Discrete Mathematics

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Foundational Logic and Proofs

Propositional Logic

Syllogism : A logical argument which relies on two or more propositions to come to a conclusion.
Proposition : A declarative sentence that can be definitively proved to be true or false.

Boolean Algebra

Compound propositions were first discussed by English Mathematician George Boole.
`Boolean Algebra is named after him`

Negation : If P is a proposition, the negation of P is $\neg P$.
$\neg P$ is true when P is false and vice versa.
`Same as a NOT gate`

P	$\neg P$
T	F
F	T

Conjunction : If P and Q are propositions, the conjunction of P and Q is $P \wedge Q$.
$P \wedge Q$ is true when BOTH P and Q is true.
`Same as an AND gate`

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction : If P and Q are propositions, the disjunction of P and Q is $P \vee Q$.
$P \vee Q$ is true when either P and/or Q is true.
`Same as an OR gate`

P	Q	$P \vee Q$	
T	T	T	
T	F	T	
F	T	T	
F	F	F	

Exclusive or : If P and Q are propositions, the *exclusive or* of P and Q is $P \oplus Q$.
$P \oplus Q$ is true exclusively when either P or Q is true, but false when both have the same value.
`Same as XOR gate`

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements : Denoted by $P \rightarrow Q$, false only when P is true and Q is false, and is true otherwise.

This might be confusing, giving an example will help:
If a proposition is "If it rains today, the ground will get wet", it is denoted by $P \rightarrow Q$ where,
$P \rightarrow$ "It's raining today"
$Q \rightarrow$ "The ground got wet"
Possible scenarios :

1. It rained today (P is true), and the ground got wet (Q is true). As this supports the original proposition, $P \rightarrow Q$ is true in this case.
2. It rained today (P is true), but the ground didn't get wet (Q is false). As this **contradicts** the original proposition, $P \rightarrow Q$ is false in this situation.
3. It did not rain today (P is false), but the ground still got wet because of some other reason (Q is true). This statement does not **contradict** the original proposition, $P \rightarrow Q$ is still true in this situation.
4. It did not rain today (P is false), and the ground did not get wet (Q is false). This supports the original proposition so $P \rightarrow Q$ is true in this situation.

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T

P	Q	$P \to Q$	$\neg P \vee Q$
T	F	F	F
F	T	T	T
F	F	T	T

Biconditional statement : Denoted by $P \leftrightarrow Q$, true only when both P and Q have same values and false otherwise.

If $P \leftrightarrow Q$ denotes the proposition "A shape is called a triangle only if it has three sides" where,
$P \to$ "A shape is a triangle"
$Q \to$ "The shape has three sides"
Possible scenarios :

1. The shape is called a triangle (P is true), and it has three sides (Q is true). As this supports the original proposition, $P \leftrightarrow Q$ is true here.
2. The shape is called a triangle (P is true), but it does not have three sides (Q is false). This contradicts the proposition which means $P \leftrightarrow Q$ is false.
3. The shape is not called a triangle (P is false), but it has three sides (Q is true). This also contradicts the proposition which means $P \leftrightarrow Q$ is false.
4. The shape is not called a triangle (P is false), and it does not have three sides(Q is false). This supports the original proposition which means $P \leftrightarrow Q$ is true.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Converse, Contrapositive and Inverse :

If $P \to Q$ is a proposition;

- Converse : $Q \to P$
- Contrapositive : $\neg Q \to \neg P$
- Inverse : $\neg P \to \neg Q$
  The contrapositive of a proposition is effectively the same thing as the original proposition.

