PHYS 218 - Homework 1

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Problem 3.3.2

$$\begin{aligned}\dot{E} &= \kappa(P - E)\\ \dot{P} &= \gamma_1(ED - P)\\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP)\end{aligned}$$

a) Assuming $\dot{P} \approx 0$ and $\dot{D} \approx 0$ then,

$$\gamma_2(\lambda + 1 - D - \lambda EP) = 0$$

the solutions are

$$\begin{aligned}\gamma_2 &= 0 \text{ but } \gamma_2 \text{ is not zero}\\ \text{and } \lambda + 1 - D - \lambda EP &= 0\end{aligned}$$

$$\lambda + 1 - D - \lambda EP = 0 \Rightarrow D = \lambda + 1 - \lambda EP$$

then, P is

$$\begin{aligned}\gamma_1(ED - P) &= 0\\ \gamma 1 &= 0 \text{ but } \gamma_1 \text{ is not zero}\\ \text{and } ED - P &= 0\end{aligned}$$

$$\begin{aligned}ED - P = 0 &\Rightarrow P = ED\\ P &= E\lambda + E - \lambda E^2 P\\ (1 + \lambda E^2)P &= E(1 + \lambda)\\ P &= \frac{E(1 + \lambda)}{1 + \lambda E^2}\end{aligned}$$

Then, the first-order equation for the evolution of E is,

$$\begin{aligned}\dot{E} &= \kappa(P - E)\\ &= \kappa\left(\frac{E(1 + \lambda)}{(1 + E^2)} - E\right)\\ &= \kappa E\left(\frac{1 + \lambda}{1 + \lambda E^2} - 1\right)\\ &= \kappa E\left(\frac{1 + \lambda - 1 - \lambda E^2}{1 + \lambda E^2}\right)\\ &= \kappa E\left(\frac{\lambda - \lambda E^2}{1 + \lambda E^2}\right)\\ \Rightarrow \dot{E} &= \kappa E\lambda\left(\frac{1 - E^2}{1 + \lambda E^2}\right)\end{aligned}$$

b) The fixed points of the equation for E is when $\dot{E} = 0$,

$$
\begin{aligned}
\kappa E\lambda\left(\frac{1-E^2}{1+\lambda E^2}\right) &= 0 \\
E = 0 \text{ and } \frac{1-E^2}{1+\lambda E^2} &= 0 \\
1-E^2 &= 0 \\
E^2 &= 1 \\
\Rightarrow E_1^* = 0,\ E_2^* = 1, \text{ and } E_3^* &= -1
\end{aligned}
$$

c) Let $f(E) = \kappa E\lambda\left(\frac{1-E^2}{1+\lambda E^2}\right)$, to evaluate the stability of the fixed points, we take the derivative and evaluate it at the fixed points. If it is positive, then the point is unstable, and if it is negative, then it is stable.

$$
\frac{df}{dE} = \frac{\lambda\kappa(1-3E^2)(1-\lambda E^2)}{((1+\lambda E^2)^2}
$$

For the fixed point $E_1^* = 0$,

$$
\frac{df}{dE}(E = E_1^* = 0) = \frac{\lambda\kappa(1-0^2)(1-\lambda 0^2)}{((1+\lambda 0^2)^2} = \kappa\lambda
$$

So, if $\lambda > 0$, then E_1^* is unstable, and if $\lambda < 0$, then the point is stable. For the fixed points $E_{1,2}^* = \pm 1$,

$$
\frac{df}{dE}(E = E_{1,2}^* = \pm 1) = \frac{\lambda\kappa(1-3(\pm 1)^2)(1-\lambda(\pm 1)^2)}{((1+\lambda(\pm 1)^2)^2} = -\frac{2\kappa\lambda(1-\lambda)}{(1+\lambda)^2}
$$

So, if $\lambda < 1$ and $\lambda \neq -1$, then $E_{2,3}^*$ is stable, and if $\lambda > 1$, then the point is unstable.

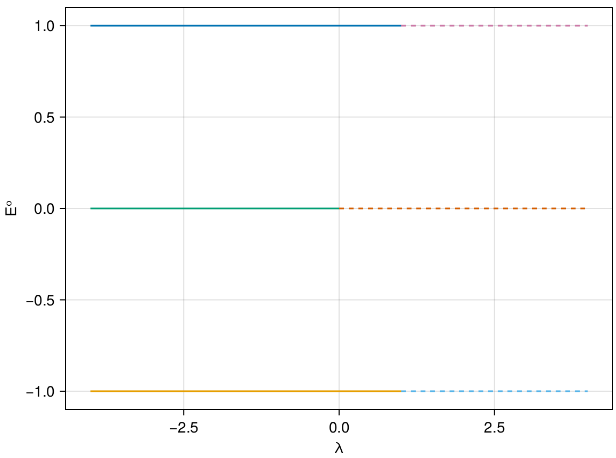

Figure 1: Bifurcation Diagram E^* vs. λ

Problem 3.5.4

Consider a system comprised of a bead of mass m that is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire. The motion of the bead is opposed by a viscous damping force with damping constant b.

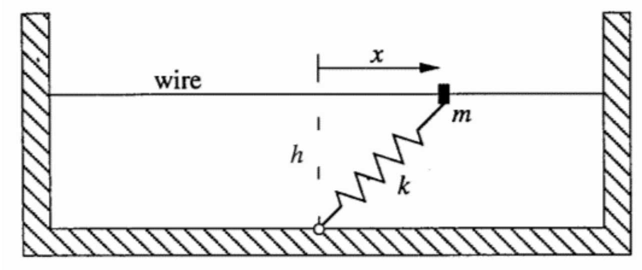

Figure 2: The Setup of the System

a) Let θ be the angle between the spring and wire. There are three forces acting on the mass: the tension force from the spring, the damping force, and the gravitational force. We project the forces on the x-direction.
The projected tension force is,

$$\begin{aligned} T &= -k\Delta x cos\theta \\ \Delta x &= (L - L_0) = \sqrt{x^2 + h^2} - L_0 \\ cos\theta &= \frac{x}{\sqrt{x^2 + h^2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow T &= -k\Delta x cos\theta \\ &= -k\left(\sqrt{x^2 + h^2} - L_0\right)\frac{x}{\sqrt{x^2 + h^2}} \\ &= -kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) \end{aligned}$$

The projected damping force is,

$$F_{fr} = -b\dot{x}$$

The projected gravitational force is,

$$F_g = 0$$

So, Newton's third law for the bead is

$$\begin{aligned} m\ddot{x} &= \Sigma F_{ext} \\ m\ddot{x} &= -b\dot{x} - kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) \\ \Rightarrow m\ddot{x} + b\dot{x} + kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) &= 0 \end{aligned}$$

b) Equilibrium is achieved when $\ddot{x} = \dot{x} = 0$, so

$$\begin{aligned} kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) &= 0 \\ x = 0 \text{ and } \left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) &= 0 \\ \left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) = 0 \Rightarrow \frac{L_0}{\sqrt{x^2 + h^2}} &= 1 \\ L_0 &= \sqrt{x^2 + h^2} \\ x^2 + h^2 &= L_0^2 \\ x^2 &= L_0^2 - h^2 \\ x &= \pm\sqrt{L_0^2 - h^2} \\ \Rightarrow E_1^* = 0,\ E_{2,3}^* &= \pm\sqrt{L_0^2 - h^2} \end{aligned}$$

Since fixed points supposed to be real, they would appear when $L_0 > h$ since the opposite would yield imaginary points.

c) Let $f(x) = k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)$, the derivative is

$$\frac{df}{dx}(x) = k\left(1 - \frac{L_0 h^2}{(x^2+h^2)^{\frac{3}{2}}}\right)$$

For the fixed point $x_1^* = 0$,

$$\frac{df}{dx}(x_1^* = 0) = k\left(1 - \frac{L_0 h^2}{(0^2+h^2)^{\frac{3}{2}}}\right) = k\left(1 - \frac{L_0}{h}\right)$$

So, if $h > L_0$, then x_1^* is unstable, and if $h < L_0$, then the point is stable.

