Triple Integral

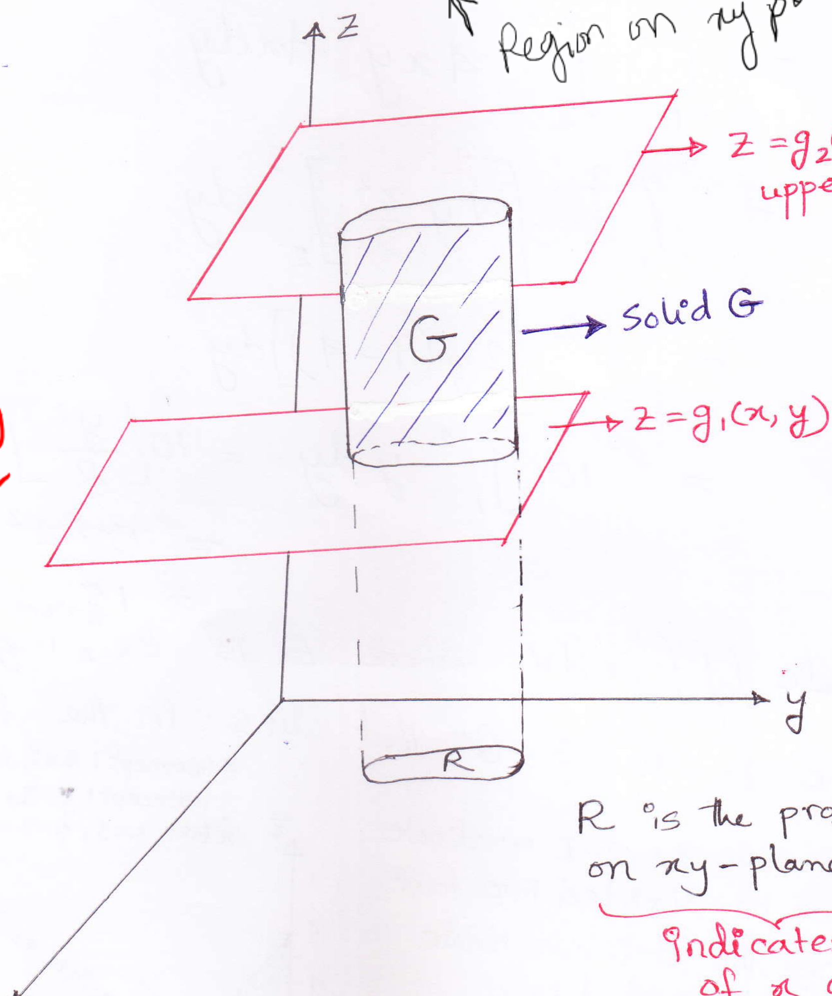
$\boxed{\text{Week 6}}$

G is a simple xy-solid with $z = g_2(x,y)$ as upper surface and $z = g_1(x,y)$ as lower surface. R is the projection of G on xy-plane. If $f(x,y,z)$ is continuous on G then

$$\iiint_{G} f(x,y,z)\, dV = \iint_{R} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] \underbrace{dA}_{dx\,dy \text{ or } dy\,dx}$$

solid

Region on xy plane



$$dV = dz\underbrace{dA}_{dx\,dy \text{ or } dy\,dx}$$

R is the projection of G on xy-plane — indicates limits of x and y on xy-plane.

Examples ① Evaluate the following integral

$$\iiint_G 8xyz\,\boxed{dV} \qquad G = [2,3]\times[1,2]\times[0,1]$$

$$G = \underbrace{[2,3]}_{x} \times \underbrace{[1,2]}_{y} \times \underbrace{[0,1]}_{z}$$

