Likelihood Functions

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Maximum Likelihood Estimation for Different Distributions

Introduction

This assignment explores the maximum likelihood estimation (MLE) of parameters for several probability distributions: Binomial, Poisson, Exponential, Normal, Gamma, and Beta. The likelihood and log-likelihood functions are derived, and numerical results with visualizations are presented. For the Normal distribution, both parameters are unknown, while for the Gamma and Beta families, one parameter is assumed to be known.

Common Helper Function

A reusable plotting function is quite handy.

```r
library(tidyverse)

# General plotting function
plot_likelihood <- function(x, lik, mle_x, mle_y, xlab, ylab, title) {
  tibble(x = x, y = lik) %>%
    ggplot(aes(x, y)) +
    geom_line() +
    geom_segment(aes(x = mle_x, xend = mle_x, y = min(y), yend = mle_y),
                 col = "red", linewidth = 1) +
    labs(x = xlab, y = ylab, title = title) +
    theme_minimal()
}
```

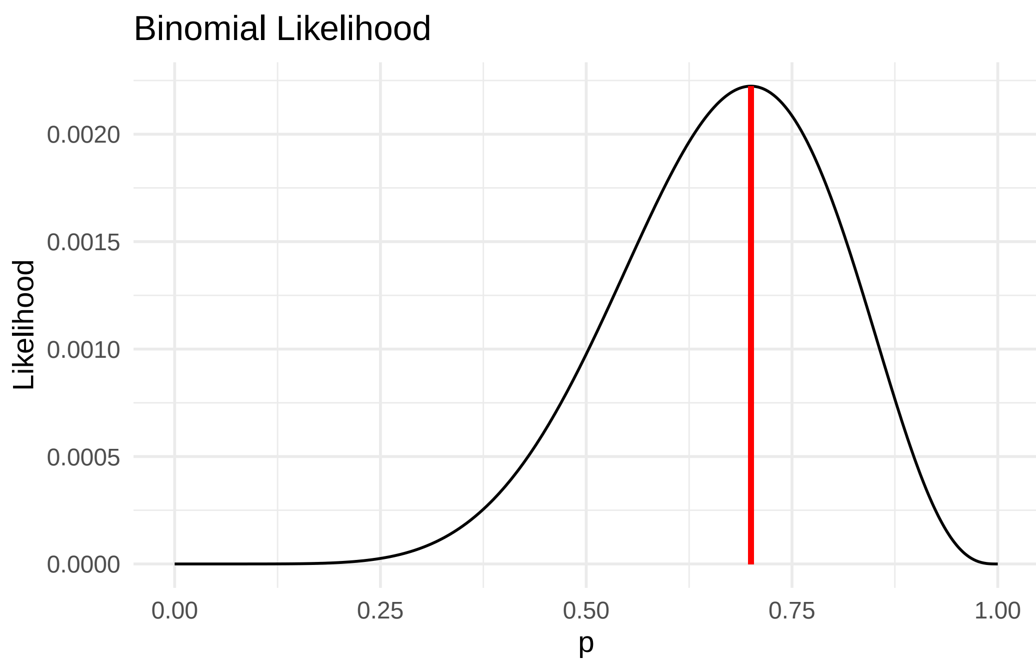
1. Binomial Distribution

Likelihood function:

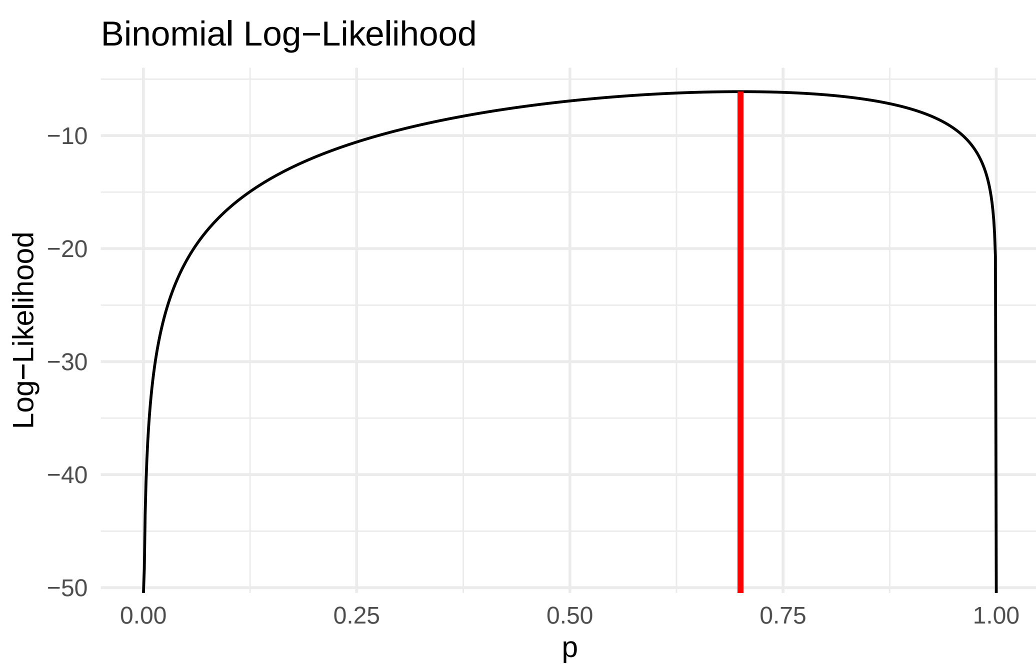
$$L(p \mid x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\ell(p) = x \log p + (n-x) \log(1-p)$$

```
# Parameters
n <- 10; k <- 7

# Likelihood
binom_lik <- function(p, n, k) {
  ifelse(p <= 0 || p >= 1, 0, p^k * (1 - p)^(n - k))
}

p_grid <- seq(0, 1, 0.001)
lik_vals <- sapply(p_grid, binom_lik, n = n, k = k)
loglik_vals <- log(lik_vals)

# MLE
mle_index <- which.max(loglik_vals)
mle_p <- p_grid[mle_index]

# Plots
plot_likelihood(p_grid, lik_vals, mle_p, lik_vals[mle_index],
                "p", "Likelihood", "Binomial Likelihood")
```




```r
plot_likelihood(p_grid, loglik_vals, mle_p, loglik_vals[mle_index],
                "p", "Log-Likelihood", "Binomial Log-Likelihood")
```