For the fixed point $x_{2,3}^* = \pm\sqrt{L_0^2 - h^2}$,

$$\frac{df}{dx}\left(x_{2,3}^* = \pm\sqrt{L_0^2 - h^2}\right) = k\left(1 - \frac{L_0 h^2}{\left(\left(\pm\sqrt{L_0^2 - h^2}\right)^2 - h^2\right)^{\frac{3}{2}}}\right) = k\left(1 - \frac{h^2}{L_0^2}\right)$$

So, if $h < L_0$, then $x_{2,3}^*$ are unstable.

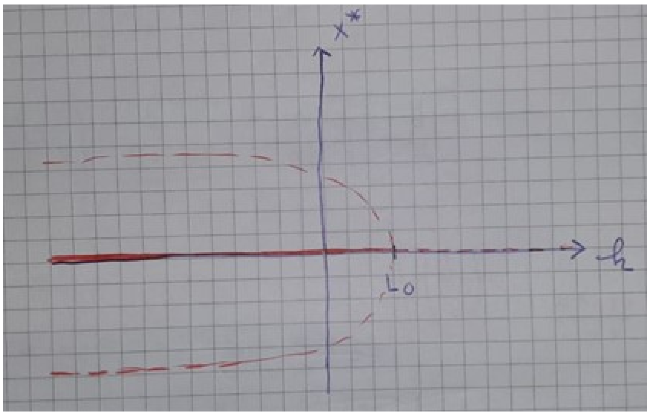

Figure 3: Bifurcation Diagram

d) Let $m \neq 0$, and the dimensionless transformation $\tau = \frac{t}{T}$, then the equation of motion becomes,

$$\frac{m}{T^2}\frac{d^2x}{d\tau^2} + \frac{b}{T}\frac{dx}{d\tau} + kx\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right) = 0$$
$$\frac{m}{kT^2}\frac{d^2x}{d\tau^2} + \frac{b}{T}\frac{dx}{d\tau} + x\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right) = 0$$

What multiply the first derivative should be on the orders of unity, then

$$\frac{b}{kT} \approx 1 \Rightarrow T \approx \frac{b}{k}$$

Then,

$$\frac{m}{kT^2} \ll 1 \Rightarrow m \ll kT^2 \Rightarrow m \ll \frac{kb^2}{k^2} \Rightarrow m \ll \frac{b^2}{k}$$

The mass is negligible in the sense that the damping force compact and win over the tension and the inertial acceleration.

Problem 3.6.5

Consider a bead of mass m constrained to slide along a straight wire inclined at an angle θ with respect to the horizontal. The mass is attached to a spring of stiffness k and relaxed length L_0, and is also acted on by gravity.

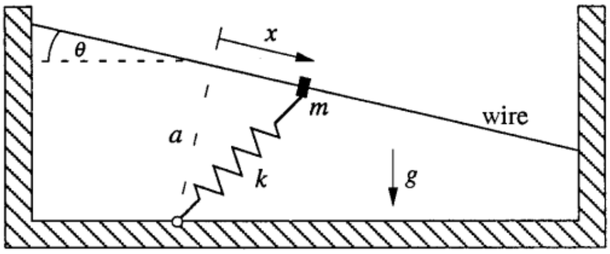

Figure 4: The Mechanical System

a) The forces acting on the bead are tension and gravity. So, their projection on the x-direction are,

$$F_g = mgcos(90^\circ - \theta) = mgsin\theta$$
$$T = k\left(\sqrt{x^2+a^2} - L_0\right)\frac{x}{\sqrt{x^2+a^2}} = kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

Equilibrium is achieved when the acceleration is zero and the forces are equal to each other, so the equilibrium positions of the bead are given by,

$$F_g = T$$
$$mgsin\theta = kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

b) Consider the equilibrium equation,

$$kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right) = mgsin\theta$$
$$1 - \frac{L_0}{\sqrt{x^2+a^2}} = \frac{mgsin\theta}{kx}$$
$$1 - \frac{L_0}{a\sqrt{1+\frac{x^2}{a^2}}} = \frac{mgsin\theta}{kx}$$
$$1 - \frac{mgsin\theta}{kx} = \frac{L_0}{a\sqrt{1+\frac{x^2}{a^2}}}$$

Let $u = \frac{x}{a}$, then the equilibrium equation becomes,

$$1 - \frac{mgsin\theta}{kau} = \frac{L_0}{a\sqrt{1-u^2}}$$
$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$$

where $h = \frac{mgsin\theta}{ka}$ and $R = \frac{L_0}{a}$

c) Consider the dimensionless equation

$$\frac{R}{\sqrt{1+u^2}} - \left(1 - \frac{h}{u}\right) = 0$$

For $R < 1$, we have one equilibrium point and it is stable (denoted by a filled circle),

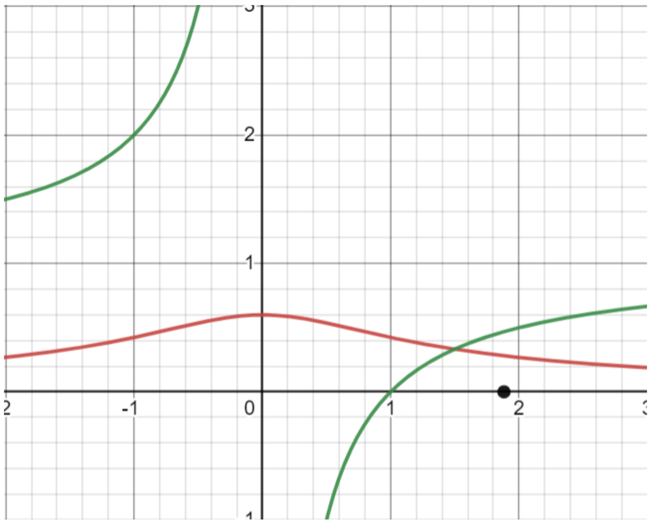

Figure 5: $R < 1$

For $R > 1$ and the two curves are tangential, we have two equilibria, one is still stable while the other is a saddle point (denoted by an x),

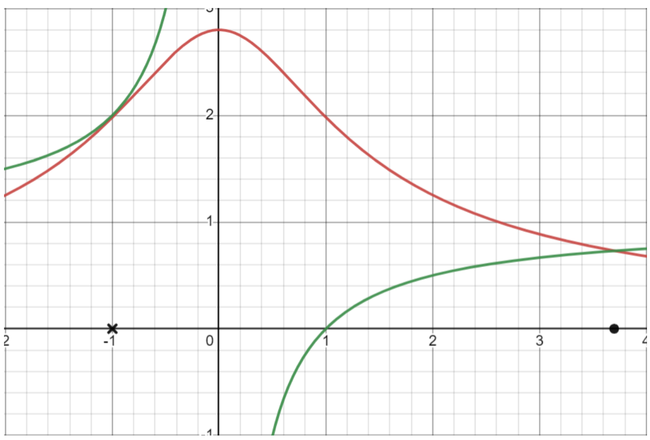

Figure 6: $R > 1$ and the curves are tangential

While for $R > 1$ and not tangential, we have three equilibria, the stable one remains stable while the saddle point splits into one stable point and another unstable point (denoted by an empty circle),

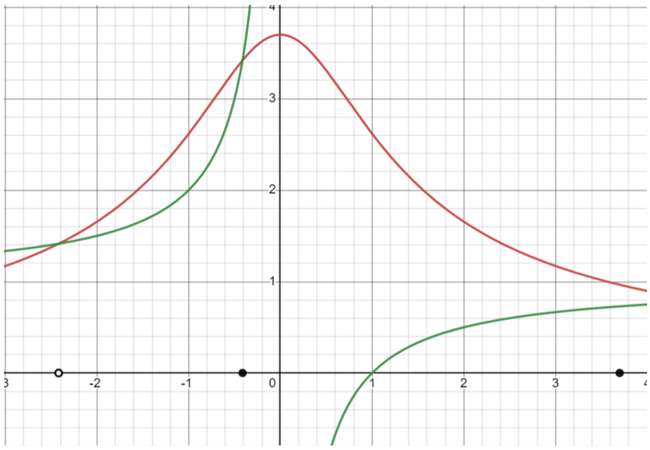

Figure 7: $R > 1$ and the curves are not tangential

d) Let $r = R - 1 \Rightarrow R = r + 1$.

