Introduction to Random Processes

Lecture 12

Spring 2002

Random Process

- A random variable is a function $X(e)$ that maps the set of experiment outcomes to the set of numbers.

- A *random process* is a rule that maps every outcome e of an experiment to a *function* $X(t, e)$.

- A random process is usually conceived of as a function of time, but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

- The function $X(u, v, e)$ would be a function whose value depended on the location (u, v) and the outcome e, and could be used in representing random variations in an image.

Random Process

- The domain of e is the set of outcomes of the experiment. We assume that a probability distribution is known for this set.

- The domain of t is a set, $\mathcal{T}$, of real numbers.

- If $\mathcal{T}$ is the real axis then $X(t, e)$ is a *continuous-time* random process

- If $\mathcal{T}$ is the set of integers then $X(t, e)$ is a *discrete-time* random process

- We will often suppress the display of the variable e and write $X(t)$ for a continuous-time RP and $X[n]$ or X_n for a discrete-time RP.

Random Process

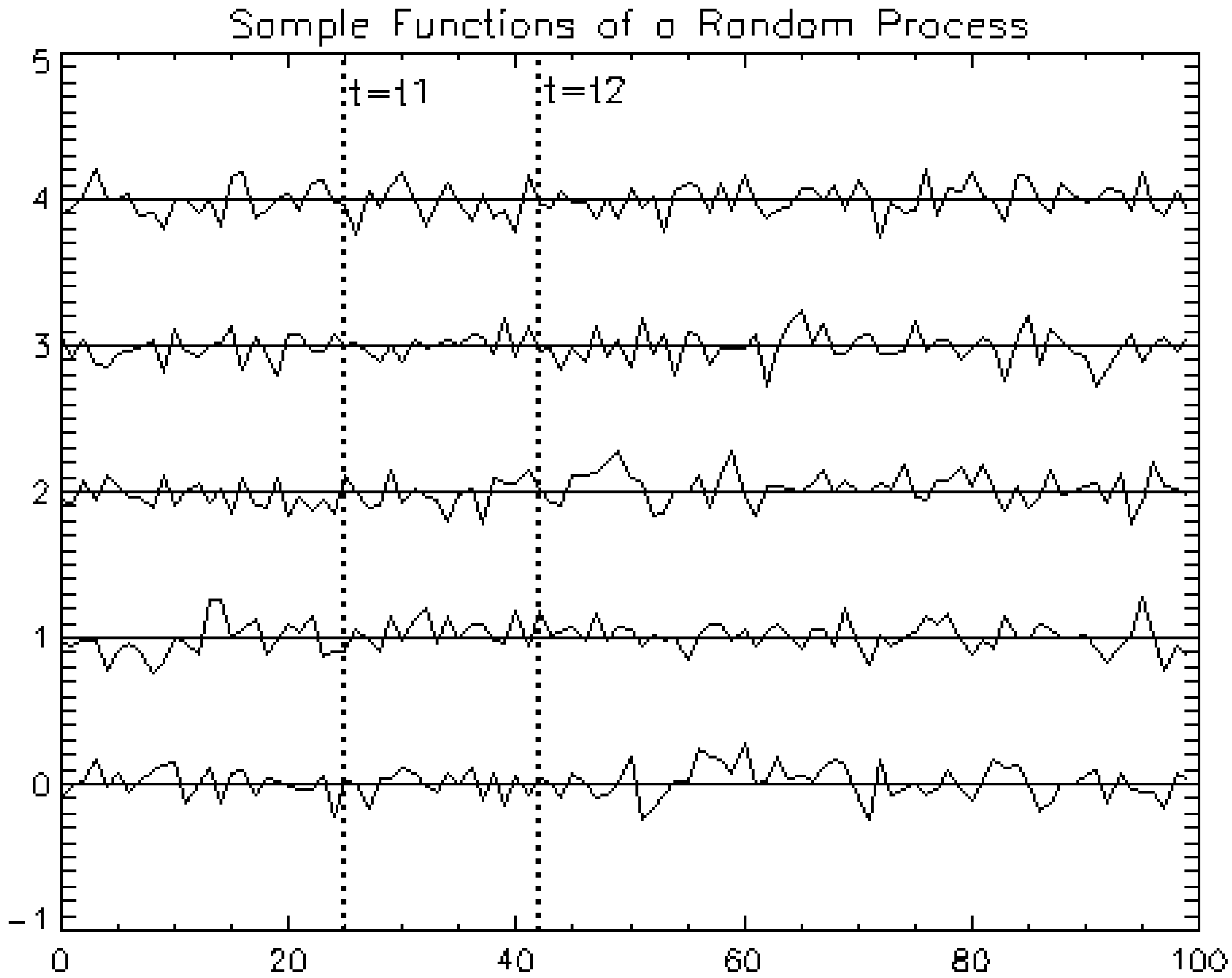
- A RP is a family of functions, $X(t, e)$. Imagine a giant strip chart recording in which each pen is identified with a different e. This family of functions is traditionally called an *ensemble*.

