# Baker's Map: A Chaotic World Within the Hands of a Baker

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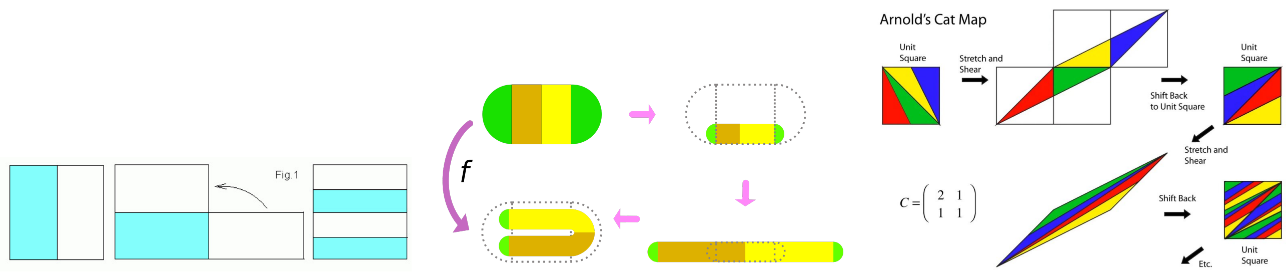
**Abstract**

The Baker's Map, inspired by the process of kneading dough, serves as a captivating example of a chaotic system. This paper delves into the mathematical formulation of the map and explores its diverse properties. We formally define the map and analyze its fixed points and stability. The study of its mixing behavior reveals its ability to thoroughly mix points within its domain and its ergodic behavior. Also, we introduce symbolic dynamics as a tool to represent the map's iterations and explore its potential for random number generation. Further analysis focuses on the map's sensitivity to initial conditions, a key characteristic of chaos, using the Lyapunov exponent. We also explore the formation of the Cantor Set, deduce that it is the map's attractor, and calculate its fractal dimension. Then, we discuss dissipation and area preservation to further highlight the chaotic nature of the map.
*Keywords:* Baker's Map, Chaotic System, Fixed Points, Stability, Mixing Behavior, Ergodicity, Symbolic Dynamics, Random Number Generation, Lyapunov Exponents, Sensitivity to Initial Conditions, Cantor Set, Attractor, Fractal Dimension, Dissipation, Area Preservation

## 1 Introduction

Have you ever watched a baker knead a dough? The stretching, folding, and compressing of the dough space creates an interesting transformation. This simple act of working the dough holds a surprising connection to chaos. The chaotic dynamical map, the Baker's map takes its name from this very process. Just like how the baker stretches and folds the dough, the map iteratively stretches and folds a square. The stretching and folding transformation exhibits fascinating properties, and the most prominent of them is that it is a chaotic system. A chaotic system display a unique behavior where small changes in the initial conditions can lead to drastically different outcomes over time. While the underlying rules of a chaotic system might be relatively simple, its long-term behavior becomes unpredicatable and appears random. In the study of dynamical systems, two maps are considered topologically conjugate if there exists a special kind of function, a homeomorphism, that makes them equivalent while preserving the overall chaotic behavior. For Baker's map us topological conjugate to the Smale Horseshoe map and Arnold Cat map. These maps involve stretching, folding, and shearing points that would lead to a chaotic scrambling effect.




(a) Baker's Map (b) Smale Horseshow Map (c) Arnold Cat Map

## 2 Mathematical Formulation

The Baker's map is a prototype for two dimensional chaotic maps. The map can be thought of as a discrete dynamical system operating on the unit square $[0,1] \times [0,1]$. It applies a set of deterministic transformation rules to the points in the domain. The transformation consists of two main operations: stretching and folding. So, the Baker's map is defined as,

$$(x_{n+1}, y_{n+1}) = \begin{cases} (cx_n,\ 2y_n) & \text{for } y_n \leqslant \frac{1}{2} \\ (cx_n + (1-c),\ 2y_n - 1) & \text{for } y_n > \frac{1}{2} \end{cases}$$

where $x_n$ and $y_n$ represent the coordinates of the point in the unit square at iteration $n$, $x_{n+1}$ and $y_{n+1}$ represent the coordinates of the point after applying the Baker's map at iteration $n+1$, and $c$ is a parameter between 0 and $\frac{1}{2}$, $0 < c \leqslant \frac{1}{2}$, that determines the folding factor.

The map's behavior depends on the $y$ coordinate of the point being transformed. If $y_n$ is less than $\frac{1}{2}$, the map stretches the$y$ coordinate by a factor of 2, effectively doubling its value while keeping it within the unit interval by taking the modulo 1. Simultaneously, it multiplies the $x$ coordinate with $c$, effectively folding it. Conversely, if the $y$ coordinate is greater than or equal to $\frac{1}{2}$, the map applies the same stretching operation to $y$ but then shifts the $x$ coordinate by 1 before multiplying it with $c$. This sequence of stretching and folding operations defines the trajectory of points in the unit square as they undergo successive iterations of the map.

