$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

Multiple Integral "Jacobians"

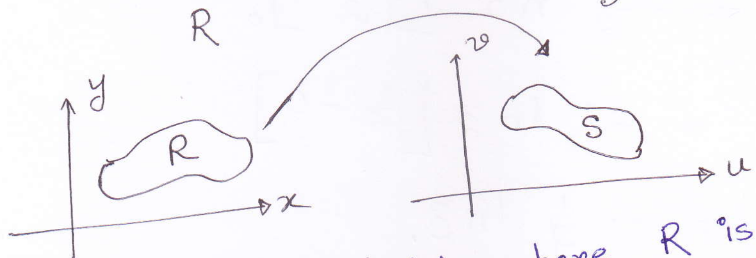
$\boxed{\text{Week 6}}$

If T is a transformation from the uv-plane to $y=y(u,v)$ the xy-plane defined by the equations $x=x(u,v)$, $y=(u,v)$, then the Jacobian of T is denoted by $J(u,v)$ or by $\frac{\partial(x,y)}{\partial(u,v)}$ and is denoted by

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}$$

And hence $\iint_R f(x,y)\,dA_{xy}=\iint_S f(u,v)\,|J|\,dA_{uv}$



Examples: ① Evaluate $\iint_R \frac{x-y}{x+y}\,dA$ where R is the region enclosed by the lines $x-y=0$, $x-y=1$, $x+y=1$, $x+y=3$ using transformation.

Let $u=x+y$, $v=x-y$

$\therefore u=1,3$ $\quad\because x+y=1,\ x+y=3$

$v=1,0$ $\quad\because x-y=1,\ x-y=0$

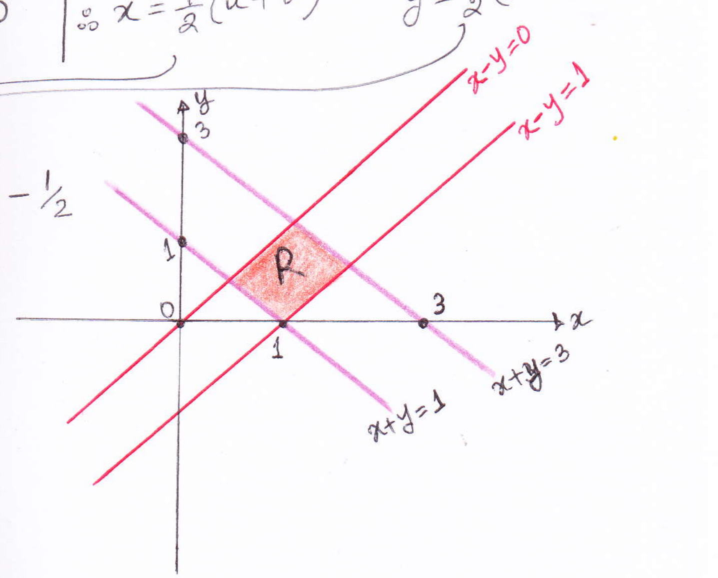
$$\begin{aligned} u&=x+y\\ v&=x-y\\ \hline u+v&=2x \quad \text{Add}\end{aligned}$$

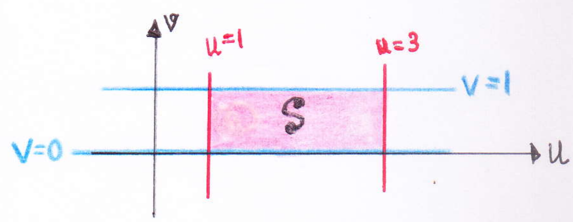
$\therefore x=\frac{1}{2}(u+v)$

$$\begin{aligned} u&=x+y\\ v&=x-y\\ (-)\ &\ \ (-)\ \ (+)\\ \hline u-v&=2y \quad \text{subtract}\end{aligned}$$