Example :

Original proposition : $P \to Q$; "If it is raining, the ground will get wet"
Contrapositive proposition : $\neg Q \to \neg P$; "If the ground is not wet, it is not raining"
Converse proposition : $Q \to P$; "If the ground is wet, it is raining"
Inverse proposition : $\neg P \to \neg Q$; "If it is not raining, the ground will not get wet"

Truth tables

P	Q	$\neg P$	$\neg Q$	$P \land Q$	$P \lor Q$	$P \oplus Q$	$P \to Q$	$P \leftrightarrow Q$
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

Practice: Find the truth table for $(P \lor \neg Q) \to (P \land Q)$

P	Q	$\neg Q$	$P \lor \neg Q$	$P \land Q$	$(P \lor \neg Q) \to (P \land Q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Operator precedence

Operator	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\to$	4
$\leftrightarrow$	5

Propositional Equivalence

Tautology : A compound proposition that is always true and independent of the truth values of its constituent propositions
Contradictions : A compound proposition that is always false and independent of the truth values of its constituent propositions
Contingency : A compound proposition that is neither a tautology nor a contradiction.

Logical Equivalence

When two compound propositions have the same truth values for all possible cases, they are logically equivalent. This is denoted by $P \equiv Q$.
If $P \leftrightarrow Q$ is a tautology then $P \equiv Q$.
example:

P	Q	$\neg P$	$P \rightarrow Q$	$\neg P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$(P \rightarrow Q) \equiv (\neg P \vee Q)$

De Morgan's Law is used to convert conjunctions to disjunctions and vice versa.

1. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
2. $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
   Similarly, for n propositional variables;
3. $\neg(P_1 \vee P_2 \vee P_3 \ldots \vee P_n) \equiv \neg P_1 \wedge \neg P_2 \wedge \neg P_3 \ldots \wedge \neg P_n$
4. $\neg(P_1 \wedge P_2 \wedge P_3 \ldots \vee P_n) \equiv \neg P_1 \vee \neg P_2 \vee \neg P_3 \ldots \vee \neg P_n$

Practice :
Prove $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

A tautology is always true.
$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$
$\neg(p \wedge q) \vee (p \vee q) \equiv \neg p \vee \neg q \vee (p \vee q)$
$\neg p \vee \neg q \vee (p \vee q) \equiv (\neg p \vee p) \vee (\neg q \vee q)$
$(\neg p \vee p) \vee (\neg q \vee q) \equiv T \vee T$
$T \vee T \equiv T$
Some important logical equivalencies:

Equivalence	**Name**
$P \wedge T \equiv P$ $P \vee F \equiv P$	Identity laws
$P \wedge F \equiv F$ $P \vee T \equiv T$	Domination laws

Equivalence	Name
$P \wedge P \equiv P$ $P \vee P \equiv P$	Idempotent laws
$\neg(\neg P) \equiv P$	Double negation law
$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative laws
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative laws
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	Distributive laws
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P_1 \vee P_2 \vee P_3 \ldots \vee P_n) \equiv \neg P_1 \wedge \neg P_2 \wedge \neg P_3 \ldots \wedge \neg P_n$ $\neg(P_1 \wedge P_2 \wedge P_3 \ldots \vee P_n) \equiv \neg P_1 \vee \neg P_2 \vee \neg P_3 \ldots \vee \neg P_n$	De Morgan's laws
$(P \to Q) \equiv (\neg P \vee Q)$	Definition of Implication
$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	Absorption laws
$P \vee \neg P \equiv T$ $P \wedge \neg P \equiv F$	Negation laws
$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	Definition of biconditional

Equivalence	Name

S

Predicate Quantifiers

Predicates

Predicates: Statements that, due to lack of information, are neither true or false.
`x>3 is a predicate as without knowing the value of x we cannot definitively say if it is true or false.`
Predicates are denoted as a propositional function;

$$P(x) = x > 3$$

When we assign a value to x and **quantify** it, the propositional function becomes a proposition that is either TRUE or FALSE.