$dV = dz\,dA$

$= dz\,dx\,dy$

or $dz\,dy\,dx$

$$= \iint_R \left[\int_{z=0}^{1} 8xyz\,dz\right] dx\,dy.$$

$$= \int_{y=1}^{2}\int_{x=2}^{3.} \left[\cancel{8}^{4}xy\frac{z^2}{\cancel{2}}\right]_0^1 dx\,dy$$

$$= \int_1^2\int_2^3 4xy\,dx\,dy$$

$$= \int_1^2 \left[\cancel{4}^{2}y\frac{x^2}{\cancel{2}}\right]_2^3 dy$$

$$= \int_1^2 \left[2y(9-4)\right]dy$$

$$= 10\int_1^2 y\,dy = 10\left[\frac{y^2}{2}\right]_1^2$$

$$= 5\left[2^2 - 1^2\right] = 15$$

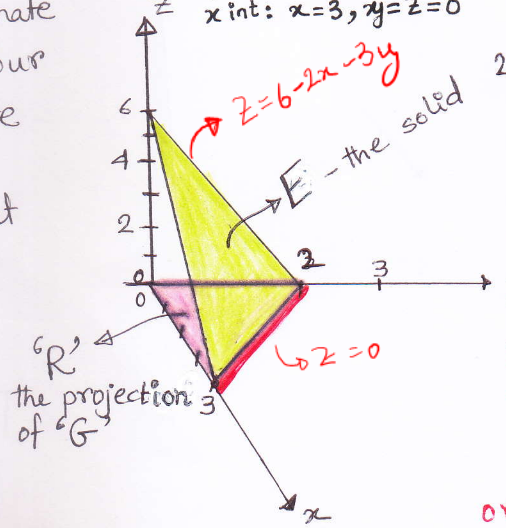
② Evaluate $\iiint_E 2x\,dV$ where E is the triangular shape region under the plane $2x+3y+z=6$ that lies in the first octant

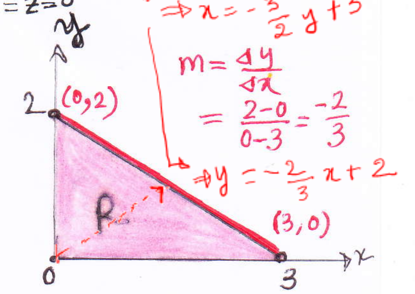
Octant: Just as the 2-D coordinate system can be divided into four quadrants, the 3-D coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

z intercept: $z=6,\ x=y=0$

y intercept: $y=2,\ x=z=0$

x int: $x=3,\ y=z=0$





Alternatively: R: $z=0$ xy-plane

$\therefore\ 2x+3y=6$

$\Rightarrow x = -\frac{3}{2}y+3$

$m = \frac{dy}{dx} = \frac{2-0}{0-3} = \frac{-2}{3}$

$\Rightarrow y = -\frac{2}{3}x+2$

Eqn of the red line $(x_1,y_1) = (0,2)$ or $(3,0)$

$y - y_1 = m(x - x_1)$

$y - 2 = -\frac{2}{3}(x-0)$

$y = -\frac{2}{3}x + 2$

or $x = -\frac{3}{2}y + 3$

$x\in[0,3],\ y\in\left[0,-\frac{2}{3}x+2\right]$

OR: $y\in[0,2),\ x\in\left[0,-\frac{3}{2}y+3\right]$

We need to determine the region R in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region R in xy-plane. So R will be the triangle with vertices at (0,0), (3,0) and (0,2).

Now we need the limits of integration. Since we are in the 1st octant (so we are under the plane and above the plane $z=0$). $\therefore$ $0 \le z \le 6-2x-3y$.

We can integrate the double integral over R using either of the following two sets of inequalities.

$$0 \le x \le 3 \qquad 0 \le y \le -\frac{2}{3}x+2$$

OR

$$0 \le x \le -\frac{3}{2}y+3 \qquad 0 \le y \le 2$$

Note: If you consider your limits as $0 \le x \le 3$, $0 \le y \le 2$, you will end up considering a region of a reclangle rather than a triangle.

Recall example 4 from "double integral."

