downturn recovery rate that was proposed by supervisors (Basel Committee on Banking Supervision 2004).

1.2. Modeling Rating Transitions

We assume that a bank's rating system has n grades where the n-th grade is the default grade. There are typically two ways how banks extract statistical information about defaults from their rating systems to estimate multi-year default probabilities.

In the first approach, banks directly estimate a term structure of default probabilities, i.e. for each rating grade k a function $p_k(t)$ is estimated where $p_k(t)$ is the probability that a debtor in rating grade k will default within the next t years. This could be done using techniques from survival analysis. From the term structure of default probabilities given today, conditional default probabilities at future times u can be computed easily. The probability $p_k(t|u)$ that a debtor in rating grade k will default up to time t conditional that he is alive at time u is given by

$$p_k(t|u) = 1 - \frac{1 - p_k(t)}{1 - p_k(u)}, \; u < t \tag{13}$$

In the second approach a one-year transition matrix is estimated. This matrix is denoted with $\mathbf{P}(1)$. The entries of the matrix are denoted with $p_{kl}, k, l = 1, \ldots, n$ where p_{kl} is the probability that a debtor in rating grade k moves to grade l within one year. The matrix $\mathbf{P}(1)$ has the following properties:

1. The entries of $\mathbf{P}(1)$ are nonnegative, i.e. $p_{kl} \geq 0$, $k, l = 1, \ldots, n$.
2. All rows of $\mathbf{P}(1)$ sum to one, i.e. $\sum_{l=1}^{n} p_{kl} = 1$, $l = 1, \ldots, n$.

3. The last column p_{kn}, $k = 1, \ldots, n-1$ contains the one-year default probabilities of the rating grades $1, \ldots, n-1$.
4. The default state is absorbing, i.e. $p_{nl} = 0$, $l = 1, \ldots, n-1$ and $p_{nn} = 1$.

If we assume that rating transitions are Markovian, i.e. they depend on the debtor's current rating grade only, and that transition probabilities are time-homogeneous, i.e. the probability of a rating transition between two time points depends on the length of the time interval only, then it is possible to apply the theory of Markov chains to construct transition matrices $\mathbf{P}(t)$ for arbitrary time lengths t. This is done from the generator matrix $\mathbf{H}$ which is computed as the matrix logarithm of $\mathbf{P}(1)$

$$\mathbf{H} = \log(\mathbf{P}(1)) = \mathbf{P}(1) - \mathbf{I} - \frac{1}{2}(\mathbf{P}(1) - \mathbf{I})^2 + \frac{1}{3}(\mathbf{P}(1) - \mathbf{I})^3 \pm \ldots. \tag{14}$$

The matrix $\mathbf{I}$ is the $n \times n$ unit matrix. Transition matrices for arbitrary time lengths are computed by matrix exponentials as

$$\mathbf{P}(t) = \exp(\mathbf{H} \cdot t) = \mathbf{I} + \mathbf{H} \cdot t + \frac{1}{2}(\mathbf{H} \cdot t)^2 + \frac{1}{6}(\mathbf{H} \cdot t)^3 + \ldots. \tag{15}$$

The matrix $\mathbf{P}(t)$ fulfills the conditions of a transition matrix for arbitrary values of t only if the matrix $\mathbf{H} = (h_{kl})_{k,l=1,\ldots,n}$ is a generator matrix, i.e. if it satisfies the conditions:

1. $h_{kk} \leq 0$, $k = 1, \ldots, n$.
2. $h_{kl} \geq 0$, $k, l = 1, \ldots, n$, $k \neq l$.
3. $\sum_{l=1}^{n} h_{kl} = 0$, $k = 1, \ldots, n$.

More details about properties and conditions for the existence of generator matrices and estimation techniques for either the transition matrix or the generator matrix can be found in the books by Bluhm, Overbeck, and Wagner (2003) and Schönbucher (2003). We remark that in most practical cases generator matrices do not exist and some regularization algorithm has to be used in the calculation process for the generator matrix. Examples of regularization algorithms are given in Kreinin and Sidelnikova (2001).

Transition matrices allow the construction of rating trees. An example of a rating tree is shown in Figure 1. Suppose we are interested in the pricing of a financial instrument that depends on the rating grade of a debtor at several time points T_0, $T_1, \ldots, T_m$. The probability that a debtor in rating grade k at time T_i moves to rating grade l at time T_{i+1} is given by $p_{kl}(T_{i+1}|T_i)$. Since we have assumed that transition probabilities are time homogeneous, these probabilities can be read from the transition matrix $\mathbf{P}(T_{i+1} - T_i)$, which can be computed from (15). To specify all probabilities in the tree, we have to compute transition matrices for all time intervals $T_{i+1} - T_i$. In Figure 1 the lowest node is the default state. Once a debtor has reached this state he will stay there forever.

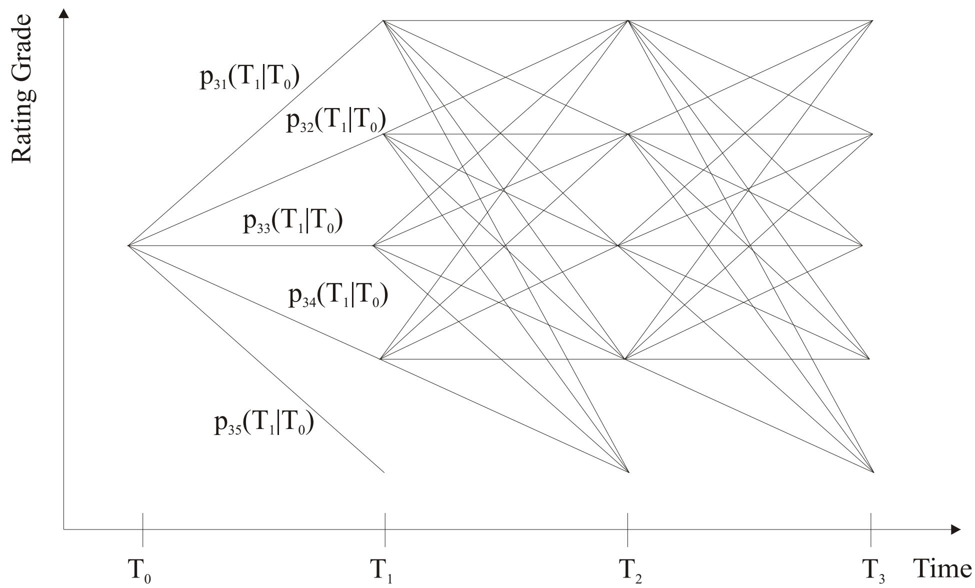

Figure 1. Rating tree for a rating system with five grades. Grade 5 is the default state.

