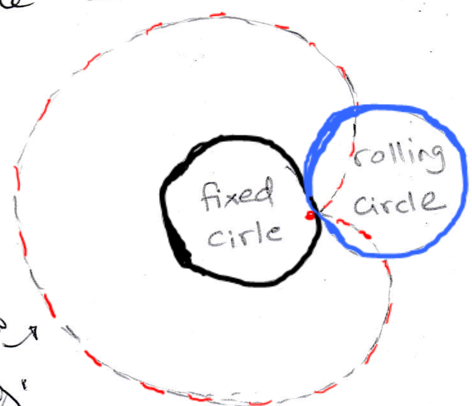
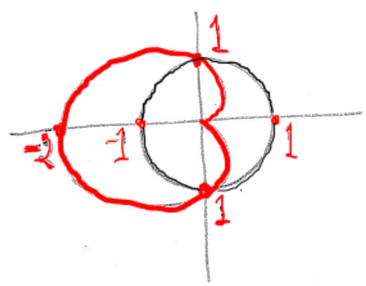
Double Integral: Concept of Cardioid

A cardioid is a plane curve traced by a pt on the perimeter of the circle that is rolling around a fixed circle of the same radius.

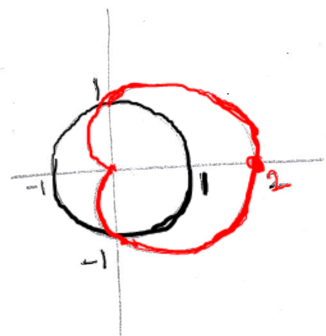


by completing the tour pt creates this pathway.

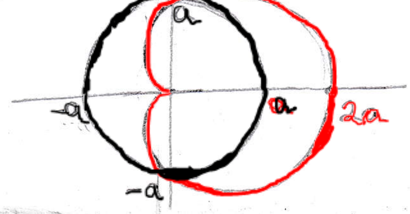
$r = 1 - \cos\theta$



$r = 1 + \cos\theta$



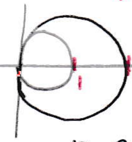
$r = a(1 + \cos\theta) = a + a\cos\theta$



Rose petals

$r = \text{ } \cos n\theta$ ⟶ no. of petals

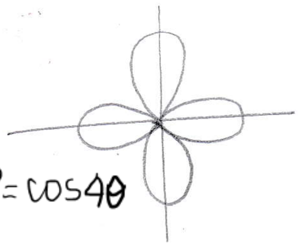
$r = a\cos\theta$ $a=1, 2$



$a=1$ $r=\cos\theta$

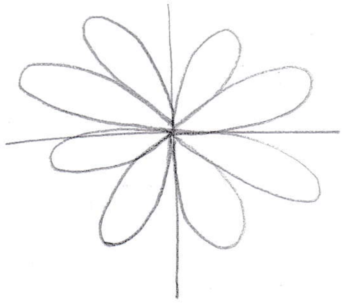
$\because x = r\cos\theta$ on x-axis

$r = 2\cos\theta$ $a=2$

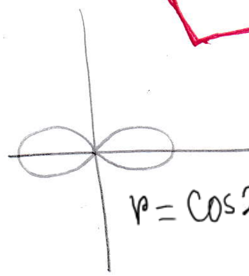


$r = \cos 4\theta$

4 (c)

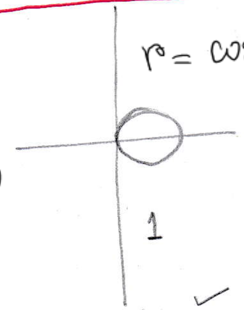


8 (c)



$r = \cos 2\theta$

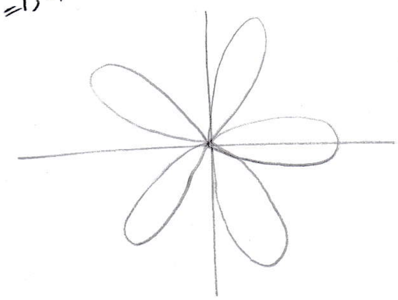
2 (a)