$\therefore$ neither region holds an advantange over the other, we will use the first one.

$$\iiint_E 2x\,dV = \iint_R \left[\int_0^{6-2x-3y} 2x\,dz\right] dA$$

$$= \iint_R 2x\left[z\right]_0^{6-2x-3y} dA$$

$$= \int_{x=0}^{3}\int_{y=0}^{-\frac{2}{3}x+2} 2x(6-2x-3y)\,dy\,dx$$

$$= \int_0^3 \left[12xy - 4x^2y - \cancel{6}^{3}x\frac{y^2}{\cancel{2}}\right]_0^{-\frac{2}{3}x+2} dx$$

$$= \int_0^3 \left[12x\left(-\tfrac{2}{3}x+2\right) - 4x^2\left(-\tfrac{2}{3}x+2\right) - 3x\left(-\tfrac{2}{3}x+2\right)^2\right] dx - 0 + 0 + 0$$

$$= \int_0^3 \left(\frac{4}{3}x^3 - 8x^2 + 12x\right) dx$$

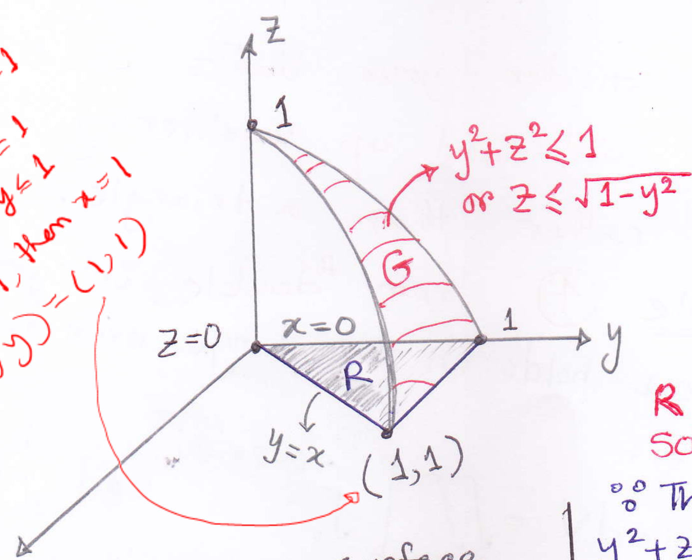
$$= \left[\frac{4}{3}\cdot\frac{x^4}{4} - 8\frac{x^3}{3} + 12\frac{x^2}{2}\right]_0^3$$

$$= \frac{(3)^4}{3} - \frac{8}{3}(3)^3 + 6(3)^2 - 0 + 0 - 0$$

$$= 9.$$

$\boxed{3}$ Let G be the wedge in the 1st Octant cut from the cylindrical solid $y^2+z^2 \le 1$ by the planes $y=x$ & $x=0$. Evaluate $\iiint_G z\, dv$.

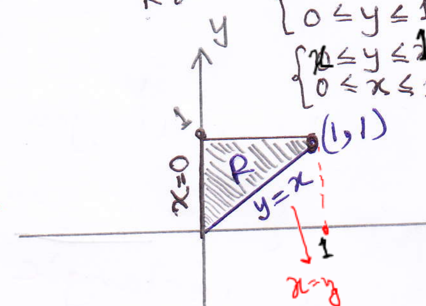
Given $y^2+z^2 \le 1$
in xy plane: $z=0$
$\therefore y^2+0^2 \le 1$
$y^2 \le 1$
$y \le 1$
$\because x=y$ Given, then $x=1$
$\therefore$ when $y=1$, $(x,y)=(1,1)$



For double integral over R:

$\begin{cases} 0 \le x \le y \\ 0 \le y \le 1 \end{cases}$ type II Region

$\begin{cases} x \le y \le 1 \\ 0 \le x \le 1 \end{cases}$ type I Region



type I & II are found in week 5

R is the pojection of solid G on xy-plane.

The upper surface of the solid is formed by cylinder (i.e. $y^2+z=1$) and lower surface by xy plane.