MLE: $\hat{p} = 0.7$

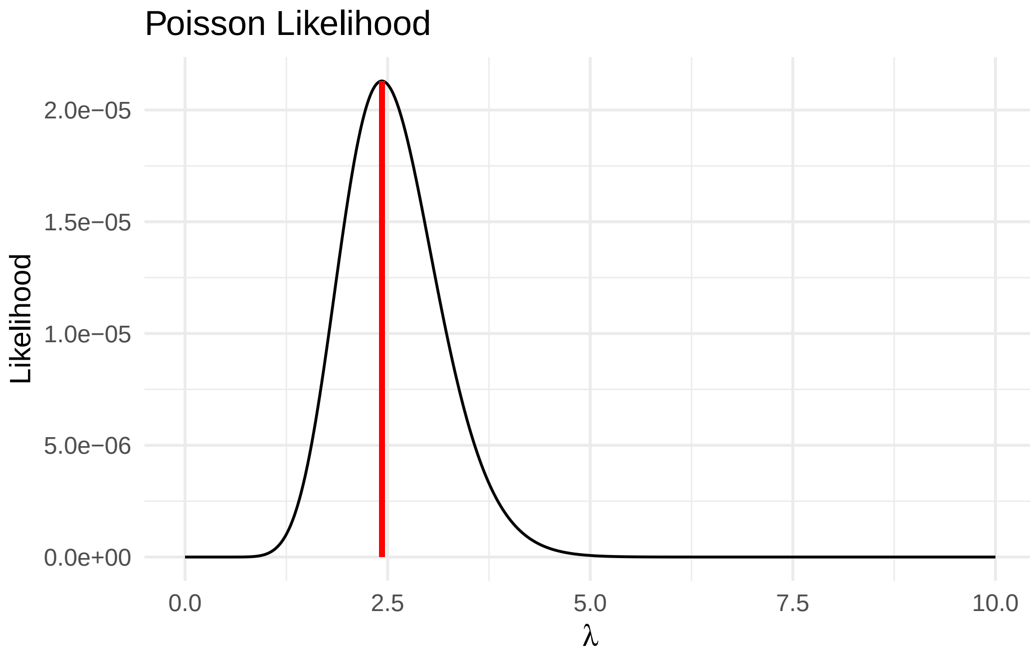
2. Poisson Distribution

$$L(\lambda \mid x) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

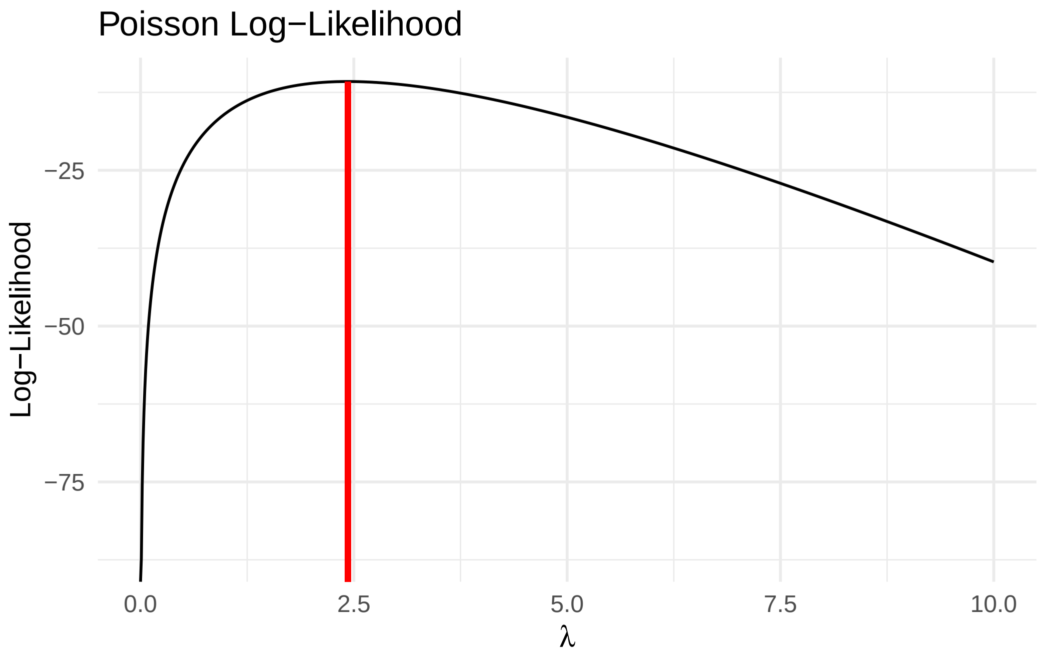
Taking natural logarithm,

$$\ell(\lambda) = -n\lambda + (\sum x_i)\log\lambda$$

```r
# Sample data
x <- c(2, 3, 4, 2, 1, 3, 2)
n <- length(x)
sum_x <- sum(x)

pois_lik <- function(lambda, x){
  ifelse(lambda <= 0, 0, prod(dpois(x, lambda = lambda)))
}

lambda_grid <- seq(0, 10, 0.01)
lik_vals <- sapply(lambda_grid, pois_lik, x = x)
loglik_vals <- log(lik_vals)

mle_index <- which.max(loglik_vals)
mle_lambda <- lambda_grid[mle_index]

plot_likelihood(lambda_grid, lik_vals, mle_lambda, lik_vals[mle_index],
                expression(lambda),
                "Likelihood", "Poisson Likelihood")
```


Poisson Likelihood
Likelihood
2.0e−05
1.5e−05
1.0e−05
5.0e−06
0.0e+00
0.0
2.5
5.0
7.5
10.0
λ

```r
plot_likelihood(lambda_grid, loglik_vals, mle_lambda, loglik_vals[mle_index],
                expression(lambda),
                "Log-Likelihood", "Poisson Log-Likelihood")
```

