— Exact Equations & Integrating Factors —

$M(x,y)\,dx + N(x,y)\,dy = 0$ is an Exact differential eqn if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solve for $f(x,y) = c$ by the following steps:

1 $\boxed{a}$ $\dfrac{\partial f}{\partial x} = M(x,y)$ or 1 $\boxed{b}$ $\dfrac{\partial f}{\partial y} = N(x,y)$

2 $\displaystyle\int \frac{\partial f}{\partial x} = \int M(x,y)$

$$\underbrace{\int \partial}\, f = \int M(x,y)\,\partial x$$

3 $\boxed{f = \underbrace{f(x,y)}_{\text{Result of integration}} + \underbrace{\phi(y)}_{\text{integration constant}}}$ ...... final eqn while we evaluate $\phi(y)$

fundamental law of calculus: if we derive & integrate a particular function, then the function is unchanged

integration constant while $x \to$ variable, $y \to$ constant

4 $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[f(x,y) + \phi(y)\right]$ derive step 3 with respect to 'y'

$N(x,y) = \dfrac{\partial}{\partial y} f(x,y) + \phi'(y)$ from 1 $\boxed{b}$ $\dfrac{\partial f}{\partial y} = N(x,y)$

$\phi'(y) = N(x,y) - \dfrac{\partial}{\partial y} f(x,y)$

$\int \phi'(y)\,dy = \phi(y)$ $\longrightarrow$ $\phi(y) = \displaystyle\int \left[N(x,y) - \frac{\partial}{\partial y} f(x,y)\right]\partial y$

Substitute $\phi(y)$ into step 3 and hence solve $f(x,y) = C$.

Examples:

$\boxed{1}$ $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

$M dx + N dy = 0$

$M = \sin y - y\sin x \qquad N = \cos x - x\cos y - y$

$\frac{\partial M}{\partial y} = \cos y - \sin x \qquad \frac{\partial N}{\partial x} = -\sin x + \cos y$

$\therefore$ Exact DE

we know,

$\frac{\partial f}{\partial x} = M = \sin y - y\sin x \qquad \frac{\partial f}{\partial y} = N = \cos x - x\cos y - y$

$\int \partial f = \int (\sin y - y\sin x)\,\partial x$

$f = x\sin y + y\cos x + \phi(y)$ —— (*)

$\frac{\partial f}{\partial y} = x\cos y + \cos x + \phi'(y)$ derived w.r.t. 'y'

$N = x\cos y + \cos x + \phi'(y)$

$\cancel{\cos x} + \cancel{x\cos y} - y = \cancel{x\cos y} + \cancel{\cos x} + \phi'(y)$

$\phi'(y) = -y$

$\phi(y) = -\int y\,dy = \frac{-y^2}{2} + C_0$

substitute $\phi(y)$ into (*)

$f(x,y) = x\sin y + y\cos x - \frac{y^2}{2} + C_0 = C$

$\therefore f(x,y) = C$

$\boxed{2}$ $(x - y^3 + y^2 \sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$

$(x - y^3 + y^2 \sin x)\,dx + \{-(3xy^2 + 2y\cos x)\}\,dy = 0$

$M = x - y^3 + y^2 \sin x \qquad N = -3xy^2 - 2y\cos x$

$\dfrac{\partial M}{\partial y} = -3y^2 + 2y\sin x \qquad \dfrac{\partial N}{\partial x} = -3y^2 + 2y\sin x$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, It is exact DE.

$\dfrac{\partial f}{\partial x} = M = x - y^3 + y^2 \sin x \qquad \dfrac{\partial f}{\partial y} = N = -3xy^2 - 2y\cos x$

$\int \partial f = \int (x - y^3 + y^2 \sin x)\,\partial x$

$f = \dfrac{x^2}{2} - xy^3 - y^2\cos x + \phi(y)$ —— (*)

$\dfrac{\partial f}{\partial y} = 0 - x \cdot 3y^2 - 2y \cdot \cos x + \phi'(y)$ derived w.r.t. 'y'

$N = -3xy^2 - 2y\cos x + \phi'(y)$

$\cancel{-3xy^2 - 2y\cos x} = \cancel{-3xy^2 - 2y\cos x} + \phi'(y) \quad \because N = \dfrac{\partial f}{\partial y}$

$\phi'(y) = 0$

$\phi(y) = \int 0\,dy = C_0$

Substitute $\phi(y)$ into (*)

$f(x,y) = \dfrac{x^2}{2} - xy^3 - y^2\cos x + C_0 = C$

$\therefore f(x,y) = C$.