Consider a set of points $M = \left\{0 \leqslant x \leqslant 1;\ 0 \leqslant y \leqslant \frac{1}{2}\right\}$,

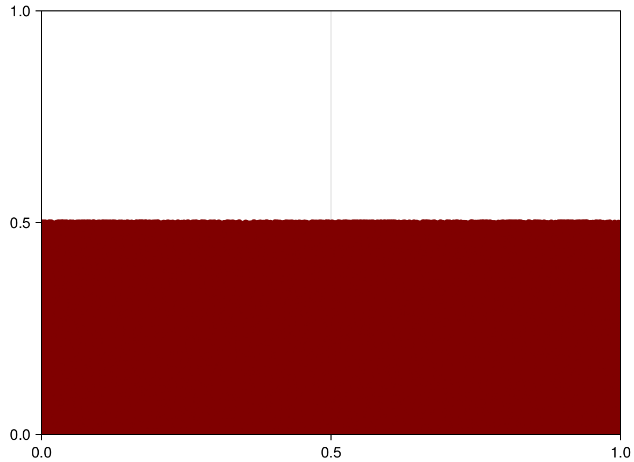
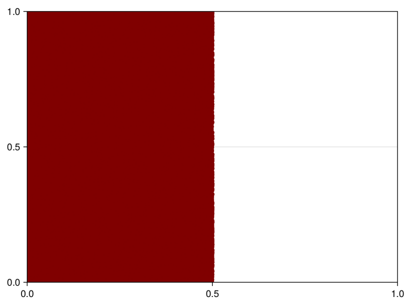
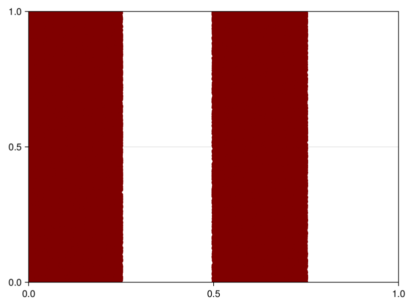



Figure 2: The Set of Points M

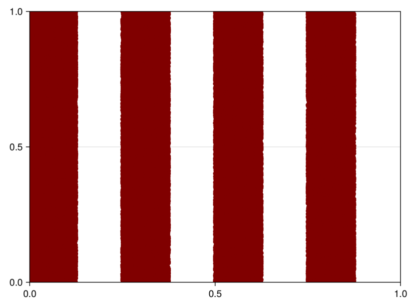
Then, we transform the points using the Baker's map



(a) The Set of Points M After One Iteration of the Baker's Map



(b) The Set of Points M After Two Iteration of the Baker's Map



(c) The Set of Points M After Three Iteration of the Baker's Map

Similarly, consider another set of points such that $B = \left\{0 \leqslant x \leqslant 1;\ \frac{1}{2} \leqslant y \leqslant 1\right\}$,

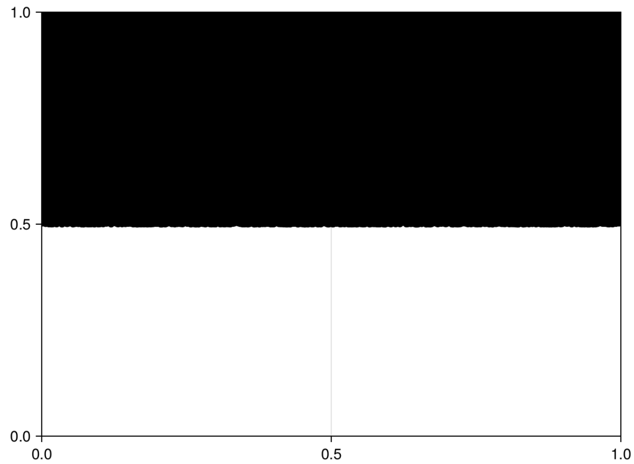
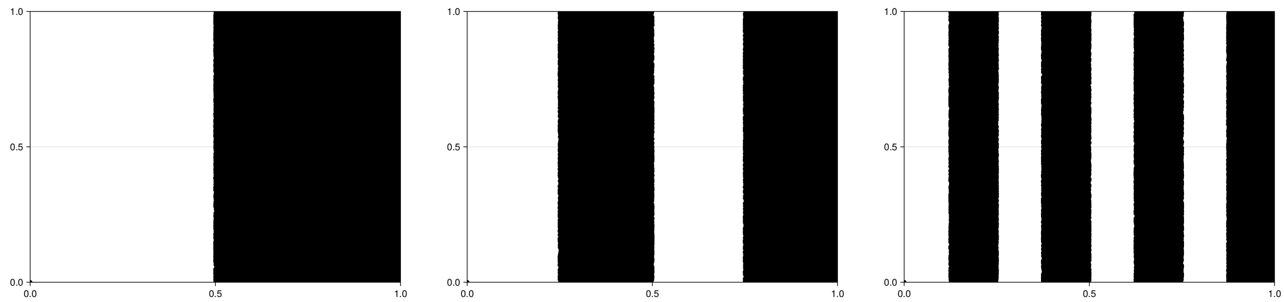



Figure 4: The Set of Points B

Then, we transform the points using the Baker's map



(a) The Set of Points B After One Iteration of the Baker's Map

(b) The Set of Points B After Two Iteration of the Baker's Map

(c) The Set of Points B After Three Iteration of the Baker's Map

We can clearly see the stretching in the $y$ coordinates, and the cutting and folding in the $x$ coordinates.