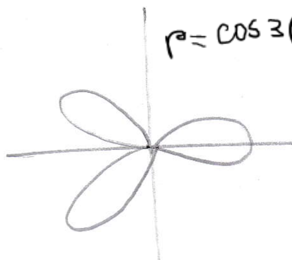
$r = \cos 1\theta$

1 (a) horizontal

Show up to $\cos 3\theta$, $\sin 3\theta$ $n = 1, 2, 3$ only

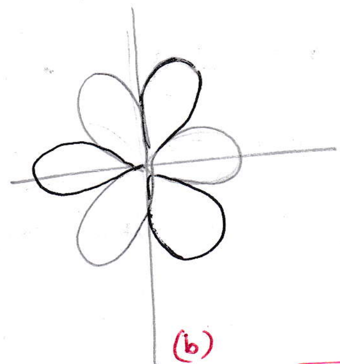


5 (b)



$r = \cos 3\theta$

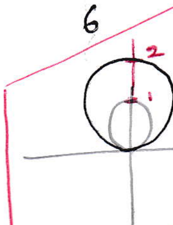
3 (b)



(b) 6

$r = a\sin\theta$ on y axis $y = r\sin\theta$

$a = 1, 2$

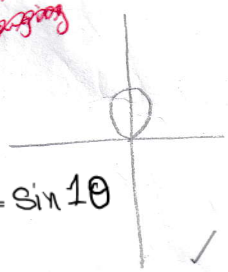


$r = 2\sin\theta$

$r = \sin\theta$

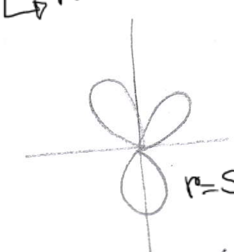
vertical

$r = \cancel{a} \sin n\theta$ ⟶ number of petals



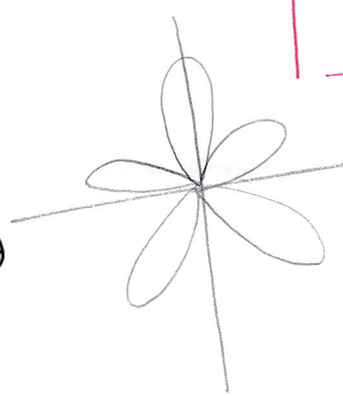
$r = \sin 1\theta$

1

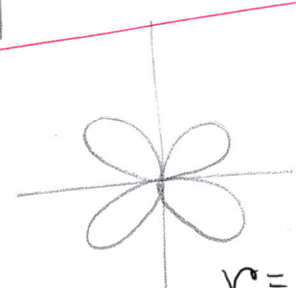


$r = \sin 3\theta$

3



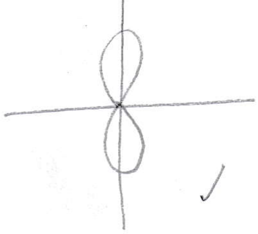
5



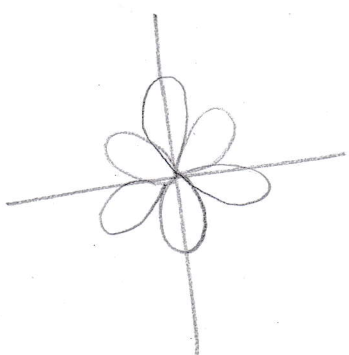
$r = \sin 4\theta$

4

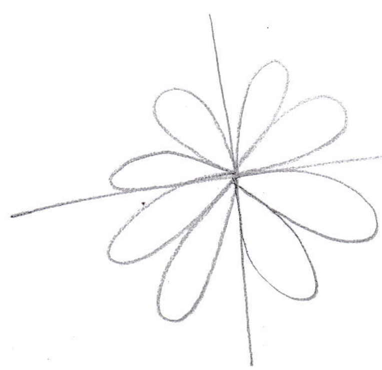
$r = \sin 2\theta$



2



6

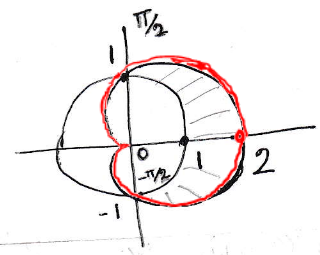


8

Double integrals in Polar form

① Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{r=1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$



$$= \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} 1\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[(1+\cos\theta)^2 - 1^2\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[1 + 2\cos\theta + \cos^2\theta - 1\right] d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[2\cos\theta + \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\cos\theta + \frac{1}{4} + \frac{\cos 2\theta}{4}\right) d\theta$$

$$= \left[\sin\theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8}\right]_{-\pi/2}^{\pi/2}$$

$$= \sin\frac{\pi}{2} + \frac{\pi/2}{4} + \frac{\sin 2(\pi/2)}{8} - \sin\left(-\frac{\pi}{2}\right) - \frac{\left(-\frac{\pi}{2}\right)}{4} - \frac{\sin 2(-\pi/2)}{8}$$

$$= 1 + \frac{\pi}{8} + 0 - (-1) + \frac{\pi}{8} - 0$$

$$= 2 + \frac{\pi}{4}.$$

② $\iint_R e^{-(x^2+y^2)}\, dA$, where R is the region bounded by the circle $x^2+y^2=1$

