

# Variation of Parameters

4.6

$$a_2(x)y'' + a_1(x)y' + a_b(x)y = g(x)$$

↳ solution  $y = y_c + y_p$

$$\Rightarrow y'' + P(x)y' + Q(x)y = f(x)$$

Step 1:  $y'' + Py' + Qy = 0$

$$y_c = C_1 y_1 + C_2 y_2$$

Step 2:  $y_p = u_1 y_1 + u_2 y_2$

$$u_1' = \frac{\omega_1}{\omega}$$

$$u_2' = \frac{\omega_2}{\omega}$$

$$\therefore \omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$



$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}$$

comes from the

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$$

standard form of  
equation

Example 01: Solve  $y'' - 4y' + 4y = (x+1)e^{2x}$  ————— (1)  
by the method of

a) Variation of parameters

b) Undetermined Co-efficients (Annihilation)

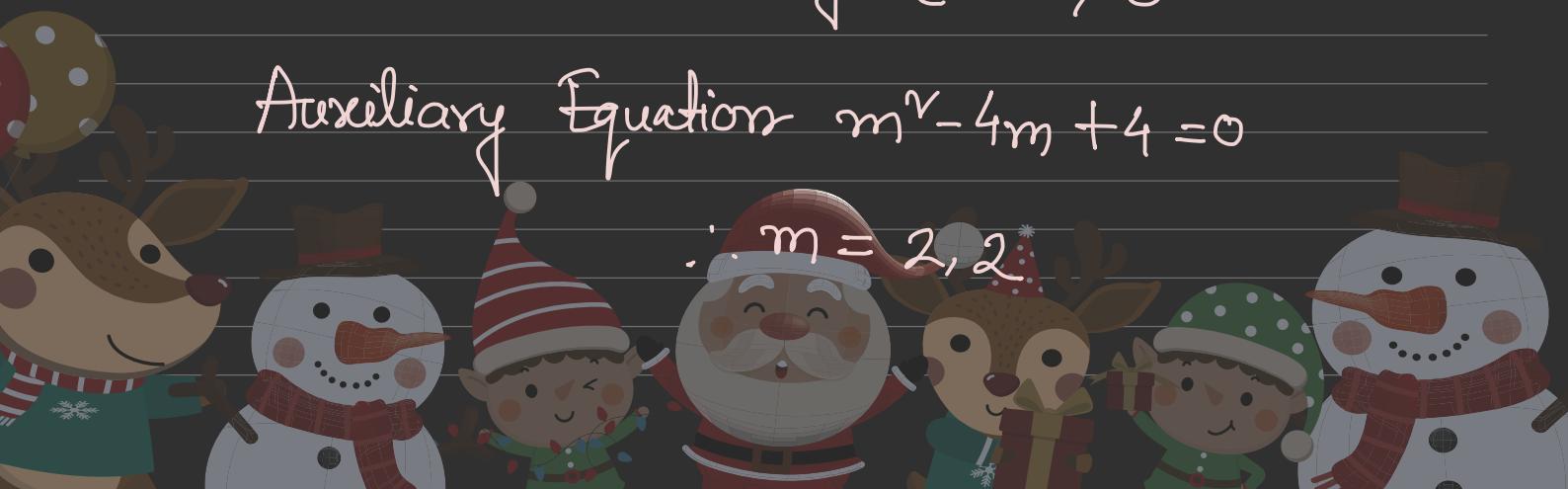
(a)

Solution: Let,  $y'' - 4y' + 4y = 0$  ————— (II)

Let,  $y = e^{mx} \neq 0$

Auxiliary Equation  $m^2 - 4m + 4 = 0$

$$\therefore m = 2, 2$$



$$y_1 = e^{2x}$$

$$y_2 = x e^{2x}$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{4x} \neq 0$$

$\Rightarrow y_1 + y_2$  are linearly independent

Thus 
$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

↪ This solution is complementary  
solution for eq<sup>n</sup> ①.

↪ This solution is general solution  
for eq<sup>n</sup> ②.

Let,  $y_p = u_1(x) e^{2x} + u_2(x) x e^{2x}$

$$\therefore \omega_1 = \begin{vmatrix} 0 & x e^{2x} \\ (x+1) e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} = - (x^2 + x) e^{4x}$$

$$w_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

$$u_1' = \frac{w_1}{\omega} = \frac{-(x^2+x)e^{4x}}{e^{4x}} = -(x^2+x)$$

$$u_1 = \int -(x^2+x)dx = -\int x^2+x dx$$

$$\therefore u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2' = \frac{w_2}{\omega} = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

$$\therefore u_2 = \int (x+1)dx = \frac{x^2}{2} + x$$

$$\therefore y_p = \left( -\frac{x^3}{3} - \frac{x^2}{2} \right) e^{2x} + \left( \frac{x^2}{2} + x \right) xe^{2x}$$

$$= e^{2x} \left( -\frac{x^3}{3} - \frac{x^2}{2} + x \left( \frac{x^2}{2} + x \right) \right)$$

$$= e^{2x} \left( -\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^3}{2} + x^2 \right)$$



$$= e^{2x} \left( \frac{-2x^3 - 3x^2 + 3x^3 + 6x^2}{6} \right)$$

$$= e^{2x} \left( \frac{x^3 + 3x^2}{6} \right)$$

$\therefore$  General Solution

$$y = y_c + y_p = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{6} x^3 e^{2x} + \frac{1}{2} x^2 e^{2x}$$

(Ans.)