$\boxed{3}$ $x\dfrac{dy}{dx} = 2xe^{x} - y + 6x^2$

Reading

$x\,dy = (2xe^{x} - y + 6x^2)\,dx$

$(2xe^{x} - y + 6x^2)\,dx - x\,dy = 0$

$M = 2xe^{x} - y + 6x^2$; $N = -x$

$\dfrac{\partial M}{\partial y} = -1$, $\dfrac{\partial N}{\partial x} = -1$

$\therefore$ Exact DE

$\dfrac{\partial f}{\partial x} = M = 2xe^{x} - y + 6x^2$

$\int \partial f = \int (2xe^{x} - y + 6x^2)\,\partial x$

$f = 2\left[x\int e^{x}dx - \int\left[\frac{\partial}{\partial x}(x)\int e^{x}dx\right]dx\right] - xy + 6\cdot\frac{x^3}{3} + \phi(y)$

$= 2xe^{x} - 2\int e^{x}dx - xy + 2x^3 + \phi(y)$

$f = 2xe^{x} - 2e^{x} - xy + 2x^3 + \phi(y)$ — (*)

derived w.r.t. 'y'

$\dfrac{\partial f}{\partial y} = 0 - 0 - x + 0 + \phi'(y)$ $\therefore N = \dfrac{\partial f}{\partial y}$

$N = -x + \phi'(y)$

$-x = -x + \phi'(y)$

$\phi'(y) = 0$

$\phi(y) = c_0$

Substitute $\phi(y)$ into (*)

$f(x,y) = 2xe^{x} - 2e^{x} - xy + 2x^3 + c_0 = c_1$

Reading

$\boxed{4}$ $(\tan x - \sin x \sin y)dx + \cos x . \cos y \, dy = 0$

$M = \tan x - \sin x . \sin y$; $N = \cos x . \cos y$

$\frac{\partial M}{\partial y} = -\sin x \cos y$ $\qquad \frac{\partial N}{\partial x} = -\sin x . \cos y$

$\therefore$ Exact DE

$\frac{\partial f}{\partial x} = M = \tan x - \sin x . \sin y$

$\int \partial f = \int (\tan x - \sin x . \sin y) \partial x$

$f = \ln|\sec x| + \cos x . \sin y + \phi(y)$ —— $*$

$\frac{\partial f}{\partial y} = 0 + \cos x \cos y + \phi'(y)$ derived w.r.t. 'y'

$N = \cos x \cos y + \phi'(y)$ $\qquad \therefore N = \frac{\partial f}{\partial y}$

$\cos x \cos y = \cos x \cos y + \phi'(y)$

$\phi'(y) = 0$

$\phi(y) = \int 0 \, dy = c_0$

substitute $\phi(y)$ into $*$

$f(x,y) = \ln|\sec x| + \cos x . \sin y + c_0 = C.$

$\boxed{5}$ $(e^x+y)dx+(2+x+ye^y)dy=0$; $y(0)=1$

$x=0$

$y=1$

$M=e^x+y$; $N=2+x+ye^y$

$\frac{\partial M}{\partial y}=1$ $\qquad$ $\frac{\partial N}{\partial x}=1$

$\therefore$ exact D.E.

