Practical 01

Topic: Limits & Continuity.

1] $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$= \lim_{x \to a} \left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{3a+x} + 3\sqrt{x})} \right]$

$= \lim_{x \to a} \left[\frac{((a+2x) - (3x))(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})((3a+x) - 4x)} \right]$

$= \lim_{x \to a} \left[\frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})(3a - 3x)} \right]$

$= \lim_{x \to a} \left[\frac{\sqrt{3a+x} + 2\sqrt{x}}{3(\sqrt{a+2x} + \sqrt{3x})} \right]$

$= \lim_{x \to a} \left[\frac{\sqrt{3a+5x}}{(\sqrt{a+5x})3} \right]$

$\Rightarrow \frac{\sqrt{3a+5a}}{3(\sqrt{a+5a})}$

$\Rightarrow \frac{\sqrt{8a}}{3(\sqrt{6a})}$

$\Rightarrow \frac{2}{3\sqrt{3}}$

2] $\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$\Rightarrow \lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$\Rightarrow \lim_{y \to 0} \left[\frac{a + y - a}{y\sqrt{a+y}(\sqrt{a+y} + \sqrt{a})} \right]$

$$\Rightarrow \frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$$

$$\Rightarrow \frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a}}$$

$$\Rightarrow \frac{1}{\sqrt{a}\,(\sqrt{a}+\sqrt{a})}$$

$$\Rightarrow \frac{\cancel{1}}{\cancel{\sqrt{a}(\sqrt{a}+\sqrt{a})}}\ \frac{1}{\sqrt{a}\,(2\sqrt{a})}$$

$$\Rightarrow \frac{1}{2a}$$

③ $$\lim_{x\to\pi/6}\left[\frac{\cos x-\sqrt{3}\sin x}{\pi-6x}\right]$$

→ Put $x-\pi/6=h$, $x\to h+\pi/6$ $h\to 0$

$$\Rightarrow \lim_{h\to 0}\frac{\cos(h+\pi/6)-\sqrt{3}\sin(h+\pi/6)}{\pi-6(h+\pi/6)}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos h.\cos\pi/6-\sin h.\sin\pi/6-\sqrt{3}\sin h.\cos\pi/6+\cos h.\sin\pi/6}{\pi-6(\frac{6h}{\pi}\cancel{+\pi})}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos h.\frac{\sqrt{3}}{2}.\sin 1/\sqrt{2}-\sqrt{3}(\sin h\,\sqrt{3}/2+\cos h.1/\sqrt{2}.}{-6h.}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos h.\sqrt{3}/2\,h.\sin h/2-\sin 3h/2-\cos\sqrt{3}/2\,h.}{-6h.}$$

$$\Rightarrow \lim_{h\to 0}\frac{-\sin 4h/2}{-6h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin 4h}{12h}$$

$$\Rightarrow 1/3\lim_{h\to 0}\frac{\sin h}{h}\quad\Rightarrow 1/3.$$

(4) $\lim\limits_{x\to\infty}\left[\dfrac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

$= \lim\limits_{x\to\infty}\left[\dfrac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \dfrac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}} \times \dfrac{\sqrt{x^2+5}+\sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2+1}}\right]$

$= \lim\limits_{x\to\infty} \dfrac{(x^2+5-x^2+3)(\sqrt{x^2+3}+\sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5}+\sqrt{x^2+3})}$

$= \lim\limits_{x\to\infty} \dfrac{8(\sqrt{x^2+3}+\sqrt{x^2+1})}{2(\sqrt{x^2+5}+\sqrt{x^2+3})}$

$= 4\lim\limits_{x\to\infty} \dfrac{(\sqrt{x^2(1+3/x^2)}+\sqrt{x^2(1+1/x^2)})}{\sqrt{x^2(1+5/x^2)}+\sqrt{x^2(1-3/x^2)})}$

$= 4$

5]

(i) $f(x) = \begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}} & 0 < x > \pi/2 \\[2ex] = \dfrac{\cos x}{\pi - 2x} & \pi/2 < x < \pi \end{cases}$ $x = \pi/2$.

Sol. $f(\pi/2) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$

$f(\pi/2) = 0$

L.H.S :-

$\lim\limits_{x\to\pi/2^+} f(x) = \lim\limits_{x\to\pi/2^-} \dfrac{\cos x}{\pi - 2x}$

put $x - \pi/2 = h$, $x = h + \pi/2$ $h \to 0$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{\cos(h+\pi/2)}{\pi - 2(h+\pi/2)}$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{\cos(h+\pi/2)}{-2h}$

$$\Rightarrow \lim_{h\to 0} \frac{\cosh \cos \pi/2 - \sinh . \sin \pi/2.}{-2h}$$

$$\Rightarrow \lim_{h\to 0} \cosh . 0 - \frac{\sin h}{-2h} = 1/2 .$$

R.H.S

$$\Rightarrow \lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$$

$$\Rightarrow \lim_{x\to \pi/2^-} \frac{2 \sin x \cos x}{\sqrt{2 \sin^2 x}}$$

$$\Rightarrow \lim_{x\to \pi/2^-} \frac{2 \cancel{\sin x} \cos x}{\sqrt{2}\ \cancel{\sin x}}$$

$$\Rightarrow \lim_{x\to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$$

$$\Rightarrow \frac{2}{\sqrt{2}} \lim_{x\to \pi/2^-} \cos x$$

$$= 0$$

L.H.S $\neq$ R.H.S.

$\therefore$ $f(x)$ is not continuous at $x = \pi/2$.

ii]
$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & 0 < x < 3 \\ = x+3 & 3 < x < 6 \\ = \dfrac{x^2-9}{x+3} & 6 < x < 9 \end{cases} \quad \begin{matrix} x = 3 \\ x = 6 \end{matrix}$$

Sol. at $f(x) = 3$

L.H.S. = R.H.S

$\Rightarrow \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

$\Rightarrow \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^+} x + 3$

$\Rightarrow \lim_{x \to 3^-} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \to 3^+} x + 3$

$\Rightarrow (3+3) = (3+3)$

$6 = 6$

$\therefore$ $f(x)$ is continuous at $x = 3$.

at $x = 6$

L.H.S = R.H.S

$\Rightarrow \lim_{x \to 6^-} f(x) = \lim_{x \to 6^+} f(x)$

$\Rightarrow \lim_{x \to 6^-} x + 3 = \lim_{x \to 6^+} \frac{x^2 - 9}{x + 3}$

$\Rightarrow \lim_{x \to 6^-} x + 3 = \lim_{x \to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\Rightarrow 6 + 3 = 6 - 3$

$\Rightarrow 9 \neq 3$

$\therefore$ $f(x)$ is not continuous at $x = 6$.

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i] $f(x) = \frac{1 - \cos 4x}{x^2} \quad x < 0$

$= k \quad x = 0$

$\Bigg\} \; x = 0$

S.E.Q.

Sol. $\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow \lim_{x \to 0} \dfrac{1 - \cos 4x}{x^2} = k$

$\Rightarrow \dfrac{2\sin^2 2x}{x^2} = k$

$\Rightarrow \boxed{k = 8}$

ii) $f(x) = \begin{cases} (\sec^2 x)^{\cot^2 x} & x \neq 0 \\ 2k & x = 0 \end{cases} \quad x = 0$

Sol. $f(x) = (\sec^2 x)^{\cot^2 x}$

$= \lim_{x \to 0} (1 + \tan^2 x)^{1/\tan^2 x}$

$\boxed{k = e}$

iii) $f(x) = \begin{cases} \dfrac{\sqrt{3} - \tan x}{\pi - 3x} & x \neq \pi/3 \\ = k & x = \pi/3 \end{cases} \quad x = \pi/3$

Sol. put $x - \pi/3 = h \quad x = \pi/3 + h \quad h \to 0$

$x \to \pi/3$

$f(x) = \dfrac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$\Rightarrow \lim_{h \to 0} \dfrac{\dfrac{\sqrt{3} - \sqrt{3} \times \sqrt{3} \tan h - \sqrt{3} - \tan h}{1 - \sqrt{3} \tan h}}{-3h}$

$$\Rightarrow \lim_{h\to 0} \frac{\dfrac{-4\tan h}{1-\sqrt{3}\tan h}}{-3h}$$

$$\Rightarrow \lim_{h\to 0} \frac{-4\tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$\Rightarrow 4/3 \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1-\sqrt{3}\tan h)}$$

$$\Rightarrow 4/3 \frac{1}{1-\sqrt{3}(0)}$$

$$\Rightarrow 4/3$$

(7) Discuss the continuity of the following function which of these function have removeable discontinuity? Redefine functions.