↳ Variation of parameter method

(b)

$$\text{Let, } y'' - 4y' + 4y = 0 \quad (2)$$

$$\text{Let, } y = e^{mx} \neq 0$$

Auxiliary equation:  $m^2 - 4m + 4 = 0$

$$\therefore m = 2, 2$$

$$y_1 = e^{2x}; \quad y_2 = xe^{2x}$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{4x} \neq 0$$

$\Rightarrow y_1 + y_2$  are linearly independent

Thus,  $y_c = C_1 e^{2x} + C_2 xe^{2x}$

now,  $(x+1)e^{2x}$  annihilating operator.

$$(Ax^3 + Bx^2 + Cx + D) e^{2x}$$

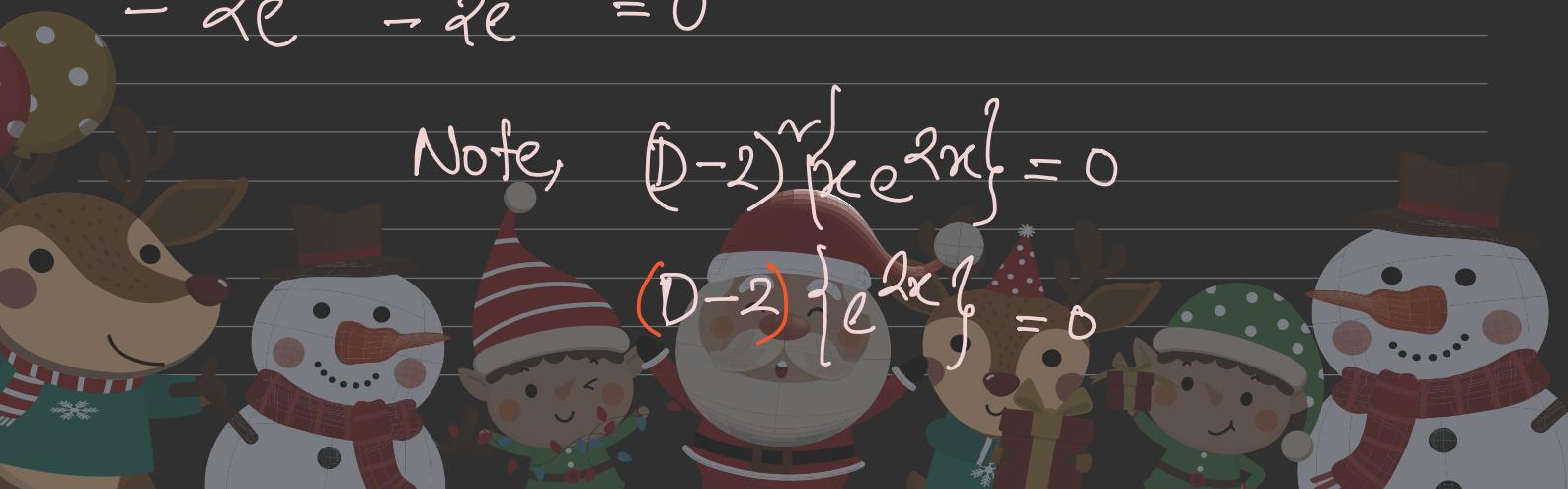
$$(D-2)(D-2) \left\{ (x+1)e^{2x} \right\}$$

