PHYS 218 - Homework 2

Fahed Abu Shaer

April 2024

Problem 4.5.1

In the firefly model, the sinusoidal form of the firefly's response function was chosen somewhat arbitrarily. Let's consider the alternative model

$$\begin{cases} \dot{\Theta} = \Omega \\ \dot{\theta} = \omega + Af(\Theta - \theta) \end{cases}$$

where f is given now by a triangle wave, not a sine wave. Specifically, let

$$f(\phi) = \begin{cases} \phi, & -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2} \\ \pi - \phi, & \frac{\pi}{2} \leqslant \phi \leqslant \frac{3\pi}{2} \end{cases}$$

on the interval $-\frac{\pi}{2} \leqslant \phi \leqslant \frac{3\pi}{2}$, and extend f periodically outside this interval.

a) insert photo

b)

$$\phi = \Theta - \theta$$
$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - Af(\phi)$$

Finding the fixed points of ϕ,

$$\dot{\phi} = 0$$
$$\Omega - \omega - f(\phi) = 0$$
$$\Omega - \omega = Af(\phi)$$

Finding the maximum and minimum of $f(\phi)$,

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$$
$$f\left(\frac{3\pi}{2}\right) = \pi - \frac{3\pi}{2} = \frac{2\pi - 3\pi}{2} = -\frac{\pi}{2}$$
$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \text{ and } = \pi - \frac{\pi}{2} = \frac{2\pi - \pi}{2} = \frac{\pi}{2}$$

So, the maximum of $f(\phi)$ is $\frac{\pi}{2}$ and the minimum is $-\frac{\pi}{2}$. So, the range of entrainment is,

$$min\left(Af(\phi)\right) \leqslant \Omega - \omega \leqslant max\left(Af(\phi)\right)$$
$$-\frac{A\pi}{2} \leqslant \Omega - \omega \leqslant \frac{A\pi}{2}$$
$$\omega - \frac{A\pi}{2} \leqslant \Omega \leqslant \omega + \frac{A\pi}{2}$$

c) Assuming that the firefly is phased-locked to the stimulus, the formula for the phase difference, fixed points, ϕ^* is,

$$\dot{\phi} = 0$$
$$\Omega - \omega - Af(\phi^*) = 0$$

if $-\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}$, then

$$\begin{aligned} \Omega - \omega - A\phi^* &= 0 \\ A\phi^* &= \Omega - \omega \\ \phi^* &= \frac{\Omega - \omega}{A} \end{aligned}$$

if $\frac{\pi}{2} \leqslant \phi \leqslant \frac{3\pi}{2}$, then

$$\begin{aligned} \Omega - \omega - A\left(\pi - \phi^*\right) &= 0 \\ A\left(\pi - \phi^*\right) &= \Omega - \omega \\ \phi^* &= \frac{\omega - \Omega}{A} + \pi \end{aligned}$$

Therefore, the phase difference is,

$$\begin{cases} \frac{\Omega-\omega}{A}, & -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2} \\ \frac{\omega-\Omega}{A} + \pi & \frac{\pi}{2} \leqslant \phi \leqslant \frac{3\pi}{2} \end{cases}$$

d)

$$\begin{aligned} T_{drift} &= \int dt \\ &= \int_0^{2\pi} \frac{dt}{d\phi} d\phi \\ &= \int_0^{2\pi} \frac{1}{\frac{d\phi}{dt}} d\phi \\ &= \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - Af(\phi)} \\ &= \int_0^{\pi/2} \frac{d\phi}{\Omega - \omega - A\phi} + \int_{\pi/2}^{3\pi/2} \frac{d\phi}{\Omega - \omega - A(\pi - \phi)} + \int_{3\pi/2}^{2\pi} \frac{d\phi}{\Omega - \omega - A\phi} \end{aligned}$$

consider the following substitutions,

$$\begin{aligned} u &= \Omega - \omega - A\phi \\ \frac{du}{d\phi} &= -A \\ d\phi &= -\frac{du}{A} \\ u(0) &= \Omega - \omega \\ u(\pi/2) &= \Omega - \omega - \frac{A\pi}{2} \\ u(3\pi/2) &= \Omega - \omega - \frac{3A\pi}{2} \\ u(2\pi) &= \Omega - \omega - 2A\pi \end{aligned}$$

$$\begin{aligned} v &= \Omega - \omega - A\pi + A\phi \\ \frac{dv}{d\phi} &= A \\ d\phi &= \frac{dv}{A} \\ u(\pi/2) &= \Omega - \omega - \frac{A\pi}{2} \\ u(3\pi/2) &= \Omega - \omega + \frac{A\pi}{2} \\ u(2\pi) &= \Omega - \omega - 2A\pi \end{aligned}$$

Therefore, the integral becomes

$$
\begin{aligned}
T_{drift} &= \frac{1}{A}\int_{\Omega-\omega}^{\Omega-\omega-A\pi/2}\frac{du}{u} + \frac{1}{A}\int_{\Omega-\omega-A\pi/2}^{\Omega-\omega+A\pi/2}\frac{dv}{v} + \frac{1}{A}\int_{\Omega-\omega-3A\pi/2}^{\Omega-\omega-2A\pi}\frac{du}{u} \\
&= \frac{1}{A}ln\left(\Omega-\omega-\frac{A\pi}{2}\right) - \frac{1}{A}ln\left(\Omega-\omega\right) + \frac{1}{A}ln\left(\Omega-\omega+\frac{A\pi}{2}\right) - \frac{1}{A}ln\left(\Omega-\omega-\frac{A\pi}{2}\right) \\
&+ \frac{1}{A}ln\left(\Omega-\omega-2A\pi\right) - \frac{1}{A}ln\left(\Omega-\omega-\frac{3A\pi}{2}\right) \\
&= \frac{1}{A}\left[ln\left(\frac{\Omega-\omega+\frac{A\pi}{2}}{\Omega-\omega}\right) + ln\left(\frac{\Omega-\omega-2A\pi}{\Omega-\omega-\frac{3A\pi}{2}}\right)\right]
\end{aligned}
$$

Problem 4.5.3

Suppose you stimulate a neuron by injecting it with a pulse of current. If the stimulus is small, nothing dramatic happens: the neuron increases its membrane potential slightly, and then relaxes back to its resting potential. However, if the stimulus exceeds a certain threshold, the neuron will "fire" and produce a large voltage spike before returning to rest. Surprisingly, the size of the spike doesn't depend much on the size of the stimulus- anything above threshold will elicit essentially the same response. Similar phenomena are found in other types of cells and even in some chemical reactions. These systems are called excitable. The term is hard to define precisely, but roughly speaking, an excitable system is characterized by two properties: (1) it has a unique, globally attracting rest state, and (2) a large enough stimulus can send the system on a long excursion through phase space before it returns to the resting state.

Let $\dot{\theta} = \mu + sin\theta$ where μ is slightly less than 1.

a) Let's plot the function $\dot{\theta} = \mu + sin\theta$, and assess the fixed points.