Note that it is also possible to build a rating tree from a term structure of default probabilities. Here the tree consists of two rating states only, the alive state and the default state. The default probability from time T_i to time T_{i+1} is computed by (13). The probability of staying alive is simply $1 - p_k(T_{i+1}|T_i)$.

1.3. Modeling Interest Rates

A rating transition model is only a part of a loan pricing model. As we want to include prepayment rights into the model which depend on the level of interest rates, we have to make interest rates stochastic. In the loan pricing model (2) we have introduced two discount curves, the funding curve $\delta(t)$ and the swap curve $\delta_M(t)$. We assume that the dynamics of interest rates is entirely captured by the swap curve which is described by a short rate model. We choose this model class because it allows the use of tree algorithms for the pricing which are well suited for the valuation of exercise rights. The model for the short rate $r_M(t)$ describing the swap curve dynamics is given by

$$dr_M(t) = \mu(r_M,t)dt + \sigma(r_M,t)dW(t), \tag{16}$$

where $\mu(r_M,t)$ is the risk-neutral drift, $\sigma(r_M,t)$ the instantaneous volatility, and $W(t)$ the driving Wiener process. Both functions μ and σ are deterministic. The funding curve is also described by a short rate r which is defined by simply adding a deterministic spread $\beta(t)$ to the swap curve, i.e.

$$r(t) = r_M(t) + \beta(t). \tag{17}$$

As shown in Brigo and Mercurio (2006), a standard recombining trinomial tree that approximates the short rate distribution implied by (17) can be constructed for any choice of model parameters.

The trinomial tree is fully specified by the value of the short rate $r_{i,j}$ at each node of the tree where i is the time index and j is the location index which starts from zero at the bottom of the tree. At each node three probabilities for moving to one of three possible states in the subsequent time node, $q_{u,i,j}$, $q_{m,i,j}$, and $q_{d,i,j}$, are given where u stands for up, m for mid and d for down. The indices i and j have the same meaning as for the short rate $r_{i,j}$. The three probabilities sum to one. The risk-neutral probabilities and the short rate levels of the tree are calibrated to the funding discount curve and a set of European swaptions. This ensures that the pricing of basic market instruments is done correctly. For details on the calibration process, we refer to Brigo and Mercurio (2006). We note that in practical applications one would rather use a PDE solver instead of the trinomial tree because of its superior convergence properties (Randall and Tavella 2000). The tree is used here because it is more intuitive than a PDE solver.

This section is concluded with a remark on the assumption of deterministic funding spreads β. In reality, of course, this assumption is not valid and funding spreads are stochastic. However, from a risk management perspective this is not a problem as long as a bank is trying to match the maturity term structure of its funding instruments with the maturity term structure of its loan portfolio. This policy could be viewed as a static hedge of funding spread risk which justifies the modeling of the funding spread as deterministic. In practice this strategy cannot be followed strictly and there will be always some open funding spread risk. Considering this in a pricing context, however, is meaningless because there are no hedging instruments in the market that allow banks to risk manage their funding spread risk. Of course, funding spread risk has to be measured in the context of liquidity risk management where a stochastic funding spread is required for this purpose. Since this is beyond the scope of this article we stick to the simpler deterministic model.

1.4. Loan Pricing Framework

To build a loan pricing framework, we have to combine the rating model and the term structure model. We do this by assuming that rating changes and short rate changes are independent. This kind of model was already introduced in a different context in Schönbucher (2003).

Combining the rating tree and the interest rate tree leads to a three-dimensional tree. From each node it is possible to attain $3n$ nodes where n is the number of rating grades. The probabilities of each branch are given by the products of the probabilities in the interest rate and rating sub-trees because we have assumed independence of interest rate and rating changes. The price of a loan is given as the discounted expected value of all cash flows under the probability measure computed from the risk-neutral probabilities of the interest rate tree and the real-world probabilities of rating changes. We will explain in Section 2 how this combination of risk-neutral and real-world probabilities fits into the frameworks of both the theory of derivatives pricing and credit risk modeling.

To illustrate the pricing framework in detail, we apply it to the bullet loan that was already used in Section 1.1. In this section we add prepayment rights to the loan. For simplicity, to avoid the inclusion of accrued interest, we assume that prepayment is possible only on certain interest rate payment dates which are specified in the loan contract, i.e. the loan has Bermudan style prepayment rights. Since a loan is a much more complex product than an interest rate derivative, we have to make additional assumptions to price the prepayment rights:

1. We assume that the debtor needs the money until the loan's maturity. If he is able to get a cheaper loan on prepayment dates, he will prepay and enter the cheaper loan.
2. If the debtor prepays and enters a new loan, the cost components s_{UL} and s_C will be the same as for the old loan.[2]
3. If the debtor prepays and enters a new loan, this new loan will not have any embedded options.
4. Prepaying a loan involves transaction costs $c\,N$ where c is the transaction cost margin. A debtor will prepay a loan only if the gain in prepaying covers transaction costs.
5. The irrational exercise behavior of debtors will be modeled by an exercise probability p_{ex}. This is the probability that a debtor will prepay a loan when the conditions are favorable for him.
6. All banks have the same opinion on default probabilities and the recovery rate.

We will explain in detail how these assumptions enter the pricing algorithm.