$\frac{\partial f}{\partial x}=M=e^x+y$ $\qquad$ $\frac{\partial f}{\partial y}=N=2+x+ye^y$

$\int \partial f=\int(e^x+y)\partial x$

$f=e^x+xy+\phi(y)$ ——— (*)

$\frac{\partial f}{\partial y}=x+\phi'(y)$ derived w.r.t. 'y'

$N=x+\phi'(y)$ $\qquad$ $\therefore N=\frac{\partial f}{\partial y}$

$2+\cancel{x}+ye^y=\cancel{x}+\phi'(y)$

$\phi'(y)=2+ye^y$

$\phi(y)=\int(2+ye^y)dy$

$\int ye^y dy$
$=y\int e^y dy-\int\{\frac{d}{dy}y\int e^y dy\}dy$
$=ye^y-\int 1 e^y dy$
$=ye^y-e^y$

$\phi(y)=2y+\underbrace{ye^y-e^y}+c_0$

Substitute $\phi(y)$ into (*)

$f(x,y)=e^x+xy+2y+ye^y-e^y+c_0=c$

$\Rightarrow e^x+xy+2y+ye^y-e^y=c-c_0=c$

Relabel the Constant $c-c_0=c$

$e^x+xy+2y+ye^y-e^y=c$ ——— (**)

substitute $x=0,\ y=1$ into $**$

$$e^0+(0)(1)+2(1)+(1)e^1-e^1=C$$

$$1+0+2+e-e=C$$

$$C=3$$

$$\therefore f(x,y)=e^x+xy+2y+ye^y-e^y=C$$

$$\Rightarrow e^x+xy+2y+ye^y-e^y=3.$$

$\boxed{6}$ $$\left(\frac{1}{1+y^2}+\cos x-2xy\right)\frac{dy}{dx}=y(y+\sin x);\quad y(0)=1$$

$x=0$

$y=1$

$$\left(\frac{1}{1+y^2}+\cos x-2xy\right)dy=y(y+\sin x)\,dx$$

$$\left(\frac{1}{1+y^2}+\cos x-2xy\right)dy=0$$

$$y(y+\sin x)\,dx-\left(\frac{1}{1+y^2}+\cos x-2xy\right)$$

$$M=y(y+\sin x)=y^2+y\sin x$$

$$N=-\frac{1}{1+y^2}-\cos x+2xy$$

$$\frac{\partial N}{\partial x}=-0+\sin x+2y$$

$$\frac{\partial M}{\partial y}=2y+\sin x$$

$\therefore$ exact D.E.

$$\frac{\partial f}{\partial x}=M=y^2+y\sin x$$

$$\int\partial f=\int(y^2+y\sin x)\,\partial x$$

$$f=xy^2-y\cos x+\phi(y) \text{ ——— } *$$

derived w.r.t. 'y'

$$\frac{\partial f}{\partial y}=2xy-\cos x+\phi'(y) \qquad \therefore N=\frac{\partial f}{\partial y}$$

$$N=2xy-\cos x+\phi'(y)$$

$$N = 2xy - \cos x + \phi'(y)$$

$$-\frac{1}{1+y^2} - \cancel{\cos x} + \cancel{2xy} = \cancel{2xy} - \cancel{\cos x} + \phi'(y)$$

$$\phi'(y) = -\frac{1}{1+y^2}$$

$$\phi(y) = -\int \frac{1}{1+y^2}\,dy = -\tan^{-1} y + C_0$$

substitute $\phi(y)$ into $*$

$$f = xy^2 - y\cos x - \tan^{-1} y + C_0 = C \qquad \because f(x,y) = C$$

$$\Rightarrow xy^2 - y\cos x - \tan^{-1} y = C - C_0 = C \quad \text{Relabel Constant } C - C_0 = C$$

$**$

Substitute $x = 0$, $y = 1$ into $**$

$$(0)(1)^2 - 1\cos(0) - \tan^{-1}(1) = C$$

$$0 - 1 - \frac{\pi}{4} = C$$

$$C = -1 - \frac{\pi}{4}$$

Substitute C into $*$

$$xy^2 - y\cos x - \tan^{-1} y = C$$

$$\therefore f(x,y) = xy^2 - y\cos x - \tan^{-1} y = -1 - \frac{\pi}{4}.$$

$$\Rightarrow xy^2 - y\cos x$$