Quantifiers

Types of quantifications:

- Universal quantification: Where a predicate is true for every element in a set. Denoted by

$$\forall x P(x)$$

- Existential quantification: Where a predicate is true for at least one element in a set. Denoted by

$$\exists x P(x)$$

$\forall$ and $\exists$ quantifiers have a higher precedence than all other logical operators

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

Negating quantified expressions

Let $P(x)$ denote 'It has rained today'
Thus, $\forall x P(x)$ denotes 'It has rained everyday'
Negating this function would give $\neg\forall x P(x)$ 'It is not the case that it has rained everyday'
This statement is the same as 'There is at least one day where it has not rained' which is denoted by $\exists x \neg P(x)$

$$\therefore \neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\therefore \neg\exists x P(x) \equiv \forall x \neg P(x)$$

Uniqueness quantifier

When it is stated that 'There exists a unique singular value of x such that $P(x)$ is true", it is denoted as $\exists! x P(x)$

Nested Quantifiers

When one quantifier is within the scope of another quantifier it is called a nested quantifier.

Statement	When True	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y	There is a pair x, y for which $P(x, y)$ is false
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true	There is an x such that $P(x, y)$ is false for every y
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y	For every x there is a y for which $P(x, y)$ is false
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true	$P(x, y)$ is false for every pair x, y.

Rules of Inference

Inference : The process of reaching a conclusion using premises

You must have a **valid argument** to prove anything in mathematics.

An argument is a sequence of statements that end with a conclusion, it is only valid when the conclusion follows the truth of the preceding premises/statements.

Fallacies : Invalid arguments that are based on incorrect reasoning.

Example: If an argument is denoted by $p \to q$, it has 3 truth values (p,q,$p \to q$). If we have only 2 out of these 3 values we can *infer* the value of the third proposition.

When $p \to q$ and p are both true, the only possible value of q is true . Hence we can *infer* q is true.

Inference Laws

Modus Ponens

`Mode that affirms`

$$(p \land (p \to q)) \to q$$

If p is *true*, and p implies q, $\therefore$ we can infer that q is also *true*

p

$p \to q$

$\therefore q$

Modus Tollens

`Mode that denies`

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

If q is false, and p implies q is true, $\therefore$ we can infer that p is also *false*

$\neg q$

$p \rightarrow q$

$\therefore \neg p$

Hypothetical syllogism

`Syllogistic reasoning using two different hypothesis`

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

If p implies q, and q implies r, $\therefore$ we can infer that p implies r

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Disjunctive syllogism

`Syllogistic reasoning using disjunction`

$$((p \vee q) \wedge \neg p) \rightarrow q$$

If p OR q is *true*, and p is *false*, $\therefore$ we can infer that q is *true*

$p \vee q$

$\neg p$

$\therefore q$

Addition

$$p \rightarrow (p \vee q)$$

If p is *true*, $\therefore$ disjunction of p and q must be *true*

p

$\therefore p \vee q$

Simplification

$$(p \wedge q) \rightarrow p$$

If both p AND q are *true*, $\therefore$ p must also be *true*

$p \wedge q$

$\therefore p$

Conjunction

$$((p) \land (q)) \to (p \land q)$$

If p is *true*, and q is also *true*, $\therefore$ Conjunction between p and q must also be *true*
p
q
$\therefore p \land q$

Resolution

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$$

If p OR q is true, and negation of p or r is also true, $\therefore$ Disjunction between q and r must be *true*
$p \lor q$
$\neg p \lor r$
$\therefore q \lor r$

Validity of an argument

If an argument consists of n number of p premises and a conclusion q, the argument is only valid when $(p_1 \land p_2 \land p_3 \land ... \land p_n) \to q$ is a tautology

Proof Techniques

Theorem: A statement that can be shown to be true

Axioms: Statements that we assume to be true

Lemma: A less important theorem that is helpful in the proof of other results

Corrolary: It is a theorem that can be established directly from a proven theorem

Conjecture: A statement that is proposed as true statement, usually on the basis of some partial evidence.

