PHAS1240 session 2: Defining convergence criteria

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We claimed that if we set our tolerance criterion (i.e. the quantity that the final term in our series approximation) to 10E-l, then our approximation will be correct to l decimal places. Is this a mathematically valid claim?

First we'll consider the first series we looked at—the series expansion

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n} \tag{1}$$

This series converges mathematically (otherwise it wouldn't be equal to 1/3 !). But how well does it converge?

For any convergent infinite series, we can write the actual value of the series as

$$s = \sum_{n=0}^{\infty} a_n, \tag{2}$$

where the terms of the series are represented by a_n. We can also define a *partial sum* s_k as all the terms in the series up to and including the kth term:

$$s_k = a_0 + a_1 + a_2 + \ldots + a_k \tag{3}$$

$$= \sum_{n=0}^{k} a_n \tag{4}$$

Then we can define the error E_k as the difference between s_k and the true value s:

$$E_k = s - s_k \tag{5}$$

$$= \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{k} a_n \tag{6}$$

$$= a_{k+1} + a_{k+2} + \ldots \tag{7}$$

$$= \sum_{n=k+1}^{\infty} a_n. \tag{8}$$

Our series expansion of 1/3 is not an alternating series, so we need to use the integral test to determine an upper bound on this error. This will give us

$$E_k < \int_k^{\infty} f(x)dx, \tag{9}$$

where $f(x)$ is a function representing each term in the series. For our expansion then,

$$E_k < \int_k^\infty \frac{3}{10^x} dx = -3 \left[\frac{10^{-x}}{\ln 10}\right]_k^\infty \approx 1.302 \times 10^{-k}. \tag{10}$$

We can also look at the next term in the series. If we have k terms in the series, then the next term will be

$$\text{next term} = \frac{3}{10^{k+1}} = 3 \times 10^{k+1} = 0.3 \times 10^{-k}. \tag{11}$$

Note that this is less than $|E_k|$ for any integer k.

Therefore, for this series, we can put the following quantities in order:

$$\text{next term} < |E_k| < \frac{1.302}{10^{-k}}. \tag{12}$$

For our original claim to be true, we need to be able to show that the actual error E_k in our series approximation is less than the tolerance. Thus we can say that if

$$\text{next term} < \text{tolerance}, \tag{13}$$

where tolerance $= 0.5 \times 10^{-(l+1)}$, then our calculation is correct to l decimal places.

For the approximation of $\pi/4$, bounding the error is much simpler, as

$$|E_k| \leq |a_{k+1}| \tag{14}$$

for an alternating series. However this means we can't make the same ordering of quantities as above, and so our claim is harder to justify mathematically. Instead, we can turn the problem around—how many terms k of the series are required to be certain that the error E_k is less than our defined tolerance (again $= 0.5 \times 10^{-(l+1)}$)?

$$|\frac{(-1)^{k+1}}{2k+1}| < \frac{10^{-(l+1)}}{2} \tag{15}$$

$$\frac{1}{2k+1} < \frac{10^{-(l+1)}}{2} \tag{16}$$

$$k \gtrsim 10^{l+1} \tag{17}$$

Clearly, we need lots of terms to converge to a reasonable number of decimal places. In this case we're relying more on approximations in any case, so we choose to continue to use this as our convergence criterion.

In most cases in physics, we're trying to converge on an *unknown* quantity, so in any case our convergence criterion becomes a rather more nebulous "good enough for our practical purposes"— which is what makes us physicists rather than mathematicians!