Birds on a Wire

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The model considers birds landing randomly on a wire. These birds can be described as sociable but skittish. So, they prefer not to be crowded and will maintain a minimum distance between themselves. Also, if a new bird lands within this minimum distance, that we call an exclusion zone, of an existing bird, the existing bird will fly away. The goal is to analyze the system in a steady state, where the average arrival rate of birds equals the average departure rate.

1 The Void Density

Let V_k be the probability of finding a void of length k. The outflux represents the rate at which birds leave a void of length k to form a void of length $k-1$. Similarly, the influx represents the rate at which birds enter a void of length k from a void of length $k+1$. So, we need to find expression for the outflux and influx to find the probability.

$$\dot{V}_k = -kV_k - 2V_k + 2V_{k-1} + 2\Sigma_{j \geqslant k+1} V_j$$

where $\dot{V}_k$ denotes the rate of change of V_k with respect to time. The term $-kV_k$ represents the outflux of birds from voids of length k due to individual birds leaving the void. The term $-2V_k$ represents the outflux of birds from voids of length k due to neighboring birds leaving the void. The term $2V_{k-1}$ represents the influx of birds into voids of length k from voids of length $k-1$. The term $2\sum_{j \geqslant k+1} V_j$ represents the influx of birds into voids of length k from longer void lengths $j \geqslant k+1$.

The summation of V_k is the density ρ.

$$\sum_{k \geqslant 1} V_k = \rho$$
$$\text{and } \sum_{k \geqslant 1} (k+1) V_k = 1$$

Therefore, the probability of finding a void of length k becomes,

$$\begin{aligned}
\dot{V}_k &= -kV_k - 2V_k + 2V_{k-1} + 2 \sum_{j > k+1} V_j \\
&= -kV_k - 2V_k + 2V_{k-1} - 2V_k - 2V_{k-1} + 2V_k + 2V_{k-1} + 2 \sum_{j > k+1} V_j \\
&= -kV_k - 4V_k + 2 \sum_{j \geqslant k-1} V_j \\
\Rightarrow \dot{V}_k &= -(k+4)V_k + 2 \sum_{j \geqslant k-1} V_j
\end{aligned}$$

Now, let's express the above using the density ρ,

$$\sum_{k \geqslant 1} \dot{V}_k = \dot{\rho}$$

$$\begin{aligned}
-\sum_{k \geqslant 1} kV_k &= -\sum_{k \geqslant 1} kV_k - V_k + V_k \\
` &= -\sum_{k \geqslant 1} (k+1)V_k + \sum_{k \geqslant 1} V_k \\
&= -1 + \rho \\
-\sum_{k \geqslant 1} 2V_k &= -2\rho
\end{aligned}$$

$$\sum_{k\geqslant 1} 2V_{k-1} = 2V_0 + 2\sum_{l\geqslant 2} V_{k-1} = 2\sum_{k\geqslant 2} V_{k-1} = 2\sum_{k'\geqslant 1} V_{k'} = 2\rho$$

$$\begin{aligned}
2\sum_{k\geqslant 1}\sum_{j\geqslant k+1} V_j &= \sum_{j\geqslant 2} V_j + \sum_{j\geqslant 3} V_j + \sum_{j\geqslant 4} V_j + \sum_{j\geqslant 5} V_j + ... \\
&= -V_1 + \rho - V_1 - V_2 + \rho - V_1 - V_2 - V_3 + \rho - ... \\
&= -\sum_{j\geqslant 1}(N-j)V_j + (N-1)\rho \\
&= -\rho + 1 - \rho \\
&= 1 - 2\rho
\end{aligned}$$

Therefore, the change of density as a function of time is,

$$\begin{aligned}
\dot{\rho} &= -1 + \rho - 2\rho + 2\rho + 2 - 4\rho \\
&= 1 - 3\rho
\end{aligned}$$

$$\Rightarrow \dot{\rho} = 1 - 3\rho \Rightarrow \rho(t) = \frac{1}{3}(1 - e^{-3t})$$

So, at $t \to \infty$, $\rho \to \frac{1}{3}$.