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$$
$$1 - \frac{h}{u} - \frac{R}{\sqrt{1+u^2}} = 0$$
$$1 - \frac{h}{u} - \frac{r+1}{\sqrt{1+u^2}} = 0$$
$$\frac{u\sqrt{1+u^2} - h\sqrt{1+u^2} - ru - u}{u\sqrt{1+u^2}} = 0$$
$$u\sqrt{1+u^2} - h\sqrt{1+u^2} - ru - u = 0$$

For small u,

$$\sqrt{1+u^2} = 1 + \frac{1}{2}u^2 + O(u^4)$$

So, the equation becomes,

$$u\left(1 + \frac{1}{2}u^2\right) - h\left(1 + \frac{1}{2}u^2\right) - ru - u = 0$$
$$u + \frac{1}{2}u^3 - h - \frac{h}{2}u^2 - ru - u = 0$$
$$\frac{1}{2}u^3 - h - \frac{h}{2}u^2 - ru$$
$$h + ru + \frac{h}{2}u^2 - \frac{1}{2}u^3 = 0$$

e) The equation from the point of tangency is,

$$\frac{\partial}{\partial u}\left(h + ru + \frac{h}{2}u^2 - \frac{1}{2}u^3\right) = 0$$
$$r + hu - \frac{3}{2}u^2 = 0$$
$$r = \frac{3}{2}u^2 - hu$$

We plug this r back into the equation,

$$
\begin{aligned}
h + ru + \frac{h}{2}u^2 - \frac{1}{2}u^3 &= 0 \\
h + \left(\frac{3}{2}u^2 - hu\right) + \frac{h}{2}u^2 - \frac{1}{2}u^3 &= 0 \\
h + \frac{3}{2}u^3 - hu^2 + \frac{h}{2}u^2 - \frac{1}{2}u^3 &= 0 \\
u^3 - \frac{h}{2}u^2 + h &= 0 \\
\frac{h}{2}u^2 - h &= u^3 \\
h\left(\frac{u^2}{2} - 1\right) &= u^3 \\
h = \frac{u^3}{\frac{u^2-2}{2}} &\Rightarrow h = \frac{2u^3}{u^2-2} \\
\Rightarrow r &= \frac{3}{2}u^2 - \frac{2u^4}{u^2-2}
\end{aligned}
$$

So, an approximate formulae for the saddle-node bifurcation curves in the limit of small r, h, and u are,

$$
\begin{aligned}
h &= \frac{2u^3}{u^2-2} \\
r &= \frac{3}{2}u^2 - \frac{2u^4}{u^2-2}
\end{aligned}
$$

f) Consider the dimensionless equation,

$$
1 - \frac{h}{2} = \frac{R}{\sqrt{1+u^2}}
$$

The tangency condition gives us the following,

$$
\begin{aligned}
\frac{\partial}{\partial u}\left(1 - \frac{h}{u} - \frac{R}{\sqrt{1+u^2}}\right) &= 0 \\
\frac{h}{u^2} + \frac{Ru}{(1+u^2)^{\frac{3}{2}}} &= 0 \\
h &= -\frac{Ru^3}{(1+u^2)^{\frac{3}{2}}}
\end{aligned}
$$

The intersection condition gives us,

$$
\begin{aligned}
1-\frac{h}{u} &= \frac{R}{\sqrt{1+u^2}}\\
1+\frac{Ru^3}{u(1+u^2)^{\frac{3}{2}}} &= \frac{R}{\sqrt{1+u^2}}\\
R\left(-\frac{u^2}{(1+u^2)^{\frac{3}{2}}}+\frac{1}{(1+u^2)^{\frac{1}{2}}}\right) &= 1\\
R\left(\frac{-u^2+1+u^2}{(1+u^2)^{\frac{3}{2}}}\right) &= 1\\
R\left(\frac{1}{(1+u^2)^{\frac{3}{2}}}\right) &= 1\\
\Rightarrow R &= (1+u^2)^{\frac{3}{2}}\\
\Rightarrow h &= -\frac{(1+u^2)^{\frac{3}{2}}u^3}{(1+u^2)^{\frac{3}{2}}} = -u^3
\end{aligned}
$$

For small u,

$$
\begin{aligned}
h &= \frac{2u^3}{u^2-2} \approx -u^3\\
r &= \frac{3}{2}u^2 - \frac{2u^4}{u^2-2} = R-1
\end{aligned}
$$

Problem 3.6.6

a) Consider the Landau equation,

$$\tau\dot{A} = \varepsilon A - gA^3$$

At steady state, $\dot{A}=0$, so the fixed points are,

$$
\begin{aligned}
\varepsilon A - gA^3 &= 0\\
A(\varepsilon - gA^2) &= 0\\
A = 0 \text{ and } \varepsilon - gA^2 &= 0\\
A^2 &= \frac{\varepsilon}{g}\\
A &= \pm\sqrt{\frac{\varepsilon}{g}}\\
\Rightarrow A_1^* = 0 \text{ and } A_{2,3}^* &= \pm\left(\frac{\varepsilon}{g}\right)
\end{aligned}
$$

So, the Landau equation predicts the same power law that Dubois and Bergé showed experimentally.

b) Now consider the following equation and the case $g=0$,

$$\tau\dot{A} = \varepsilon A - gA^3 - kA^5 = \varepsilon A - kA^5$$

The fixed points that this equation posses are,

$$\varepsilon A - kA^5 = 0$$
$$A(\varepsilon - kA^4) = 0$$
$$A = 0 \text{ and } \varepsilon - kA^4 = 0$$
$$A^4 = \frac{\varepsilon}{k}$$
$$A^2 = \pm\sqrt{\frac{\varepsilon}{k}}$$
$$A = \pm\sqrt{\pm\sqrt{\frac{\varepsilon}{k}}}$$
$$\Rightarrow A_1^* = 0 \text{ and } A_{2,3,4,5}^* = \pm\sqrt{\pm\sqrt{\frac{\varepsilon}{k}}}$$

So, the relation between the fixed points and ε is

$$A* \propto \varepsilon^{0.25}$$

c) Assuming that the equation is modified to $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$ where $h > 0$,
In the case $g = 0$, $\varepsilon A - (kA^5 - h) = 0$,

