Vector Tools for Computer Graphics

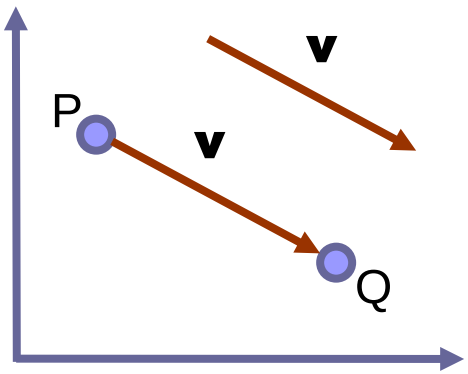
Computer Graphics

Basic Definitions

- Points specify location in space (or in the plane).
- Vectors have magnitude and direction (like velocity).

$$\boxed{\text{Points} \neq \text{Vectors}}$$

Basics of Vectors



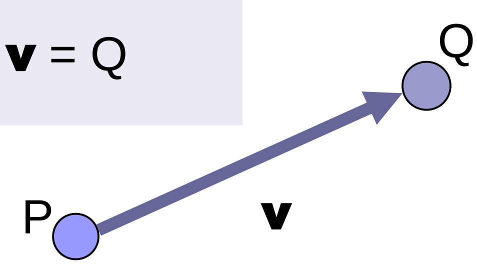
Vector as displacement:

v is a vector from point P to point Q.

The **difference** between two points is a vector: **v** = Q - P

Another way:

The **sum** of a point and a vector is a point : P + **v** = Q



Operations on Vectors

Two operations

Addition

$\mathbf{a} + \mathbf{b}$

$\mathbf{a} = (3,5,8)$, $\mathbf{b} = (-1,2,-4)$

$\mathbf{a} + \mathbf{b} = (2,7,4)$

Multiplication be scalars

$s\mathbf{a}$

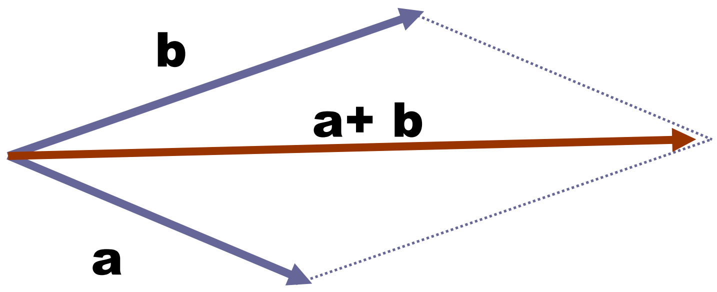
$\mathbf{a} = (3,-5,8)$, $s = 5$

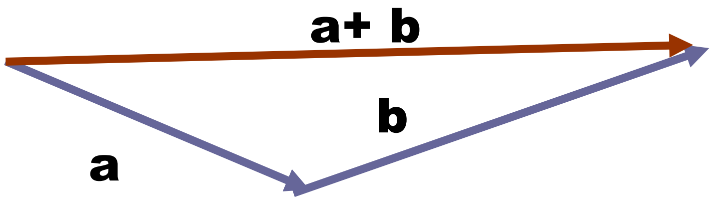
$5\mathbf{a} = (15,-25,40)$

operations are done ***componentwise***

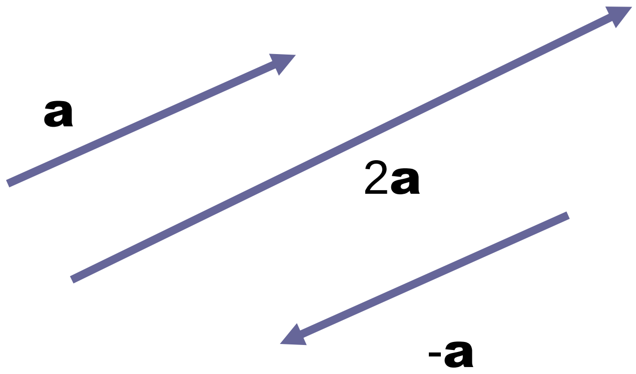
Operations on vectors

Addition



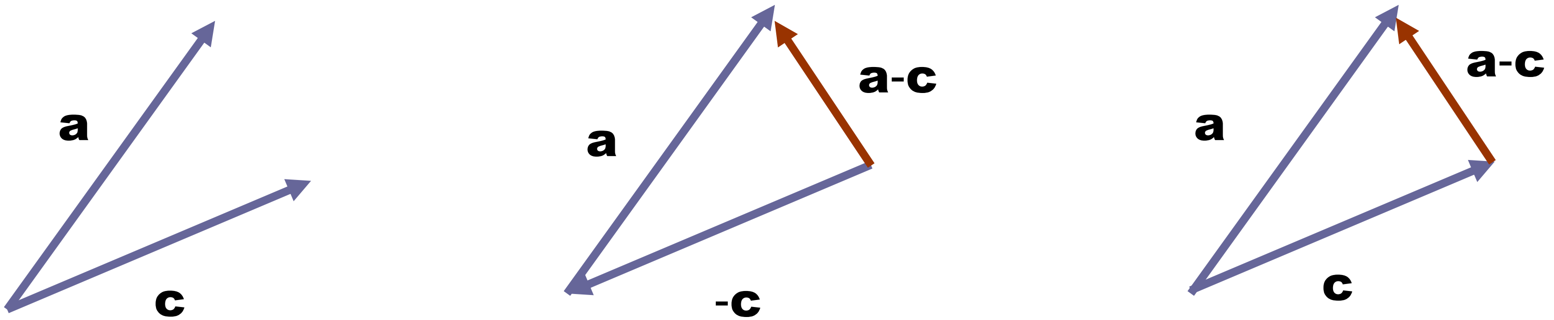


Multiplication by scalar



Operations on vectors

Subtraction



Properties of vectors

Length or size

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)$$

$$\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$$

Unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

- The process is called **normalizing**
- Used to refer **direction**

The **standard unit vectors**: $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$