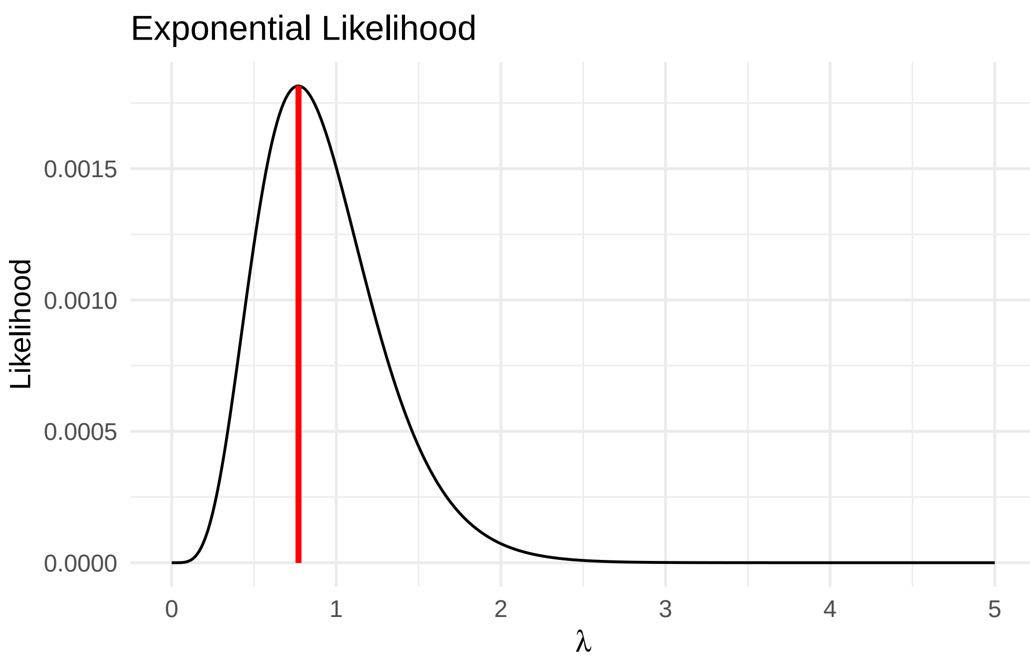


MLE: $\hat{\lambda} = 2.43$

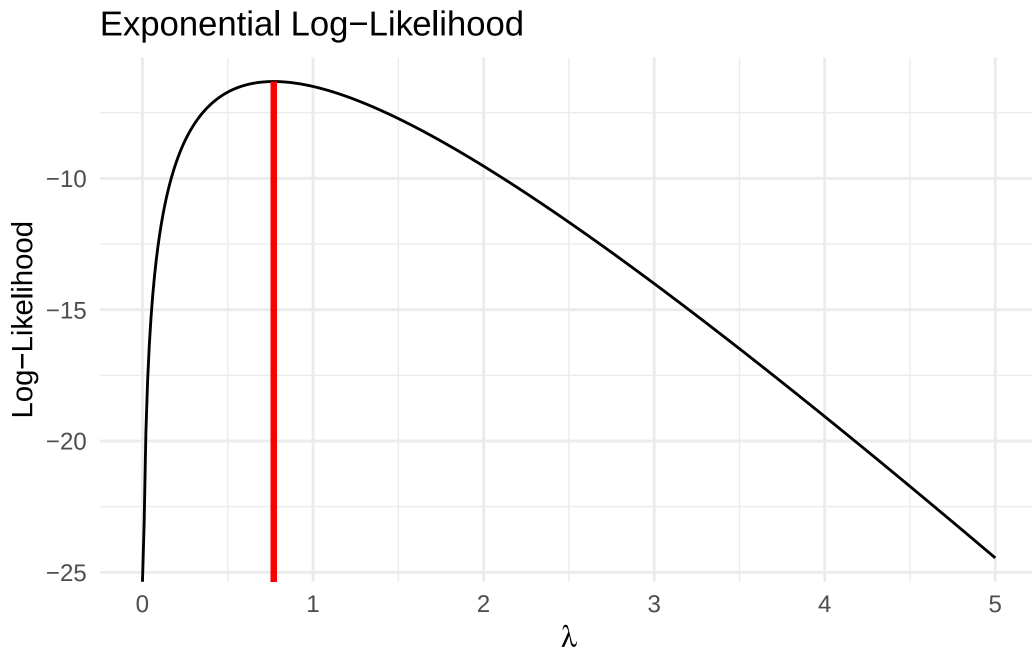
3. Exponential Distribution

$$L(\lambda \mid x) = \lambda^n e^{-\lambda \sum x_i}, \quad \ell(\lambda) = n \log \lambda - \lambda \sum x_i$$

```
x <- c(2.1, 0.9, 1.5, 0.8, 1.2)
n <- length(x)
sum_x <- sum(x)

exp_lik <- function(lambda, x) {
  ifelse(lambda <= 0, 0, lambda^length(x) * exp(-lambda * sum(x)))
}

lambda_grid <- seq(0, 5, 0.01)
lik_vals <- sapply(lambda_grid, exp_lik, x = x)
loglik_vals <- log(lik_vals)

mle_index <- which.max(loglik_vals)
mle_lambda <- lambda_grid[mle_index]

plot_likelihood(lambda_grid, lik_vals, mle_lambda, lik_vals[mle_index],
                expression(lambda),
                "Likelihood", "Exponential Likelihood")
```



```r
plot_likelihood(lambda_grid, loglik_vals, mle_lambda, loglik_vals[mle_index],
                expression(lambda),
                "Log-Likelihood", "Exponential Log-Likelihood")
```



MLE: $\hat{\lambda} = 0.77$

4. Normal Distribution

Probability density function:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Likelihood function:

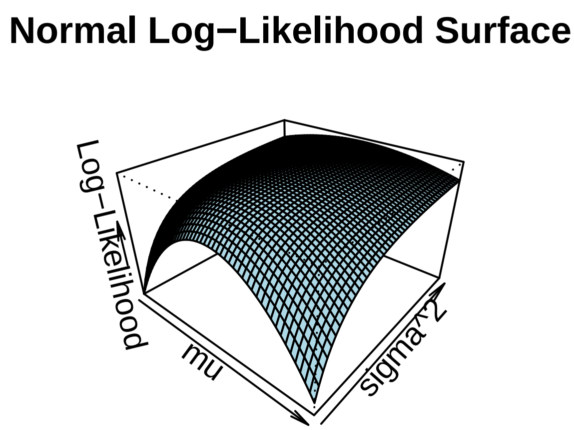
$$L(\mu, \sigma^2 \mid x) = \prod_{i=1}^{n} f(x_i \mid \mu, \sigma^2)$$