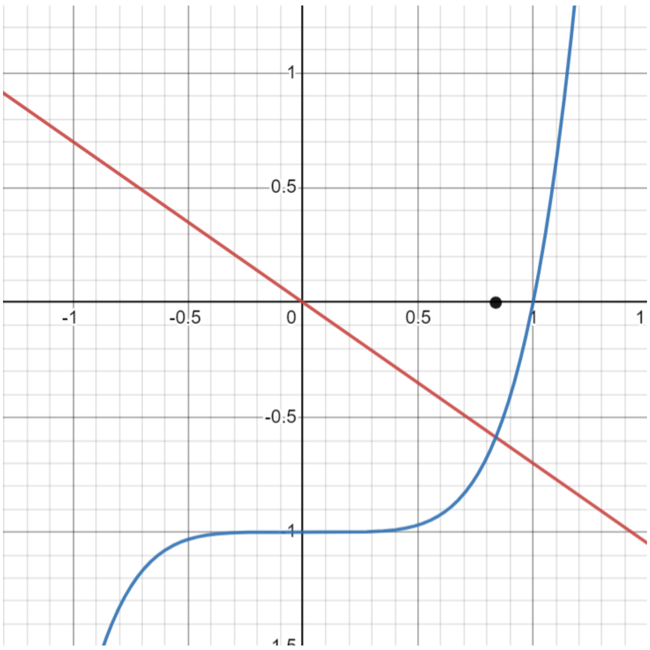

Figure 8: $h = 1, g = 0, \varepsilon < 0$

Figure 9: $h = 1, g = 0, \varepsilon = 0$

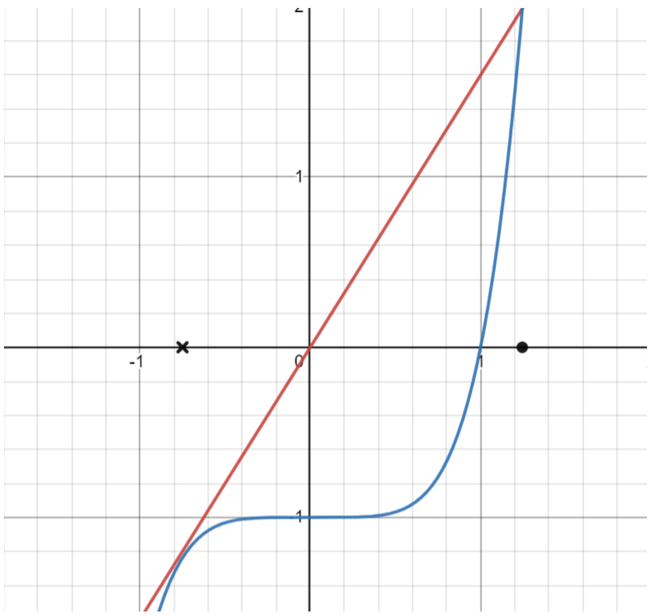

Figure 10: $h = 1, g = 0, \varepsilon > 0$ and tangential

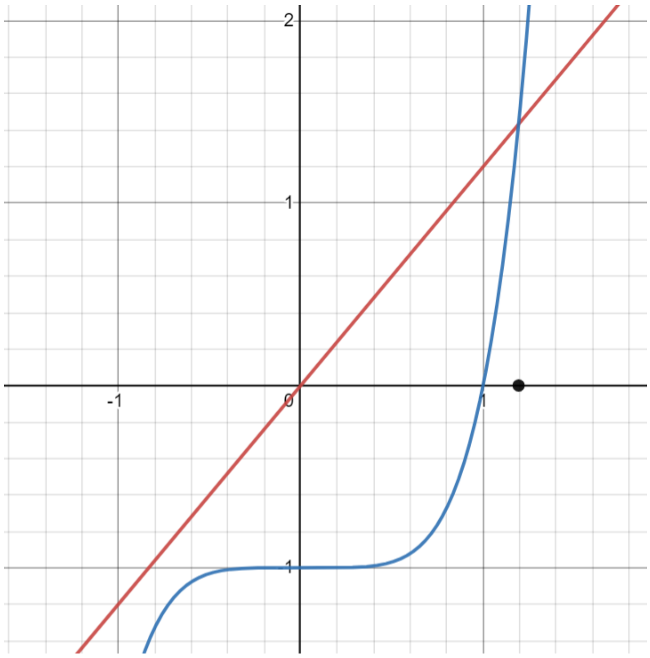

Figure 11: $h = 1, g = 0, \varepsilon > 0$

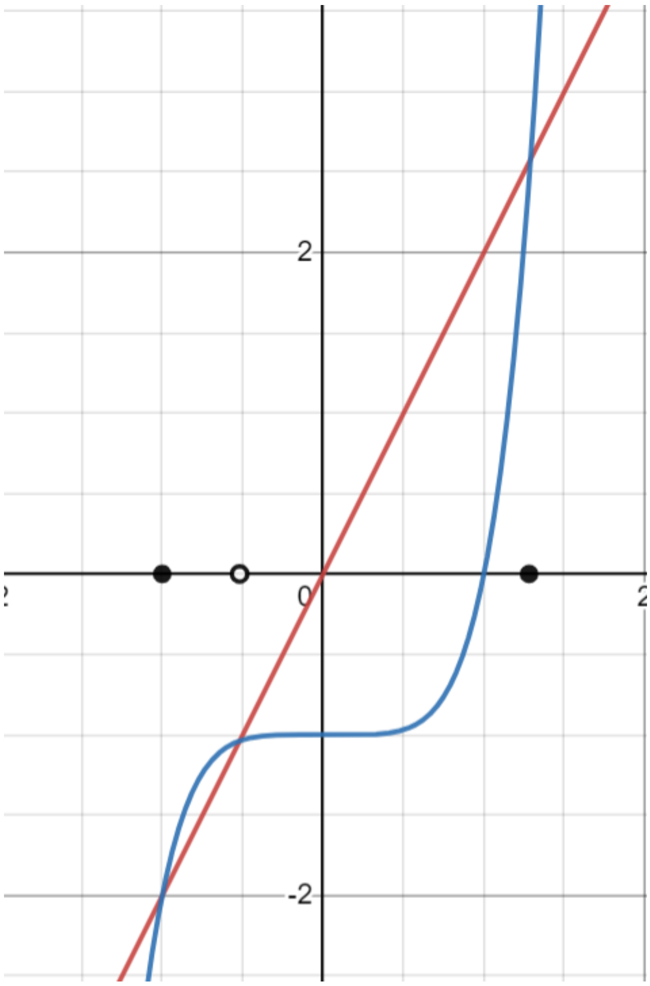

Figure 12: $h = 1, g = 0, \varepsilon > 0$

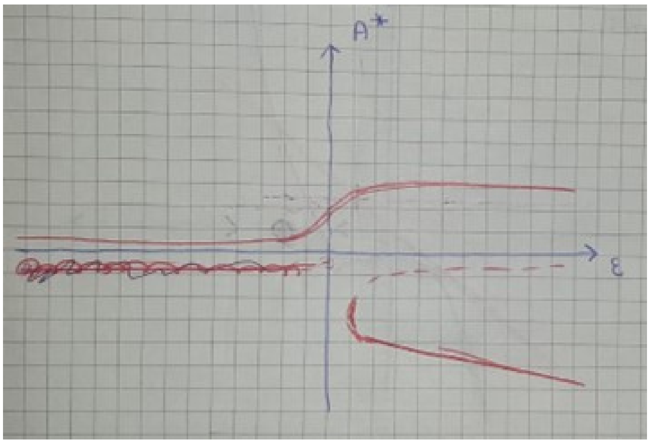

Figure 13: A^* vs ε bifurcation diagram for $g = 0$

For the case $g > 0$, $\varepsilon A - (gA^3 + kA^5 - h) = 0$, it is similar to that when $g = 0$, what differs is the values of the fixed points. The bifurcation diagram for $g > 0$ is

