18.03.24

# Variation of Parameters

4.6

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

$\hookrightarrow$ solution $y = y_c + y_p$

$$\Rightarrow \quad y'' + P(x)y' + Q(x)y = f(x)$$

**Step 1:** $y'' + Py' + Qy = 0$

$$y_c = c_1 y_1 + c_2 y_2$$

**Step 2:** $y_p = u_1 y_1 + u_2 y_2$

$$u_1' = \frac{W_1}{W}$$

$$u_2' = \frac{W_2}{W}$$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}$$

$f$ → comes from the standard form of equation

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$$

**Example 01:** Solve $y'' - 4y' + 4y = (x+1)e^{2x}$ —— (I)
by the method of

a) Variation of parameters

b) Undetermined Co-efficients (Annhiliation)

(a)

**Solution:** Let, $y'' - 4y' + 4y = 0$ —— (II)

Let, $y = e^{mx} \neq 0$

Auxiliary Equation $m^2 - 4m + 4 = 0$

$\therefore m = 2, 2$

$$y_1 = e^{2x}$$

$$y_2 = x e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{4x} \neq 0$$

$\Rightarrow$ $y_1$ + $y_2$ are linearly independent

Thus $\boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$

↳ This solution is complementary solution for eqn ①.

↳ This solution is general solution for eqn ②.

Let, $y_p = u_1(x) e^{2x} + u_2(x) x e^{2x}$

$$\therefore W_1 = \begin{vmatrix} 0 & x e^{2x} \\ (x+1) e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix}$$

$$= -(x^2 + x) e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

$$u_1' = \frac{W_1}{W} = \frac{-(x^2+x)e^{4x}}{e^{4x}} = -(x^2+x)$$

$$u_1 = \int -(x^2+x)\,dx = -\int x^2+x\,dx$$

$$\therefore u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

$$\therefore u_2 = \int (x+1)\,dx = \frac{x^2}{2} + x$$

$$\therefore y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

$$= e^{2x}\left(-\frac{x^3}{3} - \frac{x^2}{2} + x\left(\frac{x^2}{2} + x\right)\right)$$

$$= e^{2x}\left(\frac{-x^3}{3} - \frac{x^2}{2} + \frac{x^3}{2} + x^2\right)$$

$$= e^{2x}\left(\frac{-2x^3-3x^2+3x^3+6x^2}{6}\right)$$

$$= e^{2x}\left(\frac{x^3+3x^2}{6}\right)$$

$\therefore$ General Solution

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}$$

(Ans)

→ Variation of parameter method

(b) Let, $y'' - 4y' + 4y = 0$ —— (2)

Let, $y = e^{mx} \neq 0$

Auxiliary equation: $m^2 - 4m + 4 = 0$

$\therefore m = 2, 2$

$$y_1 = e^{2x} \; ; \; y_2 = xe^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{4x} \neq 0$$

$\Rightarrow$ $y_1$ + $y_2$ are linearly independent

Thus, $y_c = c_1 e^{2x} + c_2 x e^{2x}$

now, $(x+1)e^{2x}$ annhiliating operator,

$$(Ax^3 + Bx^2 + Cx + D)e^{2x}$$

$$(D-2)(D-2)\{(x+1)e^{2x}\}$$

$$= (D-2)\left[(1+0)e^{2x} + 2(x+1)e^{2x} - 2(x+1)e^{2x}\right]$$

$$= 2e^{2x} - 2e^{2x} = 0$$

Note, $(D-2)^2\{xe^{2x}\} = 0$

$$(D-2)\{e^{2x}\} = 0$$

now, $(D^2 - 4D + 4)y = (x+1)e^{2x}$

$\Rightarrow (D-2)^2 (D^2 - 4D + 4)y = (D-2)^2 \{(x+1)e^{2x}\}$

$\therefore (D-2)^2 (D^2 - 4D + 4)y = 0$

Let, $y = e^{mx} \neq 0$

$\therefore (m-2)^2 (m^2 - 4m + 4) = 0$

$\therefore m = 2, 2, 2, 2$

$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 x^3 e^{2x}$

Let, $y_p = Ax^2 e^{2x} + Bx^3 e^{2x}$

$\Rightarrow y_p' = A(2xe^{2x} + 2x^2 e^{2x}) + B(3x^2 e^{2x} +$
$2x^3 e^{2x})$

$\Rightarrow y_p' = A2xe^{2x} + A2x^2 e^{2x} + B3x^2 e^{2x} +$
$B2x^3 e^{2x}$

$\Rightarrow y_p'' = 2A(e^{2x} + 2e^{2x}) + 2A(2xe^{2x} + 2x^2 e^{2x})$

$$+3B\left(2xe^{2x}+2x^2e^{2x}\right)+2B\left(3x^2e^{2x}+2x^3e^{2x}\right)$$

$$=6Ae^{2x}+4Axe^{2x}+4Ax^2e^{2x}+6Bxe^{2x}$$

$$+6Bx^2e^{2x}+6Bx^2e^{2x}+4Bx^3e^{2x}$$

$$=e^{2x}\left(6A+4Ax+4Ax^2+6Bx+6Bx^2+\right.$$

$$\left.6Bx^2+4Bx^3\right)$$

now, $\quad y_p''-4y_p'+4y_p=(x+1)e^{2x}$

$$\Rightarrow 6Ae^{2x}+4Axe^{2x}+4Ax^2e^{2x}-4\left(2Axe^{2x}\right.$$

$$\left.+2Ax^2e^{2x}\right)-4\left(3Bx^2e^{2x}+2Bx^3e^{2x}\right)$$

$$+4Ax^2e^{2x}+4Bx^3e^{2x}=(x+1)e^{2x}$$

$$\Rightarrow 6Ae^{2x}+4Axe^{2x}+4Ax^2e^{2x}-8Axe^{2x}-8Ax^2e^{2x}$$

$$-12Bx^2e^{2x}-8Bx^3e^{2x}+4Ax^2e^{2x}+4Bx^3e^{2x}$$

$$=(x+1)e^{2x}=xe^{2x}+e^{2x}$$

$\Rightarrow e^{2x}\,(6A$ $\qquad A=\frac{1}{2}\,,\ B=\frac{1}{6}$

$xe^{2x}\,(4A+4Ax-8A-8Ax-12Bx-8Bx^2+$

$4Ax+4Bx^2)$

Quiz -02 : Exactly Same question but diffrent equation,

Chapter : 4.6