(i) $$f(x) = \left.\begin{array}{ll} \dfrac{(e^{3x}-1)\sin x}{x^2} & x \neq 0 \\ = \pi/60 & x = 0 \end{array}\right\} x = 0$$

Sol. $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(e^{3x}-1)\sin x}{x^2}$$

$$= \lim_{x\to 0} \frac{(e^{3x}-1)}{3x} \times 3 \lim_{x\to 0} \frac{\sin x^\circ}{x}$$

$$\therefore \lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} \times \pi/180$$

$$\Rightarrow 3 \log e \times \pi/180$$

$$\Rightarrow \pi/60 \qquad \therefore \log e = 1$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

$$\text{iii)}\quad f(x) = \begin{cases} \dfrac{1-\cos 3x}{x \tan x} & x \neq 0 \\ = 9 & x = 0 \end{cases} \quad x = 0$$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1-\cos 3x}{x \tan x}$

$= \lim\limits_{x \to 0} \dfrac{3 \sin^2 3/2\, x}{x \tan x}$

$= \lim\limits_{x \to 0} \dfrac{\dfrac{2 \sin^2 3/2\, x}{x^2}}{\dfrac{x + \tan x}{x^2}}$

$= 2 \times 9/4$

$= 9/2$

$\therefore f(x) = 9/2$ is not continuous at $x = 0$

$\therefore$ Redefine function : $x = 0$ $f(0) = 9/2$.

removeable function $f(x) = f(0)$ discontinuity $x = 0$.

(8) If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ $x \neq 0$ is continuous $x = 0$, find $f(0)$.

Sol. $f(x) = \lim\limits_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2}$

$= \lim\limits_{x \to 0} \dfrac{e^{x^2} - 1 - \cos x + 1}{x^2}$

$= \lim\limits_{x \to 0} \dfrac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$

$$\Rightarrow \lim_{x\to 0} \frac{(e^{x^2}-1)}{x^2} + \lim_{x\to 0} \frac{(1-\cos x)}{x^2}$$

$$\Rightarrow \lim_{x\to 0} \frac{(e^{x^2}-1)}{x^2} + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$$

$$\Rightarrow \log_e 2 + 2 \ \frac{1}{4} = 1 + 2\times \frac{1}{2} = \frac{3}{2} = f(0)$$

(9) If $f(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$, $x \neq \pi/2$ continous $x = \pi/2$. find $f(\pi/2)$

Soln $\lim_{x\to \pi/2} f(x) = \lim_{x\to\pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$

put $x - \pi/2 = h$, $x = \pi/2 + h$, $x \to \pi/2$, $h \to 0$

$$\Rightarrow \lim_{x\to\pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} - \sqrt{1+\sin x}}{\sqrt{2} - \sqrt{1+\sin x}}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{2-(1-\sin x)}{\cos^2 x(\sqrt{2} - \sqrt{1+\sin x})}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{2-1+\sin x}{(1+\sin^2 x)(\sqrt{2} - \sqrt{1+\sin x}}$$

$$\Rightarrow \lim_{x\to\pi/2} \frac{1+\sin x}{(1+\sin x)(1-\sin x)(\sqrt{2} - \sqrt{1+\sin x}}$$

$$\Rightarrow \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$\Rightarrow \frac{1}{4\sqrt{2}}$$

$$\Rightarrow f(\pi/2).$$

Practical 02

Topic : Derivatives.

Q1. Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are different.

i) $\cot x$.

Sol. $f(x) = \cot x$

$$f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

$$= \lim_{x \to a} \frac{\tan a - \tan x}{(x-a) \tan x \tan a}$$

put $x - a = h$, $x = a + h$, $x \to a$, $h \to 0$

$$f(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a) \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \cdot \tan(a+h) \tan a}$$

formula : $\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \cdot \tan b}$

$\tan A - \tan B = \tan(A-B)(1 + \tan A \cdot \tan B)$

$$= \lim_{h \to 0} \frac{\tan(a-a-h) - (1 + \tan a + \tan(a+h))}{h \cdot \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} -\frac{\tan h}{h} \times \frac{1 - \tan a \tan(a+h)}{\tan(a+h) \tan a}$$

$$= -1 \times \frac{1 + \tan^2 a}{\tan^2 a}$$

$= \dfrac{-\sec^2 a}{\tan^2 a}$

$= -1/\cos^2 a \times \dfrac{\cos^2 a}{\sin^2 a}$

$= -\operatorname{cosec}^2 a$

$\therefore f(a) = -\operatorname{cosec}^2 a$

$\therefore f$ is differential $\forall a \in \mathbb{R}$.

iii) $\operatorname{cosec} x$

Sol. $f(x) = \operatorname{cosec} x$

$f(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$= \lim_{x \to a} \dfrac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$

$= \lim_{x \to a} \dfrac{1/\sin x - 1/\sin a}{x - a}$

$= \lim_{x \to a} \dfrac{\sin a - \sin x}{(x-a)\sin a . \sin x}$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$f(h) = \lim_{h \to 0} \dfrac{\sin a - \sin(a+h)}{(a+h-a)\sin a . \sin(a+h)}$

formula : $\sin c - \sin d = 2\cos\left(\frac{c+D}{2}\right)$

$\Rightarrow \lim_{h \to 0} \dfrac{2\cos\frac{(a+a+h)}{2} . \sin\frac{(a-a-h)}{2}}{h . \sin a . \sin(a+h)}$

$\Rightarrow \lim_{h \to 0} \dfrac{-\sin h/2}{h/2} \times 1/2 \times \dfrac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a . \sin(a+h)}$

$$\Rightarrow -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$\Rightarrow \frac{-\cos a}{\sin^2 a} = \boxed{-\cot a \ \mathrm{cosec}\, a.}$$

iii) sec x

Sol. $f(x) = \sec x$

$$f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\cos x - 1/\cos a}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)\cos a \cos x}$$

put $x - a = h$, $x = a+h$, as $x \to a$, $h \to 0$

$$f(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a . \cos(a+h)}$$

formula :- $-2\sin\left(\frac{c+d}{2}\right)\sin\left(\frac{c-d}{2}\right)$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a+h}{2}\right)}{h \times \cos a . \cos(a+h)}$$

$$\Rightarrow \lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin h/2}{\cos a \cos(a+h) \times -h/2} \times -1/2$$

$$= -1/2 \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a . \cos(a+0)}$$

$$\Rightarrow -\frac{1}{2} \times -2\,\frac{\sin a}{\cos a \times \cos a}$$

$\Rightarrow \tan a \sec a.$

Q2] If $f(x) = 4x+1 \quad x \leq 2$
$= x^2+5 \quad x > 0$

at $x=2$, then find function differentiate or not.

Sol. L.H.D :-

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4 \times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4.$$

R.H.D :-

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$= 2+2$

$= 4$

L.H.D = R.H.D $\Rightarrow$ f is differentiable at $x=2$.

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Q3. If $f(x) = 4x+7 \quad x < 3$
$\quad\quad = x^2+3x+1 \quad x \geq 3$ } $x=3$ then, find

F is differentiable or not.

Sol. R.H.D:

$$Df(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3+3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 9.$$

L.H.D:

$$Df(3^-) = \lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$$

$Df(3^+) = 4$

RHD $\neq$ LHD

$\therefore f$ is not differentiable at $x = 3$.

Q4. If $f(x) = 8x - 5 \quad x \leq 2$
$= 3x^2 - 4x + 7 \quad x > 2$ } $x = 2$

Sol. $f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

R.H.D

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 8$$

$Df(2^+) = 8$

L.H.D :

$$Df(2^+) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

$Df(2^-) = 8.$

L.H.D = R.H.D

$\therefore$ f is differentiable at $x = 3$.

Practical 03

Topic : Application of Derivatives.

Q1. Find the intervals in which function is increasing and decreasing.

(i) $f(x) = x^3 - 5x - 11$

(ii) $f(x) = x^2 - 4x$

(iii) $f(x) = 2x^3 + x^2 - 20x + 4$

(iv) $f(x) = x^3 - 27x + 5$

(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

Q2. Find the intervals in which function is concave upwards.

(i) $y = 3x^2 - 2x^3$

(ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

(iii) $y = x^3 - 27x + 5$

(iv) $y = 69 - 24x + 9x^2 - 2x^3$

(v) $y = 2x^3 + x^2 - 20x + 4.$

Sol.

Q1.