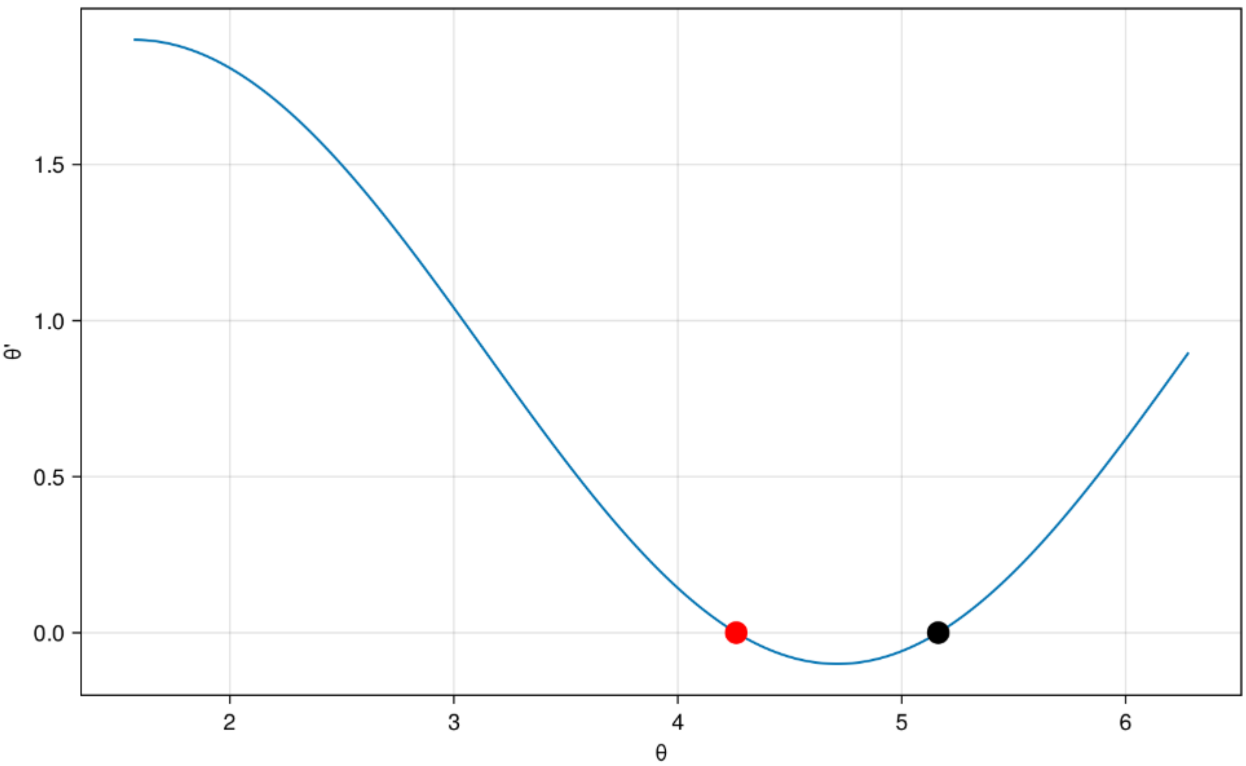

Figure 1: $\dot{\theta}$ as a function of θ

We can see that there are two fixed points, the red one is a stable fixed point where everything tends to it, and the black one is an unstable fixed points where things move away from it. So, the attracting "rest state" is the stable point, and the "threshold" is the unstable fixed point. So, the system satisfies the two properties mentioned above.

b) Let $V(t) = cos\theta(t)$, for different initial conditions, the function looks like the figure below,

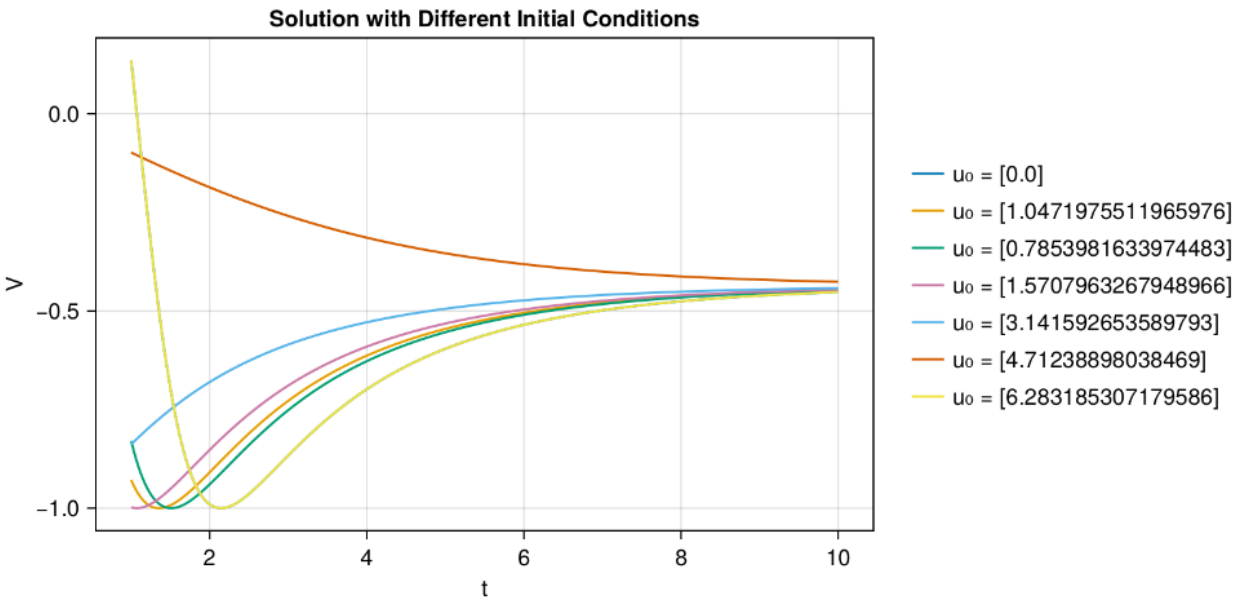

Figure 2: $V(t)$ as a Function of t for Different Initial Conditions

Problem 5.1.10

Consider a fixed point x^* of a system $\dot{x} = f(x)$. We say that x^* is attracting if there is a $\delta > 0$ such that $lim_{t\to\infty}x(t) = x^*$ whenever $||x(0) - x^*|| < \delta$. In other words, any trajectory that starts within a distance δ if x^* is guaranteed to converge to x^* eventually. Trajectories that start nearby are allowed to stray from x^* in the short run, but they must approach x^* in the long run. In contrast, Liapunov stability requires that nearby trajectories remain close for all time. We say that x^* is Liapunov stable if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $||x(t) - x^*|| < \varepsilon$ whenever $t \geqslant 0$ and $||x(0) - x^*|| < \delta$. Thus, trajectories that start within δ pf x^* remain within ε for x^* for all positive time. Finally, x^* is asymptotically stable if it is both attracting and Liapunov stable.

a)

$$\begin{cases} \dot{x} = y \\ \dot{y} = -4x \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

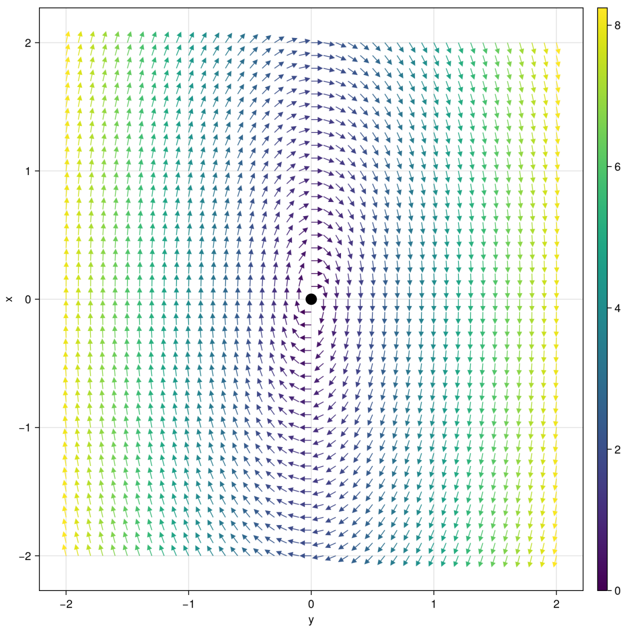

Figure 3: Velocity Field

From the figure, we can see and conclude that the origin is Liapunov stable.

b)

$$\begin{cases} \dot{x} = 2y \\ \dot{y} = x \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

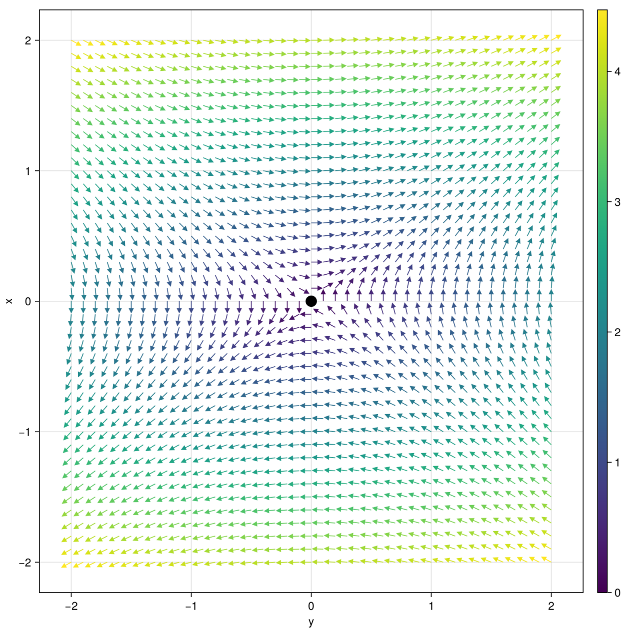

Figure 4: Velocity Field

From the figure, we can see and conclude that the origin is none of the above.

c)

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = x \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