Dot Product

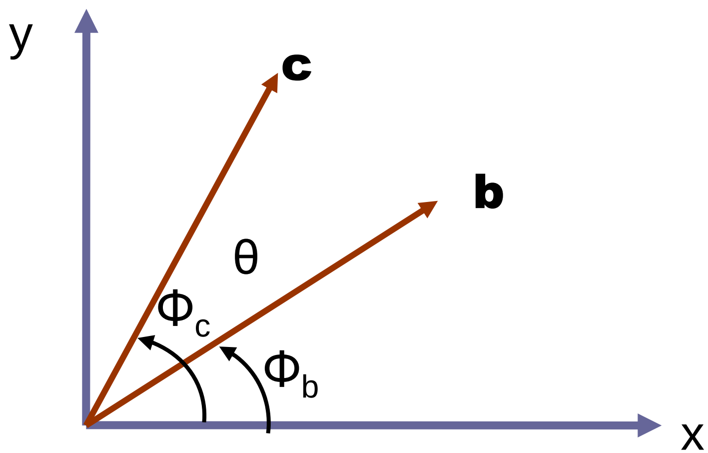
The dot product $\boldsymbol{d}$ of two vectors $\mathbf{v} = (v_1,v_2,\ldots,v_n)$ and $\mathbf{w} = (w_1,w_2,\ldots w_n)$:

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a}+\mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$

Application of Dot Product

Angle between two unit vectors **b** and **c**



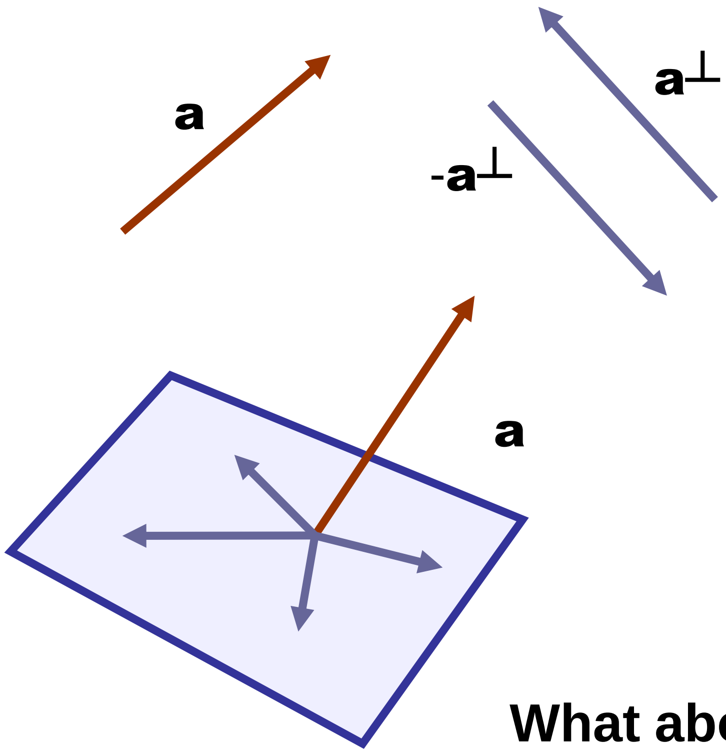
$$\cos(\theta) = \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$$

Two vectors **b** and **c** are perpendicular (**orthogonal/normal**) if
$\mathbf{b} \cdot \mathbf{c} = 0$

2D perp Vector

Which vector is perpendicular to the 2D vector **a** = (a_x,a_y)?

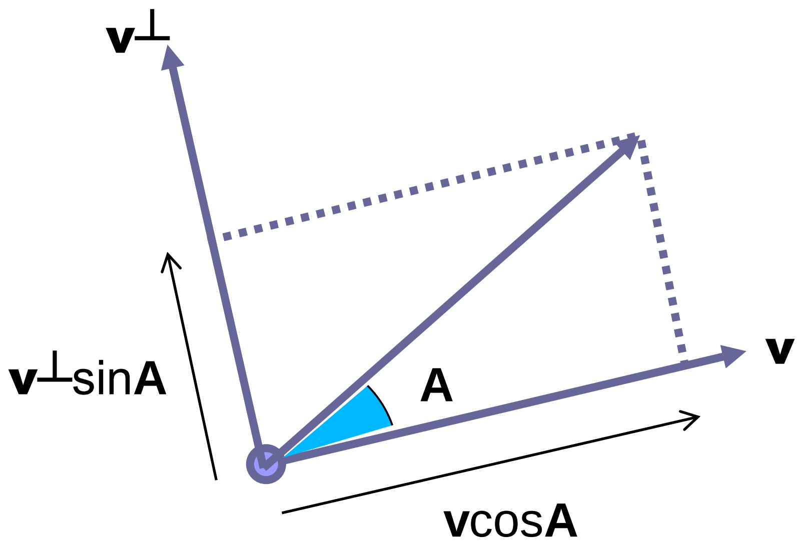
Let **a** = (a_x,a_y).
Then
$\mathbf{a}^{\perp} = (-a_y,a_x)$
is the
counterclockwise perpendicular to **a**.



What about 3D case?