For convergence reasons the tree that is used for pricing the loan has to contain more time points than $T_0, \ldots, T_m$. We denote the time points of the tree with $t_0 = T_0,\ t_1, \ldots,\ t_l = T_m$. All start and end times of interest rate periods have to be contained in the tree's time grid. A quantity z_{fix} is needed in the algorithm below which allows us to describe the pricing of a fixed-rate and a floating-rate loan simultaneously. It refers to the fixed part of the interest rate which is $z_{fix} = y$ for a fixed-rate loan and $z_{fix} = s$ for a floating-rate loan. We compute the price[3] $V(k, r_{i,j}, t_i)$ of the loan depending on the rating grade k of the debtor, the short rate $r_{i,j}$ and time t_i using the algorithm:

[2] This is a simplifying assumption since the unexpected loss margin s_{UL} will be in general a function of both the debtor's rating grade and the loan's maturity. This assumption can be relaxed at the price of a more complex numerical algorithm (in this case several auxiliary variables V_{ex} have to be defined in the algorithm below). Since this is more a technical detail than a fundamental property of the pricing algorithm, we leave it out here.

[3] The dimension of this matrix is $n \times n_{SR}$ in every time point t_i where n is the number of rating grades and n_{SR} is the number of grid points in the short rate grid slice at t_i.

1. At t_l: Initialize $V(k, r_{l,j}, t_l)$ and $V_{ex}(k, r_{l,j}, t_l)$ with N.
2. At t_l: Add $z_{fix}\, \tau_m\, N$ to $V(k, r_{l,j}, t_l)$ and add $(z_{fix} - s_{UL} - s_C)\, \tau_m\, N$ to $V_{ex}(k, r_{l,j}, t_l)$. If the loan has a floating interest rate initialize a matrix $V_f(k, r_{l,j}, t_l)$ with 1.
3. At t_{l-1}: Compute $V(k, r_{l-1,j}, t_{l-1})$ from the values of V at the succeeding nodes as

$$V(k, r_{l-1,j}, t_{l-1}) = e^{-r_{l-1,j}\Delta t_l} \Big(\sum_{g=1}^{n-1} p_{kg}(t_l|t_{l-1}) \hat{V}_{g,l-1,j} + p_{kn}(t_l|t_{l-1}) R\, N \Big),$$

where $\Delta t_l = (t_l - t_{l-1})$ and $\hat{V}_{g,l-1,j} = \hat{V}(g, r_{l-1,j}, t_{l-1})$ is computed by taking the expectation of the tree's short rate part

$$\hat{V}_{g,l-1,j} = q_{u,l-1,j} V(g, r_{l,j+2}, t_l) + q_{m,l-1,j} V(g, r_{l,j+1}, t_l) + q_{d,l-1,j} V(g, r_{l,j}, t_l).$$

Repeat this step for V_{ex}. In the case of a floating-rate loan repeat this step for V_f.
4. Repeat Step 3 until reaching time $T_{m-1} = t_o$.
5. At T_{m-1}: Add $z_{fix}\, \tau_{m-1}\, N$ to $V(k, r_{o,j}, T_{m-1})$ and $(z_{fix} - s_{UL} - s_C)\, \tau_{m-1}\, N$ to $V_{ex}(k, r_{o,j}, T_{m-1})$. If the loan has a floating interest rate additionally add the expected present value of the floating rate payment paid in T_m

$$\frac{\delta(T_m)\, \delta_M(T_{m-1})}{\delta(T_{m-1})\, \delta_M(T_m)} \left(1 - p_{kn}(T_m|T_{m-1})\right) - V_f(k, r_{o,j}, T_{m-1}) \qquad (18)$$

to $V(k, r_{o,j}, T_{m-1})$ and to $V_{ex}(k, r_{o,j}, T_{m-1})$, and initialize $V_f(k, r_{o,j}, T_{m-1})$ again with 1.[4]
6. At T_{m-1}: If T_{m-1} is a prepayment time replace $V(k, r_{o,j}, T_{m-1})$ after adding the floating-rate payment to both V and V_{ex} and before adding the fixed-rate payment by

$$V(k, r_{o,j}, T_{m-1}) = p_{ex}\, N + (1 - p_{ex})\, V(k, r_{o,j}, T_{m-1}),$$

if the condition $V_{ex}(k, r_{o,j}, T_{m-1}) > (1 + c)\, N$ is fulfilled.
7. Repeat steps 3-7 until T_0 is reached.

The variables V_{ex} and V_f are auxiliary variables. The former is used to decide about optimality of the prepayment decision while the latter is used to compute the expected discounted value of each floating-rate payment. Step 6 of the above algorithm deserves more explanation. If the debtor pays back an unfavorable loan at a prepayment date and enters a new loan, this operation includes some transaction costs c. Only if the gain of prepaying the loan is big enough to cover the transaction costs, the debtor will refinance his loan. Further, it is known from practice that debtors do not act rational. They will not prepay their loans in all cases where it is favorable for them. Either they are not sophisticated enough or they do not want to finish the relationship with their house bank. Therefore we have introduced an exercise

[4] The expression (18) for the present value of the floating rate payment is derived in the appendix.

probability p_{ex} which has to be estimated from statistical data. The exercise condition reflects the expected assets equals liabilities condition $V = N$. If the loan excluding the cost components s_{UL} and s_C, which is computed by V_{ex}, is more valuable than N this means that the expected loss margin is too conservative and that a more favorable loan should be available for the borrower in the market.

This section is concluded with a remark on the modeling of prepayment. Using a simple prepayment probability is not consistent with empirical evidence. In reality, the prepayment probability is a function of the ”moneyness” of the prepayment option. The larger the difference between the loan's interest margin and the margin prevailing in the market, the higher is the prepayment probability, see for instance Peristiani (1998). The model presented above could be extended to capture this behavior by computing the minimum margin (12) from the future discount curve implied by the short rate $r_{i,j}$ and the future survival probabilities derived from the future rating k in each node at a prepayment date and calculating the difference to the loan's current margin. The exercise probability could then be modeled as a function of this difference.