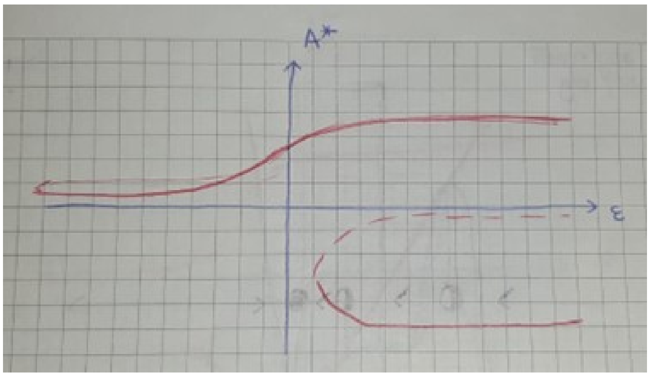

Figure 14: A^* vs ε bifurcation diagram for $g > 0$

For the case $g < 0$, $\varepsilon A - (gA^3 + kA^5 - h) = 0$,

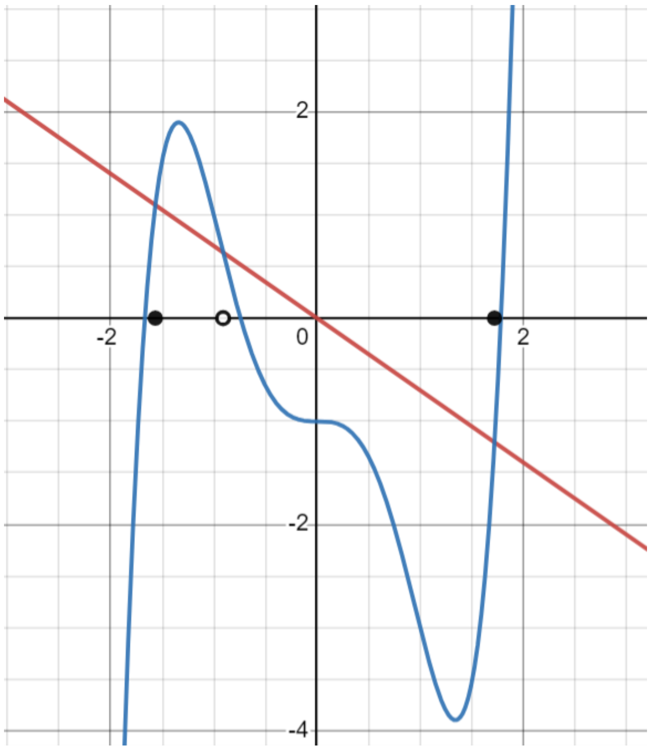

Figure 15: $h = 1, k = 1, g < 0, \varepsilon < 0$

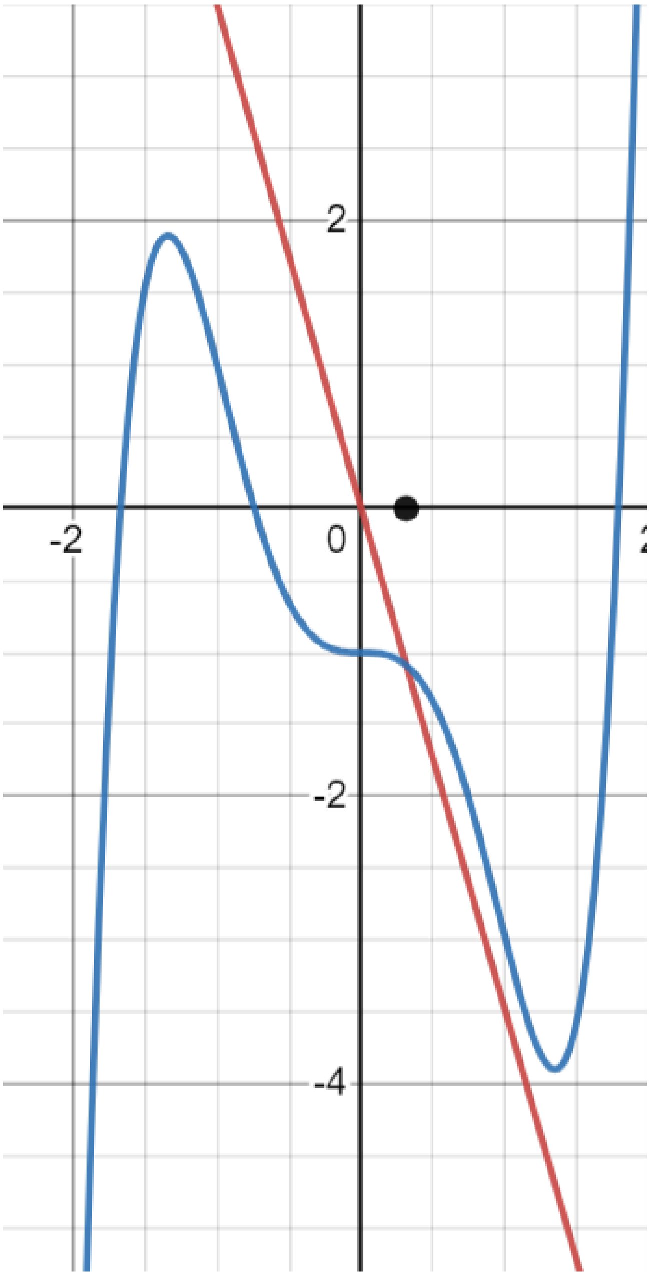

Figure 16: $h = 1, k = 1, g < 0, \varepsilon < 0$ and tangential

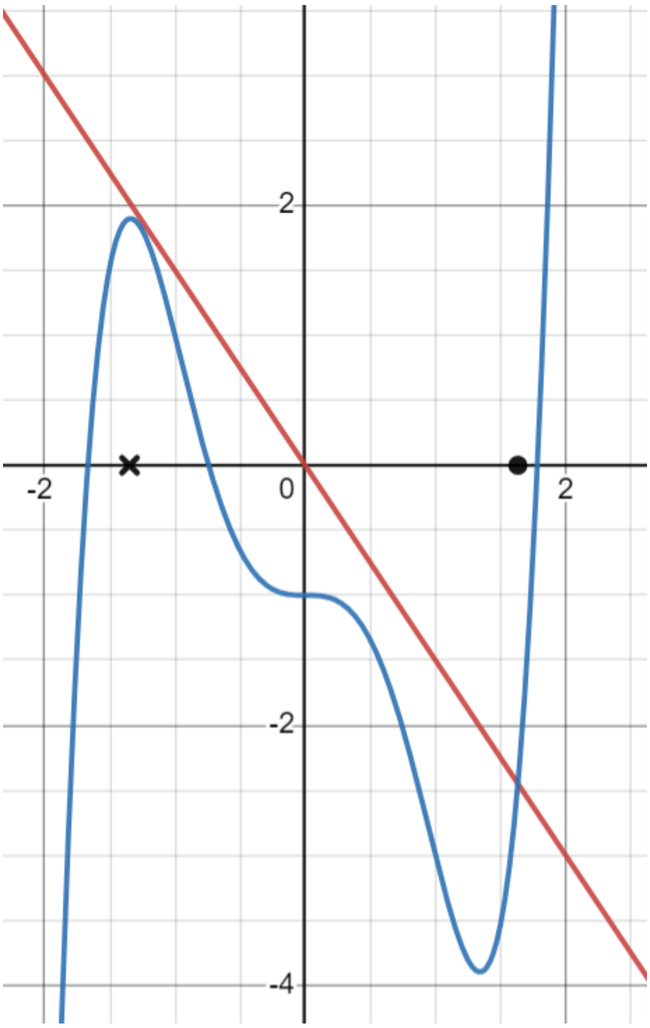

Figure 17: $h = 1, k = 1, g < 0, \varepsilon < 0$

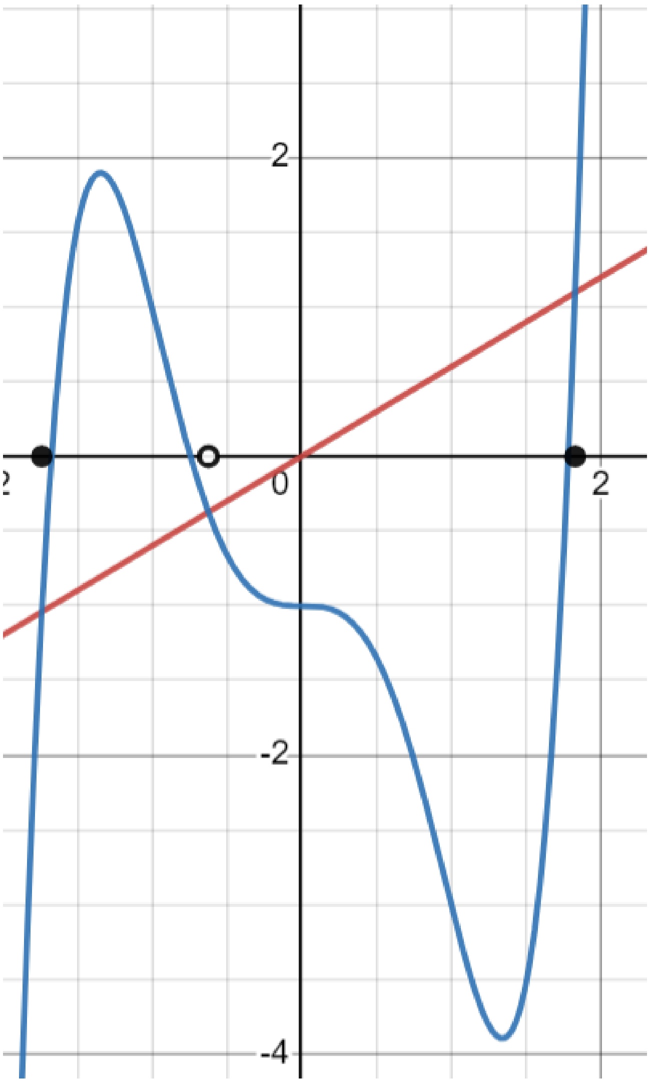

Figure 18: $h = 1, k = 1, g < 0, \varepsilon > 0$

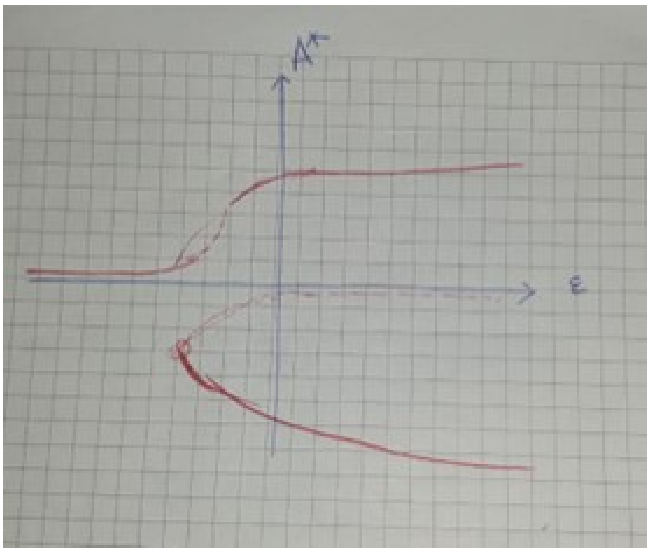

Figure 19: A^* vs ε bifurcation diagram for $g < 0$

Problem 3.7.6

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

a)

$$\begin{aligned}
\frac{dg}{dt} &= k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2} \\
\frac{d\frac{gk_4}{k_4}}{dt} &= k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4\left(1 + \frac{g^2}{k_4^2}\right)} \\
k_4 \frac{d\frac{g}{k_4}}{dt} &= k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4\left(1 + \frac{g^2}{k_4^2}\right)} \\
\frac{k_4}{k_3} \frac{d\frac{g}{k_4}}{dt} &= \frac{k_1 s_0}{k_3} - \frac{k_2}{k_3} g + \frac{g^2}{k_4\left(1 + \frac{g^2}{k_4^2}\right)}