Log-likelihood function:

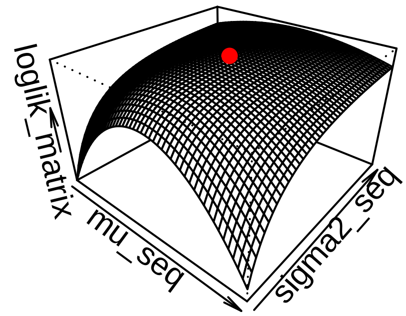
$$\ell(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

```
set.seed(123)
x <- rnorm(50, mean = 5, sd = 2)
n <- length(x)

# Log-likelihood function
norm_loglik <- function(mu, sigma2, x) {
  if (sigma2 <= 0) return(-Inf)
  -n/2*log(2*pi) - n/2*log(sigma2) - sum((x - mu)^2)/(2*sigma2)
}

# Parameter grid
mu_seq <- seq(3, 7, length.out = 50)
sigma2_seq <- seq(1, 6, length.out = 50)

loglik_matrix <- outer(mu_seq, sigma2_seq,
                       Vectorize(function(mu, s2) norm_loglik(mu, s2, x)))

# MLE estimates
mle_mu <- mean(x)
mle_sigma2 <- var(x)

# 3D Log-likelihood surface
persp(mu_seq, sigma2_seq, loglik_matrix,
      theta = 40, phi = 30,
```

```
      expand = 0.6, col = "lightblue",
      xlab = expression(mu), ylab = expression(sigma^2), zlab = "Log-Likelihood",
      main = "Normal Log-Likelihood Surface")
```




```
# Add MLE point in red
points(trans3d(mle_mu, mle_sigma2,
               norm_loglik(mle_mu, mle_sigma2, x),
               persp(mu_seq, sigma2_seq, loglik_matrix,
                     theta = 40, phi = 30, expand = 0.6)),
       col = "red", pch = 19)
```