Types of proof

Direct proof : Use of standard rules of inference to draw a conclusion without changing the problem statement.
Example: Prove that if m is even and n is odd, their sum is always odd
$n = 2k + 1 \ , \ k \in \mathbb{Z}$
$m = 2j \ , \ j \in \mathbb{Z}$
$\therefore m + n = 2j + 2k + 1 = 2(j + k) + 1$
As both j and k are integers ($\mathbb{Z}$), $j + k$ must also be an integer. Therefore $m + n$ is odd by definition.

Indirect proof : Changing the problem statement and proving the new statement

Types of Indirect Proof

- Contrapositive : if we cant directly prove $p \rightarrow q$, we can prove the contrapositive of the statement which is $\neg q \rightarrow \neg p$
  Example: Given $n \in \mathbb{Z}$ and $3n+2$ is odd, show that n is odd.
  p : $3n+2$ is odd
  q : n is odd
  Problem statement in propositional terms ; $p \rightarrow q$
  The contrapositive of this statement is $\neg q \rightarrow \neg p$
  Which translates to : If n is even, $3n+2$ is even (Proving this will prove the original statement).
  $n = 2k\ ,\ k \in \mathbb{Z}$
  $\therefore 3n+2 = 3(2k)+2 = 2(3k+1)$ (Even number definition).
  As $\neg q \rightarrow \neg p$ has been proven, this indirectly proves $p \rightarrow q$
- Proof by cases : Given x is an integer, prove that $x^2 + x$ is even
  Here there are two cases where x can either be even or odd, if both of these individual cases can be proven then the problem statement can be proven to be true.
  Case 1
  $p : x = 2n$ (even)
  $x^2 + x = (2n)^2 + 2n = 4n^2 + 2n = 2(2n^2 + n)$ (even)
  Case 2
  $q : x = 2n+1$ (odd)
  $x^2 + x = (2n+1)^2 + 2n + 1 = (4n^2 + 6n + 2) = 2(2n^2 + 3n + 1)$ (even)
  As both cases are true, the problem statement is also proved.
- Proof by contradiction : Instead of proving the statement, we can prove the negation of the statement to be false.
  Example: Prove that $\sqrt{2}$ is irrational.
  $P : \sqrt{2} \in \mathbb{I}$
  $\neg P : \sqrt{2} \in \mathbb{Q}$
  All rational numbers can be expressed in the form $\frac{P}{Q}$ where it is in its lowest terms.
  $\sqrt{2} = \frac{P}{Q}$
  $2 = \frac{P^2}{Q^2}$
  $P^2 = 2Q^2$ (P^2 is in even form , $\therefore P$ is also even)
  $(2k)^2 = 2Q^2 \equiv 2k^2 = Q^2$($\therefore Q$ is also even)
  As both P and Q are even, both of them have a common factor of 2 , which contradicts with the fact that $\frac{P}{Q}$ has to be in its lowest terms.
  As we proved the negation of P to be false, this proves P to be true $[\neg(\neg P) \equiv P \equiv T]$

Sets, Functions

Sets

Describing a Set

A set is an unordered collection of distinct non-repeating elements.
Ways to describe a set:

- **Roster Method :** Listing out all the elements of the set

$$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

- **Set Builder Notation :** Instead of listing out all the elements the characteristics

$$A = \{x | x \text{ is an odd positive integer less than } 10\}$$

Cardinality

The **cardinality** of a set is the number of elements in that set.
Sets with a **cardinality** of 0 are called **Null** sets

$$\varnothing = \{\} \ ; \ |\varnothing| = 0$$

Sets with a **cardinality** of 1 are called **singleton** sets

Subsets

If set A is a *subset* of set B, then set B is a superset of set A. Where every element in set A is an element of set B.

$$A \subseteq B$$

Power Sets

For a set A, the *power set* of A is the set of all subsets of the set S. Denoted by $\mathcal{P}(A)$
$A = \{0, 1, 2\}$
$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
$|\mathcal{P}(A)| = 2^{|A|}$