# 3 Fixed Points and Stability Analysis

Consider an iterative map $x_{n+1} = f(x_n)$. To find the fixed points of the map, we let $x_{n+1} = x_n$, which indicate that the value of $x$ remains unchanged under iteration, and we solve for $x$. Then, to check the stability, we linearize the map around the fixed points by finding the Jacobian matrix and evaluating it at the fixed points. Then, we calculate the eigenvalues of the Jacobian matrix. If the eigenvalues are both real and $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then the map is expanding. Orbits are exponentially diverging away from the fixed point with distance along directions determined by the eigenvectors. This type of fixed point is also called a node repeller. If both eigenvalues are real and absolute value less than 1. The fixed point is called a node attractor. Orbits are exponentially decaying toward the fixed point. If both eigenvalues are real but one has absolute value greater than 1 and the other has absolute value less than 1, then the point is a saddle point. There is one eigenvector along which the orbit decays to the fixed point. Along the other direction the trajectory moves outwards. If both eigenvalues are complex then the fixed point is said to be elliptic. Orbits circulate about the fixed point. If the fixed points have complex parts but their real part has absolute value less than 1 then the node is a spiral attractor. If the fixed points have complex parts but their real part has absolute value greater than 1 then the node is a spiral repeller.
Applying the stability analysis to the Baker's map, for $y_n \leqslant \frac{1}{2}$,

$$x_n = cx_n$$
$$y_n = 2y_n$$

Then, $(0, 0)$ is a fixed point of the map when $y_n \leqslant \frac{1}{2}$. The Jacobian of the map is,

$$J(x_n, y_n) = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix}$$

The eigenvalues are,

$$|J(0,0) - \lambda I| = \begin{vmatrix} c-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (c-\lambda)(2-\lambda) \Rightarrow \lambda_1 = c \text{ and } \lambda_2 = 2$$

The absolute value of one eigenvalues is less than one, and the other is greater than one, $|\lambda_1| = |c| < 1$ and $|\lambda_2| = 2 > 1$. Therefore, the fixed point $(0,0)$ is a saddle point.
For $y_n > \frac{1}{2}$,

$$\begin{aligned} x_n &= cx_n + (1-c) \\ x_n - cx_n &= (1-c) \\ x_n(1-c) &= (1-c) \\ x_n &= 1 \end{aligned}$$

$$\begin{aligned} y_n &= 2y_n - 1 \\ y_n - 2y_n &= -1 \\ -y_n &= -1 \\ y_n &= 1 \end{aligned}$$

Therefore, $(1,1)$ is a fixed point of the map when $y_n > \frac{1}{2}$. The Jacobian of the map is,

$$J(x_n, y_n) = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix}$$

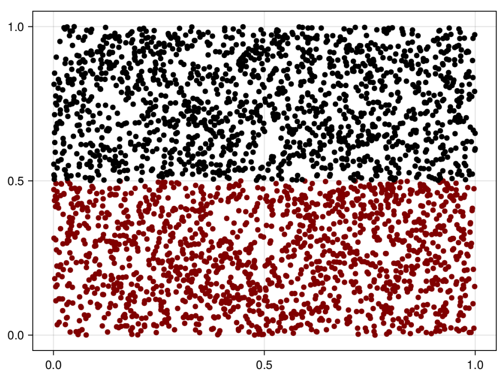
The eigenvalues are,

$$|J(1,1) - \lambda I| = \begin{vmatrix} c-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (c-\lambda)(2-\lambda) \Rightarrow \lambda_1 = c \text{ and } \lambda_2 = 2$$

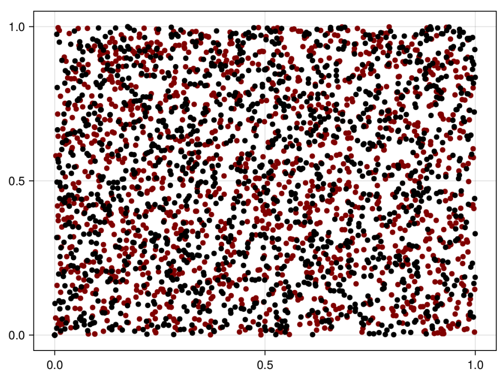
The absolute value of one eigenvalues is less than one, and the other is greater than one, $|\lambda_1| = |c| < 1$ and $|\lambda_2| = 2 > 1$. Therefore, the fixed point $(1,1)$ is a saddle point.

# 4 Mixing Behavior and Ergocity

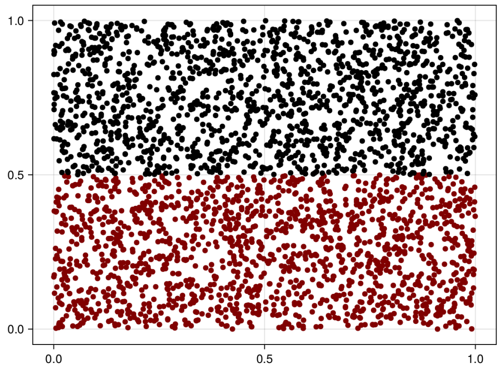
In the Baker's map, the stretching and folding operations cause the points in the square to spread out and mix together in a chaotic manner. Mixing refers to the property where nearby points in the phase space become arbitrarily separated after a finite number of iterations. Imagine starting with a cluster of points close to each other in the square. Through iterations, these points will undergo stretching and folding, causing them to disperse and become entangled with each other. Eventually, they become so intermingled that it becomes challenging to infer their original positions and predict the future behavior of the individual points in the phase space. This phenomenon is known as chaotic mixing.