In this pricing algorithm, we have used an interest rate tree with risk-neutral probabilities that are calibrated from traded instruments like European swaptions and a rating tree that is calibrated from statistical data, i.e. its probabilities are real-world probabilities. Further, in the valuation of prepayment rights we have introduced an exercise probability which is also a real-world probability. It remains to answer the question how this algorithm fits into the theories of derivatives pricing and credit risk modeling. This is done in the next section.

2. Model Properties

This section is split into two subsections. In the first subsection the theoretical properties of the loan pricing model are analyzed and it is explained how risk is defined and measured on an abstract level. The second subsection outlines in detail how to calculate risk measures for the loan pricing model of Section 1.4. Furthermore, it is discussed how the model can be extended to more general modeling frameworks.

2.1. Theoretical Model Properties

To start analyzing the theoretical properties of this loan pricing framework, we take the position of a trader who is hedging the embedded options risks of a loan portfolio by a dynamic hedging strategy. Since we have assumed that there are no market instruments linked to the credit quality of a debtor available, the trader can use interest rate derivatives, like swaps, futures, European swaptions, or caps only to manage the risks of his portfolio.

We consider two extreme cases. In the first case, the trader's portfolio consists of a single loan. When hedging this loan's embedded options risks the trader is exposed to the default risk of the debtor. He has no chance to set up a dynamic trading strategy which hedges all risks of the loan perfectly. He can hedge the interest rate risk perfectly but this hedge breaks down when the debtor defaults. In the second case the trader's portfolio consists of a large number of identical loans and we assume that defaults in this portfolio are independent. If there is no dependence between defaults, the law of large numbers implies that the variance of realized default rates becomes arbitrarily small when the number of debtors in the portfolio is increased. In the limit the realized default rates are identical with the default probabilities. This means that default rates and losses are deterministic in this case. Therefore, only randomness in interest rates is left which can be hedged perfectly.

These considerations demonstrate that the position of a trader hedging embedded options in loan portfolios is very similar to the position of a credit risk manager. He has some expectations about the hedge ratios but he also faces uncertainty about his hedges stemming from uncertainty about the true number of defaults in his portfolio. Furthermore, the riskiness of hedging a portfolio of embedded options is influenced by the dependence structure and the granularity of the underlying loan portfolio. This is in sharp contrast to the risk management of pure interest rate derivatives where each instrument can be considered in isolation (at least in theory). For this reason, the techniques that have been developed in credit risk modeling should be carried over to the risk management of embedded options.

We define the price $V(t)$ of a loan at time t as its expected hedging costs under the real-world rating migration probabilities which is consistent with (2) where hedging means replication by zero bonds with zero bond prices $\delta(t)$. Proposition 1 states a very general formula for this price.

Proposition 1 *The price of a loan defined as the expected costs of hedging under the real-world rating migration probability measure is given by*

$$\frac{V(t)}{Y(t)} = E^{G_H}\left[E^{Q}\left[\sum_{t<T_i} \frac{c\,(T_i|g(T_i|t))}{Y(T_i)}\right]\right], \tag{19}$$

where $g(T_i|t)$ is the path of rating grades of a debtor conditional that he is alive in t and $c\,(T_i|g(T_i|t))$ are the loan's cash flows conditional on the rating path, Y is a suitable numéraire, Q is the risk-neutral measure of the interest rates dynamics corresponding to the numéraire Y, and G_H is the real-world distribution of future rating paths driven by a Markov chain with generator matrix H.[5]

The proof of this proposition is simple. Conditional on a rating path, the loan is a pure interest rate derivate and the hedging costs are known from interest rate derivatives pricing theory. The

[5] Note that the cash flow $c\,(T_i|g(T_i|t))$ at time T_i could be either a regular interest or amortization payment in case of survival or a suitably discounted recovery cash flow in case of default.

expected value of hedging costs is computed by taking expectations over rating paths. In the case of the pricing algorithm in Section 1.4 the numéraire Y was the money market account, the risk-neutral measure Q is obtained from the suitably calibrated short rate model, and the distribution of rating paths can be inferred from the Markov chain generated by the one-year transition matrix.

The expected value in Proposition 1 is not the price of a trading strategy in tradable securities. It only reflects the expected cost of all possible replication strategies conditional on rating paths given real-world migration probabilities. The choice of real-world migration probabilities has been done for two reasons. First, the expected losses implied by this model are forecasts of the losses that are expected in reality which is an important quantity in credit risk management. Second, in the (unrealistic) limiting case of a large portfolio with independent defaults, default risk can be diversified away and the expected losses computed from real-world probabilities will be observed with certainty. In this limiting case real-world migration probabilities are the theoretically uniquely correct probabilities for pricing.

If the independence assumption is not fulfilled, it is not possible to derive a unique pricing measure from derivatives pricing theory. Since by assumption the only tradable instruments in the market are interest rate instruments any generator matrix H leads to a valid pricing measure which prices these instruments correctly. There is no tradable security containing the default risk of an obligor available in the market which allows to derive a limitation of the admissible range of generator matrices H by trading strategies involving the no-arbitrage criterion. For this reason, the choice of real-world migration probabilities is neither justified nor enforced by derivatives pricing theory but a judgmental decision motivated by credit risk modeling.

In credit risk modeling, risk is typically defined as the uncertainty about the number of defaults within one year which leads to a loss distribution from which a risk measure can be computed. The measurement of risk by a loss distribution at a future point in time is not suitable in this framework. Instead, we use a different approach.

Definition 1 *Risk is defined as uncertainty in today's present value of future cash flows.*

The reason is that losses can occur in two ways in our setup. They can stem from defaults of debtor's and from losses in embedded options due to inappropriate hedge ratios. For this reason, the sole consideration of credit losses in insufficient.