2 Simulation

Using Python, we ran a simulation of our model to check that at steady state the density converges t0 $\frac{1}{3}$ and that the void density distribution is a power law. We need a lattice and birds. The lattice is represented as an array initialized with zeros, indicating empty sites. The simulation proceeds through a series of iterations, during each of which a random lattice site is selected to simulate the movement of a bird. They follow the rules of the game. If neighboring sites are occupied by birds, the central site becomes empty, simulating bird movement. The simulation also considers boundary conditions, ensuring particles behave appropriately when reaching the edges of the lattice. Then, after the interactions, we calculated the steady-state density of birds. Finally, we calculated the distribution of void sizes, gaps between birds, and visualized them using a histogram. We found out that the steady state density is equal to 0.32, and the distribution of void sizes is a power law.

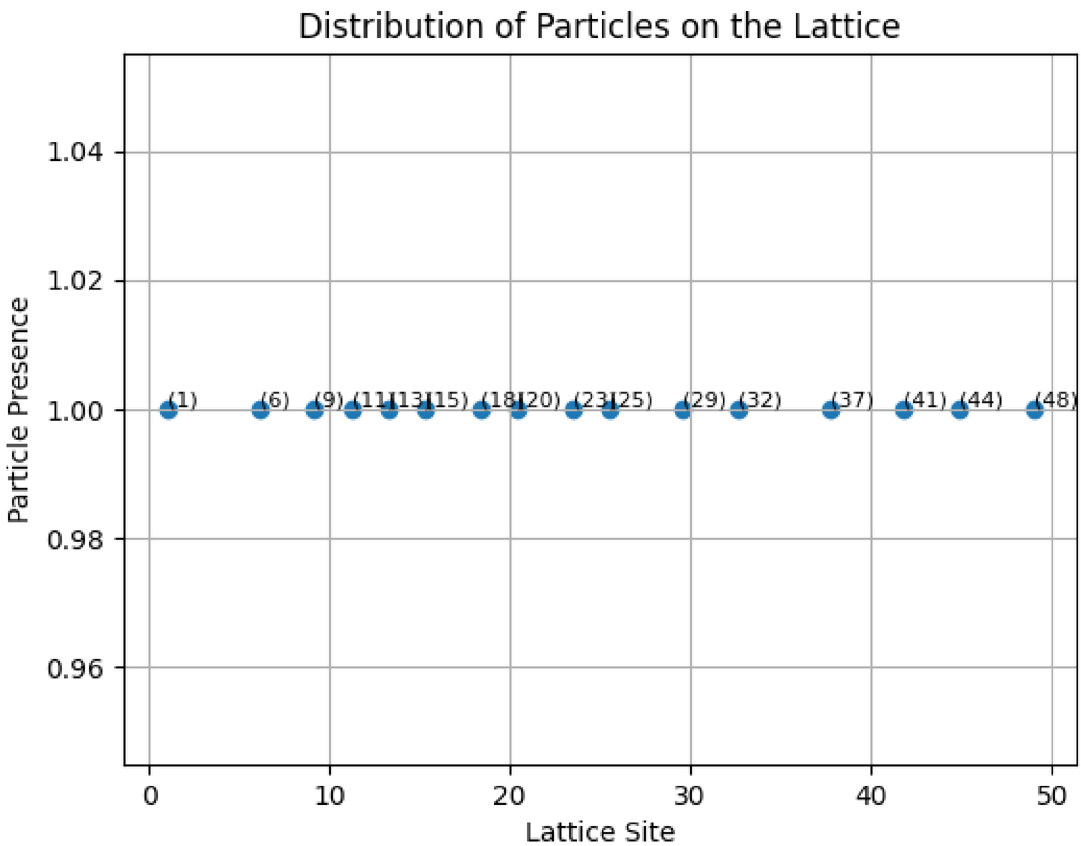

Figure 1: Distribution of birds on the lattice

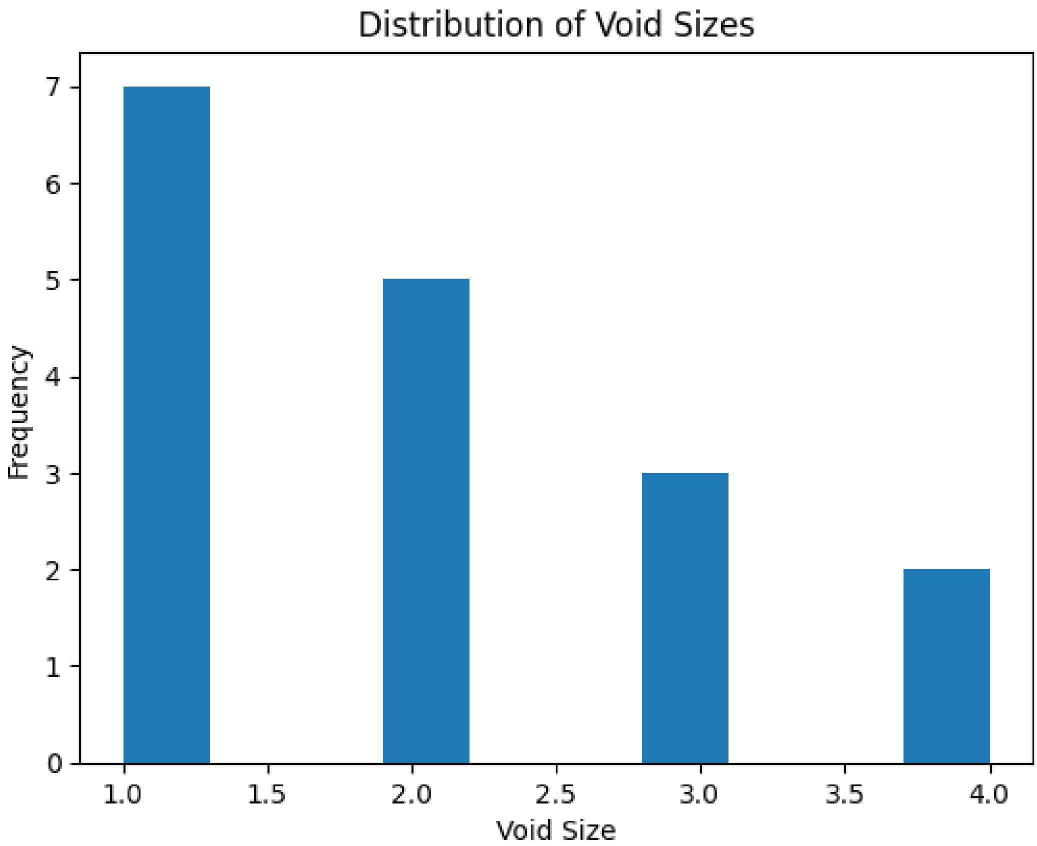

Figure 2: Distribution of void sizes between birds

Appendix: Code

```
N = 50
line = np.zeros(N)
tracker = 0
for i in range(10000):
    n = np.random.randint(N)
    if n != 0 and n != N-1 and line[n + 1] == 1 and line[n - 1] == 1:
        line[n], line[n + 1], line[n - 1] = 1, 0, 0