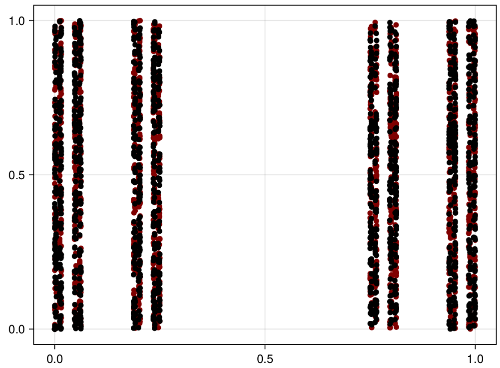

(a) The Set of Points Before the Stretching and Folding for $c = \frac{1}{2}$



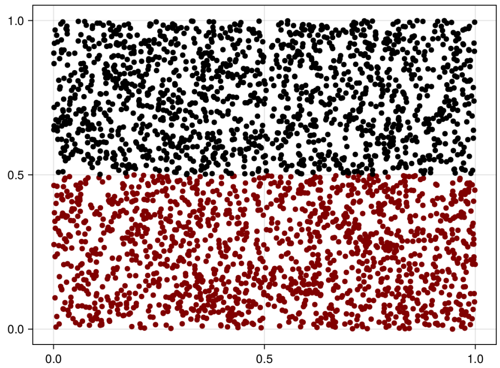

(b) The Set of Points After the Stretching and Folding for $c = \frac{1}{2}$



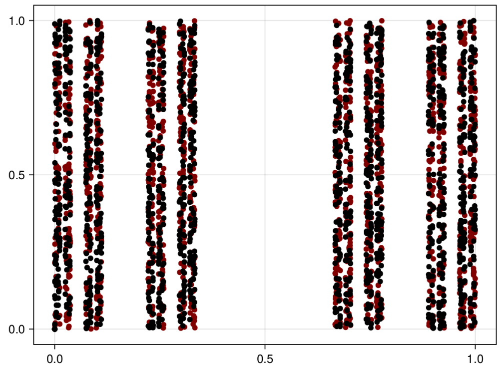

(a) The Set of Points Before the Stretching and Folding for $c = \frac{1}{4}$



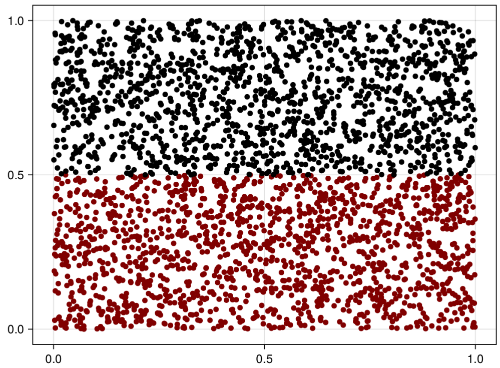

(b) The Set of Points After the Stretching and Folding for $c = \frac{1}{4}$



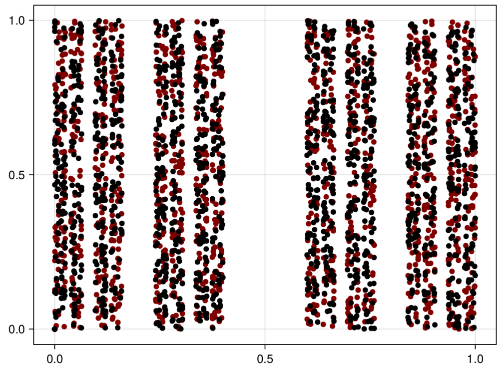

(a) The Set of Points Before the Stretching and Folding for $c = \frac{1}{3}$




(b) The Set of Points After the Stretching and Folding for $c = \frac{1}{3}$




(a) The Set of Points Before the Stretching and Folding for $c = 0.4$




(b) The Set of Points After the Stretching and Folding for $c = 0.4$

An ergodic system is one in which, over a sufficiently long period, the system explores and visits all possible states or configurations available to it. In simpler terms, an ergodic system thoroughly explores its entire phase space. In an ergodic system, trajectories originating from different initial conditions eventually visit every accessible point in the phase space. This property ensures that the system fully explores its entire configuration space, exhibiting a sort of uniformity in its exploration. In many chaotic systems such as the Baker's map, the process of mixing often leads to ergodic behavior. As nearby points become increasingly intertwined through chaotic mixing, trajectories originating from various initial conditions tend to explore the phase space uniformly. This convergence towards uniform exploration is a hallmark of ergodicity in chaotic systems. The chaotic stretching and folding operations in the Baker's map induce mixing, which make the system exhibit an ergodic.

## 5 Symbolic Dynamics

Symbolic dynamics translates the dynamics of a dynamical system down to a topological sequence space that represents the trajectories of those dynamics systems. We define $\Sigma$ as the set of infinite symbolic sequences. A symbolic sequence is an infinite sequence of symbols. We keep track of the iterates by recording where the point ended up in the space. The point can be written as $a = a_1a_2a_3...a_n$. A mapping from the space back to itself, and when it is applied to one of these sequences, it takes off the first element of the sequence and move everything down.

$$\sigma : \Sigma \to \Sigma \text{ where } \sigma(a_1a_2a_3...a_n) = a_2a_3a_4...a_n$$

A natural distance between the sequences can be defined as follows,

$$d(a,b) = \sum_{n\geqslant 1} \frac{|a_n - b_n|}{2^n}$$

a and b are going to be close if and only if the beginning part of their sequence is the same. The more elements at the beginning of their sequence that are the same, the closer those things will be. The further into the sequences we are, the less we care about how close those two things are. This distance puts more preference at the beginning of your sequence than as we go through it. If two points are close together with respect to our metric d then they remain close together after they are shifted by the shift map. But if the shift map is reapplied many times then we eventually expect the orbits to differ.