- A single function $X(t, e_k)$ is selected by the outcome e_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.

- If t is fixed, say $t = t_{1,}$ then $X(t_1, e)$ is a random variable. Its value depends on the outcome e.

- If both t and e are given then $X(t, e)$ is just a number.

Random Processes



Moments and Averages

$X(t_1, e)$ is a random variable that represents the set of samples across the ensemble at time t_1

If it has a probability density function $f_X(x; t_1)$ then the moments are

$$m_n(t_1) = E[X^n(t_1)] = \int_{-\infty}^{\infty} x^n f_X\,(x; t_1)\,dx$$

The notation $f_X(x; t_1)$ may be necessary because the probability density may depend upon the time the samples are taken.

The mean value is $\mu_X = m_1$, which may be a function of time.

The central moments are

$$E[(X(t_1) - \mu_X(t_1))^n] = \int_{-\infty}^{\infty} (x - \mu_X(t_1))^n\, f_X\,(x; t_1)\,dx$$

Pairs of Samples

The numbers $X(t_1, e)$ and $X(t_2, e)$ are samples from the same time function at different times.

They are a pair of random variables (X_1, X_2).

They have a joint probability density function $f(x_1, x_2; t_1, t_2)$.

From the joint density function one can compute the marginal densities, conditional probabilities and other quantities that may be of interest.

Covariance and Correlation

The covariance of the samples is

$$\begin{aligned} C(t_1, t_2) &= E[(X_1 - \mu_1)(X_2 - \mu_2)^*] \\ &= \iint_{-\infty}^{\infty} (x_1 - \mu_1)(x_2 - \mu_2)^* f(x_1, x_2; t_1, t_2) dx_1 dx_2 \end{aligned}$$

The correlation function is

$$R(t_1, t_2) = E[X_1 X_2^*] = \iint_{-\infty}^{\infty} x_1 x_2^* f(x_1, x_2; t_1, t_2) dx_1 dx_2$$

$$C(t_1, t_2) = R(t_1, t_2) - \mu_1 \mu_2$$

Note that both the covariance and correlation functions are conjugate symmetric in t_1 and t_2. $C(t_1, t_2) = C^*(t_2, t_1)$ and $R(t_1, t_2) = R^*(t_2, t_1)$

Mean-Squared Value

The "average power" in the process at time t is represented by

$$R(t,t) = E[|X(t)|^2]$$

and $C(t,t)$ represents the power in the fluctuation about the mean value.

Example: Poisson Random Process

Let $N(t_1, t_2)$ be the number of events produced by a Poisson process in the interval (t_1, t_2) when the average rate is λ events per second.

The probability that $N = n$ is

$$P[N = n] = \frac{(\lambda\tau)^n \, e^{-\lambda\tau}}{n!}$$

where $\tau = t_2 - t_1$. Then $E[N(t_1, t_2)] = \lambda\tau$.

A random process can be defined as the number of events in the interval $(0, t)$. Thus, $X(t) = N(0, t)$. The expected number of events in t is $E[X(t)] = \lambda t$.