        tracker = tracker - 1 - 1 + 1

    elif n == 0 and line[n + 1] == 1:
        line[n], line[n + 1] = 1, 0
        tracker = tracker - 1 + 1

    elif n == N-1 and line[n - 1] == 1:
        line[n], line[n - 1] = 1, 0
        tracker = tracker - 1 + 1

    elif n != 0 and n!= N-1 and line[n + 1] == 1 and line[n - 1] == 0:
        line[n], line[n + 1] = 1, 0
        tracker = tracker - 1 + 1

    elif n != 0 and n!= N-1 and line[n + 1] == 0 and line[n - 1] == 1:
        line[n], line[n - 1] = 1, 0
        tracker = tracker + 1 - 1

    else:
        if line[n] == 0:
            tracker = tracker + 1
```

```python
        line[n] = 1

x = np.linspace(0, N, N)
xs = []
ys = []

for k in range(len(line)):
    if line[k] == 1:
        xs.append(x[k])
        ys.append(line[k])
plt.scatter(xs, ys)
for xi, yi in zip(x[line == 1], line[line == 1]):
    plt.text(xi, yi, f'({int(xi)})', fontsize=8, ha='left', va='bottom')
plt.grid()
plt.title('Distribution of Particles on the Lattice')
plt.xlabel('Lattice Site')
plt.ylabel('Particle Presence')
plt.show()

print("Steady State Density: ", tracker/N)

voids = []
index = 0
for j in range(1, len(line)):
    if line[j] == 1:
        voids.append(j - index)
        index = j + 1
if line[-1] == 0:
    voids.append(len(line) - index)

print("Voids = ", voids)
plt.hist(voids)
plt.title('Distribution of Void Sizes')
plt.xlabel('Void Size')
plt.ylabel('Frequency')
plt.show()
```