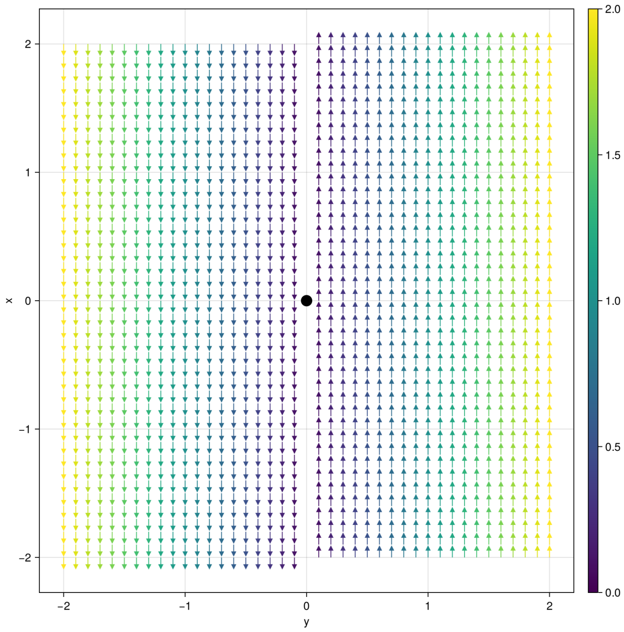

Figure 5: Velocity Field

From the figure, we can see and conclude that the origin is none of the above.

d)

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -y \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

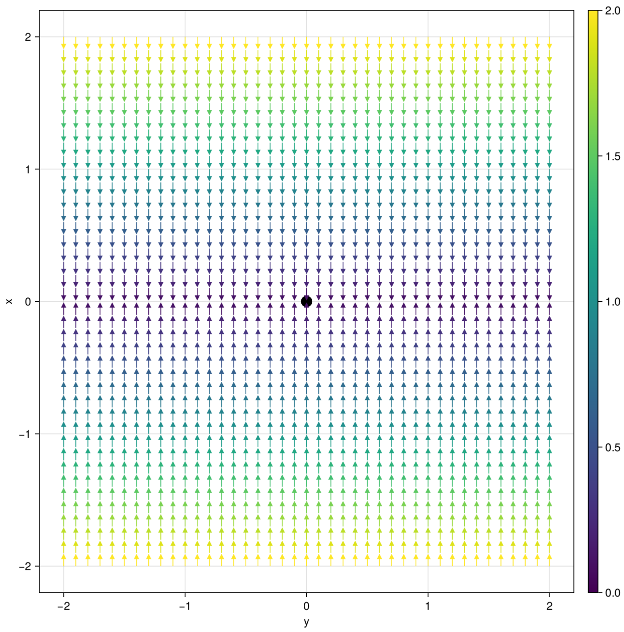

Figure 6: Velocity Field

From the figure, we can see and conclude that the origin is Liapunov stable.

e)

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -5y \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

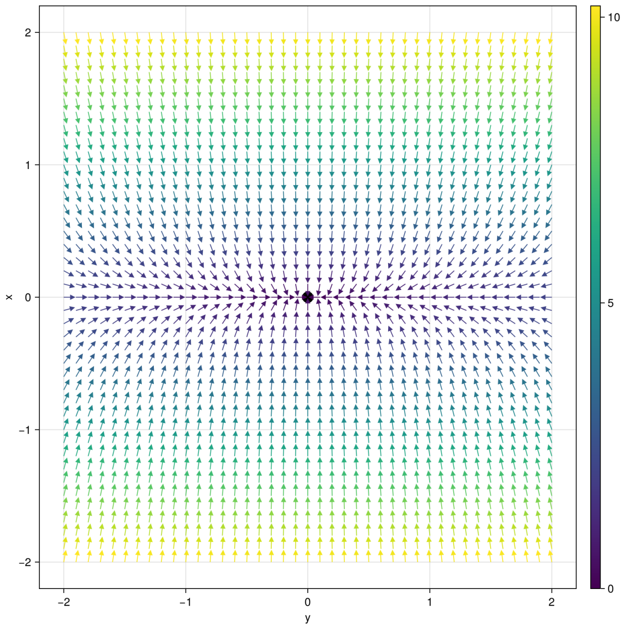

Figure 7: Velocity Field

From the figure, we can see and conclude that the origin is asymptotically stable.

f)

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$$

For the above system, when we plot the velocity field, we get the below figure,

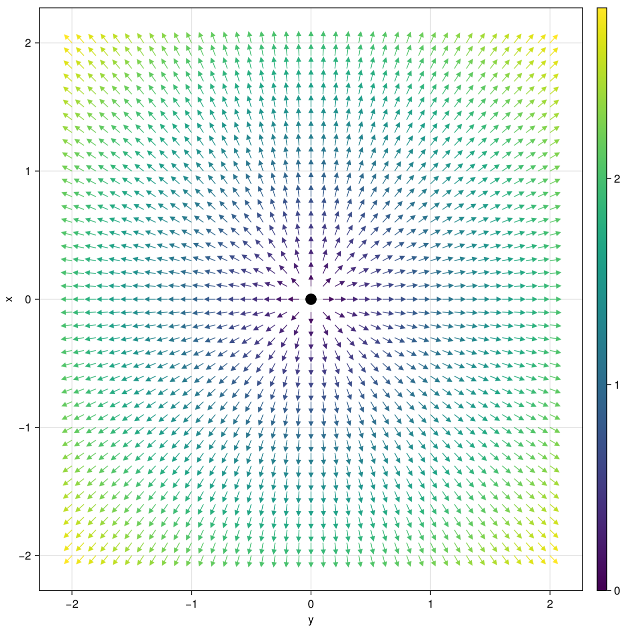

Figure 8: Velocity Field

From the figure, we can see and conclude that the origin is none of the above.

Problem 5.2.14

Suppose we pick a linear system at random. Consider the system $\dot{x} = Ax$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose we pick the entries a, b, c, d independently and at random from a uniform distribution on the interval $[-1, 1]$. Using a Monte Carlo method, code is attached, i generated millions of random matrices, -, first using a uniform distribution and then using a normal distribution, and I got the following relative probabilities,

Type	Uniform Probability	Normal Probability
Centers	0.000	0.000
Saddle Points	0.499	0.5
Unstable Nodes	0.053	0.058
Unstable Spirals	0.197	0.192
Stable Nodes	0.1971	0.192
Stable Nodes	0.053	0.058

Problem 6.5.6

Consider the Kermack-McKendrick model of an epidemic, let $x(t) \geqslant 0$ denote the size of the healthy population and $y(t) \geqslant 0$ denote the size of the sick people. Then the model is

$$\begin{cases} \dot{x} = -kxy \\ \dot{y} = kxy - ly \end{cases}$$

where $k, l > 0$, and the equation for $z(t)$, the number of deaths, plays no role in the x, y dynamics so we can omit it.

a) To find the fixed points, we set the change equal to zero.

$$\begin{aligned} \dot{x} &= 0 \\ -kxy &= 0 \\ x &= 0 \text{ or } y = 0 \end{aligned}$$

$$\begin{aligned} \dot{y} &= 0 \\ kxy - ly &= 0 \\ (kx - l)y &= 0 \\ x &= \frac{l}{k} \text{ or } y = 0 \end{aligned}$$

So, the fixed points are $(0, 0)$ and $(\frac{l}{k}, 0)$. To evaluate the stability of the fixed point, we first find the Jacobian of the system,

$$J(x, y) = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}$$

Then, we evaluate it at the fixed points and find the eigenvalues. From the eigenvalues, we might be able to determine the category of the fixed point belongs to. For $(0, 0)$

$$J(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & -l \end{pmatrix}$$

$$|J(0, 0) - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 0 & -l - \lambda \end{vmatrix}$$

So, the eigenvalues are,

$$\begin{aligned} (-\lambda)(-l - \lambda) &= 0 \\ \lambda = 0\& - l - \lambda &= 0 \\ \lambda = 0\&\lambda &= -1 \end{aligned}$$

Since one of the eigenvalues is zero, we can't determine the stability of the point using this method. Now, for $(\frac{l}{k}, 0)$,

$$J(\frac{l}{k}, 0) = \begin{pmatrix} 0 & -l \\ l & -l \end{pmatrix}$$
$$|J(0,0) - \lambda I| = \begin{vmatrix} -\lambda & -; \\ l & -l-\lambda \end{vmatrix}$$

So, the eigenvalues are,

$$(-\lambda)(-\lambda - l) + l^2 = 0$$
$$\lambda^2 + l\lambda + l^2 = 0$$
$$\lambda_{1,2} = \frac{-l \pm il\sqrt{3}}{2}$$