## 6 Random Number Generator

Using symbolic dynamics, we constructed a number generator. We divided the domain into two parts. If the point landed in the range $0 < x_n \leqslant \frac{1}{2}$, then we record that instant with the 0, while if it landed in the range $\frac{1}{2} < x \leqslant 1$, then we record it with 1. So, after $n$ iterations, each trajectory has a sequence made of random 0s and 1s, a binary number. If we convert the random sequence of the trajectory from a binary number to a decimal number, using the following expression, we have a random number.

$$N = \sum_{n\geqslant 1} \frac{a_n}{2^n}$$

where $a = a_1a_2a_3...a_n$. Using numerical analysis, one trajectory had the sequence 0100011011000101, so the random number is 18117, while another trajectory has the sequence 0011111111111111 and gave us the number 16383.

## 7 Lyapunov Exponent and Sensitivity to Initial Conditions

Sensitive dependence on initial conditions means that two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different future and end points. The practical implication is that prediction becomes impossible on the long-term in system who highly depend on initial conditions. A chaotic system should show sensitive dependence on initial conditions, in the sense that neighboring trajectories separate exponentially fast on average.

Given some initial condition $x_0$, consider a nearby point $x_0 + \delta_0$, where the initial separation $\delta_0$ is extremely small. Let $\delta_n$ be the separation after $n$ iterates. If $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then $\lambda$ is called the Lyapunov exponent. A positive Lyapunov exponent is a signature of chaos.

Noting that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$,

$$\begin{aligned}
|\delta_n| &\approx |\delta_0|e^{n\lambda} \\
e^{n\lambda} &= \left|\frac{\delta_n}{\delta_0}\right| \\
ln\left(e^{n\lambda}\right) &= ln\left|\frac{\delta_n}{\delta_0}\right| \\
n\lambda &= ln\left|\frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0}\right| \\
\lambda &= \frac{1}{n}ln\left|\frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0}\right| \\
\lambda &= \frac{1}{n}ln\left|(f^n)'(x_0)\right|
\end{aligned}$$

where we've taken the limit $\delta_0 \to 0$ in the last step. The term inside of the logarithm can be expanded by the chain rule,

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

Therefore,

$$\lambda \approx \frac{1}{n} ln \left| \prod_{i=0}^{n-1} f'(x_i) \right|$$

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} ln \left| f'(x_i) \right|$$

If this expression has a limit as $n \to \infty$, we define that limit to be the Lyapunov exponent for the orbit starting at $x_0$,

$$\lambda = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} ln \left| f'(x_i) \right| \right\}$$

Note that $\lambda$ depends on $x_0$. However, it is the same for all $x_0$ in the basin of attraction of a given attractor. For stable fixed points and cycles, $\lambda$ is negative, while for for chaotic attractors, $\lambda$ is positive.
Since the Baker's map is a two dimensional map, instead of the derivative, we use the Jacobian. So, we defined a function numerically that would compute the average Lyapunov exponent for the Baker's map. The exponent for the map is $\lambda = 0.6931471805600546$. It is positive, meaning that the Baker's map has a chaotic attractor.

# 8 The Cantor Set

We start with the closed interval $S_0 = [0, 1]$ and remove its open middle third by deleting the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ and leave the endpoints behind. This produces the pair of closed intervals. Then, we remove the open middle thirds of those two intervals to produce $S_2$, and so on. The limiting set $C = S_\infty$ is called the Cantor set. The set has structure at arbitrarily small scales. If we enlarge part of $C$ repeatedly, we continue to see a complex pattern of points separated by gaps of various sizes. This structure is never-ending. The Cantor set is also self-similar. It contains smaller copies of itself at all scales. For example, if we the left part of $C$ and enlarge it by a factor of three, we get $C$ back. Similarly, the parts of $C$ in each of the four intervals of $S_2$ are geometrically similar to C, except scaled down by a factor of nine. Also,the dimension of $C$ is not an integer. $C$ has measure zero and it consists of uncountably many points.

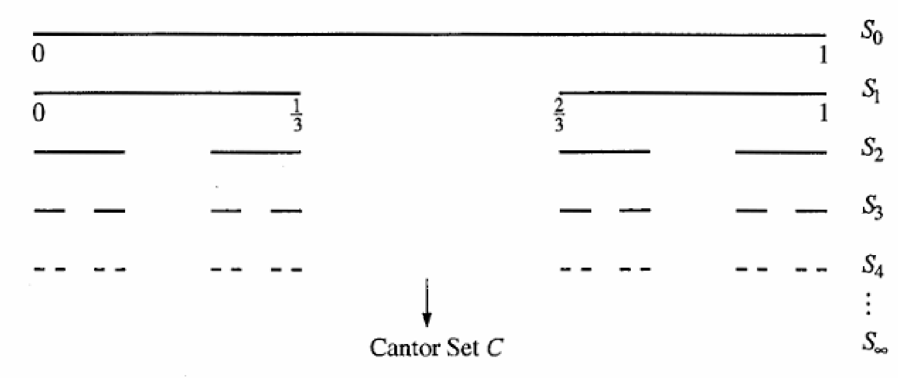
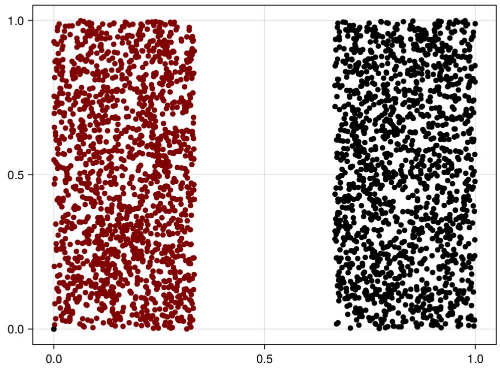