\end{aligned}$$

Let $x = \frac{g}{k_4}$

$$\begin{aligned}
\frac{k_4}{k_3} \frac{dx}{dt} &= \frac{k_1 s_0}{k_3} - \frac{k_2}{k_3} g + \frac{x^2}{(1 + x^2)} \\
\frac{dx}{d\tau} &= s - rx + \frac{x^2}{1 + x^2}
\end{aligned}$$

where $\frac{k_4}{k_3} \frac{1}{dt} = \frac{1}{d\tau}$, $\frac{k_1 s_0}{k_3} = s$, and $\frac{k_2 k_4}{k_3} = r$.

b) If $s = 0$,

$$x' = -rx + \frac{x^2}{1 + x^2}$$

So, the fixed points are when $x' = 0$,

$$\begin{aligned}
-rx + \frac{x^2}{1 + x^2} &= 0 \\
x\left(\frac{x}{1 + x^2} - r\right) &= 0 \\
x = 0 \text{ and } \frac{x}{1 + x^2} - r &= 0 \\
\frac{x}{1 + x^2} &= r \\
rx^2 - x + r &= 0 \\
x &= \frac{1 \pm \sqrt{1 - 4r^2}}{2r} \\
\Rightarrow x_1^* = 0 \text{ and } x_{2,3}^* &= \frac{1 \pm \sqrt{1 - 4r^2}}{2r}
\end{aligned}$$

The fixed points appear only when they are real, so

$$1 - 4r^2 \geqslant 0$$
$$4r^2 \leqslant 1$$
$$r^2 \leqslant \frac{1}{4}$$
$$r \leqslant \frac{1}{2}$$
$$\Rightarrow r_c = \frac{1}{2}$$

r is at most 0.5, then the fixed points are always positive for any r when s = 0.

c) Assuming that initially there is no gene product and that s is slowly increasing from zero, let's consider three cases for the following equation,