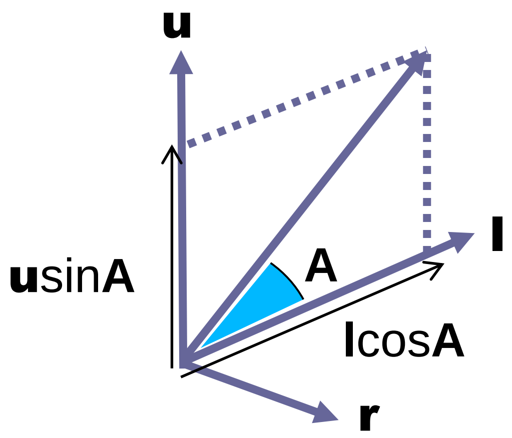
Rotation in 2d

- We want to rotate a 2d vector **v** counterclockwise by an angle **A**
- First we determine perp(**v**), $\mathbf{v}^{\perp}$
- Then we scale **v** by cos**A** and scale $v^{\perp}$ by sin**A** and take their sum



Rotation in 3d

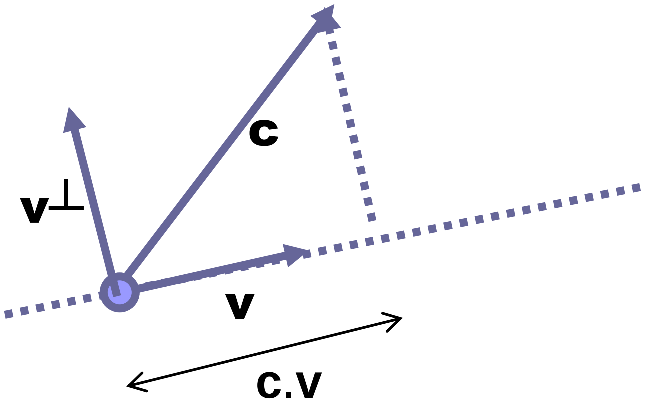
- We want to rotate a 3d vector **l** counterclockwise with respect to a 3d unit vector **r** by an angle **A,** where **l** and **r** are perpendicular to each other
- First we determine the vector **u**, that is perpendicular to both **l** and **r** and have a length equal to that of **l**
- So, **u** = **r** X **l**
- Then we scale **l** by cos**A** and scale **u** by sin**A** and take their sum



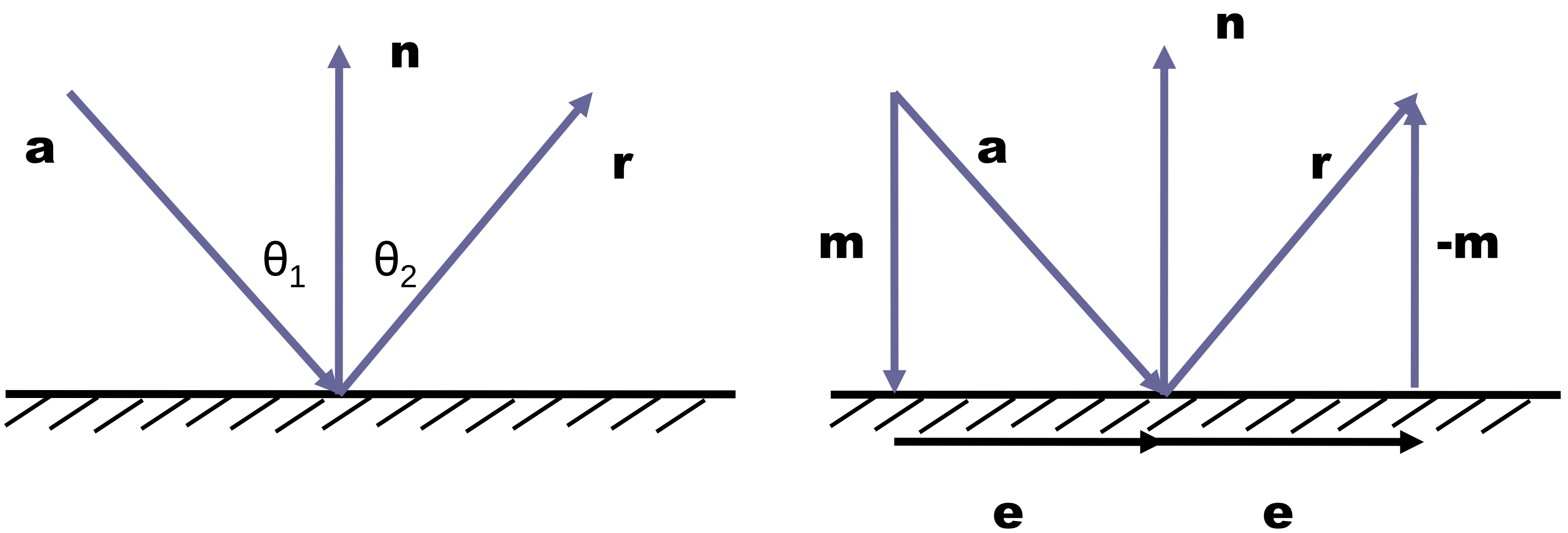
* note that, this method is applicable only in cases where the axis of rotation and the vector which is to be rotated are perpendicular to each other

Orthogonal Projection

- We want to decompose the vector **c** into two vectors, one along the direction of a unit vector **v** and another along perp(**v**)
- The length of the orthogonal projection of **c** along **v** is **c.v** (as **v** is a unit vector)
- Thus the component (or orthogonal projection) of **c** along **v** is (**c.v**)**v**
- So the component of **c** along perp(**v**) is **c**-(**c.v**)**v**