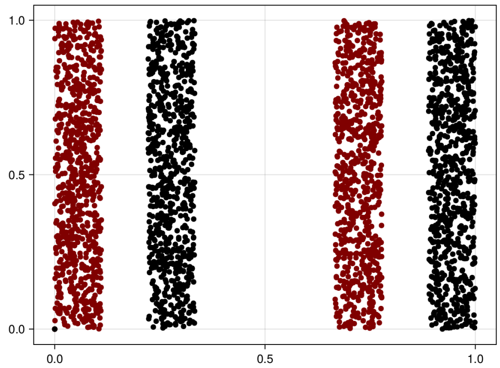

Figure 10: The Cantor Set

# 9 The Attractor of the Baker's Map

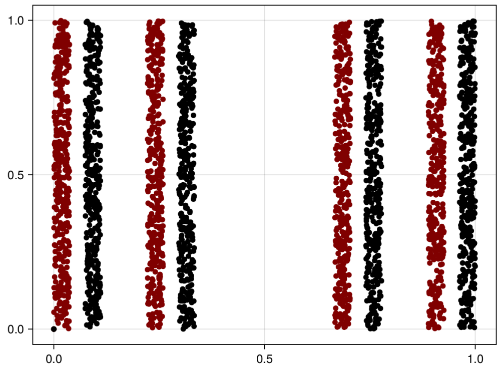
Let $S$ denote the square $[0, 1] \times [0, 1]$, which includes all possible initial conditions. The first four images of S under the Baker's map B are shown below,



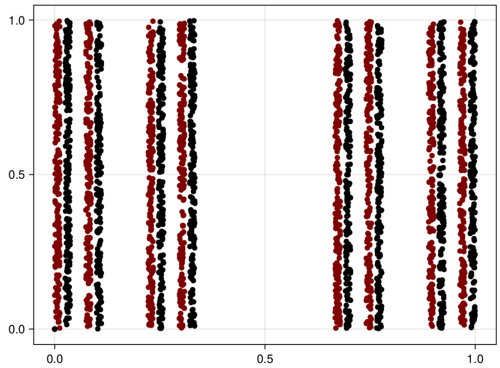

(a) $B^1(S)$




(b) $B^2(S)$

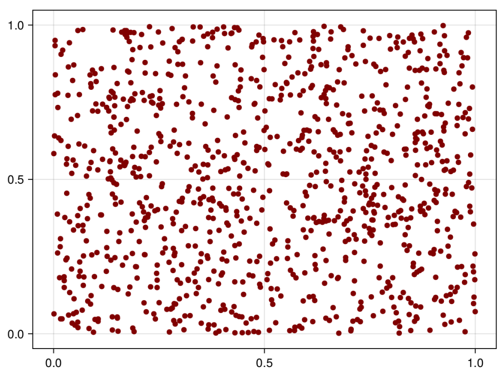



(a) $B^3(S)$

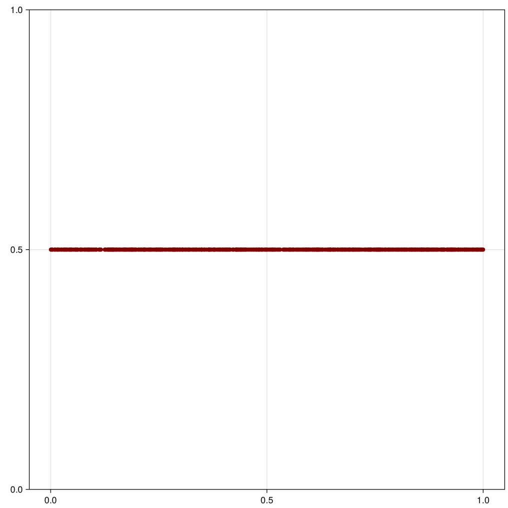



(b) $B^4(S)$

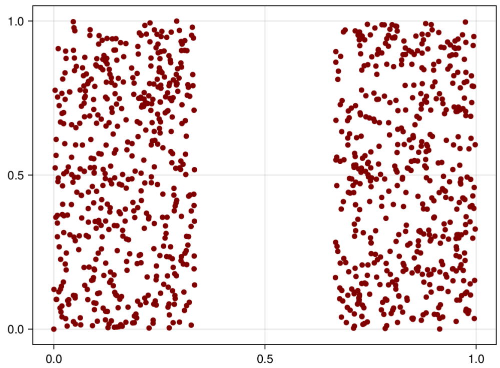
The first image $B(S)$ consists of two strips of width c. The $B(S)$ is flattened, stretched, cut, and stacked to get $B^2(S)$. Now, we have four strips of width $c^2$. Then, $B^3(S)$ would have eight strips of width $c^3$, $B^4(S)$ would have sixteen strips of width $c^4$, and so on. We see that $B^n(S)$ consists of $2^n$ vertical strips of width $c^n$. The limiting set $A = B^\infty(S)$ is fractal. Successive images of the square are nested inside of each other, $B^{n+1}(S) \subset B^n(S)$ for all $n$, so $A \subset B^n(S)$ for all n. The nesting property helps us to show that $A$ attracts all orbits. The point $B^n(x_0, y_0)$ lies somewhere in one of the strops of $B^n(S)$, and all points in these strips are within a distance $c^n$ of $A$, because $A$ is contained in $B^n(S)$. Since $c^n$ goes to zero as $n$ goes to infinity, the distance from $B^n(x_0, y_0)$ to $A$ tends to zero as $n \to \infty$.
If we take a horizontal cross-section of the Baker's map, we get the following,