Both eigenvalues are imaginary, we can conclude that this fixed point is a linear center.

b) The nullclines and vector field is

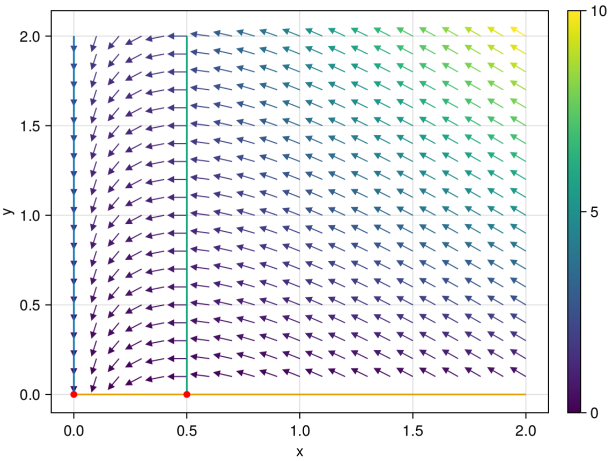

Figure 9: Nullclines and Vector Fields

c) To find a conserved quantity for the system, let's form a differential equation for $\frac{dy}{dx}$ and solve,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{kxy - ly}{-kxy}$$
$$\frac{dy}{dx} = \frac{(kx-l)y}{-kxy}$$
$$dy = -\frac{kx-l}{kx}dx$$
$$y = -\int \frac{kx-l}{lx}$$
$$= -\int \frac{kx'}{kx'}dx' + \int \frac{l}{kx'}dx'$$
$$= -\int dx' + \int \frac{l}{kx'}dx'$$
$$= -x + \frac{l}{k}lnx + C$$
$$C = y + x - \frac{l}{k}lnx$$

d) At $t \to \infty$, y goes to zero while $x < \frac{l}{k}$ which means all sick people died and there are less people in total now.

e) The epidemic occur when $\dot{y} > 0$, so

$$\begin{aligned} kx_0y_0 - ly_0 &> 0 \\ kx_0 - l &> 0 \\ kx_0 &> 0 \\ x_0 &> \frac{l}{k} \end{aligned}$$

When $x_0 > \frac{l}{k}$ and $y_0 > 0$, an epidemic occur.

Problem 6.5.7

The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$$

where $u = 1/r$ and r, θ are the polar coordinates of the planet in its plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics; the term εu^2 is Einstein's correction. Here ε is a very small positive parameter.

a) Let $v = \frac{du}{d\theta}$, the relativistic equation becomes,

$$\begin{cases} \frac{du}{d\theta} = v \\ \frac{dv}{d\theta} = \alpha + \varepsilon u^2 - u \end{cases}$$

b) The equilibrium point of the system are when the change is zero, so

$$\frac{du}{d\theta} = 0 \Rightarrow v = 0$$

$$\begin{aligned} \frac{dv}{d\theta} = 0 \Rightarrow \alpha + \varepsilon u^2 - u &= 0 \\ u^2 - \frac{1}{\varepsilon}u + \frac{\alpha}{\varepsilon} &= 0 \\ u &= \frac{-1/\varepsilon \pm \sqrt{1/\varepsilon^2 - 4\alpha/\varepsilon}}{2} \\ u &= -\frac{1}{2\varepsilon} \pm \frac{1}{2}\sqrt{\frac{1}{\varepsilon^2} - \frac{4\alpha}{\varepsilon}} \\ u &= \frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon\alpha} \end{aligned}$$

Therefore the equilibrium fixed points of the systems are,

$$\left(\frac{1}{2\varepsilon} + \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon\alpha}, 0\right)$$

$$\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon\alpha}, 0\right)$$

c) Let's evaluate the stability of the fixed points. To evaluate the stability of the points, first we need to linearize the system by computing the Jacobian of the system where the functions are,

$$\begin{aligned} f(u, v) &= v \\ g(u, v) &= \alpha + \varepsilon u^2 - u \end{aligned}$$

The Jacobian of the system is,

$$\begin{aligned} J(u, v) &= \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 2\varepsilon u - 1 & 0 \end{pmatrix} \end{aligned}$$

We plug in the fixed points and find it's eigenvalues. The eigenvalues of the linearized system are,

$$|J(u^*, v^*) - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 2\varepsilon u^* - 1 & -\lambda \end{vmatrix}$$

We find the trace τ and determined Δ of the linearized matrix, so we can determine the sign of the eigenvalues, and hence determine the stability of the point since $\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$. For the first fixed point $\left(\frac{1}{2\varepsilon} + \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon\alpha}, 0\right)$, the Jacobian is,

$$\begin{pmatrix} 0 & 1 \\ \sqrt{1 - 4\varepsilon\alpha} & 0 \end{pmatrix}$$

Then the trace and determinant of the matrix are,

$$\tau = 0$$
$$\Delta = -\sqrt{1 - 4\varepsilon\alpha}$$

Since $\varepsilon > 0$ and $\alpha > 0$, the trace is zero and the determinant is negative so the eigenvalues are real and $\lambda_+ > 0$ and $\lambda_- < 0$. So, this equilibrium point is a saddle point.
Now, determining the stability of the second equilibrium point, the Jacobian at $\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon\alpha}, 0\right)$ is,

$$\begin{pmatrix} 0 & 1 \\ -\sqrt{1 - 4\varepsilon\alpha} & 0 \end{pmatrix}$$

So, the trace and determinant are,

$$\tau = 0$$
$$\Delta = \sqrt{1 - 4\varepsilon\alpha}$$

The trace is zero and the determinant is positive, so both eigenvalues are purely imaginary. Hence, the fixed point is a linear center in the (u, v) phase plane.

We can try to see if the center is a nonlinear center using two methods. First, we see if the system is reversible and fits the reversibility condition.

$$f(u, -v) = -v = -f(u, v)$$
$$g(u, -v) = \alpha + \varepsilon u^2 - u = g(u, v)$$

So, $f(u, v)$ is odd in v and $g(u, v)$ is even in v. Hence, we can say that the system is reversible, and the center is a nonlinear center.
Another way to check if the center is nonlinear or not is by finding a conserved quantity. First, we evaluate the trace of the Jacobian,

$$Tr(J(u^*, v^*) = 0 + 0 = 0$$

The system has zero trace, so it is a conservative system. So, there must exist a conservative quantity. Since this system describes the motion of a planet in its plane of motion, then we can start by having in mind that the conserved quantity is the energy.

$$\begin{aligned} \frac{d^2u}{d\theta^2} &= \alpha + \varepsilon u^2 - u \\ u'' - \alpha - \varepsilon u^2 + u &= 0 \\ u'u'' - u'\alpha - \varepsilon u' u^2 + u'u &= 0 \\ \frac{d}{d\theta}\left(\frac{1}{2}u'^2 - u\alpha - \frac{\alpha}{3}u^3 + \frac{1}{2}u^2\right) &= 0 \\ \frac{d}{d\theta}(E) &= 0 \end{aligned}$$

Therefore, the conserved quantity is the energy,

$$E = \frac{1}{2}v^2 - \alpha u - \frac{\varepsilon}{3}u^3 + \frac{1}{2}u^2$$

In Strogatz book on Nonlinear Dynamics, theorem 6.5.2 states that if a fixed point is a local minimum of a conserved quantity, then all trajectories close to the point are closed, a nonlinear center. First, we need to check if the equilibrium point is an extreme of the energy, so we evaluate the partial derivatives at the value of the point,

$$\begin{aligned}
E_u &= -\alpha - \varepsilon u^2 + u \\
&= -\alpha - \varepsilon\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha}\right)^2 + \frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha} \\
&= -\alpha - \varepsilon\left(\frac{1}{4\varepsilon^2} - \frac{1}{4\varepsilon^2}(1-4\varepsilon\alpha) - \frac{1}{2\varepsilon^2}\sqrt{1-4\varepsilon\alpha}\right) - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha} \\
&= -\alpha - \frac{1}{4\varepsilon} + \frac{1}{4\varepsilon}(1-4\varepsilon\alpha) + \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha} \\
&= -\alpha - \frac{1}{4\alpha} + \frac{1}{4\varepsilon} + \alpha \\
&= 0 \\
E_v &= v \\
&= 0
\end{aligned}$$

Therefore, the point $\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha}, 0\right)$ is either a local minimum or maximum of the energy. To prove that the point is a local minimum of the energy, we evaluate the Hessian of E at that point.

$$\begin{aligned}
E_{uu} &= -2\varepsilon u + 1 \\
E_{vv} &= 1 \\
E_{uv} &= 0 \\
E_{vu} &= 0 \\
E_{uu}E_{vv} - E_{uv}E_{vu} &= -2\varepsilon u + 1 \\
&= -2\varepsilon\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha}\right) + 1 \\
&= -1 + \sqrt{1-4\varepsilon\alpha} + 1 \\
&= \sqrt{1-4\varepsilon\alpha} > 0
\end{aligned}$$

Since it $E_{uu}E_{vv} - E_{uv}Evu > 0$, then it is a local minimum of E, and it is a nonlinear center.

d) If the equilibrium point $\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha}, 0\right)$ corresponds to a circular planetary orbit, then the radius should be constant.

$$\begin{aligned}
u = \frac{1}{r} \Rightarrow r &= \frac{1}{u} \\
&= \frac{1}{\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon\alpha}} \\
&= \frac{2\varepsilon}{1-\sqrt{1-4\varepsilon\alpha}}
\end{aligned}$$

Since α and ε are constant quantities, then the radius is constant. Hence, it corresponds to a circular planetary orbit.