$$\frac{x^2}{1+x^2} - rx - (-s) = 0$$

If $r > \frac{1}{2}$,

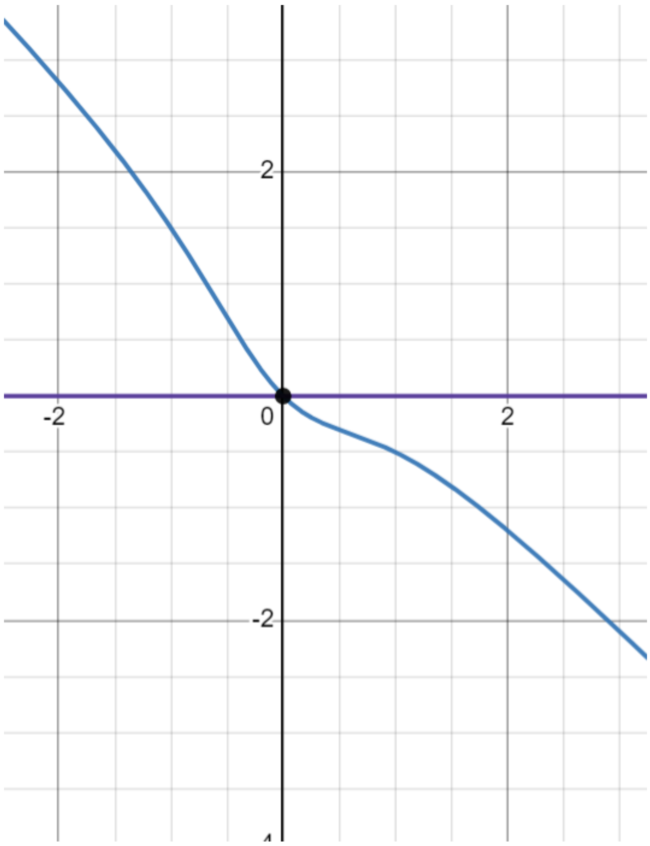

Figure 20: $r = 1, s = 0$; g(t) is going towards a stable fixed point

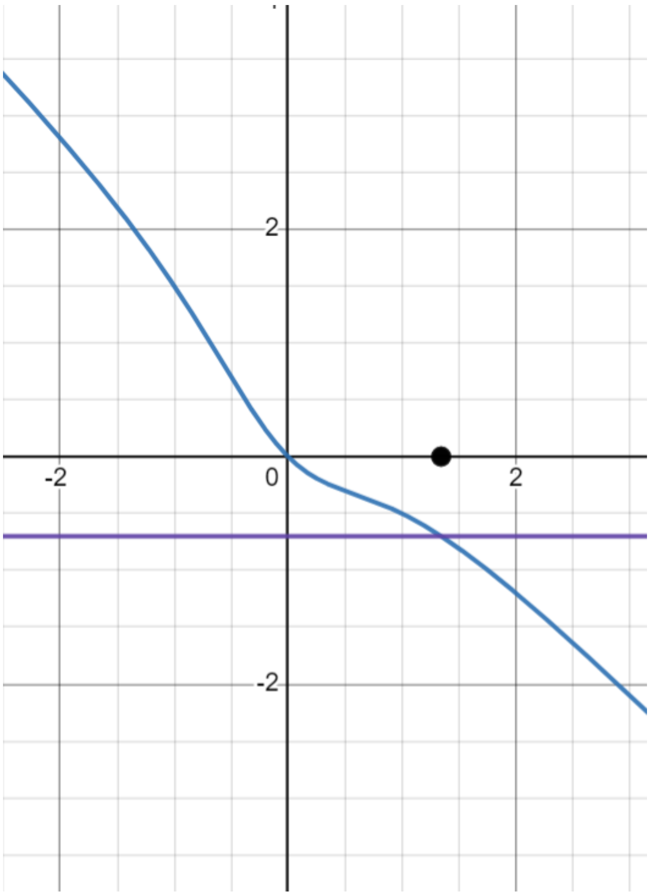

Figure 21: $r = 1, s > 0$; g(t) is going towards a shifted stable fixed point

If $r < \frac{1}{2}$,

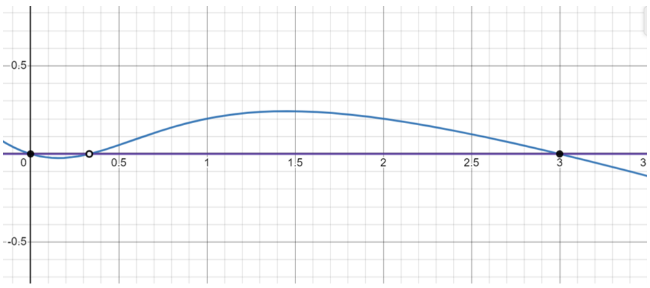

Figure 22: $r = 0.3, s = 0$; g(t) can either go to a stable fixed point, away from the unstable fixed point, or through the saddle point

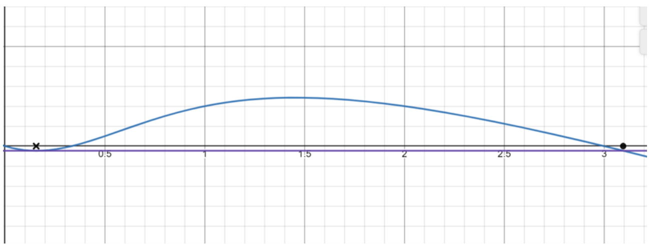

Figure 23: $r = 0.3, s > 0$ and tangential; g(t) can either go towards the stable fixed point or through the saddle point

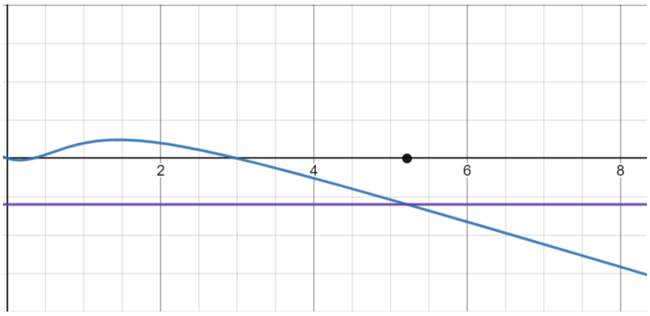

Figure 24: $r = 0.3, s > 0$; g(t) will go towards the stable fixed point

If $r = \frac{1}{2}$,

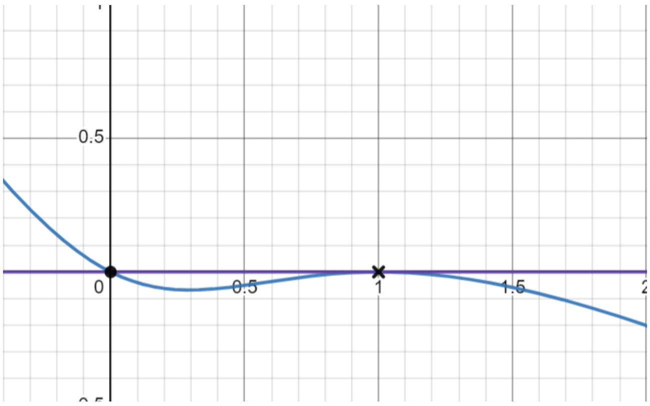

Figure 25: $r = 0.5, s = 0$; g(t) either goes towards a stable fixed point or through a saddle point

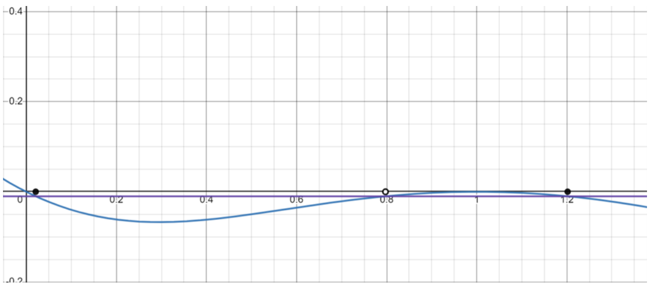

Figure 26: $r = 0.5, s > 0$; g(t) either goes towards the stable fixed points or away from the unstable fixed points

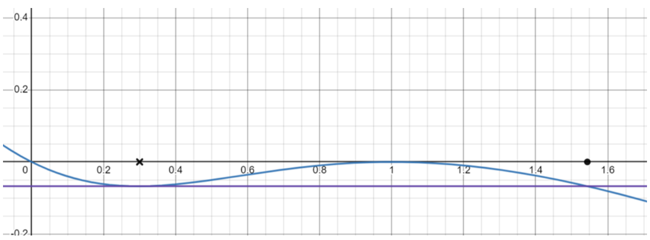

Figure 27: $r = 0.5, s > 0$ and tangential; g(t) either goes towards a stable fixed point or through a saddle point