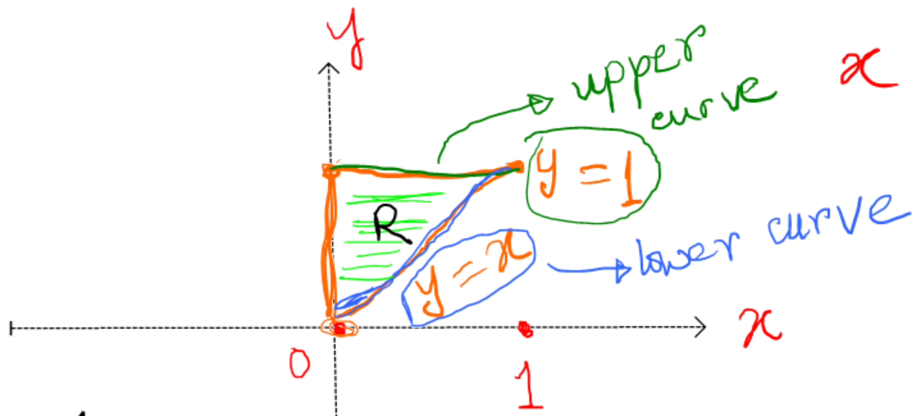
$\because$ The portion of the cylinder $y^2+z^2=1$ lies above xy-plane has eqn $z=\sqrt{1-y^2}$ and xy-plane has eqn $z=0$.

$$\iiint_G z\,dv = \iint\left\{\int_{z=0}^{\sqrt{1-y^2}} z\,dz\right\} dA$$

Graphical Explanation of Example 3 Page 4

Type I region

$x \in [0, 1]$

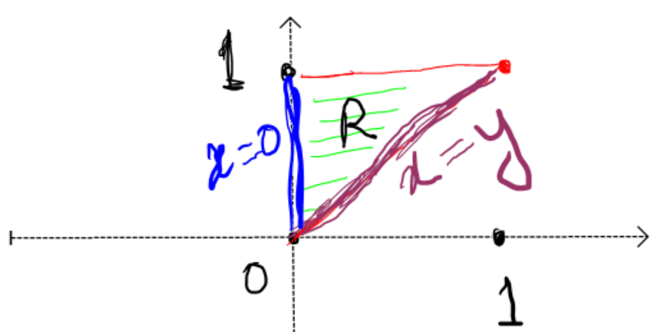


over the interval $x \in [0, 1]$ we have upper curve $y = 1$ & lower curve $y = x$

$\therefore\ x \leq y \leq 1$ or $y \in [x, 1]$

Type II region

$y \in [0, 1]$



over the interval $y \in [0, 1]$

we have outer curve $x = y$ &

inner curve $x = 0$

$\therefore\ 0 \leq x \leq y$ or $x \in [0, y]$

$$\iiint_G z\,dV = \underbrace{\int_{y=0}^{1}\int_{x=0}^{y}}_{\text{Type II region}} \int_{z=0}^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^y \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^y \left[1-y^2-0\right] dx\,dy$$

$$= \frac{1}{2}\int_0^1 \left[x - xy^2\right]_0^y dy$$

$$= \frac{1}{2}\int_0^1 \left[y-y^3\right] dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1$$

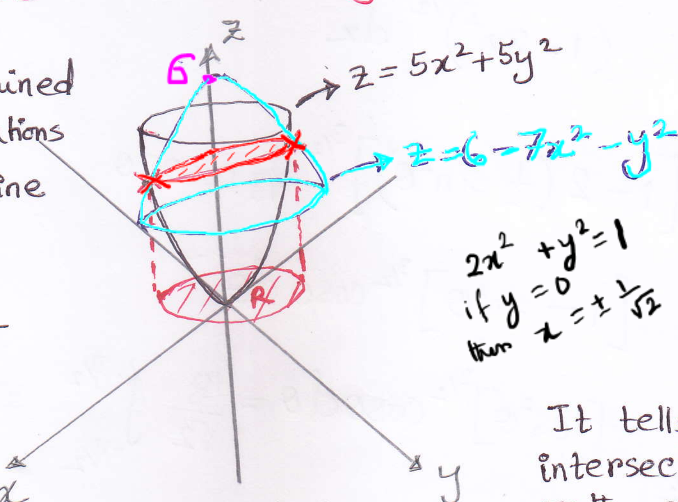
$$= \frac{1}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{1}{8}$$

$\boxed{4}$ Find the volume of the solid enclosed between the paraboloids $z = 5x^2+5y^2$ & $z = 6-7x^2-y^2$