$y=\frac{1}{2}(u-v)$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}$$





$$\iint_R \frac{x-y}{x+y}\, dA_{xy} = \iint_S \frac{v}{u} |J| \, dA_{uv}$$

$$= \int_{v=0}^{1} \int_{u=1}^{3} \frac{v}{u} \left| -\frac{1}{2} \right| du\, dv$$

$$= +\frac{1}{2} \int_{v=0}^{1} v \int_{u=1}^{3} \frac{1}{u}\, du\, dv$$

$$= +\frac{1}{2} \int_{v=0}^{1} v \left[\ln u \right]_1^3 dv$$

$$= +\frac{1}{2} \int_0^1 v \left[\ln 3 - \ln 1 \right] dv$$

$$= +\frac{1}{2} \ln 3 \left[\frac{v^2}{2} \right]_0^1$$

$$= +\frac{1}{2} \ln 3 \left[\frac{1}{2} - 0 \right]$$

$$= +\frac{1}{4} \ln 3.$$

Example: (2) Evaluate $\displaystyle\int_{y=0}^{4}\int_{x=y/2}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ by applying transformation where $u=\frac{2x-y}{2}$, $v=\frac{y}{2} \to y=2v$ and integrating over an appropriate region in uv-plane.

limit of x (given): $x=\frac{y}{2}$, $x=\frac{y}{2}+1$

limit of y (given): $y=0$, $y=4$

Given

$$u=\frac{2x-y}{2}$$

$$2u=2x-y$$

$$2x=2u+y$$

$$2x=2u+2v \quad \because v=\frac{y}{2}$$

$$\therefore \boxed{x=u+v}$$

$$v=\frac{y}{2} \text{ (given)}$$

$$\boxed{y=2v}$$

$y=0 \Rightarrow 2v=0 \Rightarrow \boxed{v=0}$

$y=4 \Rightarrow 2v=4 \Rightarrow \boxed{v=2}$

$x=\frac{y}{2} \Rightarrow x=\frac{2v}{2} \Rightarrow x=v$

$\therefore u+v=v \quad \because x=u+v$

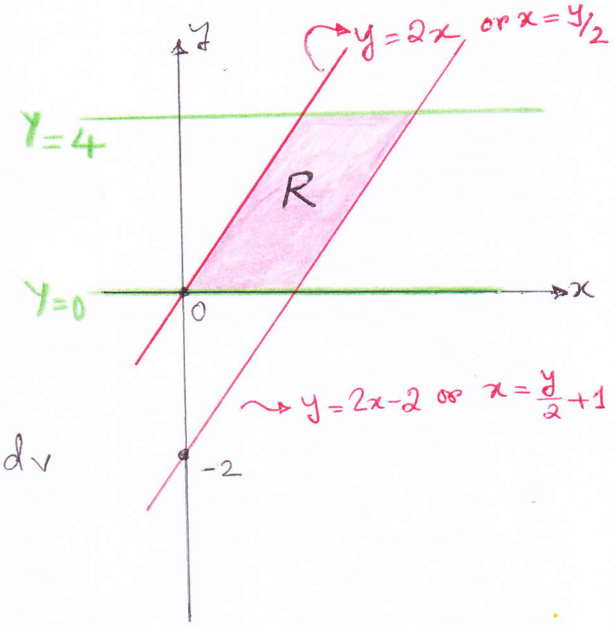
$\Rightarrow u=0$

$x=\frac{y}{2}+1 \Rightarrow x=\frac{2v}{2}+1 \Rightarrow x=v+1$

$\Rightarrow u+v=v+1$

$\Rightarrow u=1$

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$



$$\int_{y=0}^{4}\int_{x=y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$$

$$=\int_{v=0}^{2}\int_{u=0}^{1} \frac{2(u+v)-(2v)}{2}\,|J|\,du\,dv$$

$$=\int_{v=0}^{2}\int_{u=0}^{1} u\,|2|\,du\,dv$$

$$=2\int_0^2 \left[\frac{u^2}{2}\right]_0^1 dv$$

$$=\int_0^2 \left[1^2-0^2\right] dv$$

$$=\left[v\right]_0^2 = 2$$