Profile Likelihood

Ishtiaq Mahmud Fahim

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Plausibility vs Probability?

Plausibility:

- General, non-technical idea: how reasonable or believable something seems given what you know.
- Not necessarily numerical. Example: "It's plausible it will rain tomorrow because the sky is cloudy."

Probability:

- A quantitative measure of uncertainty.
- Defined before observing data: it tells how likely an event is to occur.
- Example: "The probability of getting heads when tossing a fair coin is 0.5."

Likelihood

- A **statistical concept**: the plausibility of different parameter values given the data you already observed.

Likelihood vs Probability?

Probability: Predicts the **chance of outcomes** when the model or **parameters are fixed**.

Likelihood: Judges how **plausible parameters** are once **outcomes are observed**.

Real-life Example

Suppose we toss a coin 10 times and get 7 heads.

Probability: If the coin is fair ($p = 0.5$), the probability of exactly 7 heads is

$$P(7 \mid p = 0.5) = \binom{10}{7}(0.5)^7(0.5)^3.$$

Likelihood: Now that we observed 7 heads, we ask:

"Which value of p (probability of heads) makes this data most plausible?"

The likelihood function is

$$L(p \mid 7) \propto p^7(1-p)^3.$$

This is maximized at $p = 0.7$. So 7 heads makes $p = 0.7$ the most likely parameter value.

Summary

Plausibility = informal believability.

Probability = chance of outcomes given parameters.

Likelihood = plausibility of parameters given observed outcomes.

Profile Likelihood

In simple words Profile likelihood shows how well different values of the main parameter fit the data after adjusting the other nuisance parameters in the best possible way.

Flowchart

1. **Start** : Write the likelihood (how well parameters fit the data).
2. **Pick a value** for the parameter we are interested about.
3. **Adjust other parameters** : choose their best values for that fixed choice.
4. **Record the fit** : this is the profile likelihood for that value.
5. **Repeat for many values** of the main parameter.
6. **Result** : get a curve showing which values are most supported by the data.

HW

```
library(tidyverse)

data <- c(
  39.91, 43.97, 23.19, 38.87, 39.81,
  22.70, 27.72, 44.67, 28.64, 38.58,
  37.38, 49.00, 41.48, 41.73, 51.45,
  35.72, 33.41, 76.60, 32.02, 30.35,
  38.29, 38.71, 31.39, 39.00, 26.49
)


plik <- function(alph, data){
  bet <- optim(par = 1,
               fn = function(b, data){
                 -sum(dgamma(x = data, shape = alph, scale = b,log = TRUE))
                 },
               data = data)$par
  -sum(dgamma(x = data, shape = alph, scale = bet, log=TRUE))
}

optim(15, plik, data = data)
```

```
$par
[1] 14.7334

$value
[1] 92.24078

$counts
function gradient
      21       NA

$convergence
[1] 0

$message
NULL
```

```r
m <- seq(10, 50, length.out = 250)

mplik <- NULL

for(i in 1:250){
  mplik[i] = -plik(m[i], data)
  }

alph_lik <- m[which.max(mplik)]

alph_lik
```

```
[1] 14.65863
```

The profile likelihood for α is 14.7

```r
ggplot()+
  geom_line(
    aes(
    x = m, y = mplik
    )
  )+
  theme_bw()+
  xlab(expression(alpha))+
  ylab(expression(l(alpha)))+
  geom_segment(
    aes(
      x = alph_lik,
      xend = alph_lik,
      y = min(mplik),
      yend = max(mplik)
    ),
    col = "red"
  )
```