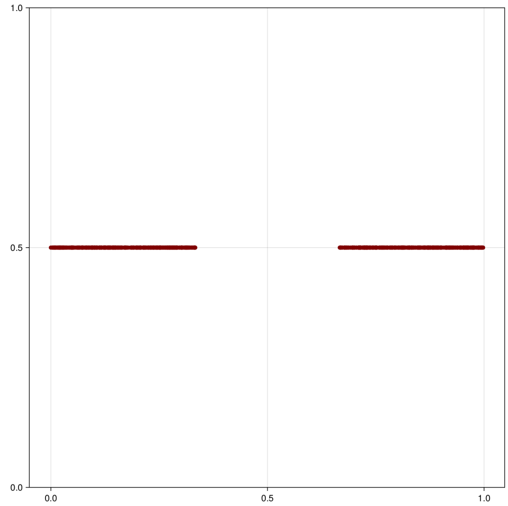

(a) $B^0(S)$



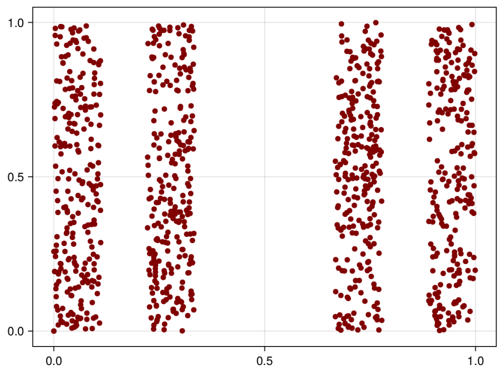

(b) A Horizontal Cross-section of $B^0(S)$



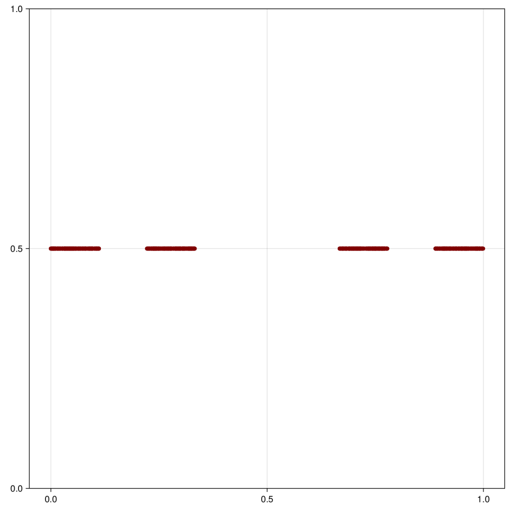

(a) $B^1(S)$



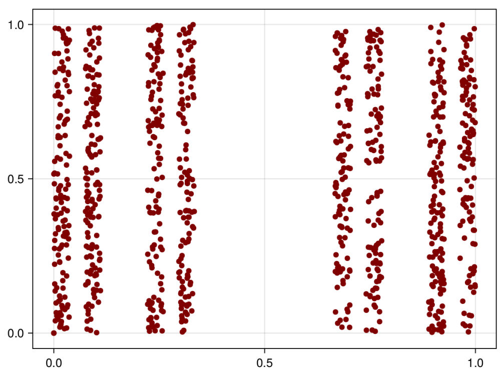

(b) A Horizontal Cross-section of $B^1(S)$



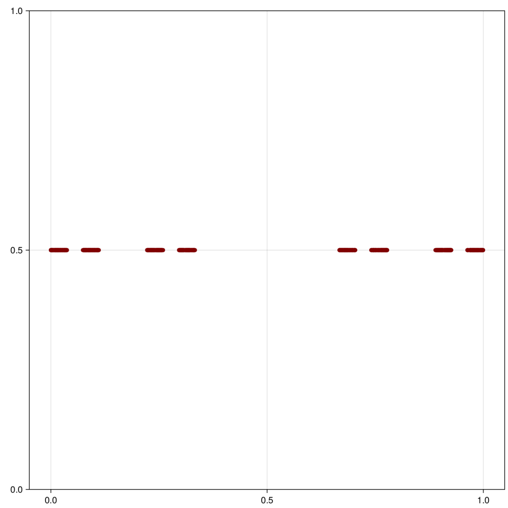

(a) $B^2(S)$



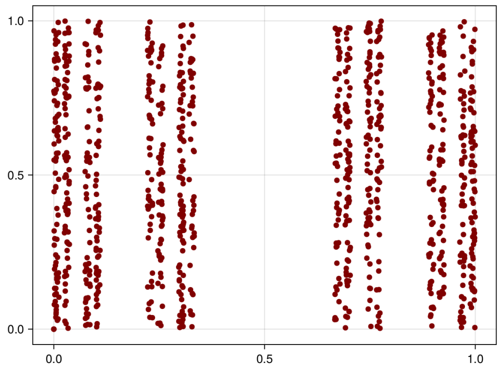

(b) A Horizontal Cross-section of $B^2(S)$



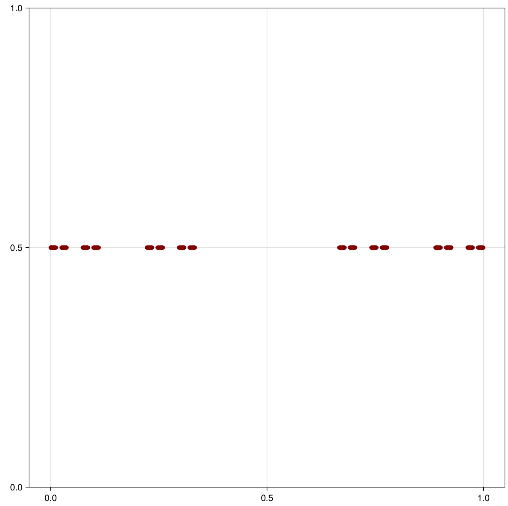

(a) $B^3(S)$



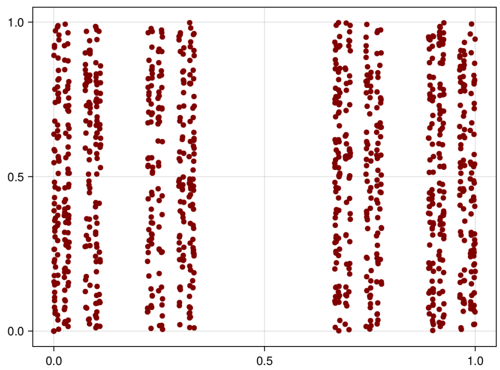

(b) A Horizontal Cross-section of $B^3(S)$



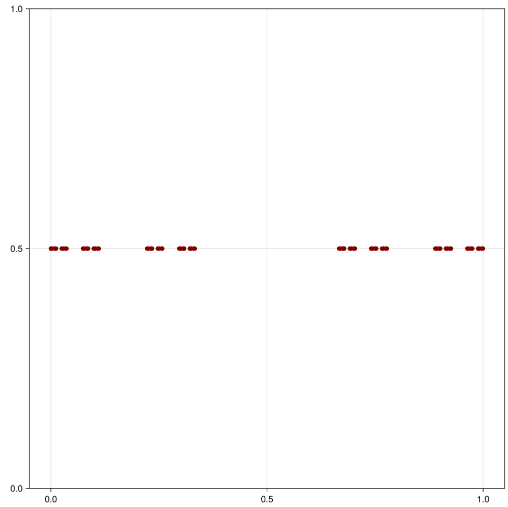

(a) $B^4(S)$




(b) A Horizontal Cross-section of $B^5(S)$




(a) $B^4(S)$




(b) A Horizontal Cross-section of $B^4(S)$

The cross-section looks like a Cantor set. So topologically, the limiting set or the attractor of the Baker's map is a Cantor set of line segments.

# 10 Fractal Dimensions

The attractor $A$ is approximated by $B^n(S)$, which consists of $2^n$ strips of width $c^n$ and length 1. Now cover $A$ with square boxes of side $\varepsilon = c^n$. Since the strips have length 1, it takes $\frac{1}{\varepsilon} = \frac{1}{c^n} = c^{-n}$ boxes to cover each of them. There are $2^n$ strips altogether, so the covering of the whole domain requires the following number of boxes, $N \approx c^{-n} \times 2^n = \left(\frac{a}{2}\right)^{-n}$. So, the box-counting dimensions of the Baker's map is,

$$d_{box} = \lim_{\varepsilon \to 0} \frac{lnN}{ln\left(\frac{1}{\varepsilon}\right)} = \lim_{n \to \infty} \frac{ln\left[\left(\frac{c}{2}\right)^{-n}\right]}{ln\left(c^{-n}\right)} = 1 - \frac{ln2}{lnc}$$

As $c \to \frac{1}{2}$, $d \to 2$. This makes sense because the attractor fills increasingly large portion of square $S$ as $c$ approaches $\frac{1}{2}$.

# 11 Dissipation and Preservation of Area of the Baker's Map

For $c < \frac{1}{2}$, the baker's map shrinks areas in phase space. Given any region $R$ in the square,

$$area(B(R)) < area(R)$$