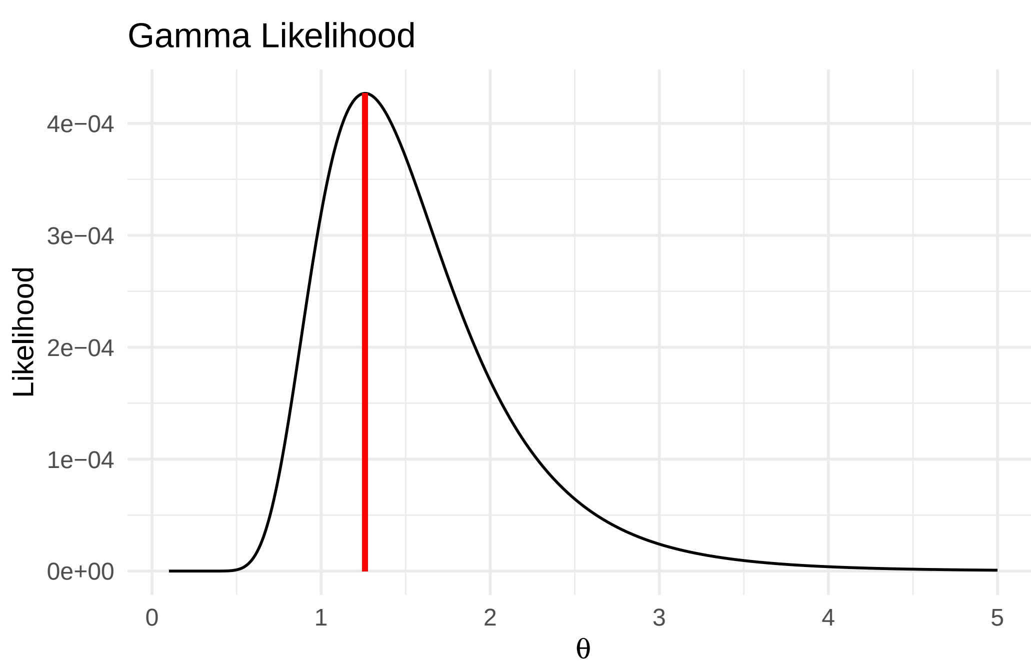


MLE: $\hat{\mu} = 5.0688071$ and $\hat{\sigma^2} = 3.4289409$

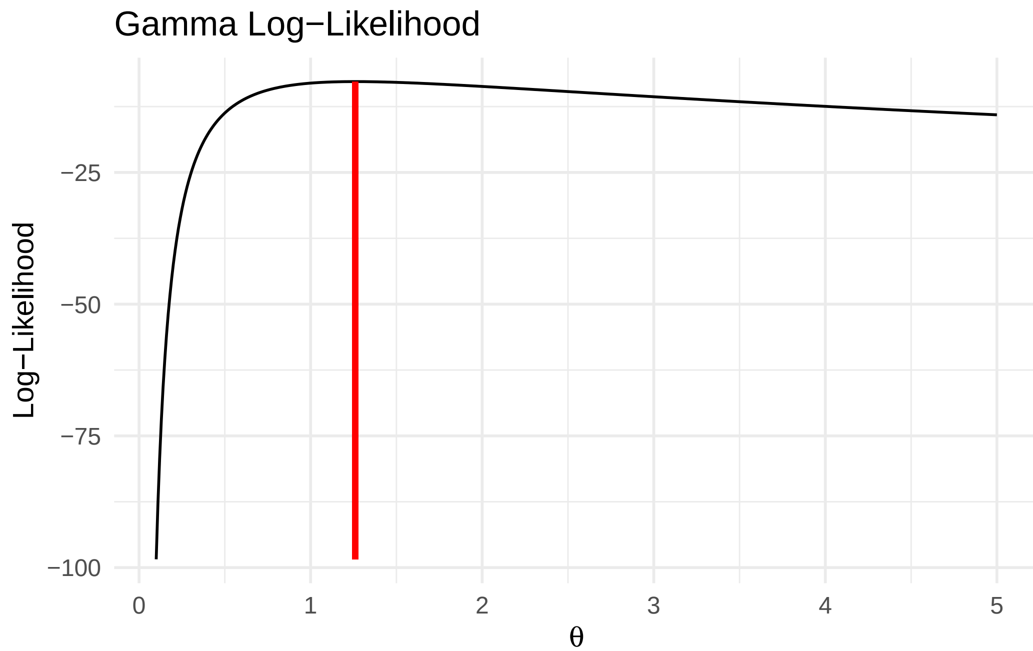
5. Gamma Distribution (known shape k)

$$L(\theta \mid x) = \prod_{i=1}^{n} \frac{1}{\Gamma(k)\theta^k} x_i^{k-1} e^{-x_i/\theta}$$

```r
x <- c(2.3, 1.9, 3.1, 2.8, 2.5)
k <- 2
n <- length(x)

gamma_lik <- function(theta, x, k){
  ifelse(theta <= 0, 0,prod(dgamma(x, shape = k, scale = theta)))
}

theta_grid <- seq(0.1, 5, 0.01)
lik_vals <- sapply(theta_grid, gamma_lik, x = x, k = k)
loglik_vals <- log(lik_vals)

mle_index <- which.max(loglik_vals)
mle_theta <- theta_grid[mle_index]

plot_likelihood(theta_grid, lik_vals, mle_theta, lik_vals[mle_index],
                expression(theta),
                "Likelihood", "Gamma Likelihood")
```



```r
plot_likelihood(theta_grid, loglik_vals, mle_theta, loglik_vals[mle_index],
                expression(theta),
                "Log-Likelihood", "Gamma Log-Likelihood")
```