$x^2+y^2=r^2$ eqn of a circle

Let
$r^2 = z$
$2r\,dr = dz$
$r\,dr = \frac{1}{2}dz$

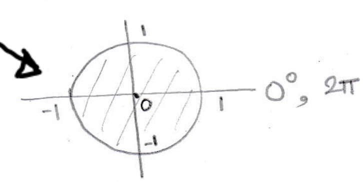
$r=0 \to z=0$
$r=1 \to z=1$

arc length of full circle 2π

$$\iint_R e^{-(x^2+y^2)}\, dA = \int_0^{2\pi}\int_0^1 e^{-r^2}\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 e^{-z}\, \frac{1}{2} dz\, d\theta$$

$dA = dx\,dy$ or $dy\,dx$ $= r\,dr\,d\theta$

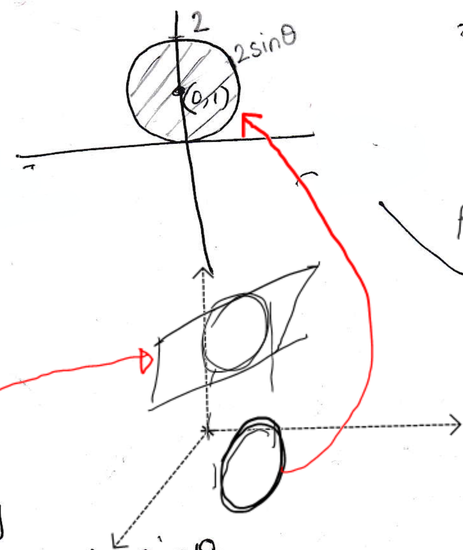


$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{e^{-z}}{-1}\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (-e^{-1}+1)\, d\theta = \frac{1}{2}\, 2\pi\,(1-e^{-1}) = \pi(1-e^{-1})$$

③ Find the ~~volume~~ area under the plane $6x+4y+z=12$ above the disk $x^2+y^2=2y$

$$x^2+y^2=2y$$
$$x^2+y^2-2y+1=1$$
$$x^2+(y-1)^2=1^2$$
$$(x-0)^2+(y-1)^2=1^2$$
Center: $(0,1)$



$$x^2+y^2=2y$$
$$r^2=2r\sin\theta$$
$$r=2\sin\theta \rightarrow \text{above } x \text{ axis } y\geq 0$$

Full disk
$$0\leq\theta\leq 2\pi \quad (1^{st}\ \&\ 2^{nd}\ \text{quadrant})$$
$$0\leq r\leq 2\sin\theta$$

Now,

$z=f(x,y)$

$$\boxed{6x+4y+z=12}$$
$$z=12-6x-4y$$
$$=12-6r\cos\theta-4r\sin\theta$$

$$I=\int_{\theta=0}^{2\pi}\int_{r=0}^{2\sin\theta}(12-6r\cos\theta-4r\sin\theta)\,r\,dr\,d\theta$$

$$\sin^2\theta=\frac{1-\cos2\theta}{2}$$
$$\cos^2\theta=\frac{1+\cos2\theta}{2}$$

$$=\int_0^{2\pi}\int_0^{2\sin\theta}(12r-6r^2\cos\theta-4r^2\sin\theta)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[6r^2-2r^3\cos\theta-\tfrac{4}{3}r^3\sin\theta\right]_0^{2\sin\theta}d\theta$$

$$=\int_0^{2\pi}\left[24\sin^2\theta-16\sin^3\theta\cos\theta-\tfrac{4}{3}\cdot 8\sin^4\theta\right]d\theta$$

$$=\int_0^{2\pi}\left[24\sin^2\theta-16\sin^3\theta\cos\theta-\frac{32}{3}\underbrace{\sin^4\theta}_{(\sin^2\theta)^2}\right]d\theta$$

$$=\int_0^{2\pi}24\cdot\tfrac{1}{2}(1-\cos2\theta)\,d\theta-16\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta-\frac{32}{3}\int_0^{\pi}\left\{\tfrac{1}{2}(1-\cos2\theta)\right\}^2d\theta$$