$$= (D-2) \left[ (1+0)e^{2x} + 2(x+1)e^{2x} - 2(x+1)e^{2x} \right]$$

$$= 2e^{2x} - 2e^{2x} = 0$$

Note,  $(D-2) \left\{ xe^{2x} \right\} = 0$

$$(D-2) \left\{ e^{2x} \right\} = 0$$



$$\text{Now, } (D^2 - 4D + 4)y = (x+1)e^{2x}$$

$$\Rightarrow (D-2)^2(D^2 - 4D + 4)y = (D-2)^2 \{(x+1)e^{2x}\}$$

$$\therefore (D-2)^2(D^2 - 4D + 4)y = 0$$

$$\text{Let } y = e^{mx} \neq 0$$

$$\therefore (m-2)^2(m^2 - 4m + 4) = 0$$

$$\therefore m = 2, 2, 2, 2$$

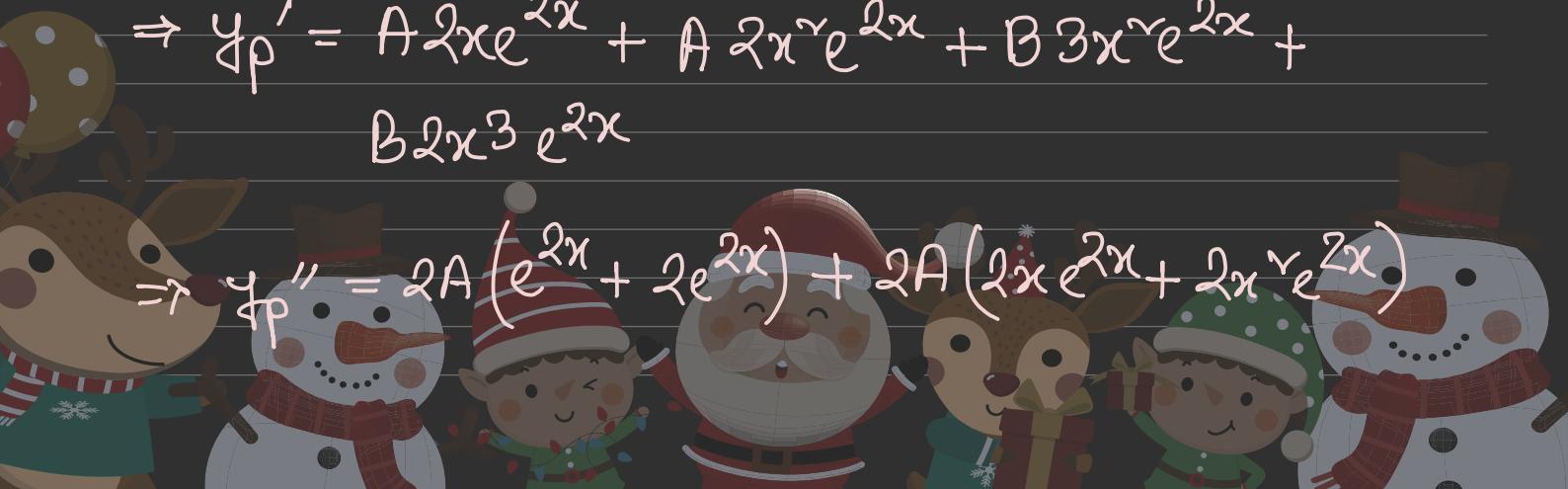
$$y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4 x^3 e^{2x}$$

$$\text{Let } y_p = Ax^2 e^{2x} + Bx^3 e^{2x}$$

$$\Rightarrow y_p' = A(2x e^{2x} + 2x^2 e^{2x}) + B(3x^2 e^{2x} + 2x^3 e^{2x})$$

$$\Rightarrow y_p' = A2x e^{2x} + A2x^2 e^{2x} + B3x^2 e^{2x} + B2x^3 e^{2x}$$

$$\Rightarrow y_p'' = 2A(e^{2x} + 2x e^{2x}) + 2A(2x e^{2x} + 2x^2 e^{2x})$$



$$+ 3B(2xe^{2x} + 2x^2e^{2x}) + 2B(3x^2e^{2x} + 2x^3e^{2x})$$

$$= 6Ae^{2x} + 4Ax^2e^{2x} + 4Ax^2e^{2x} + 6Bx^2e^{2x}$$

$$+ 6Bx^2e^{2x} + 6Bx^2e^{2x} + 4Bx^3e^{2x}$$

$$= e^{2x} (6A + 4Ax + 4Ax^2 + 6Bx + 6Bx^2 + 6Bx^2 + 4Bx^3)$$

Now,  $y_p'' - 4y_p' + 4y_p = (x+1)e^{2x}$

$$\Rightarrow 6Ae^{2x} + 4Ax^2e^{2x} + 4Ax^2e^{2x} - 4(2Ax^2e^{2x} + 2Ax^2e^{2x}) - 4(3Bx^2e^{2x} + 2Bx^3e^{2x}) + 4Ax^2e^{2x} + 4Bx^3e^{2x} = (x+1)e^{2x}$$

$$\Rightarrow 6Ae^{2x} + 4Ax^2e^{2x} + 4Ax^2e^{2x} - 8Ax^2e^{2x} - 8Ax^2e^{2x} - 12Bx^2e^{2x} - 8Bx^3e^{2x} + 4Ax^2e^{2x} + 4Bx^3e^{2x}$$

$$= (x+1)e^{2x} = xe^{2x} + e^{2x}$$

$$\Rightarrow e^{2x} (6A$$

$$A = \frac{1}{2}, B = \frac{1}{6}$$

$$xe^{2x} (4A + 4Ax - 8A - 8Ax - 12Bx - 8Bx^2 + 4Ax + 4Bx^2)$$

Quiz - 02 : Exactly Same question  
but different equations,

Chapter : 4.6