Problem 6.8.12

There;s an intriguing analogy between bifurcations of fixed points and collisions of particles and anti-particles. A two-dimensional version of the saddle-node bifurcation is given by

$$\begin{cases} \dot{x} = a + x^2 \\ \dot{y} = -y \end{cases}$$

where a is a parameter.

a) The fixed points of the system are

$$\dot{x} = 0 \Rightarrow a + x^2 = 0 \Rightarrow x = \pm\sqrt{-a}$$

$$\dot{y} = 0 \Rightarrow y = 0$$

The Jacobian is

$$J(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$$

Consider the point $(\sqrt{-a}, 0)$,

$$\tau = 2\sqrt{-a} - 1$$
$$\Delta = -2\sqrt{a}$$

If $a < 0$ and $a < -\frac{1}{4}$, $\tau > 0$ and $\Delta < 0$, then it is a saddle point, and if $a > -\frac{1}{4}$, $\tau < 0$ and $\Delta < 0$, and we also have a saddle point. So, for negative a, this fixed point is a saddle point. Now, if $a > 0$, the fixed point does not exist.
For the fixed point $(-\sqrt{a}, 0)$,

$$\tau = 2\sqrt{-a} - 1$$
$$\Delta = -2\sqrt{a}$$

if $a > 0$, there are no fixed points. For $-\frac{1}{4} < a < 0$, $\tau > 0$ and $\Delta > 0$, the point is an unstable node or spiral. If $a > -\frac{1}{4}$, $\tau < 0$ and $\Delta > 0$, then it is a stable node or spiral.

Problem 7.5.7

Tyson proposed an elegant model of the cell division cycle, based on interactions between the proteins cdc2 and cyclin. He showed that the model's mathematical essence is contained in the following set of dimensionless equations,

$$\begin{cases} \dot{u} = b(v-u)(\alpha - u^2) - u \\ \dot{v} = c - u \end{cases}$$

where u is proportional to the concentration of the active form of acdc2-cyclin complex, and v is proportional to the total cyclin concentration (monomers and dimers). The parameters $b \gg 1$ and $\alpha \ll 1$ and satisfy $8\alpha b < 1$, and c is adjustable.

a) To find the nullclines, we equate the derivatives to zero,

$$\begin{aligned} \dot{u} &= 0 \\ b(v-u)(\alpha+u^2) - u &= 0 \\ u &= b(v-u)(\alpha+u^2) \\ b(v-u) &= \frac{u}{\alpha+u^2} \\ v - u &= \frac{u}{b(\alpha+u^2)} \\ v &= \frac{u}{b(\alpha+u^2} + u^2 \\ v &= u\left(\frac{1}{b(\alpha+u^2)} + 1\right) \end{aligned}$$

$$\dot{v} = 0 \Rightarrow c - u = 0 \Rightarrow u = c$$

The nullclines are

$$\begin{cases} v = u\left(\frac{1}{b(\alpha+u^2)} + 1\right) \\ u = c \end{cases}$$

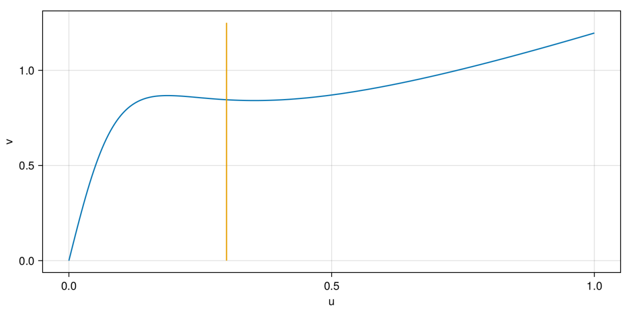

Figure 10: Nullclines

b) The maximum and minimum of the nullcline can be c_1 and c_2,

$$\frac{d}{du}(v) = \frac{d}{du}\left(u\left(\frac{1}{b(\alpha+u^2)}\right)+1\right)$$
$$\frac{1}{b(\alpha+u^2)} + 1 - \frac{2u^2}{b(\alpha+u^2)^2} = 0$$
$$\frac{(\alpha+u^2)+b(\alpha+u^2)^2-2u^2}{b(\alpha+u^2)} = 0$$
$$bu^4 + (2\alpha b - 1)u^2 + \alpha(1+b\alpha) = 0$$
$$u_1 = \sqrt{\frac{1-3\alpha b}{b}} and u^2 = \sqrt{\alpha}$$

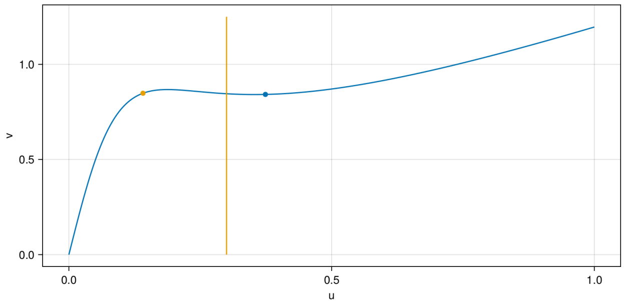

Figure 11: C1 and C2

You can also see that the system is conveying relaxation oscillations through the vector field plot,

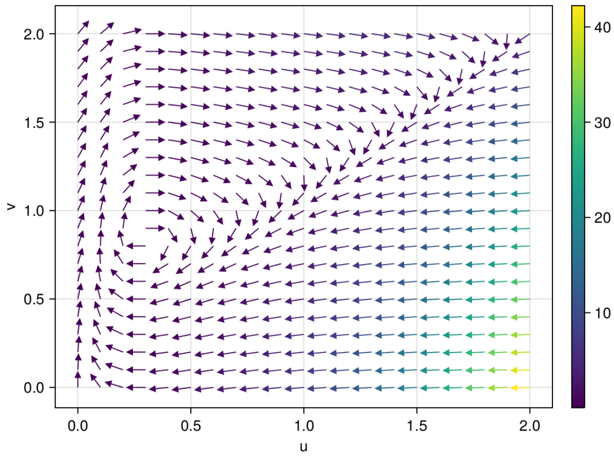

Figure 12: Vector Field Plot

Problem 7.6.2

Consider the following initial value problem,

$$\ddot{x} + x + \varepsilon x = 0$$
$$\begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases}$$

a)

$$\ddot{x} + (1+\varepsilon)x = 0$$
$$\ddot{x} = -(1-\varepsilon)x$$

The exact solution of the differential equation is given by the following,

$$x = Acos\left(\sqrt{1+\varepsilon t}\right) + Bsin\left(\sqrt{1+\varepsilon t}\right)$$
$$\dot{x} = -A\sqrt{1+\varepsilon}\ sin\left(\sqrt{1+\varepsilon t}\right) + B\sqrt{1+\varepsilon}\ cos\left(\sqrt{1+\varepsilon t}\right)$$

$$\begin{aligned} x(0) &= 1 \\ Acos(0) + Bsin(0) &= 1 \\ A &= 1 \end{aligned}$$

$$\begin{aligned} \dot{x}(0) &= 0 \\ -A\sqrt{1+\varepsilon}\ sin(0) + B\sqrt{1+\varepsilon}\ cos(0) &= 0 \\ B &= 0 \end{aligned}$$