$$=12\int_0^{2\pi}(1-\cos2\theta)\,d\theta-16\int_0^{2\pi}\sin^3\theta\cos\theta\,d\theta-\frac{32}{3}\int_0^{2\pi}\tfrac{1}{4}(1-2\cos2\theta+\cos^2 2\theta)\,d\theta$$

$$=12\underbrace{\int_0^{2\pi}(1-\cos2\theta)\,d\theta}_{(a)}-16\underbrace{\int_0^{2\pi}\sin^3\theta\cos\theta\,d\theta}_{(b)}-\frac{8}{3}\underbrace{\int_0^{2\pi}\left(1-2\cos2\theta+\frac{(1+\cos4\theta)}{2}\right)d\theta}_{(c)}$$

Now (a) $\int_0^{2\pi} (1-\cos 2\theta)\,d\theta = \theta - \frac{1}{2}\sin 2\theta \Big|_0^{2\pi} = 2\pi - 0 = 2\pi$

(b) $\int_0^{2\pi} \sin^3\theta \cos\theta\,d\theta$

let $\sin\theta = u$

$\cos\theta\,d\theta = du$

$\theta = 0 \Rightarrow u = 0$

$\theta = 2\pi \Rightarrow u = 0$

$= \int_0^0 u^3\,du$

$= 0$

(c) $\int_0^{2\pi} \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$

$= \int_0^{2\pi} \left(3/2 - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$

$= \left[3/2\,\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$

$= 3\pi - \sin 4\pi + \frac{1}{8}\sin 8\pi - 0$

$= 3\pi$

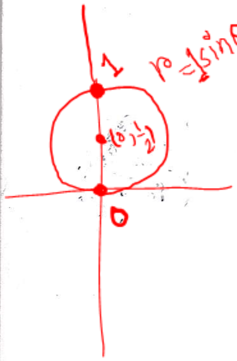
$I = 12(2\pi) - 16 \times (0) - 8/3 \times (3\pi)$

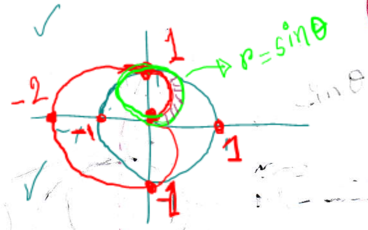
$= 24\pi - 0 - 8\pi = 16\pi$

Examples

Double Integral

(4) Find the area of the region enclosed by the inside of the circle $r=\sin\theta$ & outside of the Cardioid $r = 1 - \cos\theta$



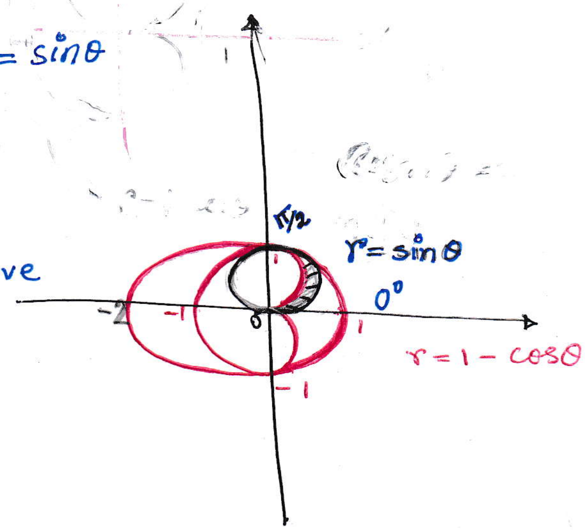


$\theta \in [0, \frac{\pi}{2}]$

Comparing the curves $r=\sin\theta$ & $r = 1-\cos\theta$

$r \in [1-\cos\theta, \sin\theta]$

inner curve — outer curve



$$A = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Big]_{1-\cos\theta}^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[-\cos 2\theta - 1 + 2\cos\theta\right] d\theta$$

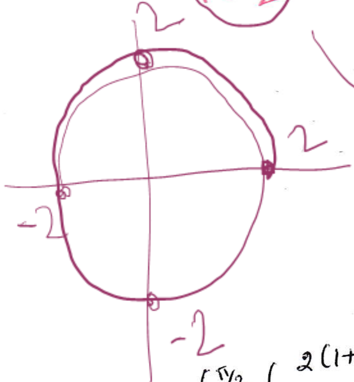
$$= \frac{1}{2}\left[-\frac{\sin 2\theta}{2} - \theta + 2\sin\theta\right]_0^{\pi/2}$$