Reflection [13]



$$\bar{r} = \bar{a} - 2(\bar{a}\cdot\hat{n})\,\hat{n}$$

- Here **m** is the orthogonal projection of **a** along **n**
- **m** equals (**a.n**)**n** as **n** is a unit vector

Cross Product

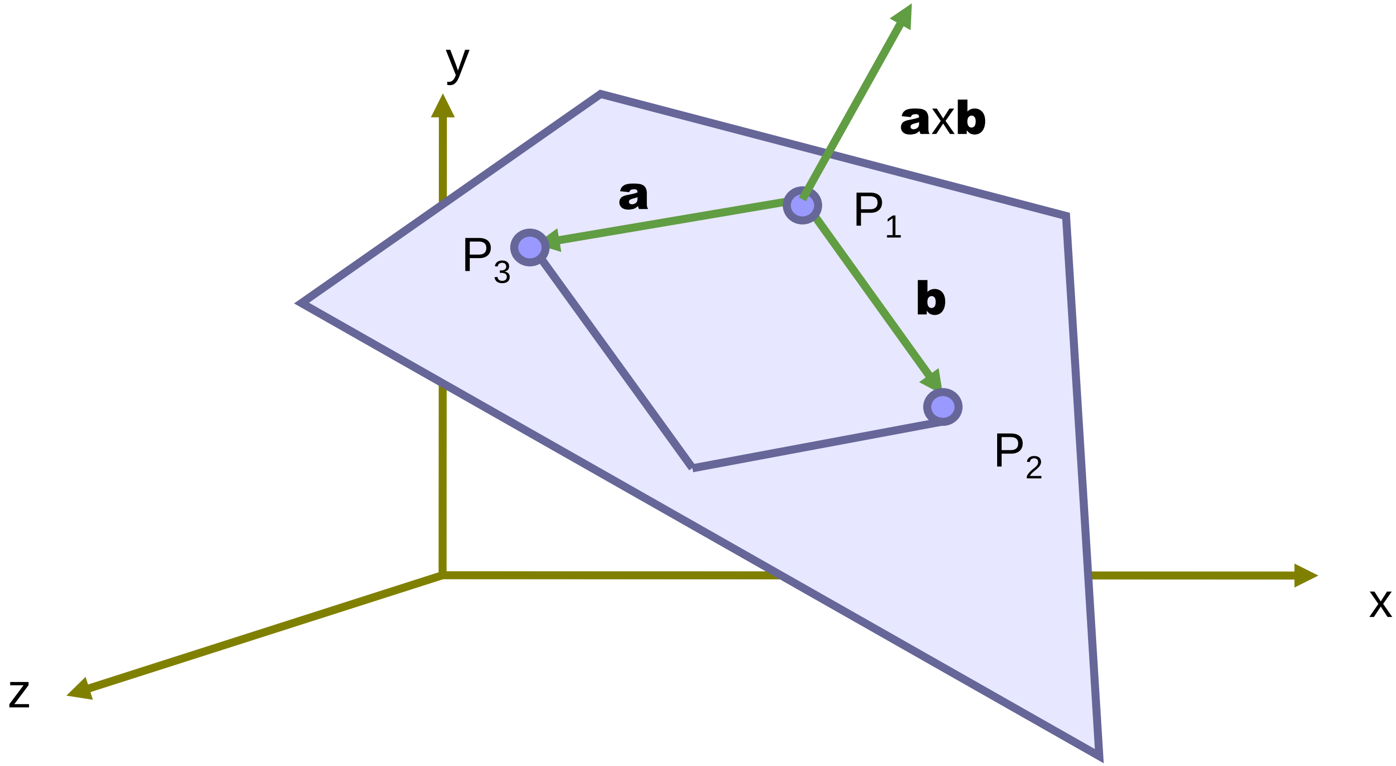
- Also called **vector product**.
- Defined for **3D** vectors only.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

Properties

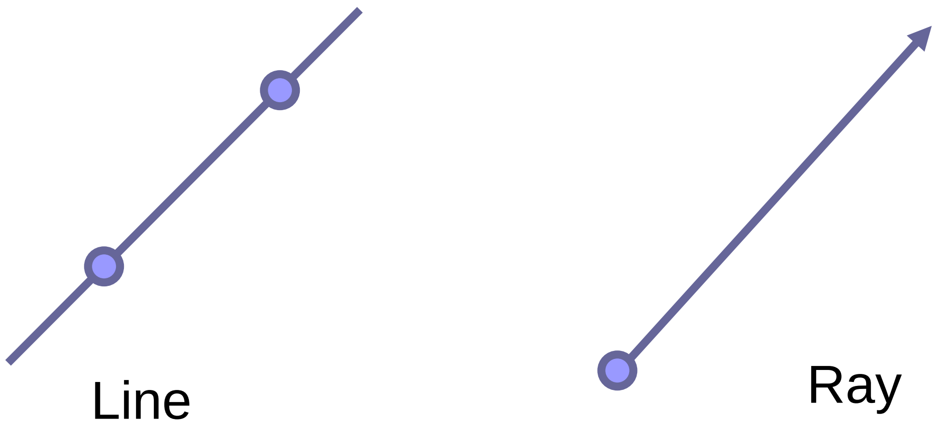
1. Antisymmetry: **a** X **b** = - **b** X **a**
2. Linearity: (**a** **+c**) X **b** = **a** X **b** + **c** X **b**
3. Homogeneity: (s**a**) X **b** = s(**a** X **b**)

Geometric Interpretation of Cross Product



1. **a**X**b** is perpendicular to both **a** and **b**
2. | **a**X**b |** = area of the parallelogram defined by **a** and **b**