Therefore, the exact solution is,

$$x = cos\left(\sqrt{1+\varepsilon} + 1\right)$$

b) Using regular perturbation theory and the following series expansion in x,

$$\begin{aligned} x(t,\varepsilon) &= x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3) \\ \frac{dx}{dt} &= \dot{(x)}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 \\ \frac{d^2x}{dt^2} &= \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 \end{aligned}$$

Now, we plug the expansion in our differential equation,

$$\begin{aligned} \ddot{x} + (1+\varepsilon)x &= 0 \\ \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + (1+\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2) &= 0 \\ \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 &= 0 \\ (\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + x_0) + \varepsilon^2(\ddot{x}_2 + x_2 + x_1) + O(\varepsilon^3) &= 0 \end{aligned}$$

This gives us three different differential equation for three different orders of ε, so we solve each one to get a good approximation of our solution.
Beginning with the zeroth order of epsilon,

$$\begin{aligned} \ddot{x}_0 + x_0 &= 0 \\ \ddot{x}_0 &= -x_0 \end{aligned}$$

The solution is,

$$\begin{aligned} x_0 &= Acost + Bsint \\ \dot{x}_0 &= -Asint + Bcost \end{aligned}$$

$$\begin{aligned} x_0(0) &= 1 \\ Acos(0) + Bsin(0) &= 1 \\ A &= 1 \end{aligned}$$

$$\begin{aligned} \dot{x}_0(0) &= 0 \\ -Asin(0) + Bcos(0) &= 0 \\ B &= 0 \end{aligned}$$

Therefore, the zeroth order of x is,

$$x_0 = cost$$

For the differential equation corresponding to the first order in ε, the equation is,

$$\begin{aligned} \ddot{x}_1 + x_1 + x_0 &= 0 \\ \ddot{x}_1 + x_1 &= x_0 \\ \ddot{x}_1 + x_1 &= -cost \end{aligned}$$

First, we solve the homogeneous part of the equation,

$$\begin{aligned}
\ddot{x}_1 &= -x_1 \\
x_{1c} &= Acost + Bsint \\
x_{1c}(0) = 1 \Rightarrow A(1) + B(0) = 1 &\Rightarrow A = 1 \\
\dot{x}_{1c}(0) = 0 \Rightarrow -A(0) + B(1) = 0 &\Rightarrow B = 0 \\
x_{1c} &= cost
\end{aligned}$$

Then, we find a particular solution of the original differential equation,

$$\begin{aligned}
x_{1p} &= Atcost + Btsint \\
\ddot{x}_{1p} &= -2Asint + 2Bcost - Atcost - Btsint
\end{aligned}$$

The differential equation becomes,

$$\begin{aligned}
-2Asint + 2Bcost - Atcost - Btsint + Atcost + Btsint &= -cost \\
-2Asint + 2Bcost &= -cost \\
-2A = 0 &\Rightarrow A = 0 \\
2B = -1 &\Rightarrow B = -\frac{1}{2} \\
x &= -\frac{1}{2}tsint
\end{aligned}$$

Therefore, the first order of x is,

$$x_1 = cost - \frac{1}{2}tsint$$

Lastly, we do the same for the equation corresponding to the second order of ε,

$$\begin{aligned}
\ddot{x}_2 + x_2 + x_1 &= 0 \\
\ddot{x}_2 + x_2 &= x_1 \\
\ddot{x}_2 + x_2 &= -cost + \frac{1}{2}tsint
\end{aligned}$$

First, we solve the homogeneous part of the equation,

$$\begin{aligned}
\ddot{x}_2 &= -x_2 \\
x_{2c} &= Acost + Bsint \\
x_{2c}(0) = 1 \Rightarrow A(1) + B(0) = 1 &\Rightarrow A = 1 \\
\dot{x}_{2c}(0) = 0 \Rightarrow -A(0) + B(1) = 0 &\Rightarrow B = 0 \\
x_{2c} &= cost
\end{aligned}$$

Then, we find a particular solution of the original differential equation,

$$\begin{aligned}
x_{2p} &= (At^2 + Bt)cost + (Ct^2 + Dt)sint \\
\ddot{x}_{2p} &= (-Ct^2 + (-D - 4A)t + 2C - 2B)sint + (-At^2 + (4C - B)t + 2D + 2A)cost
\end{aligned}$$

The differential equation becomes,

$$\begin{aligned}
(-Ct^2 + (-D - 4A)t + 2C - 2B)sint + (-At^2 + (4C - B)t + 2D + 2A)cost \\
+ (At^2 + Bt)cost + (Ct^2 + Dt)sint = -cost + \frac{1}{2}tsint \\
(-4At + 2C - 2B)sint + (2D + 2A - 4Ct)cost = -cost + \frac{1}{2}tsint
\end{aligned}$$

$$\begin{aligned}
-4A = \frac{1}{2} &\Rightarrow A = -\frac{1}{8} \\
2D + 2A = -1 &\Rightarrow 2D + 2\left(-\frac{1}{8}\right) = -1 \Rightarrow 2D = -1 + \frac{1}{4} = -\frac{3}{4} \Rightarrow D = -\frac{3}{8} \\
-4C = 0 &\Rightarrow C = 0 \\
2C - 2B = 0 &\Rightarrow B = C \Rightarrow B + 0
\end{aligned}$$

Therefore, the second order of x is,

$$x_2 = cost - \frac{1}{8}t^2cost - \frac{3}{8}tsint$$

Therefore, the solution of the initial value problem using perturbation theory is,

$$x = cost + \varepsilon\left(cost - \frac{1}{2}tsint\right) + \varepsilon^2\left(cost - \frac{1}{8}t^2cost = \frac{3}{8}tsint\right) + O(\varepsilon^3)$$

c) x_1 contains a first order of t which is a secular term because it grows and blow up as time approaches infinity, and x_2 contains a first and second order of t, which are also secular terms. So, the perturbation solution contains secular terms. However, this is unexpected because the equation can be thought of as that of a forced harmonic oscillator, so physically the system doesn't grow as time reaches infinity. This shows the inconsistency that perturbation theory has when solving differential equations of a physical system.

Problem 7.6.16

Consider the van der Pol oscillator, in the limit $\varepsilon \ll 1$,

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0$$

Assume that the limit cycle is a circle of unknown radius a about the origin. The system can be written as,

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\varepsilon y(x^2 - 1) - x \end{cases}$$

So, we can construct a vector $\vec{v} = (\dot{x}, \dot{y}$ in cartesian coordinates, Since we are working with a limit cycle, it is better to transform the vector to polar coordinates when computing the line integral over the cycle.

$$\begin{aligned}
\vec{v} &= y\vec{x} + \left(-\varepsilon y(x^2 - 1)\right)\vec{y} \\
&= rsin\theta\left(cos\theta\vec{r} - sin\theta\vec{\theta}\right) + \left(-\varepsilon rsin\theta\left(r^2cos^2\theta - 1\right) - rcos\theta\right)\left(sin\theta\vec{r} + cos\theta\vec{\theta}\right) \\
&= rsin\theta cos\theta\vec{r} - rsin^2\theta\vec{\theta} + \left(-\varepsilon r^3 sin\theta cos^2\theta + \varepsilon rsin\theta - rcos\theta\right)\left(sin\theta\vec{r} + cos\theta\vec{\theta}\right) \\
&= rsin\theta cos\theta\vec{r} - rsin^2\theta\vec{\theta} - \varepsilon r^3 sin^2\theta cos^2\theta\vec{r} + \varepsilon rsin^2\theta\vec{r} - rcos\theta sin\theta\vec{r} - \varepsilon r^3 sin\theta cos^3\theta\vec{r} + \varepsilon rsin\theta cos\theta\vec{\theta} - rcos^2\theta\vec{\theta} \\
&= \left(-\varepsilon r^3 sin^2\theta cos^2\theta + rsin^2\theta - \varepsilon r^3 sin\theta cos^3\theta\right)\vec{r} + \left(-rsin^2\theta + \varepsilon rsin\theta cos\theta - rcos^2\theta\right)\vec{\theta} \\
&= \varepsilon\left(-r^3 sin^2\theta cos^2\theta + rsin^2\theta - r^3 sin\theta cos^3\theta\right)\vec{r} + \left(-r + \varepsilon rsin\theta cos\theta\right)\vec{\theta}
\end{aligned}$$