Cartesian Product

The cartesian product of sets A and B is the set of all ordered pairs (a, b).
$A = \{1, 2, 3\}$
$B = \{a, b, c\}$
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

Set Operations

Union

Let;
$A = \{x | x \in A\}$
$B = \{x | x \in B\}$

The union of sets A and B is denoted by;
$A \cup B \equiv \{x|x \in A \vee x \in B\}$

Intersection

Let;

$$A = \{x|x \in A\}$$

$$B = \{x|x \in B\}$$

The intersection of sets A and B is denoted by;

$$A \cap B \equiv \{x|x \in A \wedge x \in B\}$$

Complement

The complement of set A is the set of all elements that is not in A
$\overline{A} = \{x|x \notin A\}$

Disjoint sets

When the intersection of multiple sets result in a null set, the sets are said to be disjoint

$$A \cap B = \varnothing$$

Difference

The difference of set A and B is the set of elements that are in A but not in B
$A - B = \{x|x \in A \cap x \notin B\}$
$= \{x|x \in A \cap x \in \overline{B}\}$
$= \{x|x \in A \cap \overline{B}\}$
$A - B = A \cap \overline{B}$

Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws

Identity	Name
$\overline{(\overline{A})} = A$	Complementation laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Functions

A function maps each element from a non-empty set A to exactly one element from a non-empty set B.

Types of functions

- Injective (One-to-One)
  `A function f is one-to-one if and only if f(a) = f(b) implies a=b for all values of $a$ and $b$ within the domain of f.`
  `For a function f(x) = x, all distinct values of x within the domain must have a distinct corresponding value of f(x)`
  For the function $f : A \to B$, if it is injective
  $|A| \leq |B|$
- Surjective (Onto)
  `A function f is onto if and only if every element in the codomain of the`

`function can be mapped from an element of the domain.`
For the function $f : A \to B$, if it is surjective
$|A| \geq |B|$

- Bijective (One-to-One correspondence)
  `A function f is bijective if both it is both one-to-one and onto`
  For the function $f : A \to B$ if it is bijective,
  $|A| = |B|$

Increasing and Decreasing functions

For a function $f(x)$,

- **INCREASING** when $x_2 > x_1$ and $f(x_2) \geq f(x_1)$
- **STRICTLY INCREASING** when $x_2 > x_1$ and $f(x_2) > f(x_1)$
- **DECREASING** when $x_2 > x_1$ and $f(x_2) \leq f(x_1)$
- **STRICTLY DECREASING** when $x_2 > x_1$ and $f(x_2) < f(x_1)$

A function reaches a stationary point when $\frac{d}{dx} = 0$
A function reaches a minima when $\frac{d^2}{dx^2} > 0$
A function reaches a maxima when $\frac{d^2}{dx^2} < 0$

Inverse Functions

For a function $f : A \to B$ which is bijective, the inverse of f assigns codomain elements to a unique domain element such $f(a) = b$ when $f^{-1}(b) = a$
ONLY BIJECTIVE FUNCTIONS CAN BE INVERSED

Floor and Ceiling Functions

Floor functions

- The floor of a variable x is denoted by $\lfloor x \rfloor$, it is defined as the largest integer that is smaller than x.
  Ceiling functions
- The ceiling of a variable x is denoted by $\lceil x \rceil$, it is defined as the smallest integer that is larger than x.

Cardinality of Sets

Countable Sets

If a set is finite then it is said to be countable.
If a set is infinite then it can also be said to be countable, with a condition that the cardinality of the set must be equal to the cardinality of positive integers ($\mathbb{N} : \{1, 2, 3, 4, 5, \ldots.\}$)

Recursion

A function/relation that repeats or uses its own previous term to calculate subsequent terms
The factorial function is a basic example of a recursive function;
$f(x) = x!$
$f(5) = 5! = 5 \times 4 \times 3 \times 2 \times 1$
$f(5) = 5 \times 4!$
$f(5) = 5 \times f(4)$
$\therefore f(x) = x \times f(x-1)$, where $f(0) = 1$