(i) $f(x) = x^3 - 5x - 11$

$\rightarrow f'(x) = 3x^2 - 5.$

$\because$ f is increasing iff $f(x) > 0$

$\therefore 3x^2 - 5 > 0$

$\therefore 3(x^2 - 5/3) > 0$

$\Rightarrow (x - \sqrt{5}/3)(x + \sqrt{5}/3) > 0$

$+ \quad - \quad +$

$\frac{-\sqrt{5}}{3} \qquad \sqrt{5}/3$

$x \in (-\infty, -\sqrt{5}/3) \cup (\sqrt{5}/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\Rightarrow 3x^2 - 5 < 0$

$3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5}/3)(x + \sqrt{5}/3) < 0$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$

$x \in (-\sqrt{5}/3, \sqrt{5}/3)$

(2) $f(x) = x^2 - 4x$

Sol. $f'(x) = 2x - 4$

$\therefore$ f is increasing iff $f(x) > 0$

$2x - 4 > 0$

$2(x-2) > 0$

$x \in (2, \infty)$

& f is decreasing iff $f(x) < 0$

$2x - 4 < 0$

$2(x-2) < 0$

$(x-2) < 0$

$x \in (-\infty, 2)$

3) $f(x) = 2x^3 + x^2 - 20x + 4$

Sol. $f'(x) = 6x^2 - 2x - 20$

$\therefore$ f is increasing iff $f'(x) > 0$

$6x^2 + 2x - 20 > 0$

$2(3x^2 + x - 10) > 0$

$(x+2)(3x-5) > 0$

$+ \quad - \quad +$

$-2 \qquad 5/3$

$x \in (-\infty, -2) \cup (5/3, \infty)$

and f is decreasing iff $f(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$\therefore 2(3x^2 + x - 10) < 0$

$\therefore (x+2)(3x-5) < 0$

$+ \quad - \quad +$

$-2 \qquad 5/3$

$x \in (-2, 5/3)$

iv) $f(x) = x^3 - 27x + 5$

Sol- $f(x) = 3x^2 - 27$

$\therefore$ f is increasing iff $f'(x) > 0$

$3(x^2 - 9) > 0$

$(x-3)(x+3) > 0$

$+ \quad - \quad +$

$-3 \qquad 3$

$x \in (-\infty, -3) \cup (3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$(x-3)(x+3) < 0$

$+ \quad - \quad +$

$-3 \qquad 3$

$x \in (-3, 3)$

(v) $f(x) = 2x^5 - 9x^2 - 24x + 69$

Sol. $f'(x) = 6x^2 - 18x - 24$

$\therefore$ f is increasing iff $f'(x) > 0$

$6x^2 - 18x - 24 > 0$

$(x-4)(x+1) > 0$

$+ \quad - \quad +$

$-1 \qquad 4$

$x \in (-\infty, -1) \cup (4, \infty)$

and f is decreasing iff $f(x) < 0$

$6x^2 - 18x - 24 < 0$

$(x-4)(x+1) < 0$

$+ \quad - \quad +$

$-1 \qquad 4$

$x \in (-1, 4)$

Q2.

(i) $y = 3x^2 - 2x^3$

Sol. $f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

f is concave upward iff $f''(x) > 0$

$(6 - 12x) > 0$

$12(6/12 - x) > 0$

$x - 1/2 > 0$

$x > 1/2$

$f''(x) > 0$

$x \in (1/2, \infty)$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

Sol. $f'(x) = 4x^3 - 18x^2 + 24x + 5$

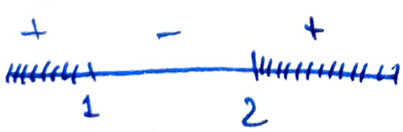
$f''(x) = 12x^2 - 36x + 24$

f is concave upward iff $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$12(x^2 - 3x + 2) > 0$

$(x-2)(x-1) > 0$



$x \in (-\infty, 1) \cup (2, \infty)$

iii) $y = x^3 - 27x + 5$

Sol. $f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$6x > 0$

$x > 0$

$x \in (0, \infty)$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

Sol. $f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\Rightarrow 12x - 18 > 0$

$12(x - 18/12) > 0$

$x - 3/2 > 0$

$x > 3/2$

$x \in (3/2, \infty)$

v) $y = 2x^3 + x^2 - 20x + 4$

Sol. $f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$x + 1/6 > 0$

$x > -1/6$

$f''(x) \not> 0$

$\therefore$ There exists no ~~interg~~ integer intervals.

$\therefore$ $x \in (-1/6, \infty)$

Practical 04

046

Topic : Application of Derivative & Newton Method.

Q1. Find Maximum & Minimum value of following function.

i) $f(x) = x^2 + 16/x^2$

$\Rightarrow f'(x) = 2x - 32/x^3$

$\Rightarrow f'(x) = 0$

$\Rightarrow 2x - \frac{32}{x^3} = 0$

$\Rightarrow 2x = 32/x^3$

$\Rightarrow x^4 = 32/2$

$\Rightarrow x = \pm 2$

$f''(x) = 2 + 96/2^4$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$.

$f(2) = 2^2 + 16/2^2$

$= 8.$

$\therefore$ f has maximum value at $x = -2$ $\therefore f''(x) = 8 < 0$

$f(-2) = (-2)^2 + 16/(-2)^2$

$= 8.$

$f''(x) = 2 + \frac{96}{(-2)^4}$

$= 2 + 96/16$

$= 2 + 6 = 8 > 0$

$\therefore$ f has minimum value at $x = -2$

function reaches minimum value at $x = 2$ & $x = -2$.

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

$f'(x) = 0$

$-15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x = \pm 1.$

ii)

$f''(x) = -30x + 60x^3$

$f''(1) = 30 > 0$

$\therefore$ f has minimum value at $x = 1$

$f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 1$

$f''(-1) = -30 < 0$

$\therefore$ f has maximum value at $x = -1$

$f(-1) = 3 - 5(-1)^3 + (3)(-1)^5$

$= 5.$

iii) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

$f'(x) = 0$

$3x^2 - 6x = 0$

$3x^2 = 6x$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$

$\therefore$ f has maximum value $x = 0$

$f(0) = (0)^3 - 3(0)^2 + 1$

$= 1.$

$f''(2) = 6(2) - 6$

$= 6 > 0$

$\therefore$ f has minimum value $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= -3.$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

$f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x = 2$ or $x = -1$

$f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 18 > 0$

$\therefore f$ minimum value $x = 2$.

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= -19$.

$f''(-1) = 12(-1) - 6$

$= -18 < 0$

$\therefore f$ has maximum value $x = -1$

$f''(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= 8$

Q2] Find the root of following equation by Newton's Method.

i) $f(x) = x^3 - 3x^2 - 55x + 95$

$f'(x) = 3x^2 - 6x - 55$

(By Newton's Method)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = 0 + \frac{95}{55}$$

$$x_1 = 0.1727.$$

$$f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 95$$

$$= -0.0829.$$

$$f'(x) = 3(0.1727)^2 - 6(0.1727) - 55$$

$$= -55.9467$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{(-0.0829)}{(-55.9467)}$$

$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 95$$

$$= 0.0011$$

$$f'(x_2) = 3(0.1712)^2 - 6(0.1712)^2 - 55$$

$$= -55.9393$$

$$x_3 = x_2 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.1712 + \frac{0.0011}{55.9393}$$

$$= 0.1712$$

$\therefore$ the root ~~of~~ is 0.1712.

ii] $f(x) = x^3 - 4x - 9$ $\quad [2, 3]$

$f'(x) = 3x^2 - 4$

$f(2) = (2)^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = (3)^3 - 4(3) - 9$

$= 6$

$x_0 = 3$ be initial approximation.

∴ By Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$= 3 - 6/23 = 2.7392$

$f(x_1) = (2.7392)^3 - 4(2.7392)^2 - 9$

$= 0.596$

$f(x_1) = 3(2.7392)^2 - 4$

$= 18.5096$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7392 - \frac{0.596}{18.5096} = 2.7071.$$

$f(x_2) = (2.7071)^3 - 4(2.7071)^2 - 4(2.7071) - 9$

$= 0.0102.$

$f'(x_2) = 3(2.7071)^2 - 4$

$= 17.9051.$

8b)...

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7071 - \frac{0.0102}{17.9851}$$

$$= 2.7051$$

$f(x_3) = (2.7051)^3 - 4(2.7051)^3 - 9$

$= -0.0901$

$f'(x_3) = 3(2.7051)^2 - 4$

$= 17.8943$

$$x_4 = 2.7015 + \frac{0.0901}{17.8943}$$

$= 2.7005$

iii] $f(x) = x^3 - 1.8x^2 - 10x + 17$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= -2.2.$

$x_0 = 2$ be initial approximation.

(By Newton's Method)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$= 2 - 22/52$

$= 1.577$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= 0.6755$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$

$= -8.2164$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.577 + \frac{0.6755}{8.2164}$

$= 1.6592$

$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$

$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= -7.7143$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6592 + \frac{0.0204}{7.7143} = 1.6618$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8618 + \frac{0.0004}{7.6977}$

$= 1.6618$

So The root of the equation 1.6618.

Practical 05

Topic : Integration.

Q1. Solve the following integration.

1) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

Sol. $I = \int \frac{1}{\sqrt{x^2+2x-3}} dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}} dx$

$= \int \frac{1}{(x+1)^2-(2)^2} dx$

put $x+1 = t$

$dx = 1/t \times dt$.