The distribution of present values of future cash flows, that is driven by the credit risk of the underlying loan portfolio, plays the role of the loss distribution in credit portfolio modeling. For this distribution, a risk measure like value-at-risk or expected shortfall can be computed and also the well-known techniques of capital allocation can be applied. The distribution of loan prices conditional on rating paths $V(t|g) := Y(t)E^{Q}\left[\sum_{t<T_i}\left(c\left(T_i|g(T_i|t)\right)/Y(T_i)\right)\right]$ is denoted with V^{G_H}. Each of these loan prices can be decomposed into the price of a loan without embedded options $V_{NO}(t|g)$ and the price of the embedded option $V_O(t|g) = V_{NO}(t|g) - V(t|g)$ with corresponding distributions $V_{NO}^{G_H}$ and $V_O^{G_H}$. For each of these distributions over rating

paths the risk measures value-at-risk (*VaR*) and expected shortfall (*ES*) corresponding to a confidence level α can be computed.

Definition 2 *The risk measures value-at-risk and expected shortfall for the distributions of loan prices V^{G_H}, loan prices without embedded options $V_{NO}^{G_H}$, and the option values $V_O^{G_H}$ are calculated for the confidence level α as*

$$
\begin{aligned}
VaR\left(V^{G_H}\right) &= \inf\left\{x \in \mathbb{R} \middle| P\left(E^{G_H}\left[V^{G_H}\right] - V^{G_H} < x\right) \geq \alpha\right\}, \\
ES\left(V^{G_H}\right) &= E^{G_H}\left[E^{G_H}\left[V^{G_H}\right] - V^{G_H} \middle| E^{G_H}\left[V^{G_H}\right] - V^{G_H} > VaR\left(V^{G_H}\right)\right], \\
VaR\left(V_{NO}^{G_H}\right) &= \inf\left\{x \in \mathbb{R} \middle| P\left(E^{G_H}\left[V_{NO}^{G_H}\right] - V_{NO}^{G_H} < x\right) \geq \alpha\right\}, \\
ES\left(V_{NO}^{G_H}\right) &= E^{G_H}\left[E^{G_H}\left[V_{NO}^{G_H}\right] - V_{NO}^{G_H} \middle| E^{G_H}\left[V_{NO}^{G_H}\right] - V_{NO}^{G_H} > VaR\left(V_{NO}^{G_H}\right)\right], \\
VaR\left(V_O^{G_H}\right) &= \inf\left\{x \in \mathbb{R} \middle| P\left(V_O^{G_H} - E^{G}\left[V_O^{G_H}\right] < x\right) \geq \alpha\right\}, \\
ES\left(V_O^{G_H}\right) &= E^{G_H}\left[V_O^{G_H} - E^{G_H}\left[V_O^{G_H}\right] \middle| V_O^{G_H} - E^{G_H}\left[V_O^{G_H}\right] > VaR\left(V_O^{G_H}\right)\right].
\end{aligned}
$$

When *VaR* and *ES* are computed for the value of the embedded option the corresponding formulas are slightly different. In this case the expected value is subtracted from the distribution of prices to measure risk because it hurts a trader if the realized hedging costs turn out to be higher than expectations. This is in contrast to the loan's value where the negative event is a realization of the present value of future cash flows that turns out to be lower than expected.

Similar to the risk management of loans without embedded options where a margin for expected losses and a margin for unexpected losses is charged to the debtor, the risk manager of embedded options in a loan portfolio should charge an expected option premium plus a fee for the capital that is needed as a buffer for unexpected hedging losses. The unexpected hedging loss can be computed by a risk measure as in Definition 2. We will see in Section 3, however, that in the case of prepayment rights risk management for a bank is not that complicated. Since prepayment rights are very difficult to trade in isolation, the prepayment options are kept together with the loan in the same bank. From the perspective of this bank, the existence of a prepayment right does not lead in general to an increase of the total amount of economic capital as will be explained in Section 3. For this reason, it is sufficient for a bank to charge an option premium for the prepayment right and to use its economic capital model for loans without embedded options.

2.2. Calculating Risk Measures in the Model of Section 1.4

In this part we show how to compute the distribution of present values of future cash flows for the loan pricing model explained in Section 1.4. We model rating states of a debtor using a simple one-factor asset value model in the spirit of Gupton, Finger, and Bhatia (1997). The model is calibrated to the transition probabilities of the given one-year transition matrix. In this model the asset value's logarithmic return A_i of debtor i at the end of one period is given by

$$A_i = \sqrt{1-\rho}\varepsilon_i + \sqrt{\rho}X, \tag{20}$$

where ε_i is the idiosyncratic risk factor and X is the systematic risk factor. Both risk factors are independent and normally distributed. The correlation of the asset values of two debtors is ρ and it is assumed that this correlation is identical for all pairs of debtors. There are thresholds $\theta_{k,1},\ldots,\theta_{k,n-1}$ which define the rating state of a debtor at the end of the period when his rating state at the beginning of the period was k. The debtors defaults if $A_i < \theta_{k,n-1}$. He migrates to rating grade l if $\theta_{k,l} < A_i < \theta_{k,l-1}$. The thresholds are computed from the transition matrix $\mathbf{P}(\tau)$, where τ is the length of the time period, by

$$\theta_{k,l} = \Phi^{-1}\left(\sum_{j=l+1}^{n} p_{kj}(\tau)\right), \tag{21}$$

and Φ is the distribution function of the standard normal distribution.

Equations (20) and (21) tell us how to simulate a rating transition over one period. To get a path of rating transitions at several time points, we repeat the simulation several times by drawing the systematic risk factor and the idiosyncratic risk factors independently in each period and compute the rating state at the end of each period using (20) and (21).

For each rating path, we know exactly if and when the debtor defaults. Therefore, we do not need the rating tree any longer and can carry out the valuation for a given scenario using the interest rate tree only where we include all cash flows that are made by the debtor until the loan's expiry or the debtor's default including the recovery payment in this case. This leads to a Monte-Carlo based pricing algorithm that is equivalent to the full three-dimensional tree approach we have introduced in Section 1.4 but has the additional advantage of calculating the distribution of loan prices depending on rating paths.