Example-continued

For a Poisson distribution we know that the variance is

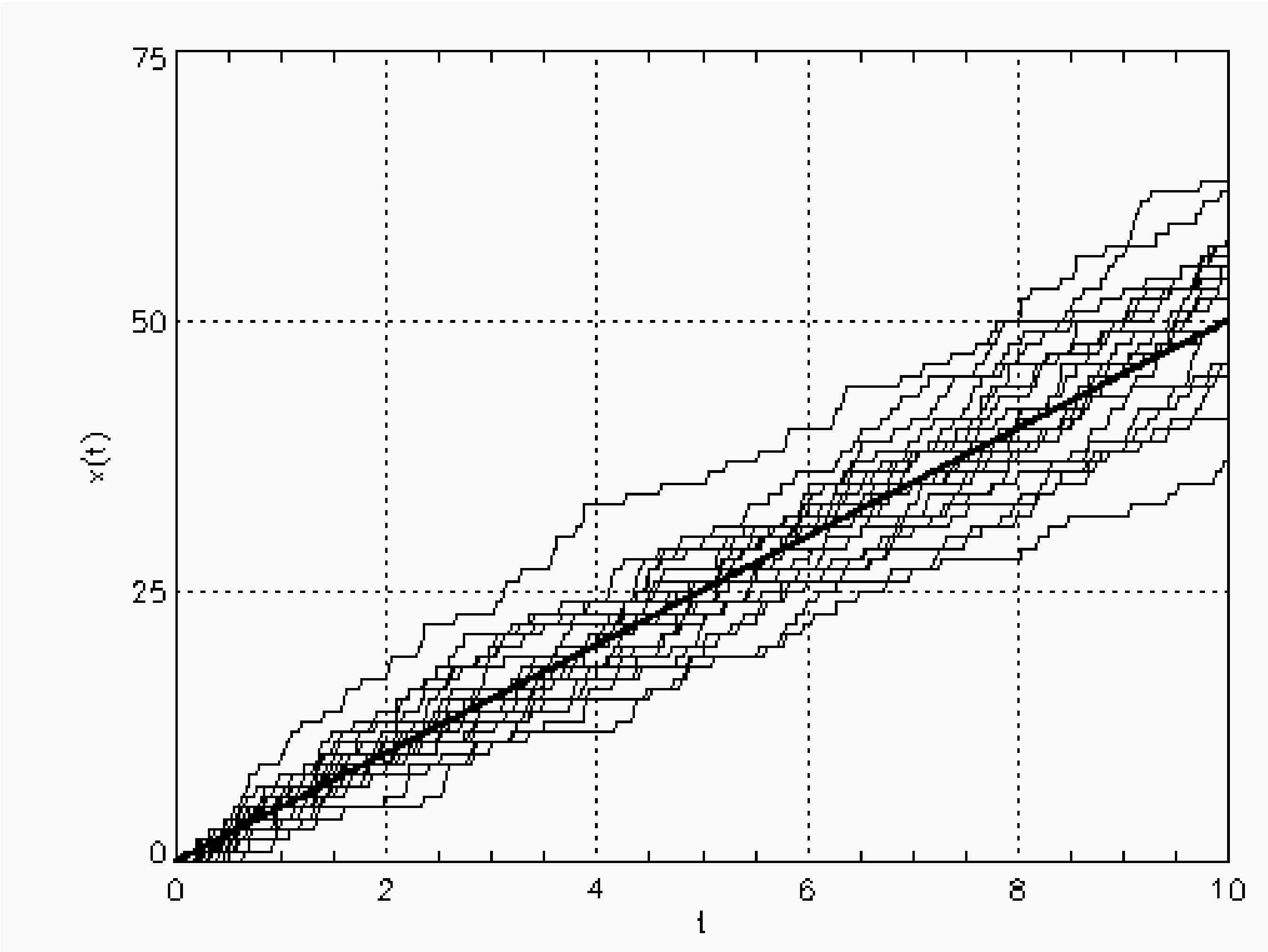
$$E[(X(t) - \lambda t)^2] = E[X^2(t)] - (\lambda t)^2 = \lambda t$$

The "average power" in the function $X(t)$ is

$$E[X^2(t)] = \lambda t + \lambda^2 t^2$$

A graph of $X(t)$ would show a function fluctuating about an average trend line with a slope λ.