Sequences & Summation

A sequence is a function which maps elements from a subset of integers to a set S.
It is a discrete structure used to represent an ordered list

Types of sequences

- Geometric
  A common ratio r exists between two subsequent terms.

$$n^{th}\ term = ar^{n-1},\ where\ n = 1\ for\ the\ first\ term$$

  Summation ;

$$S_n = \frac{ar^n - a}{r-1}\ , r \neq 1$$

- Arithmetic
  A common difference d exists between two subsequent terms.

$$n^{th}\ term = a + (n-1)d,\ where\ n = 1\ for\ the\ first\ term$$

  Summation;

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Useful summation formulae

Sum	Closed Form/Equivalent Form
$\sum_{k=0}^{n} ar^k\ (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1},\ r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$

Sum	Closed Form/Equivalent Form
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=1}^{n} c$	nc
$\sum_{k=0}^{\infty} x^k, \lVert x \rVert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lVert x \rVert < 1$	$\frac{1}{(1-x)^2}$
$\sum_{b}^{a} f(x)$	$\sum_{1}^{a} f(x) - \sum_{1}^{b-1} f(x)$
$\sum_{b}^{a} kf(x)$	$k \sum_{b}^{a} f(x)$
$\sum_{b}^{a} [f(x) + k]$	$\sum_{b}^{a} f(x) + \sum_{b}^{a} k$

Relations

Relation between two sets

A relation R from two sets A to B is a subset of $A \times B$

$$R \subseteq A \times B$$

Relation on a set

A relation R on a set A is a subset of $A \times A$

$$R \subseteq A \times A$$

Relational Matrix

A relational Matrix M_r representing a relation R on a set A looks like this:
if set A is defined as;

$$A = \{0, 1, 2\}$$

And relation R is defined as;

$$R = \{(1, 0), (0, 1), (1, 1), (1, 2), (2, 2)\}$$

The relational matrix M_r is defined as;

$$M_r = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Types of relations

Reflexive

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$

The trace of M_r will be equal to the cardinality of A , as the main diagonal will all be 1s.

$$tr(M_r) = |A|$$

Symmetric

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$

The trace of M_r will be equal to cardinality of A , as the main diagonal will all be 1s.

$$tr(M_r) = |A|$$

The matrix formed be transposing M_r will be identical to M_r

$$(M_r)^T \equiv M_r$$

Antisymmetric

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

The trace of M_r will be equal to cardinality of A , as the main diagonal will all be 1s.

$$tr(M_r) = |A|$$

The matrix formed be transposing M_r can NOT be identical to M_r[Unless M_r is an identity matrix]

$$(M_r)^T \not\equiv M_r \text{ , } M_r \not\equiv I$$

Transitive

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$

`No notable property can be observed from a transitive matrix , unlike the other types`

Composite relations

$$\text{if}\ ,\ R = \{(a, b) \in R | a \in A, b \in B\}$$

$$\text{and}\ ,\ S = \{(b, c) \in S | b \in B, c \in C\}$$

$$\text{then}\ ,\ R \circ S = \{(a, c) \in R \circ S | a \in A, c \in C\}$$

Powers of a relation

If R is a relation on set A, then the n^{th} power of R , is defined recursively as ;

$$R^n = R^{n-1} \circ R\ , n = 1, 2, 3, \ldots$$

SIDE NOTE
The relation R on set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Transitive closure

The **transitive closure** of a relation R on a set A is the smallest transitive relation T on A such that $R \subseteq T$.

If M_R is an $n \times n$ matrix, the transitive closure denoted by M_{R*} is defined as;

$$M_{R*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

where,

$$M_R^{[k]} = M_R^{[k-1]} \odot M_R$$

Equivalence relations

A relation on set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Counting

Product rule

If a task can be split into a sequence of two tasks, where there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the task.

Sum rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Subtraction rule (Inclusion-Exclusion Principle)

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$