In the next section we will illustrate the model for loans with prepayment rights. To include prepayment rights correctly in the calculation we have to carry out the valuation of the loan on the three-dimensional tree and store the indicator values $I_{ex}(k, r_{i,j}, T_{ex})$ on the nodes of a prepayment time T_{ex}, where the indicator is one if a prepayment is optimal and zero if not. The prepayment behavior of a debtor is not affected by a subsequent default which is unknown at this point in time but only by the level of the short rate and his rating grade k in T_{ex}. When we reach a node on a prepayment time of the interest rate tree during a simulation we have to replace the value $V\left(r_{i,j}, T_{ex} \middle| g\left(T_{ex}|t\right)\right)$ in the trinomial interest rate tree by

$$V\left(r_{i,j}, T_{ex} \middle| g\left(T_{ex}|t\right)\right) = \begin{cases} N, & \text{if } I_{ex}(g(T_{ex}|t), r_{i,j}, T_{ex}) = 1, \\ V\left(r_{i,j}, T_{ex} \middle| g\left(T_{ex}|t\right)\right), & \text{if } I_{ex}(g(T_{ex}|t), r_{i,j}, T_{ex}) = 0, \end{cases} \tag{22}$$

where $g(T_{ex}|t)$ is the simulated rating state in T_{ex}. If the exercise probability p_{ex} is less than one, we have to extend the simulation and draw a uniform random number x for each prepayment time and each simulated rating path. If $x \leq p_{ex}$ then the notional is paid back if prepayment is optimal. The prepayment right is not exercised in the case $x > p_{ex}$.

The calculation of the distribution of present values of future cash flows and option premia in this way considers the impact of both the granularity and the dependence structure of a loan portfolio. The model we have presented has the advantage that it is numerically very tractable. However, it uses neither an advanced interest rate model nor a state-of-the-art credit risk model. The interest rate model is a simple one-factor short rate model which is only able to calibrate to a discount curve but neither is able to explain more complex interest rate movements than parallel shifts nor can it account for market features like volatility smiles. The credit model that was used here is the simplest possible model to capture rating migrations and merely serves to illustrate the basic concepts. In a real-world application one would at least use a multi-factor model and possibly extend the model in other directions like modeling dependence between defaults and recoveries or thinking about more general dependence structures than the normal copula.

It should be clear how to extend this approach. In (19) the price of a loan is represented in a very general form. In principle the interest rate model to value the derivative conditional on a rating path can be replaced by any other interest rate derivatives pricing model while the credit risk model can be replaced by more advanced modeling approaches like replacing the one-factor model by its multi-factor extension. Apart from numerical tractability of the resulting model this is straightforward as long as interest rate and credit dynamics are modeled as independent processes. In this case already existing modeling approaches can be combined. This breaks down when dependencies between interest rates and defaults are introduced resulting in a more complex modeling framework.

3. Numerical Examples

We illustrate the concepts presented so far for a 15 year fixed-rate loan with notional $N = 1$ where the debtor has the right to fully amortize the loan after 10 years. We assume that the bank's rating system has eight grades and the one-year transition matrix is given by

$$\mathbf{P}(1) = \begin{pmatrix} 0.9700 & 0.0150 & 0.0070 & 0.0040 & 0.0020 & 0.0010 & 0.0007 & 0.0003 \\ 0.0150 & 0.9598 & 0.0100 & 0.0080 & 0.0040 & 0.0015 & 0.0010 & 0.0007 \\ 0.0100 & 0.0200 & 0.9272 & 0.0200 & 0.0100 & 0.0070 & 0.0040 & 0.0018 \\ 0.0050 & 0.0100 & 0.0200 & 0.8905 & 0.0400 & 0.0200 & 0.0100 & 0.0045 \\ 0.0030 & 0.0070 & 0.0150 & 0.0400 & 0.8700 & 0.0350 & 0.0200 & 0.0100 \\ 0.0025 & 0.0075 & 0.0150 & 0.0250 & 0.0400 & 0.8400 & 0.0500 & 0.0200 \\ 0.0020 & 0.0050 & 0.0100 & 0.0150 & 0.0350 & 0.0700 & 0.8130 & 0.0500 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \tag{23}$$

Assuming that rating transitions are homogeneous and Markovian allows us to compute a term structure of default probabilities for each rating grade. The results are shown in Table 1.

Grade	1Y	2Y	3Y	5Y	7Y	10Y	15Y
1	0.0003	0.0007	0.0012	0.0026	0.0044	0.0079	0.0157
2	0.0007	0.0016	0.0026	0.0050	0.0081	0.0137	0.0256
3	0.0018	0.0040	0.0066	0.0128	0.0202	0.0330	0.0573
4	0.0045	0.0099	0.0159	0.0297	0.0451	0.0698	0.1121
5	0.0100	0.0206	0.0316	0.0544	0.0773	0.1108	0.1623
6	0.0200	0.0398	0.0592	0.0960	0.1295	0.1738	0.2337
7	0.0500	0.0925	0.1289	0.1878	0.2333	0.2851	0.3459
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1. Term structure of default probabilities implied by the transition matrix (23)

We assume a recovery rate of $R = 20\%$ and set economic capital costs s_{UL} and all other internal costs s_C of the bank to zero for simplicity. Further, we assume that the debtor's prepayment option is compensated by a higher interest rate instead of an upfront premium. We compute the risk-adjusted interest rate of the loan for three cases. In the first case, the transition matrix (23) is used in the pricing algorithm. In the second case, we use the term structure of default probabilities of Table 1. Comparing these two cases allows us to measure the effect of rating migration on the option premium. In the final case, we compute the interest rate for the same loan without prepayment option to measure the option value.