$= \int \frac{1}{\sqrt{t^2-4}} dt$

$= \log(|t+\sqrt{t^2-4}|)$

$= \log(|x+1+\sqrt{(x+1)^2-4}|)$

$= \log(|x+1+\sqrt{x^2+2x-3}|) + c$

2) $\int (4e^{3x}+1)\, dx$.

Sol. $I = \int 4e^{3x}\, dx + \int 1\, dx$

$= 4\int e^{3x}\, dx + \int 1\, dx$

$= \frac{4e^{3x}}{3} + x$

$= \frac{4e^{3x}}{3} + x + c$

3) $\int 2x^2 - 3\sin(x) + 5\sqrt{x}\, dx$

Sol. $I = \int 2x^2 - 3\sin x + 5x^{1/2}\, dx$

$= \int 2x^2\, dx - 3\int \sin x + 5\int x^{1/2}\, dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{10\sqrt[3]{x}}{3} + c$

$= \frac{2x^3 + 10\sqrt[3]{x}}{3} + 3\cos x + c$

4) $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$

Sol. $I = \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}} \right) dx$

$= \int x^{3/2}\, dx + 3\int x^{1/2}\, dx + 4\int x^{-1/2}\, dx$

$= \frac{x^{5/2+1}}{5/2+1} + 3\,\frac{x^{1/2+1}}{1/2} + 4\,\frac{x^{-1/2+1}}{-1/2+1}$

$= \frac{2x^{7/2}}{7} + 2x^{3/2} + 8\sqrt{x} + c$

5) $\int t^7 \times \sin(2t^4)\, dt$

put $u = 2t^4$

$du = 2 \times 4t^3$

$= \int t^7 \sin(2t^4)\, {}^1/_{2t^3 4}\, du$

$= \int t^4 \sin(2t^4)\, \frac{1}{2\times4}\, du$

$= \int t^4 \sin(2t^4)\, 1/8\, du$

$= \frac{t^4 \sin(2t^4)}{8}\, du$

Substituting t^4 with $u/2$

$= \int \frac{u/2\, \sin(2u/2)}{8}\, du$

$= \int \frac{\frac{u/2\, \sin(u)}{2}}{8}\, du$

$$\int \frac{u \sin(u)}{16}\, du$$

$= 1/16 \left(u \times (-\cos(u)) - \int -\cos(u)\, du\right.$

$= 1/16 \left(u(-\cos(u)) + \sin(u)\right)$

Resubstituting $u = 2t^4$

$= 1/16 \left(2t^4(-\cos(2t^4)) + \sin(2t^4)\right)$

$= \frac{-t^4 \cos(2t^4)}{8} + \frac{\sin(2t^4)}{8} + c$

6) $\int \sqrt{x}\,(x^2 - 1)\, dx$

$I = \int \sqrt{x}\,(x^2-1)\, dx$

$= \int x^{1/2}(x^2-1)\, dx$

$= \int x^{3/2} - x^{1/2}\, dx$

$= \int x^{3/2}\, dx - \int x^{1/2}\, dx.$

$= \frac{x^{5/2+1}}{5/2+1} - \int \frac{x^{1/2+1}}{1/2+1}$

$= \frac{2x^{7/2}}{7} - \frac{2x^3}{3} + c$

7) $\int 1/x^3 \sin(1/x^2)\, dx$

$u = 1/x^2 \qquad \frac{du}{dx} = -2/x^3$

$\frac{du}{dx} = -2/x^3$

$I = \int \sin(u) \frac{du}{-2}$

$= -1/2\,(-\cos u) + c$

$= -1/2 \cos(1/x^2) + c$

8) $\int e^{\cos^2 x} \sin 2x \, dx$

put $\cos^2 x = t$

$2\cos (-\sin x) dx = dt$

$-\sin 2x \, dx = dt$

$\sin 2x \, dx = -dt.$

$= \int e^t (-dt)$

$= -\int e^t dt$

$= e^t + C$

$= e^{\cos^2 x} + C$

9) $I = \int \frac{\cos x}{\sqrt[3]{\sin x}} dx$

$= \int \frac{\cos x}{\sin x^{2/3}}$

$t = \sin(x)$

$t = \cos x$

$= \int \frac{\cos x}{\sin(x)^{3/2}} \times \frac{1}{\cos x} dt$

$= 1/\sin x^{3/2} \, dt$

$= 1/t^{2/3}$

$= \int -1/-1/3 \; t^{2/3-1} = 1/1/3 \; t^{-1/3} = 3t^{1/3}$

$= 3\sqrt[3]{t}$

$= 3\sqrt[3]{\sin x} + C$

P.T.O

10) $I = \int \frac{x^2 - 2x}{x^3 - 2x^2 + 1} \, dx$

put $x^3 - 3x^2 + 1 = t$

$(x^2 - 2x)\,dx = dt/2$

$= \int (1/t)\, dt/3$

$= 1/3 \int (1/t)\, dt$

$= 1/3 \log |t| + c$

$= 1/3 \log |x^3 + 3x^2 + 1| + c.$

—— * —— * —— * ——

Practical No. 5

Q1. Solve the following integration.

i) $I = \int \frac{dx}{\sqrt{x^2+2x-3}}$ ii) $\int (4e^{3x}+1)\,dx.$

iii) $\int (2x^2 - 3\sin x + 3\sqrt{x})\,dx.$

iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx.$ v) $\int t^7 \sin(2t^4)\,dt.$

vi) $\int \sqrt{x}(x^2-1)\,dx$ vii) $\int 1/x^3 \sin(1/x^2)\,dx.$

viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$ ix) $\int e^{\cos^2 x} \sin 2x\,dx.$

x) $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right)dx.$

Sol^n:-

i) $I = \int \frac{1}{\sqrt{x^2+2x-3}}\,dx.$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}}\,dx.$

$= \int \frac{1}{\sqrt{(x+1)-2^2}}\,dx$

$I = \ln|x+1+\sqrt{x^2+2x+3}| + c.$

ii) $I = \int (4e^{3x}+1)\,dx.$

$= 4\int e^{3x}\,dx + \int 1\,dx.$

$I = \frac{4e^{3x}}{3} + x + c.$

iii) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx.$

$= 2\int x^2 dx - 3\int \sin x\, dx + 5\int x^{1/2} dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{5\times 2}{3} x^{3/2} + c.$

$I = \frac{2x^3}{3} + 3\cos x + \frac{10}{3} x\sqrt{x} + c.$

iv) $I = \int \frac{x^3 + 3x + 4}{\sqrt{x}} dx.$

Put $\sqrt{x} = t.$

$\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$ $\therefore \frac{dx}{\sqrt{x}} = 2dt.$

$I = \int \frac{(x)^6 + 3(\sqrt{x})^2 + 4}{\sqrt{x}} dx$

$= 2\int t^6 + 3t^2 + 4\, dt.$

$= 2\left[\frac{t^7}{7} + \frac{3t^3}{3} + 4t\right] + c.$

$= 2\left[\frac{x^{7/2}}{7} + 3x^{3/2} + 4x^{1/2}\right] + c.$

v] $I = \int t^7 \sin(2t^4)\, dt.$

$= \int t^4 \sin(2t^4) \times t^3\, dt.$

Put $t^4 = x$

$4t^3 = \frac{dx}{dt}.$

$t^3 dt = 1/4 \frac{dx}{dt}$

$I = \frac{1}{4} \int x \cdot \sin(2x) dx.$

$\therefore I = 1/4 \ [x] \sin 2x dx - \int \left(\frac{d}{dx} x\right) \left(\int \sin 2x \, dx\right) dx]$

$\Rightarrow \frac{1}{4} \left[-x \cdot \frac{\cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx\right]$