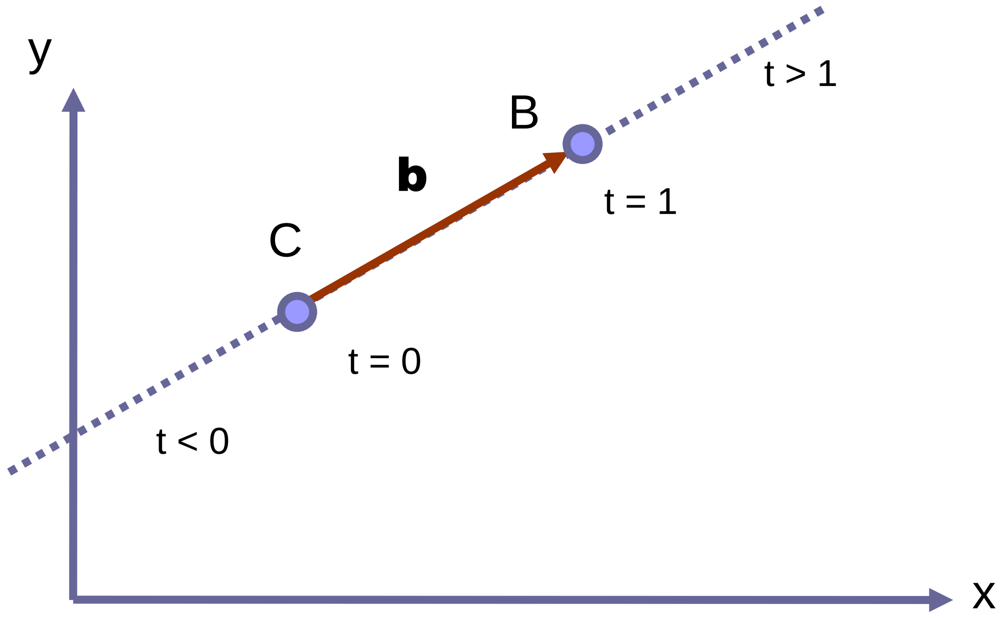
Representing Lines [11][12]



2 types of representations:

1. Two point form
2. Parametric representation

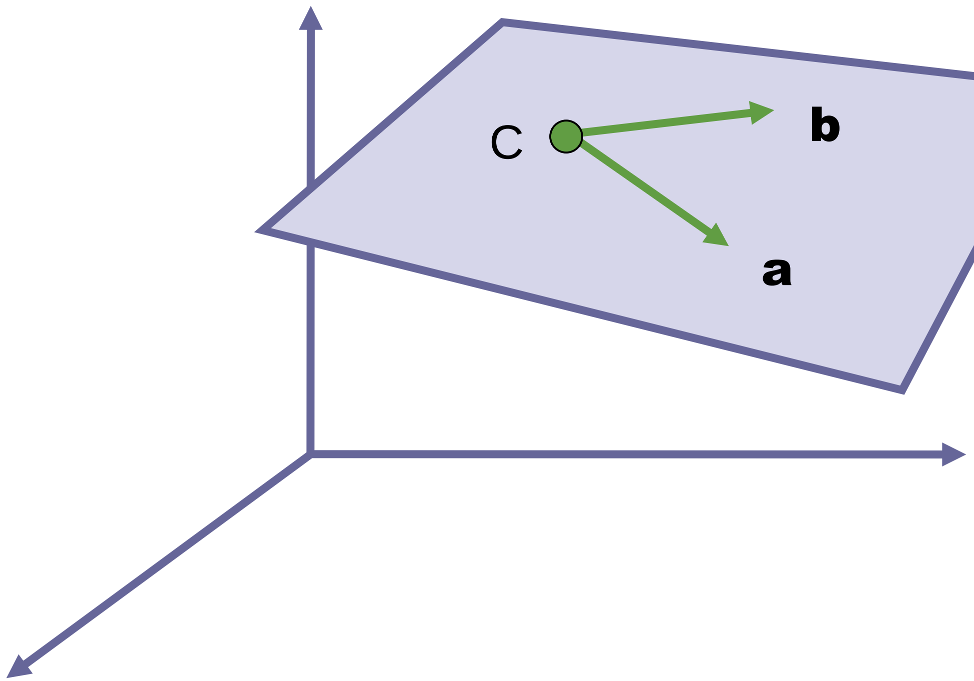
Parametric Representation of a Line



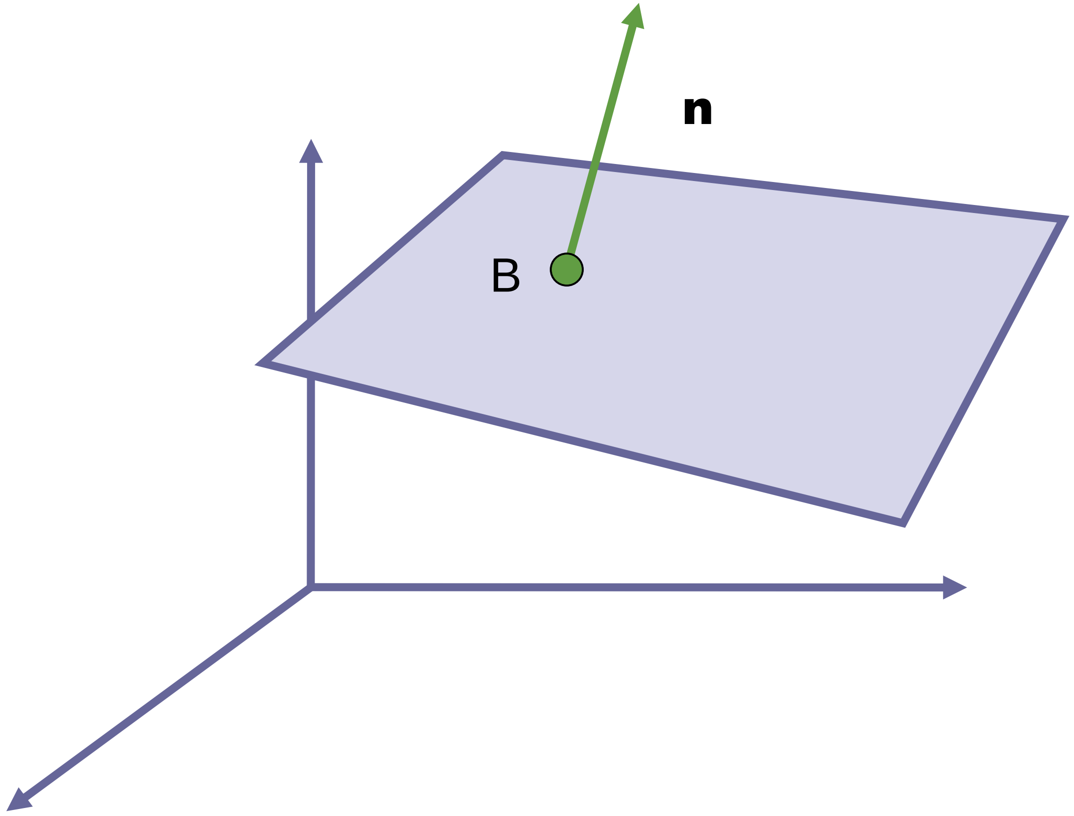
Planes in 3D

- 4 fundamental forms
  - Three-point form
  - Parametric representation
  - Point normal form
  - Equation