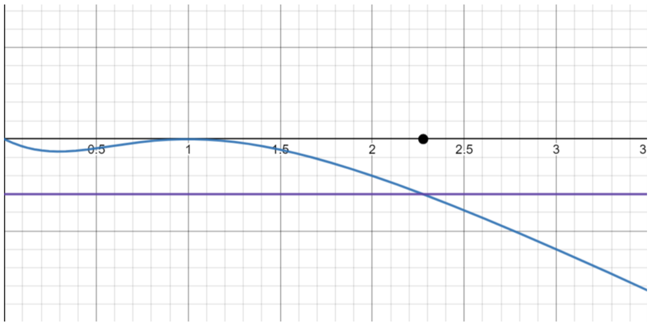

Figure 28: $r = 0.5, s > 0$; g(t) will go towards a stable fixed point

d) The parametric equations for the bifurcation curves using the tangential and intersection conditions are,

$$\begin{aligned}
s - rx - \frac{x^2}{1+x^2} &= 0 \\
\frac{d}{dx}\left(s - rx - \frac{x^2}{1_x^2}\right) &= 0 \\
-r + \frac{2x}{(1+x^2)^2} &= 0 \\
\Rightarrow r &= \frac{2x}{(1+x^2)^2} \\
s - rx + \frac{x^2}{1+x^2} &= 0 \\
s - \frac{2x^2}{(1+x^2)^2} + \frac{x^2}{1+X^2} &= 0 \\
s &= \frac{2x^2}{(1+x^2)^2} - \frac{x^2}{1+X^2} \\
s &= \frac{2x^2 - x^2 + x^4}{(1+x^2)^2} \\
s &= \frac{x^2 - x^4}{(1+x^2)^2} \\
\Rightarrow s &= \frac{x^2(1-x^2)}{(1+x^2)^2}
\end{aligned}$$

This bifurcation is a saddle node bifurcation.

Problem 3.7.7

Let $x(t)$ be the number of healthy people, $y(t)$ be the number of sick people, and $z(t)$ be the number of dead people, the model for the evolution of an epidemic is

$$\begin{aligned}
\dot{x} &= -kxy \\
\dot{y} &= kxy - ly \\
\dot{z} &= ly
\end{aligned}$$

a) N is a constant, so $\dot{N} = 0$.

$$\dot{x} + \dot{y} + \dot{z} = -kxy + kxy - ly + ly = 0 = \dot{N}$$

Therefore, $x + y + z = N$

b) Assume $x(t) = x_0 exp\left(-\frac{kz(t)}{l}\right)$,

$$\begin{aligned}\dot{x} &= x_0\left(-\frac{k}{k}\dot{z}exp\left(-\frac{kz(t)}{l}\right)\right)\\ &= x_0\left(-\frac{k}{k}lyexp\left(-\frac{kz(t)}{l}\right)\right)\\ &= -lyx_0exp\left(-\frac{kz(t)}{l}\right)\\ &= -lyx\end{aligned}$$

c)

$$\dot{z} = l\left[N - z - x_0exp\left(-\frac{kz(t)}{l}\right)\right] = l\left[x+y+z-z-x\right] = ly$$

d) Consider the above equation in z,

$$\begin{aligned}\frac{dz}{dt} &= lN - lz - lx_0exp\left(-\frac{kz(t)}{l}\right)\\ \frac{1}{x_0l}\frac{dz}{dt} &= \frac{N}{x_0} - \frac{z}{x_0} - exp\left(-\frac{kz(t)}{l}\right)\end{aligned}$$

Let $u = \frac{kz}{l}$,

$$\frac{l}{kx_0l}\frac{du}{dt} = \frac{N}{x_0} - \frac{l}{kx_0}z - e^{-u}$$

$$\begin{aligned}\frac{1}{kx_0}\frac{du}{dt} &= \frac{N}{x_0} - \frac{l}{lx_0}z - e^{-u}\\ \frac{du}{d\tau} &= a - bu - e^{-u}\end{aligned}$$

where $\tau = kx_0t$, $a = \frac{N}{x_0}$, and $b = \frac{l}{kx_0}$

e) $a = \frac{N}{x_0}$, x_0 corresponds to healthy people at $t = 0$, but there can't be more healthy people than total people, so N is always bigger than x_0.

$$N \geqslant x_0$$

$$\begin{aligned}\frac{N}{x_0} &\geqslant 1\\ a &\geqslant 1\end{aligned}$$

While since $l > 0$, $k > 0$, and $x_0 > 0$, then

$$\begin{aligned}\frac{l}{kx_0} &> 0\\ b &> 0\end{aligned}$$

f) The steady state equation is $a - bu - e^{-u} = 0$, geometrically,

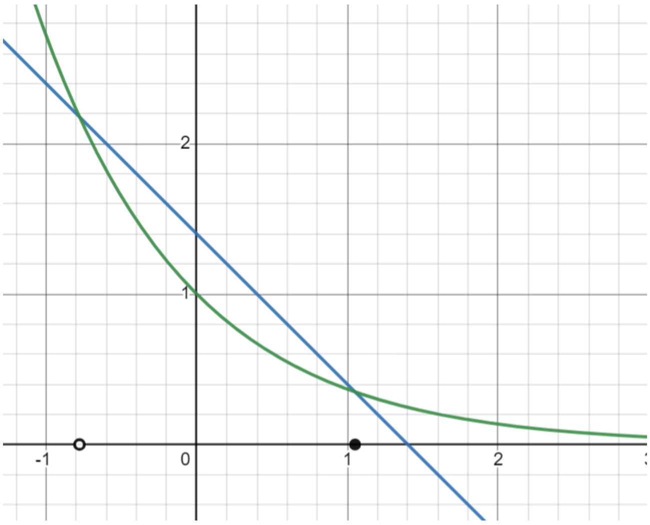

Figure 29: a = 1.4, b = 1; Two fixed points, one is stable and the other is unstable

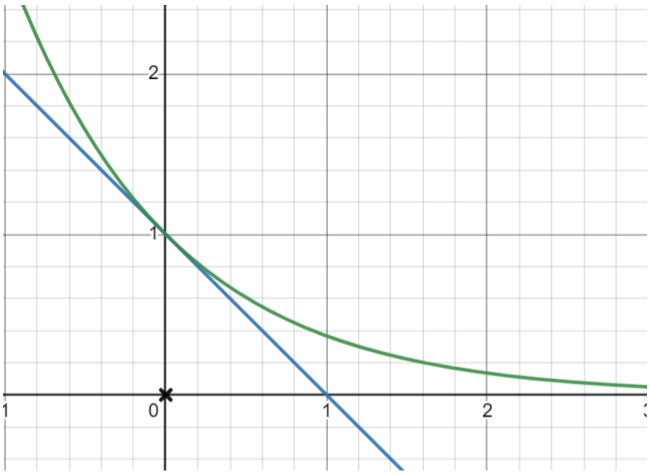

Figure 30: a = 1.4, b = 1 and tangential; One saddle point

The saddle point occurs at the tangency point,

$$
\begin{aligned}
\frac{d}{du}\left(a - bu - e^{-u}\right) &= 0 \\
-b + e^{-u} &= 0 \\
e^{-u} &= b \\
ln(e^{-u}) &= ln(b) \\
-u &= ln(b) \\
u &= -ln(b) \\
\Rightarrow &= ln\left(\frac{1}{b}\right)
\end{aligned}
$$

So, the unstable point is less that that value and the stable point is bigger.

g) The maximum of $\dot{u}(t)$ is when $\ddot{u} = 0$,

$$u = \frac{kz}{l}$$
$$\frac{du}{dt} = \frac{k\dot{z}}{l}$$
$$\frac{d^2u}{dt^2} = \frac{k\ddot{z}}{l}$$

So, the maximum of $\dot{u}$ is when $\frac{k\ddot{z}}{l} = 0$ which is also the maximum of $\dot{z}$.
In addition, $\dot{z} = ly$, so if $\dot{z}$ has the same maximum as $\dot{u}$, then y also have the same maximum as$\dot{u}$.