The short rate model has not yet been specified explicitly. In the examples, we use a Gaussian one-factor short rate model (Hull and White 1990). Its dynamics is given by

$$\begin{aligned} r(t) &= x(t) + \alpha(t), \\ dx(t) &= -\kappa\, x(t)\, dt + \sigma\, dW(t). \end{aligned}$$

The function $\alpha(t)$ is calibrated to fit the discount curve. We use a simple parameterization to ensure replicability of the results. For discounting a flat zero curve with a continuously compounded zero rate of 5% is used and a funding spread curve $\beta(t) = 0$ is assumed. The parameters of the short rate model are $\kappa = 2.0\%$ and $\sigma = 0.7\%$. In the construction of the trinomial tree for the short rate model, 50 time steps per year are used, i.e. a time step roughly corresponds to one week. Finally, we assume transaction costs of zero and $p_{ex} = 1$, i.e. that debtors act perfectly rational. The results are shown in Table 2.

As expected we find that for the good rating grades there is no difference in the interest rate when it is computed with the transition matrix or the term structure because these debtors cannot improve their rating during the maturity of the loan. They only can be downgraded. Therefore, rating migration does not increase the likelihood of a prepayment and the value of the prepayment option is fully determined by interest rate risk. For the lower rating grades we find a mild decrease in the risk-adjusted interest rate if the calculation is done with the

Grade	1	2	3	4	5	6	7
Case 1	5.34%	5.39%	5.56%	5.88%	6.24%	6.84%	8.12%
Case 2	5.34%	5.39%	5.55%	5.86%	6.22%	6.81%	8.09%
Case 3	5.14%	5.19%	5.37%	5.67%	6.03%	6.58%	7.71%

Table 2. Risk-adjusted interest rates computed for the loan with prepayment rights using the transition matrix (Case 1), the term structure of default probabilities (Case 2), and for the loan without prepayment rights (Case 3)

term structure. In this case, the debtor in Case 1 has a chance to improve his rating until his prepayment option matures. Due to the improvement in his rating he might get better terms in ten years and will prepay the option. The possibility of this event is reflected in the higher interest rate. However, the difference is not huge because of the small transition probabilities in the matrix $\mathbf{P}(1)$ in (23). The component of the option premium that is caused by interest rate risk and the reduction of default probabilities conditional on survival is much more substantial.

In the second example, we compute the distribution of present values of future cash flows for loan portfolios with and without prepayment rights. We measure risk by the expected shortfall corresponding to the 99% quantile and analyze the influence of portfolio size, dependence structure, and the existence of prepayment rights on the risk of a loan portfolio. We use a loan portfolio with identical loans. Each loan has a maturity of 15 years and a prepayment option after 10 years. Each loan has a fixed interest rate that is given in Table 2, Case 1. We use a portfolio of 700 and 7,000 debtors, i.e. 100 debtors and 1,000 debtors per rating grade, with a total portfolio notional normalized to one. We apply asset correlations of 0%, 3%, and 16%. The numbers 3% and 16% are motivated by Basel Committee on Banking Supervision (2006). They are the lowest and the highest asset correlations that are used in the internal ratings based approach to calculate capital requirements for retail loans. We use 100,000 simulations to compute the present values distributions. In this context, the expected shortfall is computed as the average of the worst 1,000 outcomes of each simulation run. Option premia are computed as the difference of the prices of two otherwise identical loans with and without prepayment right in each scenario. The results are presented in Tables 3 and 4.[6]

As expected, we find that mean values are up to small numerical errors independent of the asset correlation. Further, we find that the expected shortfall is increasing with asset correlation. Increasing the number of debtors leads to a sharp decline of the expected shortfall in the case of zero correlation while the decline is rather mild when the correlation is positive. This is also as expected because it is impossible to diversify away systemic risk. From a practical perspective, it is an important result that the expected shortfall is reduced by the introduction of prepayment rights. The reason is that debtors might pay back the loan after ten years and

[6] In both tables the price for the portfolio of loans with prepayment rights is slightly lower than 1.00. The reason is a small bias in the simulation algorithm. The recovery payment in case of a debtor's default is always paid at the end time of an interest rate period because rating grades are simulated in these time points only. This leads to a slightly decreased present value compared to the loan pricing algorithm in Section 1.4.

Asset Correlation	0%	3%	16%
Mean PV with Prepayment Rights (V)	0.9986	0.9986	0.9986
Expected Shortfall $ES(V^{G_H})$	0.0226	0.0401	0.1014
Mean PV without Prepayment Rights (V_{NP})	1.0232	1.0231	1.0232
Expected Shortfall $ES(V_{NP}^{G_H})$	0.0247	0.0440	0.1094
Mean Option Premium (V_O)	0.0246	0.0245	0.0245
Expected Shortfall $ES(V_O^{G_H})$	0.0043	0.0060	0.0100

Table 3. Expected shortfall of the present value distributions for the portfolio with 700 debtors. The mean and expected shortfall of the loan's price with and without prepayment right is computed together with the mean and expected shortfall of the option premium.

Asset Correlation	0%	3%	16%
Mean PV with Prepayment Rights (V)	0.9986	0.9986	0.9985
Expected Shortfall $ES(V^{G_H})$	0.0069	0.0338	0.0990
Mean PV without Prepayment Rights (V_{NP})	1.0231	1.0232	1.0230
Expected Shortfall $ES(V_{NP}^{G_H})$	0.0076	0.0372	0.1068
Mean Option Premium (V_O)	0.0245	0.0246	0.0245
Expected Shortfall $ES(V_O^{G_H})$	0.0014	0.0045	0.0093

Table 4. Expected shortfall of the present value distribution for the portfolio with 7,000 debtors. The mean and expected shortfall of the loan's price with and without prepayment right is computed together with the mean and expected shortfall of the option premium.

default at a later point in time. This possibility leads to a reduction in risk. There is, of course, the risk of unexpected hedging errors when prepayment rights are included in loan contracts. However, they are large when default probabilities are unexpectedly low. This does not lead to an increase in economic capital because the effect of unexpectedly high hedging costs is comparatively mild and opposite to the effect of losses coming from unexpectedly high default rates. Unexpected hedging costs and unexpected losses can therefore never be huge at the same time in this example.