Poisson Process



Program for Poisson Random Process

```
FUNCTION PoissonProcess,t,lambda,p
; S=PoissonProcess(t,lambda,p)
; divides the interval [0,t] into intervals of size
; deltaT=p/lambda where p is sufficiently small so that
; the Poisson assumptions are satisfied.
;
; The interval (0,t) is divided into n=t*lambda/p intervals
; and the number of events in the interval (0,k*deltaT) is
; returned in the array S. The maximum length of S is 10000.
;
; USAGE
; S=PoissonProcess(10,1,0.1)
; Plot,S
; FOR m=1,10 DO OPLOT,PoissonProcess(10,1,0.1)

NP=N_PARAMS()
IF NP LT 3 THEN p=0.1
n=lambda*t/p
u=RANDOMN(SEED,n,POISSON=p)
s=INTARR(n+1)
FOR k=1,n DO s[k]=s[k-1]+u[k-1]
RETURN,s
END
```

Random Telegraph Signal

Consider a random process that has the following properties:

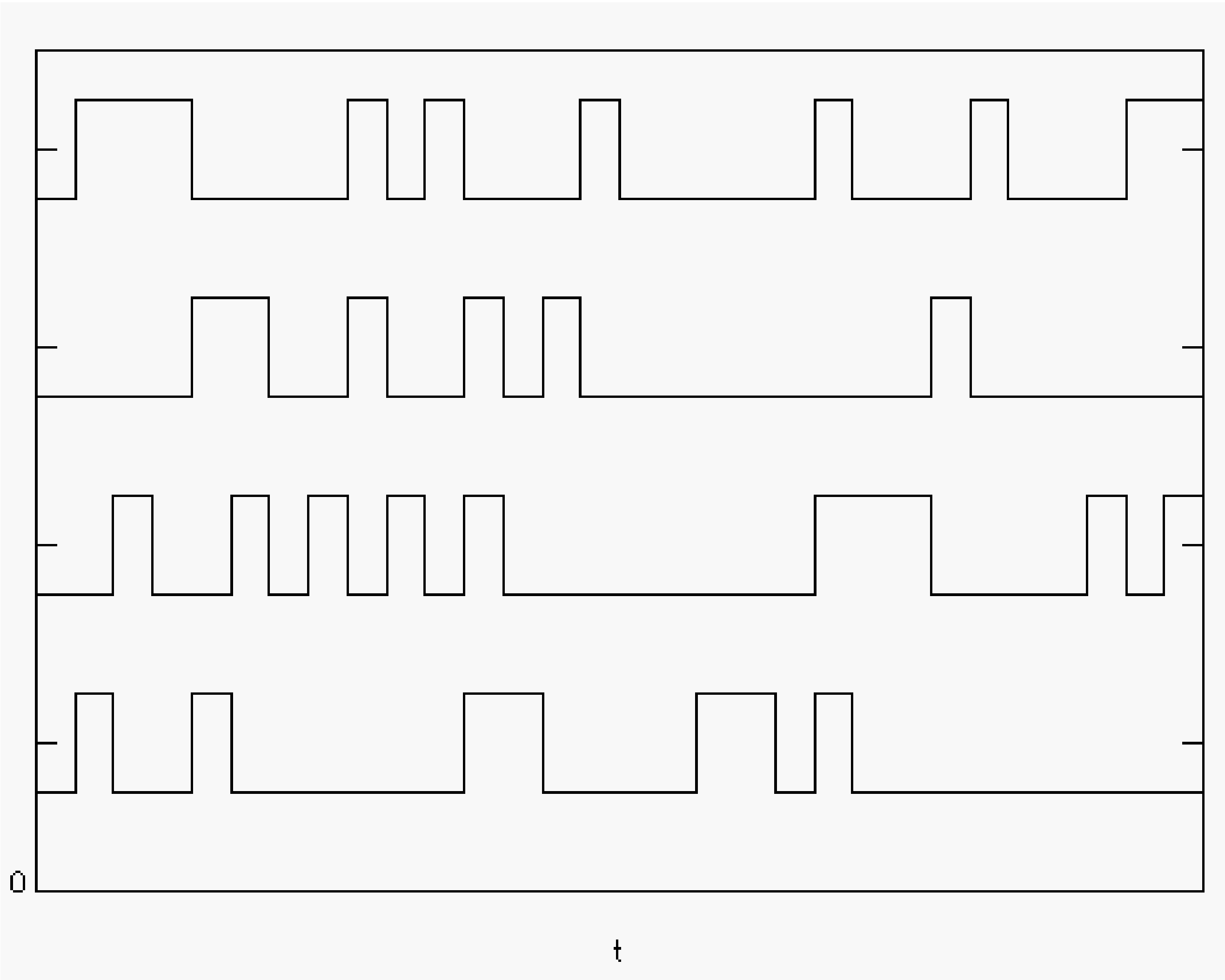
1. $X(t) = \pm 1$,

2. The number of zero crossings in the interval $(0, t)$ is described by a Poisson process

3. $X(0) = 1$. (to be removed later)

Find the expected value at time t.