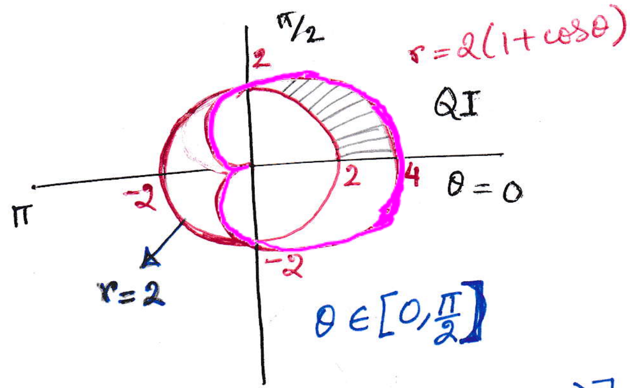
$$= \frac{1}{2}\left[-\frac{0}{2} - \pi/2 + 2(1) - 0\right]$$

$$= \frac{1}{2}\left(2 - \pi/2\right)$$

$$= 1 - \pi/4$$

⑤ Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the 1st quadrant that is outside the circle $r=2$ & inside the cardioid $r = 2(1+\cos\theta)$





$\theta \in [0, \frac{\pi}{2}]$

$r \in [2, 2(1+\cos\theta)]$

$$\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left([2(1+\cos\theta)]^2 - 4 \right) \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[4(1+2\cos\theta+\cos^2\theta)\sin\theta - 4\sin\theta \right] d\theta$$

$$= \frac{1}{2}\times 4 \int_0^{\pi/2} \left(\cancel{\sin\theta} + 2\sin\theta\cos\theta + \sin\theta\cos^2\theta - \cancel{\sin\theta} \right) d\theta$$

$$= 2\int_0^{\pi/2} \left(\sin 2\theta + \cos^2\theta \sin\theta \right) d\theta$$

$$= 2\left[\int_0^{\pi/2} \sin 2\theta \, d\theta + \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \right]$$

let $\cos\theta = u$

$-\sin\theta \, d\theta = du$

$\sin\theta \, d\theta = -du$

$\theta = 0 \to u = 1$

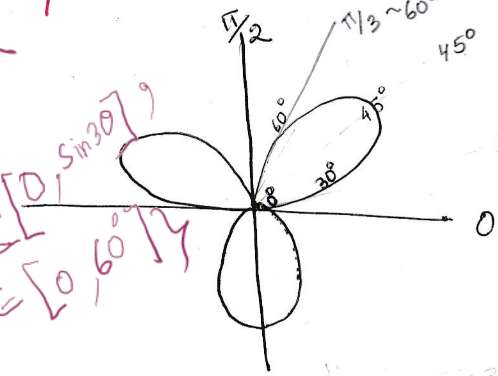
$\theta = \pi/2 \to u = 0$

$$= 2\left[-\frac{\cos 2\theta}{2}\Big|_0^{\pi/2} + \int_1^0 (-u^2)\,du \right]$$

$$= 2\left[-\left(\frac{-1}{2} - \frac{1}{2}\right) + \left[-\frac{u^3}{3}\right]_1^0 \right]$$

$$= 2\left[+1 + \left(-\frac{0}{3} + \frac{1}{3}\right) \right] = 2\left(1+\frac{1}{3}\right) = \frac{8}{3}$$

⑥ Use a polar double integral to find the area enclosed by the three petaled rose $r = \sin 3\theta$.



$R: \{ r \in [0, \sin 3\theta],\ \theta \in [0, 60°] \}$

$$A = 3 \iint_R dA$$

$$= 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r\,dr\,d\theta$$

3 petals

$$= 3 \int_0^{\pi/3} \frac{r^2}{2} \Big|_0^{\sin 3\theta} d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \left(\frac{1 - \cos 6\theta}{2} \right) d\theta$$

$$= \frac{3}{4} \int_0^{\pi/3} [1 - \cos 6\theta] \, d\theta$$

$$= \frac{3}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3}$$

$$= \frac{3}{4} \left[\frac{\pi}{3} - \frac{\sin 6\left(\frac{\pi}{3}\right)}{6} \right]$$

$$= \frac{3}{4} \left[\frac{\pi}{3} - \frac{\sin 2\pi}{6} - 0 \right]$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$r = 0$, $r = \sin 3\theta$

$\sin 3\theta = 0$

$3\theta = \sin^{-1} 0$

$= 0, 180, 360, \ldots$

$\theta = \underbrace{0, 60}_{QI}, \underbrace{120, \ldots}_{QII}$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$\therefore \theta \in [0, 60°]$