Of course, this is not a formal proof that the expected shortfall is always reduced when prepayment rights are included. In fact, it is impossible to prove this statement because it is not true in general. First, if we assume a recovery rate close to 100% the inclusion of prepayment rights increases the risk of the loan. From a practical perspective, however, this effect is not of relevance. The expected shortfalls are very low in this situation. Hedging a prepayment right is much simpler in this case because there is no risk of a reduction of the notional but only a risk in the payment time of the notional. Moreover, the assumption that no losses besides interest rate losses can occur in a loan market is not adequate and has failed often enough in history. If a very mild loss is possible in the model, e.g. if the recovery rate is chosen at 95%, the result that the expected shortfall is reduced by the inclusion of prepayment rights is already valid in this example. Second, if we extend (20) to $A_i = \sqrt{1-b_i^2}\varepsilon_i + b_i X$ and allow b_i to be positive for one half of the portfolio and negative for the other half, we would find simultaneously lower than expected default rates in one sector and higher than expected default rates in the other sector. In this case a prepayment right will lead to an increase in economic capital. However, given that negative dependence is very rare in real-world credit portfolios, this parameterization should be more of theoretical than of practical relevance.

Finally, we remark that the algorithm to compute economic capital is very simplistic from a numerical point of view and serves for illustration only. In practice one would compute the expected shortfall for a higher quantile which requires more sophisticated simulation techniques along the lines of Glasserman and Li (2005), Kalkbrener, Lotter, and Overbeck (2004), or Kloeppel, Rada, and Schachermayer (2009).

4. Conclusion

We have introduced a pricing framework for retail loans. Typically, for debtors in the retail market, no market information like bond spreads or CDS spreads can be observed. Only statistical information like transition matrices and estimations of recovery rates are available for retail debtors. Retail loans often contain embedded options like prepayment rights or caps and floor for loans with floating interest rates. In the case of prepayment options, we have seen that the probability of prepayment depends both on the future level of interest rates and the future rating of a debtor. We have considered both factors by using a model which combines a term structure model with rating transitions and have shown in detail how to value loans with prepayment rights in this context.

By defining risk in this modeling approach as the uncertainty in the present value of future cash flows, we have seen that it is possible to define credit risk measures for loan portfolios including embedded options using standard techniques of credit risk modeling. In this context, the interest rate derivatives pricing model measures the hedging costs conditional on the future rating path of a debtor while the credit portfolio model measures the uncertainty in hedging costs. In this sense, the pricing framework combines credit risk modeling with the theory of derivatives pricing. We have seen that the pricing approach in this article, a combination of a one-factor short rate model with a one-factor credit risk model, can be easily extended to a more advanced modeling framework. From a numerical perspective the resulting model will, of course, become more challenging.

In the case of prepayment rights, the most common embedded options in retail loans, we have seen in examples that, except for some extreme cases that should have no practical relevance, the overall level of economic capital is decreased compared to a portfolio without embedded options. This has important consequences for a bank selling loans with embedded prepayment rights. From a risk management perspective it is sufficient for these banks to charge a premium for the prepayment options. These options must not be ignored because we have also seen in our numerical examples that they can be of considerable value. Concerning economic capital, however, banks can stick to their framework of computing it for the case of loans without prepayment options which is in general simpler and results in more conservative values.

A. Derivation of Formula (18)

In the appendix it is shown that (18) represents indeed the present value of a floating rate that is fixed in T_{m-1} and paid in T_m conditional on the short rate $r_{o,j}$ and rating grade k. Since by assumption the short rate process and the rating process are independent and, furthermore, the floating rate payment is zero in the case of default, we can decompose V_f

$$V_f(k, r_{o,j}, T_{m-1}) = (1 - p_{kn}(T_m|T_{m-1}))\ \delta\left(T_{m-1}, T_m|r_{o,j}\right), \tag{24}$$

where the first expression is the debtor's survival probability conditional on being in rating grade k at time T_{m-1} and the second expression is the forward discount factor from time T_m to time T_{m-1} conditional on the short rate realization $r_{o,j}$. The ratio of discount factors can be simplified to

$$\begin{aligned}\frac{\delta(T_m)}{\delta_M(T_m)} &= \frac{E^Q\left[\exp\left(-\int_0^{T_m} r(u)du\right)\right]}{E^Q\left[\exp\left(-\int_0^{T_m} r_M(u)du\right)\right]} \\ &= \frac{E^Q\left[\exp\left(-\int_0^{T_m} (r_M(u)+\beta(u))\,du\right)\right]}{E^Q\left[\exp\left(-\int_0^{T_m} r_M(u)du\right)\right]} = \exp\left(-\int_0^{T_m} \beta(u)du\right)\end{aligned}$$

since $\beta(u)$ is the deterministic funding spread. Therefore, we get

$$\frac{\delta(T_m)\ \delta_M(T_{m-1})}{\delta(T_{m-1})\ \delta_M(T_m)} = \exp\left(-\int_{T_{m-1}}^{T_m} \beta(u)du\right). \tag{25}$$

Putting (18), (24), and (25) together yields

$$\begin{aligned}(18) &= (1 - p_{kn}(T_m|T_{m-1}))\left(\exp\left(-\int_{T_{m-1}}^{T_m} \beta(u)du\right) - \delta\left(T_{m-1}, T_m|r_{o,j}\right)\right) \\ &= (1 - p_{kn}(T_m|T_{m-1}))\left(\frac{1}{\delta_M\left(T_{m-1}, T_m|r_{o,j}\right)} - 1\right)\ \delta\left(T_{m-1}, T_m|r_{o,j}\right) \\ &= (1 - p_{kn}(T_m|T_{m-1}))\, f\left(T_{m-1}, T_m|r_{o,j}\right)\ (T_m - T_{m-1})\ \delta\left(T_{m-1}, T_m|r_{o,j}\right).\end{aligned}$$

The final line represents the expected present value of the forward rate that is fixed in T_{m-1} and paid in T_m at time T_{m-1} conditional on $r_{o,j}$ which is the desired quantity.

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