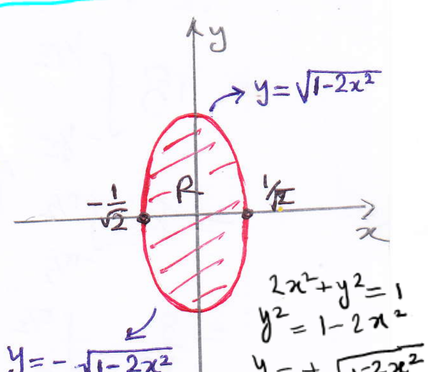
The projection R is obtained by solving the given equations simultaneously to determine whether the paraboloids intersect.

$$5x^2+5y^2 = 6-7x^2-y^2$$
$$12x^2+6y^2 = 6$$
$$2x^2+y^2 = 1$$

eqn of ellipse



$2x^2+y^2=1$
if $y=0$
then $x = \pm\frac{1}{\sqrt{2}}$



$2x^2+y^2=1$
$y^2 = 1-2x^2$
$y = \pm\sqrt{1-2x^2}$

It tells the paraboloids intersect in a curve on the elliptic cylinder $2x^2+y^2=1$

$$V = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \left[6 - 12x^2 - 6y^2\right] dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6y - 12x^2 y - 2y^3\right]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6y(1-2x^2) - 2y^3\right]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{1/\sqrt{2}} \Big[6\sqrt{1-2x^2}(1-2x^2) - 2\left(\sqrt{1-2x^2}\right)^3 - 6\left(-\sqrt{1-2x^2}\right)(1-2x^2) + 2\left(-\sqrt{1-2x^2}\right)^3\Big] dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6(1-2x^2)^{3/2} - \cancel{2(1-2x^2)^{3/2}} + 6(1-2x^2)^{3/2} - \cancel{2(1-2x^2)^{3/2}}\right] dx$$

$$= 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$$

$\therefore x^2 = \frac{1}{2}\sin^2\theta$ let $x = \frac{1}{\sqrt{2}}\sin\theta$

$dx = \frac{1}{\sqrt{2}}\cos\theta\, d\theta$

limits:

$x = -\frac{1}{\sqrt{2}} \rightarrow \theta = -90°$

$x = \frac{1}{\sqrt{2}} \rightarrow \theta = 90°$

$$= 8 \int_{-\pi/2}^{\pi/2} \left[1 - 2\left(\tfrac{1}{2}\sin^2\theta\right)\right]^{3/2} \frac{1}{\sqrt{2}}\cos\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \left[1 - \sin^2\theta\right]^{3/2} \cos\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \left[\cos^2\theta\right]^{3/2} \cos\theta\, d\theta = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int [\cos^2\theta]^2 \, d\theta$$

$$= \frac{8}{\sqrt{2}} \int \left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta$$

$$= \frac{8}{\sqrt{2}} \int \frac{1}{4} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{2}{\sqrt{2}} \int \left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \int \left[1 + 2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \int \left[\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \left[\frac{3\theta}{2} + \frac{2\sin 2\theta}{2} + \frac{1}{2}\,\frac{\sin 4\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

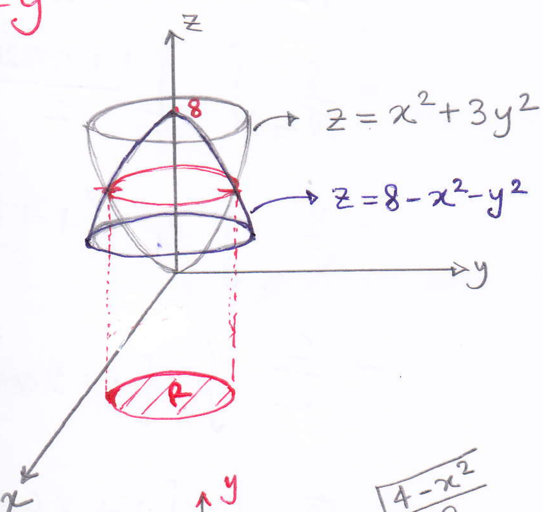
$$= \sqrt{2} \left[\frac{3}{2}\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right) + \frac{1}{8}\sin 4\left(\frac{\pi}{2}\right) - \frac{3}{2}\left(-\frac{\pi}{2}\right) - \sin 2\left(-\frac{\pi}{2}\right) - \frac{1}{8}\sin 4\left(-\frac{\pi}{2}\right)\right]$$