Since $\varepsilon \ll 1$, we can approximate $\vec{v}$ to,

$$\vec{v} \approx 0\vec{r} - r\vec{\theta} = -r\vec{\theta}$$

We evaluate the line integral over the cycle C,

$$\oint_C \vec{v} \cdot \vec{n}dl = \oint_C -r\vec{\theta} \cdot \vec{r} = 0$$

We now find the divergence of $\vec{v}$,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{v} &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}\left(-\varepsilon y(x^2 - 1) - x\right) \\
&= 0 - \varepsilon(x^2 - 1) \\
&= -\varepsilon(x^2 - 1) \\
&= -\varepsilon(r^2cos^2\theta - 1)
\end{aligned}$$

We evaluate the surface integral over the area enclosed by the cycle.

$$\begin{aligned}
\iint\limits_A &= \vec{\nabla}\cdot\vec{v}dA = \int_0^a\int_0^{2\pi} -\varepsilon(r^2cos^2\theta - 1)rdrd\theta \\
&= \int_0^a\int_0^{2\pi} -\varepsilon r^3cos^2\theta drd\theta + \int_0^a\int_0^{2\pi} \varepsilon rdrd\theta \\
&= -\varepsilon\left[\frac{r^4}{4}\right]_0^a\left[\frac{sin2\theta}{4}+\frac{\theta}{2}\right]_0^{2\pi} + \varepsilon\left[\frac{r^2}{2}\right]_0^a[\theta]_0^{2\pi} \\
&= -\varepsilon\left(\frac{a^4}{4}\right)(\pi) + \left(\varepsilon\frac{a^2}{2}\right)(2\pi) \\
&= \varepsilon\left(-\frac{a^4\pi}{4} + a^2\pi\right) \\
&= \varepsilon a^2\pi\left(-\frac{a^2}{4}+1\right)
\end{aligned}$$

The normal form of Green's theorem, divergence theorem, states

$$\oint_C \vec{v}\cdot\vec{n}dl = \iint\limits_A \vec{\nabla}\cdot\vec{v}dA$$

So, from this relation we can find a,

$$\begin{aligned}
\varepsilon a^2\pi\left(-\frac{a^2}{4}+1\right) &= 0 \\
a^2 &= 0 \Rightarrow a = 0 \\
-\frac{a^2}{4}+1 = 0 \Rightarrow a^2 &= 4 \Rightarrow a = \pm 2
\end{aligned}$$

Since a is a radius, then it should be a non-zero positive number. So, the only accepted solution for a is $a = 2$.

Problem 7.6.17

A simple model for a child playing on a swing is

$$\ddot{x} + (1+\varepsilon\gamma+\varepsilon cos2t)sinx = 0$$

where ε and γ are parameters, and $0 < \varepsilon \ll 1$. The variable x measures the angle between the swing and the downward vertical. The term $1+\varepsilon\gamma+\varepsilon cos2t$ models the effects of gravity and the periodic pumping of the child's legs at approximately twice the natural frequency of the swing.

a) For small x. the sine function can be approximated to x, so the equation becomes,

$$\ddot{x} + (1+\varepsilon\gamma+\varepsilon cos2t)x = 0$$

We rearrange the terms to obtain an equation for a weakly nonlinear oscillator,

$$\ddot{x} + x + \varepsilon(\gamma + cos2t)x = 0$$

We can read off that $h(x,\dot{x}) = (\gamma + cos2t)x$. Let $t = \theta - \phi$, so $h = (\gamma + cos(2\theta - 2\phi))rcos(\theta)$. $T = \varepsilon t$ is slow time.

$$
\begin{aligned}
\frac{dr}{dT} = r' &= \langle h sin\theta \rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\theta) sin\theta d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\gamma + cos(2\theta - 2\phi)\, r cos\theta sin\theta d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\gamma + cos2\theta cos2\phi + sin2\theta sin2\phi)\, r cos\theta sin\theta d\theta \\
&= \frac{1}{2\pi} r\left[\int_0^{2\pi} \gamma cos\theta sin\theta + \int_0^{2\pi} cos2\theta cos2\phi cos\theta sin\theta d\theta + \int_0^{2\pi} sin2\theta sin2\phi cos\theta sin\theta d\theta\right] \\
&= \frac{1}{2\pi} r\left[\gamma\left[\frac{cos^2\theta}{2}\right]_0^{2\pi} + \frac{cos2\phi}{2}\left[-\frac{cos^2\theta}{4}\right]_0^{2\pi} + \frac{sin2\phi}{2}\left[-\frac{sin4\theta - 4\theta}{8}\right]_0^{2\pi}\right] \\
&= \frac{1}{2\pi} r\left[0 + 0 - \pi sin2\phi + 0\right] \\
&= \frac{1}{2} r \frac{sin2\phi}{2} \\
\Rightarrow r' &= \frac{1}{4} r sin2\phi
\end{aligned}
$$

$$
\begin{aligned}
r\frac{d\phi}{dT} = r\phi' &= \langle h cos\theta \rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\theta) sin\theta d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\gamma + cos2\theta cos2\phi + sin2\theta sin2\phi)\, r cos^2\theta d\theta \\
&= \frac{1}{2\pi} r\left[\int_0^{2\pi} \gamma cos^2\theta + \int_0^{2\pi} cos2\theta cos2\phi cos^2\theta d\theta + \int_0^{2\pi} sin2\theta sin2\phi cos^2\theta d\theta\right] \\
&= \frac{1}{2\pi} r\left[\gamma\left[\frac{\theta}{2}\right]_0^{2\pi} + \frac{\gamma}{2}\left[\frac{sin2\theta}{2}\right]_0^{2\pi} + \frac{cos2\phi}{2}\left[\frac{sin2\theta}{2}\right]_0^{2\pi} + \frac{cos2\phi}{2}\left[-\frac{sin4\theta + 4\theta}{8}\right]_0^{2\pi}\right] \\
&+ \frac{1}{2\pi} r\left[\frac{sin2\phi}{2}\left[-\frac{cos2\theta}{2}\right]_0^{2\pi} + \frac{sin2\phi}{2}\left[-\frac{cos^2 2\theta}{4}\right]_0^{2\pi}\right] \\
&= \frac{1}{2\pi} r\left[\gamma(\pi) + 0 + 0 + \frac{\pi cos2\phi}{2} + 0 + 0\right] \\
&= \frac{1}{2} r\left(\gamma + \frac{1}{2} cos2\phi\right) \\
\Rightarrow \phi' &= \frac{1}{2}\left(\gamma + \frac{1}{2} cos2\phi\right)
\end{aligned}
$$

b)

$$
\begin{cases}
r' = \frac{1}{4} r sin2\phi \\
\phi' = \frac{1}{2}\left(\gamma + \frac{1}{2} cos2\phi\right)
\end{cases}
$$

Consider the fixed point $r = 0$, and $\phi' \gg r'$ so ϕ equilibrates relatively fast.

$$
\begin{aligned}
\phi' &= 0 \\
\frac{1}{2}\left(\gamma + \frac{1}{2} cos2\phi\right) &= 0 \\
\gamma + \frac{1}{2} cos2\phi &= 0 \\
cos2\phi &= -2\gamma
\end{aligned}
$$

The cosine function is bounded by -1 and 1, so the equation becomes,

$$
\begin{aligned}
-1 \leqslant -2\gamma \leqslant \\
\frac{1}{2} \geqslant \gamma \geqslant -\frac{1}{2} \\
-\frac{1}{2} \leqslant \gamma \leqslant \frac{1}{2} \\
|\gamma| \leqslant \frac{1}{2}
\end{aligned}
$$

Therefore, the critical value for γ is $\frac{1}{2}$, and $\phi^* = \frac{1}{2}arccps(-2\gamma)$. Now, r' at ϕ^* is

$$r' = \frac{1}{4}rsin\left(arccos(-2\gamma)\right) = \frac{1}{4}r\sqrt{1-4\sqrt{1-4\gamma^2}}$$

If we evaluate the derivative of $r' = f(r)$, and evaluate $r = 0$, we get,

$$f' = \frac{1}{4}\sqrt{1-4\gamma^2} > 0$$

Therefore, we can say that $r = 0$ at ϕ^* is unstable.

c) To find a formula for the growth rate k in terms of γ, we solve the differential equation of r at ϕ^*,

$$
\begin{aligned}
\frac{dr}{dT} &= \frac{1}{4}rsin\left(arccos(-2\gamma)\right) \\
\int_0^r \frac{4}{r'\sqrt{1-4\gamma^2}} &= \int_0^T dT' \\
\frac{4}{\sqrt{1-4\gamma^2}}\int_0^r \frac{dr'}{r'} &= \int_0^T dT' \\
\frac{4}{\sqrt{1-4\gamma^2}} ln|r| &= T \\
ln|r| &= \frac{T}{4}\sqrt{1-4\gamma^2} \\
\Rightarrow r &= e^{\frac{1}{4}\sqrt{1-4\gamma^2}T}
\end{aligned}
$$