Random Telegraph Signal



Random Telegraph Signal

Let $N(t)$ equal the number of zero crossings in the interval $(0, t)$. with $t \geq 0$.

$$P(N = n) = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!}$$

$$\begin{aligned}
P[X(t) = 1] &= P\,[N = \text{even number}] \\
&= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] \\
&= e^{-\lambda t} \cosh \lambda t
\end{aligned}$$

$$\begin{aligned}
P[X(t) = -1] &= P\,[N = \text{odd number}] \\
&= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots\right] \\
&= e^{-\lambda t} \sinh \lambda t
\end{aligned}$$

Random Telegraph Signal

The expected value is

$$E[X(t)|X(0) = 1] = e^{-\lambda t}\cosh \lambda t - e^{-\lambda t}\sinh \lambda t = e^{-2\lambda t}$$

Note that the expected value decays toward $x = 0$ for large t. That happens because the influence of knowing the value at $t = 0$ decays exponentially.

Random Telegraph Signal

The autocorrelation function is computed by finding $R(t_1, t_2) = E\left[X(t_1)\,X(t_2)\right]$. Let $x_0 = -1$ and $x_1 = 1$ denote the two values that X can attain. For the moment assume that $t_2 \geq t_1$. Then

$$R(t_1, t_2) = \sum_{j=0}^{1}\sum_{k=0}^{1} x_j x_k P[X(t_1) = x_k] P[X(t_2) = x_j | X(t_1) = x_k]$$

The first term in each product is given above. To find the conditional probabilities we take note of the fact that the number of sign changes in $t_2 - t_1$ is a Poisson process. Hence, in a manner that is similar to the analysis above,

$$\begin{aligned} P[X(t_2) = 1 | X(t_1) = 1] &= P[X(t_2) = -1 | X(t_1) = -1] \\ &= e^{-\lambda(t_2 - t_1)} \cosh \lambda(t_2 - t_1) \end{aligned}$$

Random Telegraph Signal

$$\begin{aligned} P[X(t_2) = -1 | X(t_1) = 1] &= P[X(t_2) = 1 | X(t_1) = -1] \\ &= e^{-\lambda(t_2 - t_1)} \sinh \lambda(t_2 - t_1) \end{aligned}$$

Hence

$$\begin{aligned} R(t_1, t_2) = \; & e^{-\lambda t_1} \cosh \lambda t_1 \left[e^{-\lambda(t_2-t_1)} \cosh \lambda(t_2 - t_1) - e^{-\lambda(t_2-t_1)} \sinh \lambda(t_2 - t_1) \right] \\ & - e^{-\lambda t_1} \sinh \lambda t_1 \left[e^{-\lambda(t_2-t_1)} \cosh \lambda(t_2 - t_1) - e^{-\lambda(t_2-t_1)} \sinh \lambda(t_2 - t_1) \right] \end{aligned}$$

After some algebra this reduces to

$$R(t_1, t_2) = e^{-\lambda(t_2 - t_1)} \quad \text{for } t_2 \geq t_1$$

A parallel analysis applies to the case $t_2 \leq t_1$, so that

$$R(t_1, t_2) = e^{-\lambda|t_2 - t_1|}$$

Random Telegraph Signal

The autocorrelation for the telegraph signal depends only upon the *time difference*, not the location of the time interval. We will see soon that this is a very important characteristic of stationary random processes.

We can now remove condition (3) on the telegraph process. Let $Y(t) = AX(t)$ where A is a random variable independent of X that takes on the values ± 1 with equal probability. Then $Y(0)$ will equal ± 1 with equal probability, and the telegraph process will no longer have the restriction of being positive at $t = 0$.

Since A and X are independent, the autocorrelation for $Y(t)$ is given by

$$E\left[Y(t_1)Y(t_2)\right] = E[A^2]E\left[X(t_1)X(t_2)\right] = e^{-\lambda|t_2 - t_1|}$$

since $E[A^2] = 1$.