$\Rightarrow \frac{1}{16} \sin 2x - x \cdot \frac{\cos 2x}{8} + c$

$\Rightarrow I = \frac{1}{16} \sin 2t^4 - t^4 \cdot \frac{\cos 2t^4}{8} + c.$

vi] $I = \int \sqrt{x} \, (x^2 - 1) \, dx$

$\therefore I = \int x^2 \sqrt{x} \, dx - \int \sqrt{x} \, dx.$

$= \int x^{5/2} dx - \int x^{1/2} dx.$

$= \frac{x^{7/2}}{7/2} - \frac{x^{3/2}}{3/2} + c.$

$= 2/7 \ x^3 \sqrt{x} - 2/3 \ x \sqrt{x} + c.$

vii] $I = \int 1/x^3 \sin(1/x^2) \, dx.$

$\Rightarrow 1/x^2 = t$

$\therefore -2/x^3 = \frac{dt}{dx}$

$\therefore \frac{dx}{x^3} = -1/2 \ dt.$

$I = -1/2 \int \sin t \, dt.$

$= -1/2 \cos t + c.$

$= \dfrac{\cos t}{2} + c.$

$I = \dfrac{\cos(1/x^2)}{2} + c.$

viii] $I = \int \dfrac{\cos x}{\sqrt[3]{\sin^2 x}} \, dx.$

Put $\sin x = t.$

$\therefore \cos x \, dx = dt.$

$\therefore I = \int \dfrac{1}{t^{2/3}} \, dt.$

$\Rightarrow \dfrac{t^{-2/3+1}}{-2/3+1} + c.$

$\Rightarrow 3t^{1/3} + c.$

$\Rightarrow I = 3 \sqrt[3]{\sin x} + c.$

ix] $I = \int e^{\cos^2 x} \sin 2x \, . dx.$

put $\cos^2 x = t.$

$\therefore -2 \cos x . \sin x = dt/dx.$

$\therefore \sin 2x \, dx = -dt.$

$\therefore I = -\int e^t \, dt.$

$= -e^t + c$

$\therefore I = -e^{\cos^2 x} + c.$

x] $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx.$

$\Rightarrow x^3 - 3x^2 + 1 = t.$

$\therefore 3x^2 - 6x = dt/dx.$

$\therefore (x^2 - 2x)\, dx = dt/3$

$\therefore I = 1/3 \int 1/t \; dt.$

$= 1/3 \log |t| + c$

$I = \frac{1}{3} \log |x^3 - 3x^2 + 1| + c.$

Practical No. 06

Topic: Application of integration and numerical integration.

Q1. Find the length of the following curve.

1. $x = t\sin t$, $y = 1 - \cos t$, $t \in [0, 2\pi]$

2. $y = \sqrt{4-x^2}$, $x \in [-2, 2]$

3. $y = x^{3/2}$ in $[0, 4]$

4. $x = 3\sin t$, $y = 3\cos t$ $t \in [0, 2\pi]$

5. $x = 1/6\, y^3 + 1/2y$ on $y \in [1, 2]$

Q2. Using Simpson's Rule solve the following.

1) $\int_0^2 e^{x^2}\,dx$ with $n=4$

2) $\int_0^4 x^2\,dx$ with $n=4$.

3) $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n=6$.

2) $y = \sqrt{4-x^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4-x^2}}(-2x)$

$\Rightarrow = \dfrac{-x}{\sqrt{4-x^2}}$

$\Rightarrow L = \int_{-2}^{2} \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx$

$\Rightarrow \int_{-2}^{2} \sqrt{1+\dfrac{x^2}{4-x^2}}\, dx$

$\Rightarrow \int_{-2}^{2} \sqrt{\dfrac{4}{4-x^2}}\, dx$

$\Rightarrow 2\int_{-2}^{2} \dfrac{1}{\sqrt{2-x^2}}\, dx.$

$\Rightarrow 2\left[\sin^{-1}\left(x/2\right)\right]_{-2}^{2}$

$\Rightarrow 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$

$\Rightarrow 2\left[\dfrac{\pi}{2} - \left(-\dfrac{\pi}{2}\right)\right]$

$\Rightarrow \boxed{L = 2\pi}$

720

3] $y = x^{3/2} \qquad x \in [0, 4]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{2} x^{1/2}$

$\therefore L = \displaystyle\int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \displaystyle\int_0^4 \sqrt{1 + \tfrac{9}{4} x}\; dx.$

$= \dfrac{1}{2} \displaystyle\int_0^4 \sqrt{4 + 9x}\; dx$

$= \dfrac{1}{2} \left[\dfrac{(4+9x)^{3/2}}{3/2} \times \dfrac{1}{9} \right]_0^4$

$= \dfrac{1}{27} \left[(4+9x)^{3/2} \right]_0^4$

$= -\dfrac{1}{27} \left[(4+0)^{3/2} - (4+36)^{3/2} \right]$

$L = \dfrac{1}{27} \left(40^{3/2} - 8 \right)$

4] $x = 3\sin t \;,\; y = 3\cos t$

$\dfrac{dx}{dt} = 3\cos t \;,\; \dfrac{dy}{dt} = -3\sin t.$

$\Rightarrow L = \displaystyle\int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt$

$= \displaystyle\int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\; dt$

$$\Rightarrow \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$\Rightarrow \int_0^{2\pi} 3\sqrt{1}\; dt$$

$$\Rightarrow 3\int_0^{2\pi} 1 \cdot dt.$$

$$\Rightarrow 3(2\pi)$$

$$L = 6\pi$$

5] $x = \frac{1}{6}y^3 + \frac{1}{2y}$

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$$

$$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$$

$$L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy$$

$$= \int_0^2 \sqrt{1 + \frac{(y^3-1)}{4y^2}}\; dy.$$

$$= \int_1^2 \sqrt{\frac{(y^4-1) + 2y^3}{4y^4}}\; dy.$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\; dy.$$

$$= \int_1^2 \frac{(y^4+1)^2}{(2y^2)^2}\; dy.$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\; dy.$$

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$$= \frac{1}{2}\int_1^2 y^2\,dy + \frac{1}{2}\int_1^2 y^{-2}\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right] = \frac{17}{12}.$$

Q2]

1) $\int_0^2 e^{x^2}\,dx$, $n = 4$.

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.4877.	54.5982

$$\int_0^2 e^{x^2}\,dx = \frac{0.5}{3}\left[(1 + 54.5982) + 4(1.284 + 9.4877) + 2(2.7183)\right]$$

$$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 54.36\right].$$

$$= 17.3535.$$

2] $\int_0^4 x^2 dx$ $n = 4.$

$h = \frac{4-0}{4} = 1.$

x	0	1	2	3	4
y	0	1	4	9	6

$$\int_0^4 x^2 dx = \frac{1}{3}\left[0 + 16 + 4(1+9) + 2(4)\right]$$

$\Rightarrow \frac{1}{3}\left[0 + 16 + 4(10) + 8\right]$

$\Rightarrow \frac{64}{3}$

$\Rightarrow 21.3333.$

3] $\int_0^{\pi/3} \sqrt{\sin x \, dx}$ $n = 6.$

$h = \frac{\pi/3 - 0}{6} = \pi/18.$

0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$	$6\pi/18$
0	0.4167	0.5843	0.7071	0.8017	0.8752	0.9306.

P.T.O

$$\int_0^{\pi/3} \sqrt{\sin x}\; dx = \frac{\pi/18}{3}\Big[0.4167 + 0.9306 + 4(0.4117 + 0.7071) + 2(0.5843 + 0.8017) + 0.8752\Big]$$

$$= \frac{\pi/18}{3}\Big[1.3473 + 4(1.999) + 2(1.3865)$$

$$= \frac{\pi}{54}\Big[1.3473 + 7.996 + 2.773\Big]$$

$$= \frac{\pi}{54}(12.1163)$$

$$= 0.7049.$$

Practical No. 7

Differential equation.

1) Solve the following differential equation.

i) $x\frac{dy}{dx}+y=e^x$ ii) $e^x\frac{dy}{dx}+2e^xy=1$.

iii) $x\frac{dy}{dx}=\frac{\cos x}{x}-2y$ iv) $x\frac{dy}{dx}+3y=\frac{\sin x}{x^2}$

v) $e^2x\frac{dy}{dx}+2e^{2x}y=2x$ vi) $\sec^2 x\tan xy\,dx+\sec^2 y\tan x\,dx=0$

vii) $\frac{dy}{dx}=\sin^2(x-y+1)$ viii) $\frac{dy}{dx}=\frac{2x+3y-1}{6x+9y+6}$.

Q1)

i) $x\frac{dy}{dx}+y=e^x$

$\Rightarrow \frac{dy}{dx}+\frac{1}{x}y=\frac{e^x}{x}$

$P(x)=1/x$ $q(x)=\frac{e^x}{x}$

$I.F=e\int 1/x\,dx$

$=x$

$y(IF)=\int Q(x)(IF)\,dx+c.$

$xy=\int\frac{e^x}{x}x\,dx+c$

$xy=\int e^x dx+c$

$xy=e^x+c$

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2) $e^{x}\frac{dy}{dx} + 2e^{x}y = 1.$

$$\frac{dy}{dx} + 2y = e^{-x}$$

$f(x) = 2 \qquad g(x) = e^{-x}.$

$IF = e^{\int p(x)dx}.$

$= e^{\int 2\,dx}$

$= e^{2x}.$

$y(IF) = \int g(x)\,(IF)\,dx + c.$

$y \cdot e^{2x} = \int e^{-x}\, e^{2x}dx + c$

$= \int e^{-x+2x}\,dx + c.$

$= \int e^{x}\,dx + c.$

$y \cdot e^{2x} = e^{x} + c.$

3] $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$

$f(x) = 2/x \qquad g(x) = \cos x / x^2$

$If = \int f(x)\,dx.$

$= e^{\int 2/x\,dx}$

$= e^{2/x}$

$=$

$IF = x^2$

$y(IF) = \int Q(x)(IF)\,dx + c.$

$= \int \frac{\cos 2}{x^2} x\ x^2\,dx + c.$

$= \int \cos x + c.$

$x^2 y = -\sin x + c.$

$x \cdot \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3}{x} y = \frac{\sin x}{x^2}$

