Q1a

(1) **Short-answer questions** **[2pts. each]**

(a) Let $f : A \to B$ be a function. Is the following statement <u>True or False</u>? Justify.

$$f(C) \subseteq f(D) \Rightarrow C \subseteq D \text{ for any } C, D \subseteq A.$$

False: Let $f: \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$,

let $C = \{-2, -1, 1, 2\}$. $D = \{1, 2\}$

$f(C) = \{1, 4\}$, $f(D) = \{1, 4\}$

so $f(C) \subseteq f(D)$ but $C \not\subseteq D$.

(b) Determine if the following statement is <u>True or False</u>. Justify.

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x - y > 2 \Rightarrow x > 1)$$

False. Let $x = -1$, $y = -4$;

then $x - y = -1 - (-4) = 3 > 2$,

but we don't have $x > 1$.

(c) Define sets A and B to be:

$$A = \{n \in \mathbb{Z} : 1 < n^2 < 10\} \text{ and } B = \{m \in \mathbb{N} : 33 - 6m > 20\}.$$

<u>List</u> all the elements in the following sets:

- $A =$ $\{2, 3, -2, -3\}$
- $B =$ $\{1, 2\}$
- $A \times B =$ $\{(2,1), (2,2), (3,1), (3,2), (-2,1), (-2,2), (-3,1), (-3,2)\}$

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(2) Let X and Y be sets in some universe U. Prove: **[4 pts.]**

$$\text{If } X \subseteq Y, \text{ then } X \cup (Y \setminus X) = Y.$$

Proof: we'll prove both $X \cup (Y \setminus X) \subseteq Y$ and $Y \subseteq X \cup (Y \setminus X)$.

To prove $X \cup (Y \setminus X) \subseteq Y$: take any element $a \in X \cup (Y \setminus X)$:

then either $a \in X$ or $a \in Y \setminus X$.

If $a \in X$, since $X \subseteq Y$, we have $a \in Y$;

if $a \in Y \setminus X$, we have $a \in Y$ and $a \notin X$.

Either case we have $a \in Y$. Therefore $\boxed{X \cup (Y \setminus X) \subseteq Y.}$

To prove $Y \subseteq X \cup (Y \setminus X)$: take any element $y \in Y$:

if $y \notin X$: then $y \in Y \setminus X$; since $y \in Y$ and $y \notin X$;

if $y \in X$: ~~then $y \in Y$ since $X \subseteq Y$~~. we don't need to show anything else.

So we have either $y \in X$ or $y \in Y \setminus X$,

therefore $y \in X \cup (Y \setminus X)$.

As a result $\boxed{Y \subseteq X \cup (Y \setminus X)}$

Once we combine both inclusions, we have if $X \subseteq Y$, then $X \cup (Y \setminus X) = Y$.

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(3) Let x and y be real numbers. Prove the statement **[4 pts.]**

$$\text{If } 2x^2 + y^2 > 2, \text{ then } x \neq -1 \text{ or } y \neq 0.$$

using its **contrapositive**.

Proof: Let $P: 2x^2+y^2>2$, $Q: x\neq -1$ or $y\neq 0$.

we need to prove $P \Rightarrow Q$. Instead, we'll prove $\neg Q \Rightarrow \neg P$.

Assume $x=-1$ and $y=0$:

then $2x^2+y^2 = 2(-1)^2+0^2 = 2. \leq 2$

(OR: which is not greater than 2.)

Therefore $\neg Q \Rightarrow \neg P$. As a result, if $2x^2+y^2>2$, then $x\neq -1$ or $y\neq 0$.

$\square$

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(4) Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be defined by [4 pts.]

$$f(a, b) = ab(b + 1).$$

Let $E = \{2k : k \in \mathbb{N}\}$ be the set of even natural numbers. Prove that $f(\mathbb{N} \times \mathbb{N}) = E$.

Proof: In order to prove $f(\mathbb{N}\times\mathbb{N}) = E$, we'll prove: $f(\mathbb{N}\times\mathbb{N}) \subseteq E$ and $E \subseteq f(\mathbb{N}\times\mathbb{N})$.

In order to prove $\operatorname{im} f \subseteq E$, let $y \in \operatorname{im} f$,

so $y = ab(b+1)$ for some $a, b \in \mathbb{N}$.

If b is even, then b = 2k for some integer k, and b(b+1) = 2[k(2k+1)] is even.

If b is odd, then b = 2k+1 for some integer k, and b(b+1) = (2k+1)(2k+2) = 2(2k+1)(k+1) is also even.

For the product $b(b+1)$, it's even no matter what b is.

therefore $ab(b+1)$ is always even, so $y \in E$.

As a result, $\operatorname{im} f \subseteq E$, or $f(\mathbb{N}\times\mathbb{N}) \subseteq E$.

In order to prove $E \subseteq \operatorname{im} f$, let $n \in E$:

then $n = 2k$ ($k \in \mathbb{N}$)

we can write $2k = 1 \cdot 2 \cdot k$

so when $a = k, b = 1$, $f(a,b) = n$.

therefore $n \in \operatorname{im} f$.

As a result, $E \subseteq \operatorname{im} f$, or $E \subseteq f(\mathbb{N}\times\mathbb{N})$.

Combine both proofs, we have $f(\mathbb{N}\times\mathbb{N}) = E$.