Therefore, for $|\gamma| < \frac{1}{2}$, the growth rate k is $k = \frac{1}{4}\sqrt{1-4\gamma^2}$.

d) For $|\gamma| > \frac{1}{2}$, lets find a solution for the averaged differential equation,

$$
\begin{aligned}
\frac{dr}{d\phi} = \frac{dr}{dT}\frac{dT}{\phi} = r'\frac{1}{\phi'} &= \frac{1/4rsin2\phi}{1/2(\gamma + \frac{1}{2}cos2\phi)} \\
\frac{dr}{d\phi} &= \frac{1}{2}\frac{rsin2\phi}{\gamma + \frac{1}{2}cos2\phi} \\
\int_0^r \frac{dr'}{r'} &= \int_0^\phi \frac{sin2\phi'}{2\gamma + cos2\phi'}d\phi'
\end{aligned}
$$

Consider the following substitution,

$$
\begin{aligned}
u &= 2\gamma + cos2\phi' \\
\frac{du}{d\phi'} &= -2sin2\phi \\
d\phi' sin2\phi' &= -\frac{1}{2}du \\
u(0) &= 2\gamma \\
u\phi &= 2\gamma + cos2\phi
\end{aligned}
$$

So, the integrals become,

$$\int_0^r \frac{dr'}{r'} = \int_{2\gamma}^{2\gamma+cos2\phi} -\frac{du}{2u}$$

$$ln|r| = -\frac{1}{2}\left(ln|2\gamma + cos2\phi| - ln|2\gamma|\right) + c$$

$$ln|r| = -\frac{1}{2}\left(ln\left(\frac{2\gamma + cos2\phi}{2\gamma}\right)\right) + c$$

$$\Rightarrow r = \frac{c}{\sqrt{2\gamma + cos2\phi}}$$

The solution obtained is bounded, so the trajectories that start near the origin, the fixed point, stay near the origin and are closed orbits.

Problem 7.6.19

Consider the Duffing equation,

$$\ddot{x} + x + \varepsilon x^3 = 0$$

where $0 < \varepsilon \ll 1$, $x(0) = a$, and $\dot{x}(0)$. We know from phase plane analysis that the true solution $x(t, \varepsilon)$ is periodic; our goal is to find an approximate formula for $x(t, \varepsilon)$ that is valid for all t. The key idea is to regard the frequency ω and unknown in advance, and to solve for it by demanding that $x(t, \varepsilon)$ contains no secular terms.

a) Let's define a new time $\tau = \omega t$ such that the solution has period 2π with respect to τ. The equation becomes,

$$\tau = \omega t \Rightarrow t = \frac{\tau}{\omega} \Rightarrow \frac{\omega}{d\tau} = \frac{1}{dt}$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) + x + \varepsilon x^3 = 0$$

$$\frac{\omega}{d\tau}\left(\omega\frac{dx}{d\tau}\right) + x + \varepsilon x^3 = 0$$

$$\Rightarrow \omega^2 x'' + x + \varepsilon x^3 = 0$$

b) Let $x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$ and $\omega(\varepsilon) = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$, we substitute these series into the differential equation and collect powers of ε.

$$\omega^2 x'' + x + \varepsilon x^3 = 0$$

$$(1 + \varepsilon\omega_1 + \varepsilon\omega_2)^2(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'') + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^3 = 0$$

$$(1 + 2\varepsilon\omega_1 + 2\varepsilon^2\omega_2 + \varepsilon^2\omega_1^2 + 2\varepsilon^3 w_1 w_2 + \varepsilon^4\omega_2^2)(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'') + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^3 x_2)^3 = 0$$

We collect the zeroth and first power of ε, we get,

$$O(1) : x_0'' + x_0 = 0$$

$$O(\varepsilon) : x_1'' + x_1 = -2\omega_1 x_0'' - x_0''$$

c) The initial conditions become,

$$x(0) = a$$

$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + ... + \varepsilon^k x_k = a$$

$$O(1) : x_0(0) = a$$

$$O(\varepsilon) : x_1(0) = 0$$

$$O(\varepsilon^2) : x_2(0) = 0$$

$$O(\varepsilon^3) : x_3(0) = 0$$

$$...$$

$$O(\varepsilon^k) : x_k(0) = 0$$

$$\dot{x}(0) = 0$$
$$\dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 + \varepsilon^3\dot{x}_3 + ... + \varepsilon^k\dot{x}_k = 0$$
$$O(1) : \dot{x}_0(0) = 0$$
$$O(\varepsilon) : \dot{x}_1(0) = 0$$
$$O(\varepsilon^2) : \dot{x}_2(0) = 0$$
$$O(\varepsilon^3) : \dot{x}_3(0) = 0$$
$$...$$
$$O(\varepsilon^k) : \dot{x}_k(0) = 0$$

So, the initial conditions become $x_0(0) = a,\ \dot{x}_0(0) = 0;\ x_k(0) = \dot{x}_k(0) = 0$ for all $k > 0$.

d) Consider the differential equation that corresponds to the zeroth order of ε,

$$\begin{cases} x_0'' = -x_0 \\ x_0(0) = a \\ \dot{x}(0) = 0 \end{cases}$$

$$\begin{aligned} x_0'' &= -x_0 \\ x_0 &= Acos\tau + Bsin\tau \\ x_0' &= -Asin\tau + Bcos\tau \\ x_0'' &= -Acos\tau - Bsin\tau \end{aligned}$$
$$x_0(0) = a \Rightarrow A(1) + B(0) = a \Rightarrow A = a$$
$$x_0'(0) = 0 \Rightarrow -A(0) + B(1) = 0 \Rightarrow B = 0$$
$$\Rightarrow x_0 = acos\tau$$

e)

$$\begin{aligned} O(\varepsilon) : x_1'' + x_1 &= -2\omega_1 x_0'' - x_0^3 \\ x_1'' + x_1 &= 2\omega_1 acos\tau - a^3cos^3\tau \\ x_1'' + x_1 &= 2\omega_1 acos\tau - \frac{3}{4}a^3cos\tau - \frac{1}{4}a^3cos3\tau \\ x_1'' + x_1 &= \left(2\omega_1 a - \frac{3}{4}a^3\right)cos\tau - \frac{1}{4}a^3cos3\tau \end{aligned}$$

To avoid secular terms, resonant terms, the coefficient multiplying $cos\tau$ should be zero,

$$2\omega_1 a - \frac{3}{4}a^3 = 0 \Rightarrow 2\omega_1 a = \frac{3}{4}a^3 \Rightarrow \omega_1 = \frac{3}{8}a^2$$

f)

$$x_1'' + x_1 = -\frac{1}{4}a^3cos3\tau$$

We first find the homogeneous solution,

$$\begin{aligned} x_1'' + x_1 &= 0 \\ x_1'' &= -x_1 \\ x_{1c} &= Acos\tau + Bsin\tau \\ x_{1c}' &= -Asin\tau + Bcos\tau \\ x_{1c}'' &= -Asin\tau - Bcos\tau \end{aligned}$$
$$x_{1c}(0) = 0 \Rightarrow A(1) + B(0) = 0 \Rightarrow A = 0$$
$$x_{1c}'(0) = 0 \Rightarrow A(0) + B(1) = 0 \Rightarrow B = 0$$

Now, we find a particular solution,

$$\begin{aligned}
x_1'' + x_1 &= -\frac{1}{4}a^3cos3\tau \\
x_{1p} &= Acos3\tau + Bsin3\tau \\
x_{1p}' &= -3Asin3\tau + 3Bcos3\tau \\
x_{1p}'' &= -9Acos3\tau - 9Bsin3\tau
\end{aligned}$$

$$\begin{aligned}
-9Acos3\tau - 9Bsin3\tau + Acos3\tau + Bsin3\tau &= -\frac{1}{4}a^3cos3\tau \\
\Rightarrow 8B = 0 \Rightarrow B &= 0 \\
\Rightarrow 8A = \frac{1}{4}a^3 \Rightarrow A &= \frac{1}{32}a^3 \\
\Rightarrow x_{1p} &= \frac{1}{32}a^3cos3\tau
\end{aligned}$$

Therefore, the solution for x_1 equation is,

$$x_1 = \frac{1}{32}a^3cos3\tau$$