$f(x) = 3/x \qquad Q(x) = \sin x / x^2$

$IF = e^{\int P(x)\,dx}.$

$= e^{\int e/x\,dx}.$

$= e^{h}\ x$

$IF = x^3$

$\Rightarrow y(IF) = \int Q(x)(IF)\,dx + e.$

$\Rightarrow x^2 y = \int \frac{\sin x}{x^2} x^3\,dx + c.$

$= \int \sin x\,dx + c.$

$x^3 y = -\cos x + c.$

$e^{2x} dy/dx + 2e^{2x} y = 2x.$

$\Rightarrow \frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}.$

$f(x) = 2 \qquad Q(x) = 2x e^{-2x}.$

$$IF = e^{\int 8(x)\,dx}$$
$$= e^{\int x\,dx}$$
$$= e^{2x}.$$
$$y\,(IF) = \int \emptyset(x)\,(IF)\,dx + c.$$
$$= \int 2x\,e^{2x} \cdot e^{2x}\,dx + c.$$

$$ye^{2x} = \int 2x\,dx + c.$$
$$\therefore\ ye^{2x} = x^2 + c.$$

6] $Sec^2 x \cdot \tan y\,dx + sec^2 y \tan x\,dy = 0.$

$\Rightarrow sec^2 x \cdot \tan y\,dx = sec^2 y \cdot \tan x\,dy.$

$$\Rightarrow \frac{sec^2 x\,dx}{\tan x} = \frac{-sec^2 y\,dy}{\tan y}.$$

$$\Rightarrow \int \frac{sec^2 x\,dx}{\tan x} = \int \frac{sec^2 y\,dy}{\tan xy}$$

$\Rightarrow \log|\tan x| = -\log|\tan y| + 1.$

$\Rightarrow \log|\tan x - \tan y| = c$

$\Rightarrow \tan x \cdot \tan y = c.$

7] $\frac{dy}{dx} = \sin^2(x-y-2) - x$

Differentiating both side.

$\Rightarrow x - y + 1 = Y.$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow 1 - \frac{dv}{dx} = \sin^2 x$$

$$\Rightarrow \frac{dy}{dx} = \cos^2 x.$$

$\int \sec^2 v \, dv = \int dx.$

$\tan v = x + c.$

$\tan(x + y - 1) = x + c.$

8] $\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$

put $2x + 3y = v.$

$\Rightarrow 2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\Rightarrow \frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+1}\right)$

$\Rightarrow \frac{dv}{dx} \quad \frac{v - 1 + 2v + 4}{v + 2} = \frac{3v + 3}{v + 2} = 3\left(\frac{v+1}{v+2}\right)$

$\Rightarrow \int \frac{v+1}{v+1} \, dx + \int \frac{1}{v+1} \, dv = 3x + c$

$\Rightarrow v + \log|v + 1| = 3x + c.$

$\Rightarrow 3x + 3y + \log|2x + 3y + 1| = 3x + c.$

$\Rightarrow 3y = x - \log|2x + 3y + 1| + c.$

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Practical No. 8

Euler's Method

* Using Euler's method, find the following.

1) $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2$, $h = 0.5$, find $y(2)$.

2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, find $y(1)$.

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$, find $y(1)$.

5) $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$, find $y(1.2)$ with $h = 0.2$.

4) $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, find $y(2)$ for $h = 0.5$ & $h = 0.25$.

Q1.

$f(x, y) = y + e^x - 2$

$y(0) = 2$, $x = 0.5$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.2231
2	1	3.2231	3.9414	5.1938
3	1.5	5.1938	7.6755	9.0315

$y(2) = 9.0315$

2] $\frac{dy}{dx} = 1 + y^2 = f(x, y)$

$y(0) = 0 \qquad h = 0.2$

x	x_n	y_n	$f(x_n, y_n)$	$y_n + 1$
0	0	0	1	0.2
1	0.2	0.2	1.04	0.405
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8529	1.2942.

$y(1) =$ ~~1.9~~ $1.2942.$

3] $f(x, y) = \sqrt{\frac{x}{y}}$; $y(0) = 1$; $h = 0.2.$

x	x_n	y_n	$f(x_n, y_n)$	$y_n + 1$
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2106
3	0.6	1.2106	0.704	1.3514
4	0.8	1.3514	0.7694	1.5053

$y(1) = 1.5053.$

4] $f(x,y) = 3x^2+1$; $y(1) = 2$; $h = 0.5$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
1	1	2	4	4
2	1.5	4	7.75	7.875

$y(2) = 7.875$

$h = 0.25$.

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
1	1	2	4	4
2	1.25	3	5.6875	4.4219
3	1.5	4.4219	7.75	6.3594
4	1.75	6.3594	10.1875	8.9063.

$y(1) = 8.9063$.

5] $f(x,y) = \sqrt{xy} + 2$; $y(1) = 1$, $h = 0.2$

x	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
1	1	1	3	1.6

$y(1.2) = 1.6$.

Practical No. 09

Limits and partial order derivative.

1) Evaluate the following limits.

i] $\lim\limits_{(x,y)\to(-4,-1)} \dfrac{x^3-3y+y^2-1}{xy+5}$

ii] $\lim\limits_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

iii] $\lim\limits_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2y^2}$

2) Find f_x, f_y for each of the following.

i] $f(x,y) = xye^{x^2+y^2}$

ii] $f(x,y) = e^x \cos y$.

iii] $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$.

3) Using definition find values of f_x, f_y, at $(0,0)$

for: $f(x,y) = \dfrac{2x}{1+y^2}$

4) Find all second order partial derivatives of function. Also verify whether $f_{xy} = f_{yx}$.

i] $f(x,y) = \dfrac{y^2-xy}{x^2}$ ii] $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

iii] $f(x,y) = \sin(xy) + e^{x+y}$.

5] Find the Linearization of $f(x,y)$ at given point

i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$.

ii) $f(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$

iii) $f(x,y) = \log x + \log y$ at $(1,1)$.

Q1.

i) $$\lim_{(x,y)\to(-4,-1)} \frac{x^3-3y+y^2-1}{xy+5}$$

$$\frac{(-4)^3-3(-1)+(-1)^2-1}{(-4)(-1)+5}$$

$$\frac{64+3+1-1}{4+5}$$

$$\frac{67}{9}$$

ii] $$\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$$

$$\frac{(0+1)((2)^2+(0)^2-4(2))}{2+3(0)}.$$

$$\frac{1(4+0-8)}{2} = \frac{-4}{2} = -2.$$

i] $$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$$

$$\frac{1-1}{1-1} = \frac{0}{0}$$

$\Rightarrow$ sol^n not defined.

(b) $f(x,y) = xye^{x^2+y^2}$

$= x \cdot y \cdot e^{x^2} \cdot e^{y^2}$

$f_x = \frac{d}{dx} x \cdot y \cdot e^{x^2} \cdot e^{y^2}$

$= y \cdot e^{y^2} \frac{d}{dx} x \cdot e^{x^2}$

$= y e^{y^2} \left[x \frac{d}{dx} e^{x^2} + e^{x^2} \frac{d}{dx} y \right]$

$= y \cdot e^{y^2} \left[2x^2 \cdot e^{x^2} + e^{x^2} \right]$

$= (2x^2+1)\, y e^{x^2+y^2}$

$f_y = x e^{x^2} \frac{d}{dy} y \cdot e^{y^2}$

$= x e^{x^2} \left[y \frac{d}{dy} e^{y^2} + e^{y^2} \frac{d}{dy} y \right]$

$= (2y^2+1)\, x \cdot e^{x^2+y^2}$

(i) $f(x,y) = e^x - \cos y$.

$f_x = \frac{d}{dx} e^x - \cos y$

$= e^x - \cos y$

$f_y = \frac{d}{dy} e^x - \cos y$.

$= -e^x \cdot \sin y$.

