Q16

(1) **Short-answer questions** **[2pts. each]**

(a) Define sets A and B to be:

$$A = \{n \in \mathbb{Z} : -3 < n^2 < 3\} \quad \text{and} \quad B = \{m \in \mathbb{N} : 10 < 24 - 5m\}.$$

List all the elements in the following sets:

- $A =$ $\{0, 1, -1\}$
- $B =$ $\{1, 2\}$
- $A \times B =$ $\{(0,1), (0,2), (1,1), (1,2), (-1,1), (-1,2)\}$

(b) Let $f : A \to B$ be a function. Is the following statement True or False? Justify.

$$f(C) \cap f(D) \subseteq f(C \cap D) \text{ for any } C, D \subseteq A.$$

False: let $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$.

Let $C = \{-1, -2\}$, $D = \{1, 2\}$. $C \cap D = \emptyset$.
then $f(C) = \{1, 4\}$, $f(D) = \{1, 4\}$ $f(C) \cap f(D) = \{1, 4\}$
we don't have $f(C) \cap f(D) \subseteq f(C \cap D)$ here.

(c) Determine if the following statement is True or False. Justify.

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(\underbrace{y + x = 2}_{P} \Rightarrow \underbrace{x < 0}_{Q})$$

True.

let $x = 0$, $y = 1$; then $x + y = 1 \neq 2$.

so P is false, therefore $P \Rightarrow Q$ is true.

Q1b

(2) Let X and Y be sets in some universe U. Prove: **[4 pts.]**

$$\text{If } X \subseteq Y, \text{ then } X \cup (Y \setminus X) = Y.$$

See Q1a solutions.

Q16

(3) Let x and y be real numbers. Prove the statement **[4 pts.]**

$$\text{If } x^2 + 2y^2 > 1, \text{ then } x \neq 1 \text{ or } y \neq 0.$$

using its **contrapositive**.

Proof: Let P: $x^2+2y^2>1$, Q: $x\neq 1$ or $y\neq 0$.

then $\neg P$: $x^2+2y^2\leq 1$, $\neg Q$: $x=1$ and $y=0$.

we need to prove $P\Rightarrow Q$, using its contrapositive,

we'll prove $\neg Q \Rightarrow \neg P$ instead.

Let $x=1$ and $y=0$: $x^2+2y^2=1^2+2\cdot 0^2=1\leq 1$.

So $\neg Q\Rightarrow \neg P$. As a result, $P\Rightarrow Q$, in other words.

if $x^2+2y^2>1$, then $x\neq 1$ or $y\neq 0$.

Q1b

(4) Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be defined by **[4 pts.]**

$$f(a, b) = a(a + 1)b.$$

Let $E = \{2k : k \in \mathbb{N}\}$ be the set of even natural numbers. Prove that $f(\mathbb{N} \times \mathbb{N}) = E$.

See Q1a, (4).