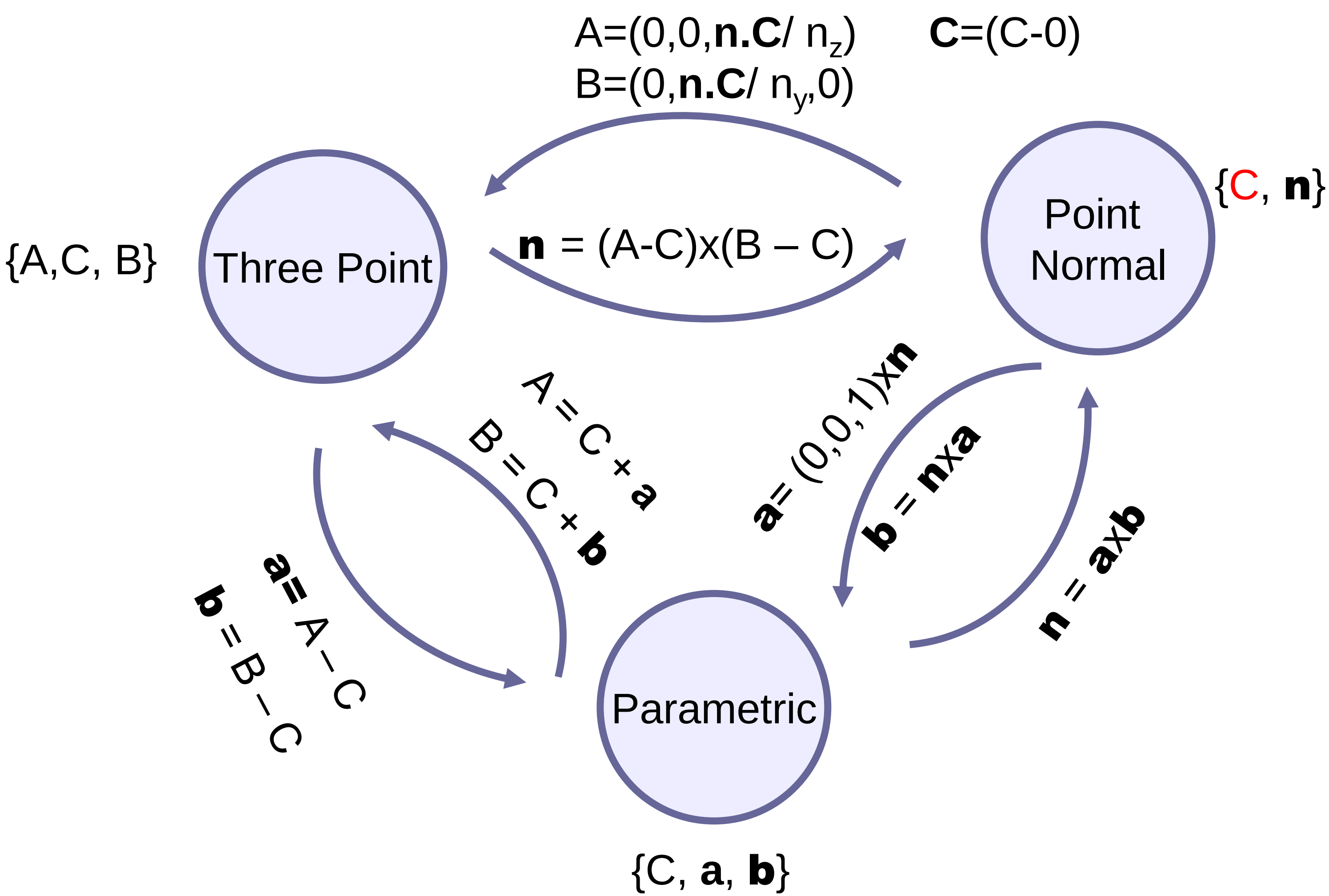
Parametric Representation of Plane



Point Normal Form of a Plane

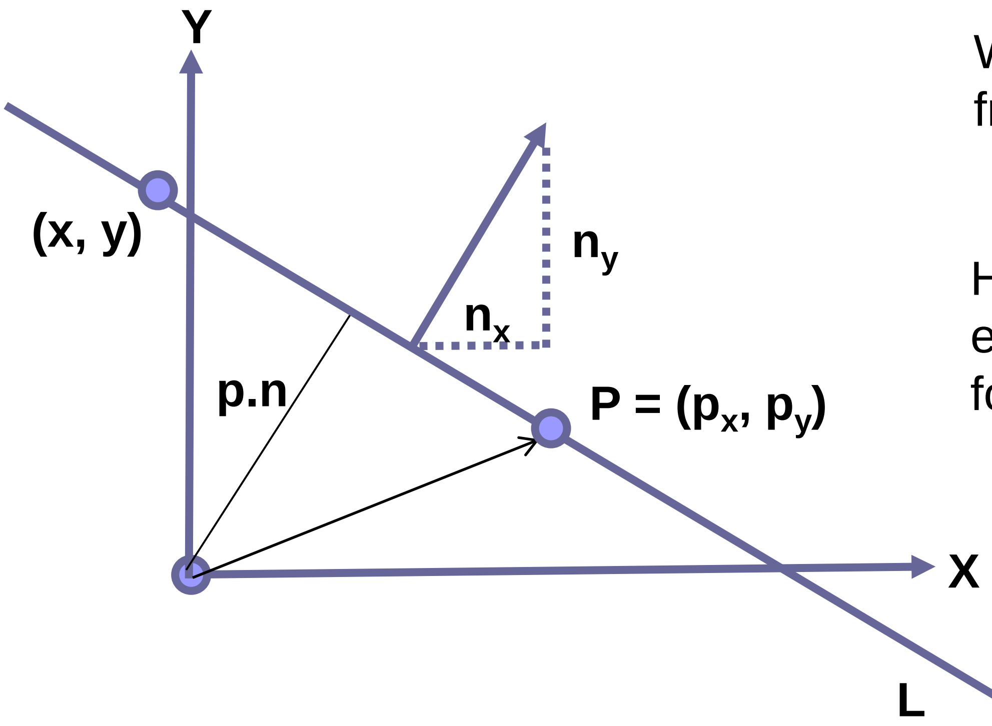


Representations of Plane [13]



Equation of a Plane [1][2][3][4]

- ax + by + cz + d = 0 is the standard equation of a plane in 3d
- If sqrt(a^2 + b^2 + c^2) = 1, then it is called the normalized form
- In the normalized form, |d| equals the distance of the plane from the origin



What about distance from an arbitrary point?

How to convert between equation and the other forms?

Line-Plane Intersection [8]

- Plane: $ax + by + cz + d = 0$
- Line: $P + tV$
- Determine the specific value of t (say t') for which the equation of the plane is satisfied, i.e., the point on the line lies on the plane

Line-Line Intersection [5][6][7]

- Four possible cases:
    - Coincident
    - Parallel
    - Not parallel and do not intersect
    - Not parallel and intersect

Line-Line Intersection

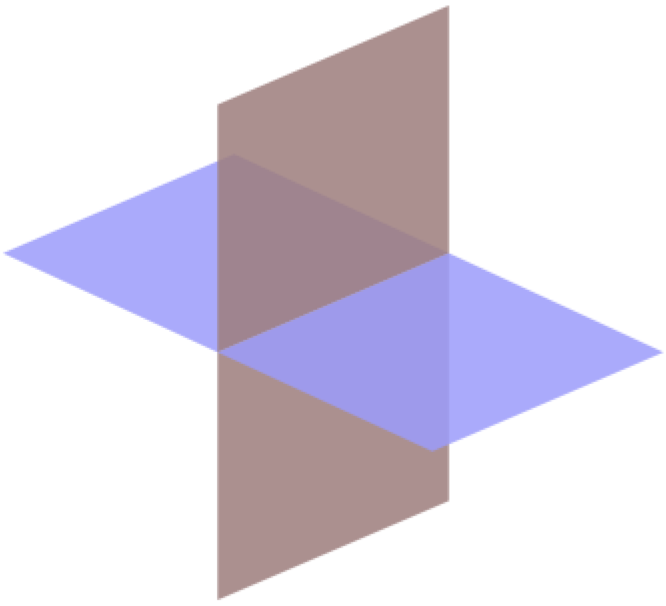
- L_1: $P_1 + tV_1$
- L_2: $P_2 + sV_2$
- Parallel if V_1 and V_2 are in the same or opposite direction (i.e., the angle between them are 0 degree or 180 degree)
- Coincident if they are parallel and have at least one point in common
- If they are not parallel, how to decide whether they intersect or not?

Line-Line Intersection

- If they are not parallel, how to decide whether they intersect or not?
- One solution
    - Generate three equations for two unknowns
    - Solve the first two equations to find a solution
    - Check whether the solution satisfies the third equation
- Another solution
    - Check whether $(P_1 – P_2).(V_1 \times V_2) = 0$
    - If lines intersect this condition must hold
    - If lines do not intersect, is it sure not to hold?

Plane-Plane Intersection [9][10]

- Intersection is a line
- So we need two points on the line, or one point and the direction
- How to get a point on the intersecting line?
- How to get the direction of the intersecting line?