Stationary Random Process

The random telegraph is one example of a process that has at least some statistics that are *independent of time.* Random processes whose statistics do not depend on time are called *stationary*.

In general, random processes can have joint statistics of any order. If the process is stationary, they are independent of time *shift.*

The first order statistics are described by the cumulative distribution function $F(x;t)$. If the process is stationary then the distribution function at times $t = t_1$ and $t = t_2$ will be identical.

If a process is strict-sense stationary then joint probability distributions of all orders are independent of time origin.

Wide-sense Stationary Processes

We often are particularly interested in processes that are stationary up to at least order $n = 2$. Such processes are called *wide-sense stationary* (wss).

If a process is wss then its mean, variance, autocorrelation function and other first and second order statistical measures are independent of time.

We have seen that a Poisson random process has mean $\mu(t) = \lambda t$, so it is not stationary in any sense.

The telegraph signal has mean $\mu = 0$, variance $\sigma^2 = 1$ and autocorrelation function $R(t_1, t_2) = R(\tau) = e^{-\lambda\tau}$ where $\tau = |t_2 - t_1|$. It is a wss process.

Correlation and Covariance

The autocorrelation function of a wss process satisfies

$$R(\tau) = E[X(t)X(t+\tau)]$$

for any value of t. Then

$$R(0) = E[X^2]$$

The covariance function is

$$C(\tau) = E[(X(t) - \mu)\,(X(t+\tau) - \mu)] = R(\tau) - \mu^2$$

$$C(0) = E[(X(t) - \mu)^2] = \sigma^2$$

Correlation and Covariance

Two random processes $X(t)$ and $Y(t)$ are called *jointly wide-sense stationary* if each is wss and their cross correlation depends only on $\tau = t_2 - t_1$. Then

$$R_{xy}(\tau) = E[X(t)Y(t+\tau)]$$

is called the cross-correlation function and

$$C_{xy}(\tau) = R_{xy}(\tau) - \mu_x\mu_y$$

is called the cross-covariance function.

Simplification with Wide-Sense Stationary

A stochastic process $x(t)$ is wss if its mean is constant

$$E[x(t)] = \mu$$

and its autocorrelation depends only on $\tau = t_1 - t_2$

$$R_{xx}(t_1, t_2) = E[x(t_1)x^*(t_2)]$$

$$E[x(t+\tau)x^*(t)] = R_{xx}(\tau)$$

Because the result is indifferent to time origin, it can be written as

$$R_{xx}(\tau) = E\left[x\left(t+\frac{\tau}{2}\right)x^*\left(t+\frac{\tau}{2}\right)\right]$$

Note that $R_{xx}(-\tau) = R_{xx}^*(\tau)$ and

$$R_{xx}(0) = E[|x(t)|^2]$$

Example

Suppose that $x(t)$ is wss with

$$R_{xx}(\tau) = Ae^{-b\tau}$$

Determine the second moment of the random variable $x(6) - x(2)$.

$$\begin{aligned} E[(x(6) - x(2))^2] &= E[x^2(6)] - 2E[x(6)x(2)] + E[x^2(2)] \\ &= R_{xx}(0) - 2R_{xx}(4) + R_{xx}(0) \\ &= 2A - Ae^{-4b} \end{aligned}$$

Determine the second moment of $x(12) - x(8)$

$$\begin{aligned} E[(x(12) - x(8))^2] &= E[x^2(12)] - 2E[x(12)x(8)] + E[x^2(8)] \\ &= R_{xx}(0) - 2R_{xx}(4) + R_{xx}(0) \\ &= 2A - Ae^{-4b} \end{aligned}$$

Ergodic Random Process

A practical problem arises when we want to calculate parameters such as mean or variance of a random process. The definition would require that we have a large number of examples of the random process and that we calculate the parameters for different values of t by averaging across the ensemble.

Often we are faced with the situation of having only one member of the ensemble–that is, one of the time functions. Under what circumstances is it appropriate to use it to draw conclusions about the whole ensemble?

A random process is *ergodic* if every member of the process carries with it the complete statistics of the whole process. Then its ensemble averages will equal appropriate time averages.

Of necessity, an ergodic process must be stationary, but not all stationary processes are ergodic.