The baker's map elongates $R$ by a factor of 2 and flattens it by a factor of $c$, so $area(B(R)) = 2c \times area(R)$. Since $c < \frac{1}{2}$, $area(B(R)) < area(R)$. So, since the area is shrinking, the attractor $A$ for the Baker's map must have zero area. Also, it cannot have any repelling fixed points, since such points would expand area elements in their neighbourhood. This is a dissipative system because the map contrasts area in phase space. These types of systems commonly arise as models of physical situations involving friction, viscosity, or some other process that dissipate energy.
In contracts, when $c = \frac{1}{2}$, the Baker's map is area-preserving,

$$area(B(R)) = area(R)$$

The square $S$ is mapped onto itself, with no gaps between the strips. The map has qualitatively different dynamics in this case. The orbits shuffle around endlessly in the square but never settle down to a lower-dimensional attractor. Area-preserving maps cannot have attractors since an attractor should attract all orbits starting in a sufficiently small open set containing it, this requirement is incompatible with area-preservation. The area-preserving maps are associated with conservative systems, particularly with the Hamiltonian systems if classical mechanics.

## 12 Conclusion

We delved into its mathematical formulation of the Baker's map, analyzed its fixed points and stability, and investigated its mixing behavior and ergodicity. We explored the concept of symbolic dynamics and its application to the Baker's Map, and using it as a random numbers generator. Then, we tested the map's sensitivity to initial conditions by calculating the Lyapunov exponent of the system. Furthermore, we investigated the Cantor Set, the map's attractor, and its fractal dimension. We discussed dissipation and area preservation to shed light on the map's long-term behavior under different folding factors.

## Bibliography

Ott, E. (2002). Chaos in dynamical systems. Cambridge University Press.
Strogatz, S. (1994). Nonlinear Dynamics and Chaos: With applications to physics, biology, chemistry, and engineering.