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iii] $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1.$

$$f_x = \frac{d}{dx}(x^3y^2 - 3x^2y + y^3 + 1)$$

$$= 3x^2y^2 - 6xy$$

$$f_y = \frac{d}{dy}(x^3y^2 - 3x^2y + y^3 + 1)$$

$$= 2x^3y - 3x^2 + 3y^2.$$

Q3.

i] $f(x,y) = \dfrac{2x}{1+y^2}$

$$f_x = \frac{1}{1+y^2}\frac{d}{dx}(2x)$$

$$= \frac{2}{1+y^2}$$

$$f_x(0,0) = \frac{2}{1+(0)^2} = 2.$$

$\therefore$ $f_y = 2x\dfrac{d}{dy}\left(\dfrac{1}{1+y^2}\right)$

$$= 2x \frac{-1}{(1+y^2)^2} \cdot 2y$$

$$= \frac{2-4xy}{(1+y^2)^2}$$

$$f_y(0,0) = \frac{-4 \times 0 \times 0}{(1+0^2)^2} = 0.$$

Q4.

i] $f(x, y) = \frac{y^2 - xy}{x^2}$

$fx = \frac{d}{dx}\left(\frac{y^2 - xy}{x^2}\right)$

$= -2y^2x^{-3} + y/x^2$

$= y/x^2 - 2y^2/x^3$

$fy = \frac{2y - x}{x^2}$

$\therefore fx\,x = \frac{d}{dx} fx.$

$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^2} - \frac{2y^2}{x^3}\right)$

$\Rightarrow \frac{6y^2}{x} - \frac{2y}{x^3}$

$\therefore fy\,y = \frac{d}{dy} fy.$

$= \frac{2}{x^2}$

$\therefore fxy = \frac{d}{dy} f(x) = \frac{d}{dy}\left(\frac{y}{x^2} - \frac{2y^2}{x^3}\right)$

$= \frac{1}{x^2} - \frac{4y}{x^3}$

$\therefore fxy = fyx.$

ii] $f(x, y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{d}{dx}(x^3 + 3x^2y^2 - \log(x^2+1))$

$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$

$f_y = \frac{d}{dy}(x^3 + 3x^2y^2 - \log(x^2+1))$

$= 0 + 6yx^2 - 0$

$= 6x^2y$

$\therefore f_{xx} = \frac{d}{dx} f(x)$

$= \frac{d}{dx}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 6x + 6y^2 - \frac{4x - 2x^2 + 2}{(x^2+1)^2}$

$\therefore f_{yy} = \frac{d}{dy} f(y)$

$= \frac{d}{dy}(6x^2y)$

$= dx^2$

$\therefore f_{xy} = \frac{d}{dy} f_x$

$= \frac{d}{dy}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 12xy.$

$\therefore f_{yx} = \frac{d}{dx} f_y$

$= \frac{d}{dx}(6x^2y)$

$= 12xy$

$\therefore f_{xy} = f_{yx}.$

(iii) $f(x, y) = \sin(xy) + e^{x+y}$

$= \sin(xy + e^x \cdot e^y$

$f_x = \frac{d}{dx}(\sin(xy) + e^x \cdot e^y)$

$= y\cos(xy) + e^x \cdot e^y$

$f_y = \frac{d}{dy}(\sin(xy) + e^x \cdot e^y)$

$= x\cos(xy) + e^x \cdot e^y.$

$f_{xx} = \frac{d}{dx} f_x$

$= \frac{d}{dx} y\cos(xy) + e^x e^y.$

$= -y^2 \sin(xy) + e^x \cdot e^y.$

$f_{yy} = \frac{d}{dy} f_y$

$= \frac{d}{dy}(x\cos(xy) + e^x e^y)$

$= -x^2 \cdot \sin^2(xy) + e^x \cdot e^y.$

$\therefore f_{xy} = \frac{d}{dy} f_x$

$= \frac{d}{dy} (y \cos(xy) + e^x . e^y)$

$= -xy \sin(xy) + \cos(xy) + e^x . e^y.$

$\therefore f_{yx} = \frac{d}{dx} f_y$

$= \frac{d}{dx} (x \cos(xy) + e^x . e^y)$

$= -xy \sin(xy) + \cos(xy) + e^x . e^y$

$\therefore f_{xy} = f_{yx}.$

Q5]

i] $f(x, y) = \sqrt{x^2+y^2}$ $(a, b) = (1, 1)$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$ $f_y = \frac{2y}{2\sqrt{y^2+x^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$ $= \frac{y}{\sqrt{x^2+y^2}}$

$f_x(1, 1) = \frac{1}{\sqrt{2}}$ $= \frac{1}{\sqrt{2}}$

$\therefore L(x, y) = f(1, 1) + f_x(1, 1)(x-1) + f_y(1, 1)(y-1)$

$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$

$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}.$

ii] $f(x,y) = 1-x+y\sin x \quad (a,b) = (\pi/2, 0)$ 066

$f(\pi/2, 0) = 1 - \pi/2 + 0 \cdot \sin \pi/2$

$= \frac{2-\pi}{2}$

$f_x = -1 + y\cos x$ $\qquad$ $f_y = \sin x$

$f_x(\pi/2, 0) = -1 + 0 \times \cos \pi/2$ $\qquad$ $f_y(\pi/2, 0) = \sin \pi/2$

$= -1$ $\qquad$ $= 1.$

$\therefore L(x,y) = f(\pi/2, 0) + f_x(\pi/2, 0)(x - \pi/2) + f_y(\pi/2, 0)(y-0)$

$= \frac{2-\pi}{2} + (-1)(x - \pi/2) + 1(y)$

$= 1 - \frac{\pi}{2} - x + \frac{\pi}{2} + y.$

$= 1 - x + y.$

iii] $f(x,y) = \log x + \log y \quad (a,b) = 1,1$

$f(1,1) = \log(1) + \log(1)$

$= 0$

~~f(x)~~ $f_x = \frac{1}{x}$ $\qquad$ $f_y = \frac{1}{y}$

$f(1,1) = 1$ $\qquad$ $f(1,1) = 1.$

$\therefore L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 + 1(x-1) + 1(y-1)$

$= x + y - 2.$

Practical No: 10

Q1. Find the directional directive of the given vector at the given points.

i) $f(x,y) = x + 2y - 3$ at $\bar{u} = 3\hat{i} - \hat{j}$, $a(1,-1)$

ii) $f(x,y) = y^2 - 4x + 1$ at $\bar{u} = \hat{i} + 5\hat{j}$, $a(3,4)$

iii) $f(x,y) = 2x + 3y$ at $\bar{u} = 3\hat{i} + 4\hat{j}$, $a(7,2)$

Q2. Find gradient vector for the following functions at the given point.

i) $f(x,y) = x^y + y^x$, $a = (1,1)$

ii) $f(x,y) = (\tan^{-1}x) - y^2$, $a = (1,-1)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

Q3. Find the equation of tangent and normal of each of the following curves.

i) $x^2 \log y + e^{xy} = 2$ at $(1,0)$

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$

Q4. Find the equation of tangent and normal line to each of the following surfaces.

i) $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

ii) $3xyz - x - y + z = -4$ at $(1,-1,2)$

Q5. Find the local maximum & minimum for the following function.

i] $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$.

ii) $f(x,y) = 2x^4 + 3x^2y - y^2$.

Q1

i) Here, $u = 3\hat{i} - \hat{j}$ is not a unit vector.

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$.

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3, -1)$

$= \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right)$

$\Rightarrow f(a+hu) = f(1, -1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$\Rightarrow f(a) = f(1, -1) = (1) + (2(-1)) - 3 = 1 - 2 - 3 = -4$.

$f(a+hu) = f(1, -1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right)\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{-4 + h/\sqrt{10} + 4}{h} = \frac{1}{\sqrt{10}}$$

ii) $f(x, y) = y^2 - 4x + 1.$

$u = \hat{i} + 5\hat{j}.$

$\therefore \bar{u} = \dfrac{\bar{u}}{|u|}, \quad \dfrac{3\hat{i} + 5\hat{j}}{\sqrt{1^2 + 5^2}}$

$\therefore \bar{u} = \left(\dfrac{1}{\sqrt{26}}, \dfrac{5}{\sqrt{26}} \right)$

$f(0) = (4)^2 - 4(3) + 1$

$= 16 - 12 + 1$

$= 5$

$f(a+hu) = f\left((3, 4) + h\left(1/\sqrt{26}, 5/\sqrt{26}\right)\right)$

$= f\left(\left(3 + h/\sqrt{26}\right), \left(4 + 5h/\sqrt{26}\right)\right)$

$= \left(4 + \dfrac{\sqrt{5h}}{26}\right)^2 - 4\left(5 + \dfrac{h}{\sqrt{21}}\right)$

$= 16 + \dfrac{40h}{\sqrt{26}} + \dfrac{25h^2}{26} - 12 - \dfrac{4h}{\sqrt{26}} + 1.$

$= \dfrac{25h^2}{26} - \dfrac{36h}{\sqrt{26}} + 5.$

$\Rightarrow D_u f(x) = \lim \dfrac{f(a+hu) - f(a)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\dfrac{25h^2}{26} - \dfrac{36h}{\sqrt{26}} + 5 - 5}{h}.$

$= \lim\limits_{h \to 0} \dfrac{25h^2}{26} - \dfrac{36}{\sqrt{26}}$

$= -\dfrac{36}{\sqrt{26}}.$

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iii) $f(x, y) = 2x + 3y$ $a(1, 2)$