$$= \sqrt{2}\left[\frac{3\pi}{4} + \sin\pi + \frac{\sin 2\pi}{8} + \frac{3\pi}{4} - \sin\pi - \frac{\sin 2\pi}{4}\right]$$

$$= \sqrt{2}\left[\frac{6\pi}{4}\right] = \sqrt{2}\left[\frac{3\pi}{2}\right] = \frac{3\pi}{\sqrt{2}}.$$

$\boxed{5}$ Find the volume of the solid enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Similar to example (4)



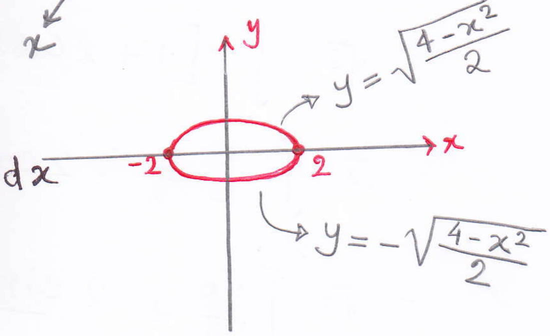
$$x^2 + 3y^2 = 8 - x^2 - y^2$$

$$x^2 + 2y^2 = 4 \longrightarrow \text{ellipse}$$

$\hookrightarrow R$

$$y = \pm \sqrt{\frac{4 - x^2}{2}}$$

if $y = 0 \Rightarrow x = \pm 2$



$$V = \int_{x=-2}^{2} \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left[8 - 2x^2 - 4y^2\right] dy\,dx$$

$$= \int_{x=-2}^{2} \left[8y - 2x^2 y - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2y\,(4 - x^2) - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2\sqrt{\frac{4-x^2}{2}}\,(4-x^2) - \frac{4}{3}\left(\sqrt{\frac{4-x^3}{2}}\right)^3 - 2\left(-\sqrt{\frac{4-x^2}{2}}\right)(4-x^2) + \frac{4}{3}\left(-\sqrt{\frac{4-x^2}{2}}\right)^3\right] dx$$

$$= \int_{-2}^{2} \left[\frac{1}{\sqrt{2}}(4-x^2)^{3/2} - \frac{\overset{2}{\cancel{4}}}{3} \cdot \frac{(4-x^2)^{3/2}}{\cancel{2}\sqrt{2}} + \frac{1}{\sqrt{2}}(4-x^2)^{3/2} - \frac{\overset{2}{\cancel{4}}}{3} \cdot \frac{(4-x^2)^{3/2}}{\cancel{2}\sqrt{2}}\right] dx$$

$$= \int_{-2}^{2} \left[\frac{2}{\sqrt{2}}(4-x^2)^{3/2} - \frac{4}{3\sqrt{2}}(4-x^2)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[\frac{6(4-x^2)^{3/2} - 4(4-x^2)^{3/2}}{3\sqrt{2}} \right] dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = -2 \rightarrow \theta = -\pi/2$

$x = 2 \rightarrow \theta = \pi/2$

$$= \frac{2}{3\sqrt{2}} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

$$= \frac{\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} (4 - 4\sin^2\theta)^{3/2}\, 2\cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int \left[4(1-\sin^2\theta)\right]^{3/2} \cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int 4^{3/2} (1-\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int (2^2)^{3/2} (\cos^2\theta)^{3/2} \cos\theta\, d\theta$$

$$= \frac{16\sqrt{2}}{3} \underbrace{\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta}_{\text{see example } 4}$$

$$= \frac{\overset{4}{\cancel{16}}\sqrt{2}}{\underset{1}{\cancel{3}}} \left[\frac{\overset{2}{\cancel{6}}\pi}{\underset{1}{\cancel{4}}} \right] = 8\sqrt{2}\pi.$$