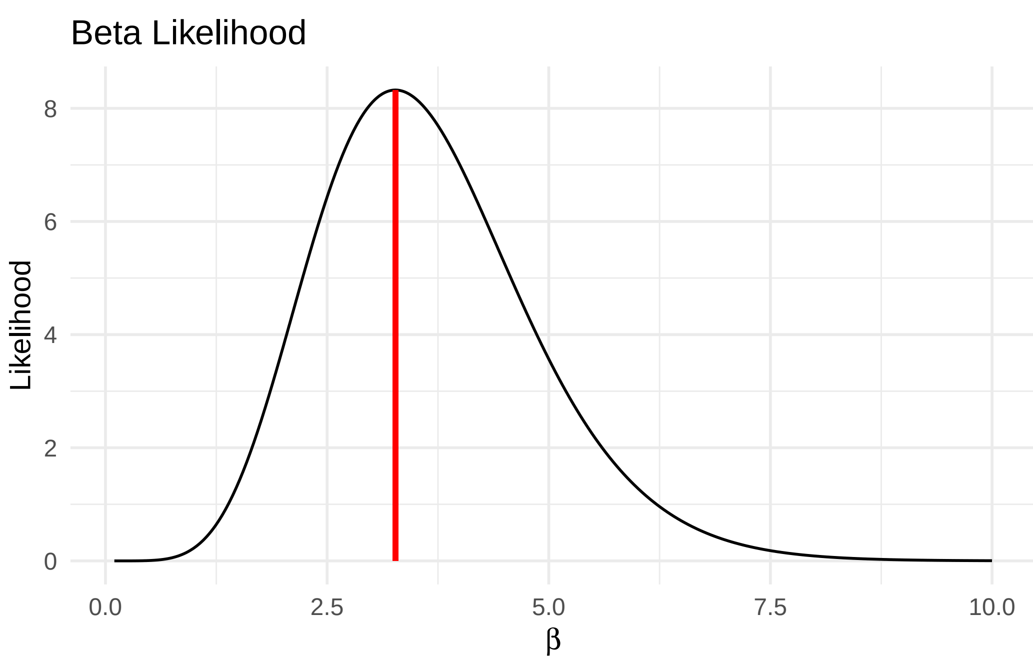


MLE: $\hat{\theta} = 1.26$

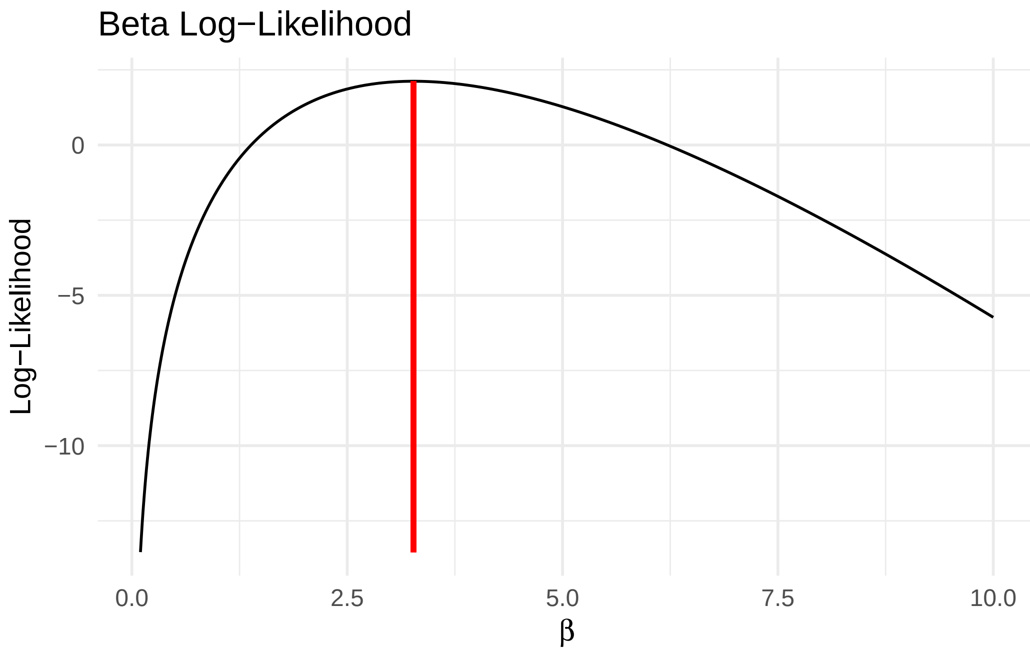
6. Beta Distribution (known α)

$$L(\beta \mid x) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}(1-x_i)^{\beta-1}}{B(\alpha, \beta)}$$

```r
x <- c(0.2, 0.4, 0.3, 0.5, 0.6)
alpha <- 2
n <- length(x)

beta_lik <- function(beta, x, alpha){
  ifelse(beta <= 0, 0, prod(dbeta(x, shape1 = alpha, shape2 = beta)))
}

beta_grid <- seq(0.1, 10, 0.01)
lik_vals <- sapply(beta_grid, beta_lik, x = x, alpha = alpha)
loglik_vals <- log(lik_vals)

mle_index <- which.max(loglik_vals)
mle_beta <- beta_grid[mle_index]

plot_likelihood(beta_grid, lik_vals, mle_beta, lik_vals[mle_index],
                expression(beta),
                "Likelihood", "Beta Likelihood")
```



```
plot_likelihood(beta_grid, loglik_vals, mle_beta, loglik_vals[mle_index],
                expression(beta),
                "Log-Likelihood", "Beta Log-Likelihood")
```



MLE: $\hat{\beta} = 3.27$