$u = 3i + 4j$

$u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3^2 + 4^2}} (3i + 4j)$

$= \frac{1}{\sqrt{25}} (3i + 4j)$

$\bar{u} = (3/25, 4/5)$

$f(a) = 2(1) + 3(2)$

$= 2+6$

$f(a) = 8.$

$f(a+h+i) = f((1, 2) + h(3/5, 4/5))$

$= f((1 + 3h/5), (2 + 4h/5))$

$= 2(1 + 2h/5) + 3(2 + 4h/5)$

$= 2 + 6h/5 + \cancel{8}6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8.$

$D_u f(a) = \lim_{x \to 12} \frac{f(a+hu) - f(a)}{2}$

$D_u f(a) = \lim_{x \to 0} \frac{18h/5 + 8 - 8}{h}$

$\lim_{x \to 10} \frac{18x/5}{x}$

$= \frac{18}{5}.$

Q2.

i) $f(x,y) = x^y + y^x$

$f_x = yx^{y-1} + y^x \cdot \log y.$

$f_y = \frac{d}{dy} x^y + y^x$

$f_y . xy^{x-1} + x^y . \log x$

$\nabla f(x,y) = f(f_y, f_x)$

$\nabla f(x,y) = (yx^{y-1} + y^x \log y, xy^{x+1}, x^y \log y)$

$\nabla f(1,1) = (1(1)^{2-1} + 1 \log 1, 1(1)^{2-1} + 1^2 \log 1$

$\nabla f(1,1) = (1,1)$

ii) $f(x,y) = (\tan^{-1} x) y^2$

$f_x = \frac{d}{dx} (\tan^{-1} x) y^2$

$f_x = \frac{y^2}{1+x^2}$

$f_y = \frac{d}{dy} (\tan^{-1} x) y^2$

$= 2y \tan^{-1} x.$

$\nabla f(x,y) = f(f_x, f_y)$

$= (\frac{y^2}{1+x^2}, 2y + \tan^{-1} x)$

$\nabla f(-1,1) = (\frac{1^2}{1+(-1)^2}, 2(-1) \tan^{-1} x)$

$= (\frac{1}{1+1}, -2 \times \pi/4)$

$\nabla f(1-1) = (\frac{1}{2}, -\pi/2)$

iii) $f(x,y,z) = xy^2 - e^{x+y+z}$

$fx = y^2 - e^{x+y+z}$

$fy = x^2 - e^{x+y+z}$

$fz = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (fx, fy, fz)$

$= (y^2 - e^{x+y+z}, x^2 - e^{x+y+z}, xy - e^{x+y+z})$

$\nabla f(1,-1,0) = (-1, -1, -2)$.

Q3.

i) $x^2 \cos y + e^{xy} - 2 = 0$.

$fx = 2x \cos y + ye^{xy}$.

$fy = -x^2 \sin y + \cancel{\ } xe^{xy}$.

$\Rightarrow fx(x-x_0) + fy(y-y_0) = 0$.

$\Rightarrow (2x \cos y + ye^{xy})(x-1) + (-x^2 \sin y + xe^{xy})(y-0) = 0$.

$\Rightarrow 2x^2 \cos y + xye^{xy} + 2x \cos y - y^{xy} - (x^2 \sin x - xe^{xy}) = 0$.

$\Rightarrow (1,0,40) = (1,0)$.

ii) $f(x,y) = x^2 + y^2 - 2x + 3y + 2 = 0$.

$fx = 2x - 2$ $\qquad fx(x,-2) = 2$.

$fy = 2y + 3$ $\qquad fy(2,-2) = 1$.

Tangent:

$fn(x_0 y_0)(x - x_0) + fy(x_0 y_0)$

$2(x-2) + 1(y+2) = 0$.

$2x - 4 + y + 2 = 0$.

$2x + y - 2 = 0$.

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Normal:

$$x - 2y + d = 0.$$
$$2 - 2(-2) + d = 0.$$
$$\therefore d = 2.$$
$$\therefore x - 2y + 2 = 0.$$

Q4]

i) $f(x, y, z) = x^2 - 2y^2 + 3y + 2x - 7.$

$f_x = 2x + 2 \quad f_x(x_0, y_0, z_0) = 2(2) + 0 = 4.$

$f_y = -2x + 3 \quad f_y(x_0, y_0, z_0) = 2(0) + 2 = 2$

$f_z = -2y + x \quad f_z(x_0, y_0, z_0) = -2(1) + 2 = 4.$

Tangent:

$f_x(x - x_0) + f_y(x_0 y_0 z_0)(y - y_0) + f_z(x_0 y_0 z_0)(z - z_0)$

$\Rightarrow 4(x-2) + 3(y-1) + 0(z-0) = 0.$

$\Rightarrow 4x - 8 + 3y - 3 = 0.$

$\Rightarrow 4x + 3y - 11 = 0.$

Normal:

$$\frac{x - x_0}{f_x(x_0 y_0 z_0)} = \frac{y - y_0}{f_y(x_0 y_0 z_0)}$$

$$\Rightarrow \frac{x-2}{4} = \frac{y-1}{1} = \frac{z-0}{0}$$

ii) $f(x,y,z) = 3xyz - x - y + z + 4 = 0.$

$fx = 3yz - 1 \quad fx(x_0 y_0 z_0) = 3(-1)(2) - 1 = -7.$

$fy = 3xz - 1 \quad fy(x_0 y_0 z_0) = 3(1)(2)(-1) = 5.$

$fz = 3xy + 1 \quad fz(x_0 y_0 z_0) = 3(1)(-1) + 1 = -2.$

Tangent :-

$fx(x_0 y_0 z_0)(x - x_0) + fy(x_0 y_0 z_0)(y - y_0) + fz(x_0 y_0 z_0) = 0.$

$-7(x-1) + 5(y+1) - 2(x-2) = 0.$

$-7x + 7 + 5y + 5 - 2x - 4 = 0.$

$-7x + 5y - 2x + 16$

$7x - 5y + 2x - 16 = 0$

Normal :

$$\frac{x - x_0}{fx(x_0 y_0 z_0)} = \frac{y - y_0}{fy(x_0 z_0 y_0)} = \frac{z - z_0}{fz(x_0 y_0 z_0)}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

iii) $f(x,y,z) = 3xyz - x - y + z + 4 = 0.$

$fx = 3x - 1 \quad fx(x_0 y_0 z_0) = 3(-1)(2) - 1 = -7$

$fy = 3x^2 - 1 \quad fy(x_0 y_0 z_0) = 3(1)(2) - 1 = 5$

$fz = 3yx + 1 \quad fz(x_0 y_0 z_0) = 3(1)(-1) + 1 = -2$

Tangent :

$fx(x_0 y_0 z_0)(x - x_0) + fy(x_0 y_0 z_0)(y - y_0) + fz(x_0 y_0 z_0).$

$-7(x-1)+5(y+1)-2(x-1)=0.$

$-7x+7+5y+5-2x+4=0.$

$-7x+5y-2z+16.$

Q5. $f(x,y) = 3x^2+y^2-3xy+6x-4y.$

$f_x = 6x-3y+6.$

$f_y = 2y-3x-4.$

$f_x = 0 \qquad f_y = 0.$

$6x-3y=-6, \quad 3x-2y=-4, \quad 6x+4y=-8.$

$\therefore y=2.$

$\therefore x=0.$

$(x,y)=(0,2).$

$r : f_x = 6.$

$r = f_{xy} = -3.$

$r = f_y = 2.$

$f-5^2 = 6(2)-(-3)^2-12-4 = 320.$

$r > 0.$

f is maximum at $(0,2)$.

$f(0,2) = 3(0)^2+(2)^2-3(0)(6)+6(0)-4(2)$

$= 4+0-0+0-8.$

$f(0,2) = -4.$

ii) $f(x,y,z) = 2x^2 + 3x^2y - y^2$

$fy = 8x^3 + 6xy$.

$fy = -2y + 3x^2$.

$fy = 0$.

$8x^3 + 6xy = 0$.

$x(8x^2 + 6y) = 0$

$(x-y) = (0,0)$ is root.

$8x^2 + 6y = 0$.

$x^2 - \frac{-2}{3} y$

$fy = 0$.

$-2y + 3(0) = 0$.

$\therefore y = 0$.

$fy = -2y - \frac{2}{3} \times 3y$.

$= -4y = 0$.

$y = 0$.

$x^6 = 0$.

$x = 0$.

$(x,y) = (0,0)$ is the only root.

$f1\,x_1 = 24x^2 + 6y = 0$.

$f1\,x_1 = 6x = 0$.

$f1\,x_1 = -2 + 0$

$= -2$

$rk - f^2 - (0)(0) - (2)^2$

$\Rightarrow rt - f^2 = -4 < 0$

$(0,0)$ is the caddle point.