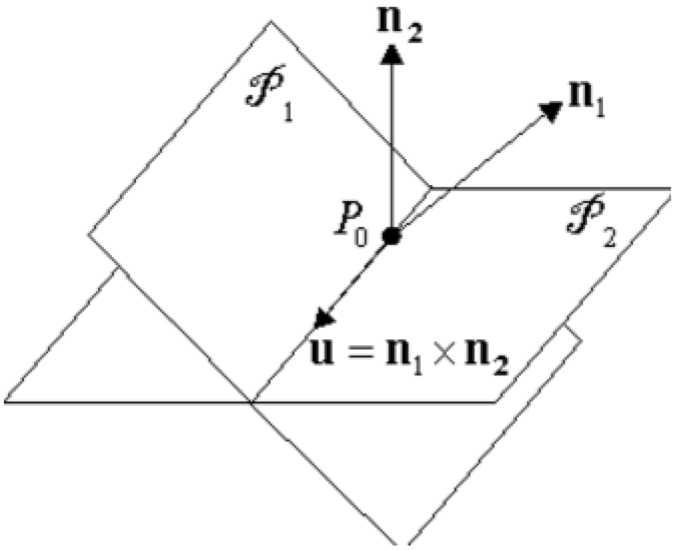


Plane-Plane Intersection

- How to get a point on the intersecting line?
  - Imagine another plane not parallel to any of the given planes, for example the plane $z = 0$ (the XY plane)
  - Now solve three plane equations to find their common intersection point

Plane-Plane Intersection

- How to get the direction of the intersecting line?
- Consider the planes in point-normal form
  - Plane 1: P_1, n_1
  - Plane 2: P_2, n_2
- n_1 and n_2 are both perpendicular to the intersecting line
- So the direction of the line of intersection is along n_1 X n_2



Linear Combination of Vectors

Definition

Linear combination of *m* vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$:

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\,\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m$$

where $a_1, a_2, \ldots a_m$ are scalars.

Types

Affine Combination

$a_1 + a_2 + \ldots + a_m = 1$

Examples:

$3\mathbf{a} + 2\mathbf{b} - 4\mathbf{c}$

$(1 - t)\mathbf{a} + (t)\mathbf{b}$

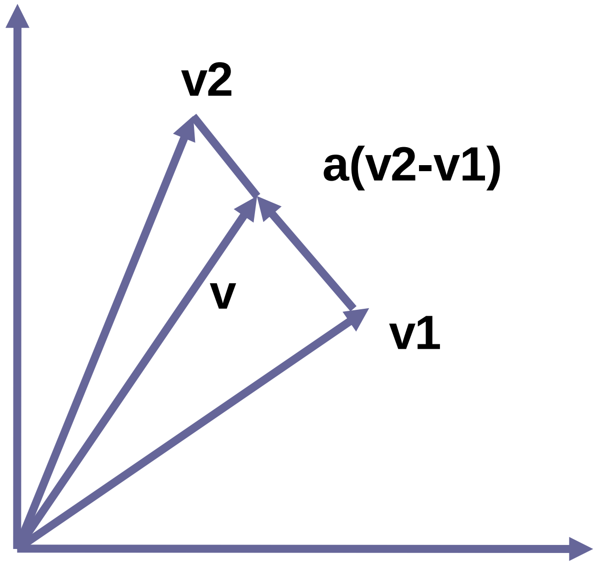
Convex Combination

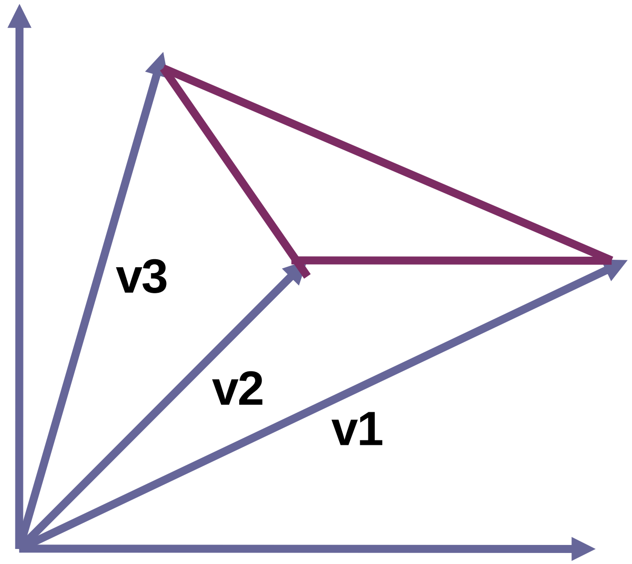
$a_1 + a_2 + \ldots + a_m = 1$

and $a_i \geq 0$, for $i = 1, \ldots m$.

Example: $.3\mathbf{a} + .2\mathbf{b} + .5\mathbf{c}$

Convex Combination of Vectors





Links

[1] http://www.songho.ca/math/plane/plane.html

[2] http://mathinsight.org/distance_point_plane

[3] https://www.youtube.com/watch?v=gw-4wltP5tY

[4] https://www.youtube.com/watch?v=7rIFO8hct9g

[5] https://www.youtube.com/watch?v=nKVCvY-FW5Q

[6] https://www.youtube.com/watch?v=bJ56Xr9081k

[7] https://www.youtube.com/watch?v=r5DwyBFxD7Q

[8] https://www.youtube.com/watch?v=Td9CZGkqrSg

[9] https://www.youtube.com/watch?v=SoSTdgqknvY

[10] https://www.youtube.com/watch?v=LpardiBTAvU

Links

[11] https://www.youtube.com/watch?v=FlLbI7DB0SM

[12] https://www.youtube.com/watch?v=nZ2mS5M4fcQ

[13] (textbook) Chapter 4, Computer Graphics using OpenGL (2nd edition) by Francis S Hill, Jr.