Integrating Factor & 1st Order Differential Eqn

Consider a differential eqn:

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

coefficient of $\frac{dy}{dx}$ should be 1

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{g(x)}{a_1(x)} \quad \div \text{ by } a_1(x)$$

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

Rename $\frac{a_0(x)}{a_1(x)} = P(x)$; $\frac{g(x)}{a_1(x)} = Q(x)$

Standard form of Differential Eqn.

Calculate integrating factor (IF) denoted by $e^{\int P(x)dx}$

$IF = e^{\int P(x)dx}$: An integrating factor (IF) is a function by which an ordinary differential eqn can be multiplied in order to make it integrable.

$$\frac{d}{dx}e^{\int P(x)dx} = e^{\int P(x)dx}\frac{d}{dx}\int P(x)dx = e^{\int P(x)dx}\cdot P(x)$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)\,y\right] = e^{\int P(x)dx}\,Q(x)$$

$$[e^{2x}]' = e^{2x}(2)$$

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)\,y = e^{\int P(x)dx}\,Q(x)$$

$$\int\left[\underbrace{e^{\int P(x)dx}}_{u}\underbrace{\frac{dy}{dx}}_{v'} + \underbrace{e^{\int P(x)dx}P(x)}_{u'}\underbrace{y}_{v}\right]dx = \int e^{\int P(x)dx}\,Q(x)\,dx$$

$v = y$

$v' = \frac{dy}{dx}$

$u = e^{\int P(x)dx}$

$u' = e^{\int P(x)dx}P(x)$

$$\frac{d}{dx}(uv) = uv' + u'v$$

$$\Rightarrow \int\left[\frac{d}{dx}\left(e^{\int P(x)dx}\,y\right)\right]dx = \int e^{\int P(x)dx}\,Q(x)\,dx$$

$$\Rightarrow y\,e^{\int P(x)dx} = \int e^{\int P(x)dx}\,Q(x)\,dx$$

Now we solve for "y"

Examples:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$\boxed{1}$ $y' + 3x^2 y = x^2$

$$\frac{dy}{dx} + (3x^2)y = x^2 \; ; \quad (1) \qquad P(x) = 3x^2 \qquad Q(x) = x^2$$

$$e^{\int P(x)dx} = e^{\int 3x^2 dx} = e^{3\frac{x^3}{3}} = e^{x^3}$$

multiply eqn (1) by $e^{\int P(x)dx}$

$$e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)y\right] = e^{\int P(x)dx} Q(x)$$

$$\Rightarrow y e^{\int P(x)dx} = \int e^{\int P(x)} Q(x)\, dx$$

we have proved it in pg 1

$$e^{x^3}\left[\frac{dy}{dx} + 3x^2 y\right] = x^2 e^{x^3}$$

$$\Rightarrow y e^{x^3} = \int x^2 \cdot e^{x^3}\, dx$$

Let $x^3 = z$

$3x^2 dx = dz$

$x^2 dx = \frac{1}{3} dz$

$$= \frac{1}{3}\int e^z\, dz$$

$$= \frac{1}{3} e^z + C$$

$$y e^{x^3} = \frac{1}{3} e^{x^3} + C$$

$$y = \frac{1}{3} + Ce^{-x^3}.$$

$$y' + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$\boxed{2}$ $y' = 2y + x^2 + 5$

$$\frac{dy}{dx} - 2y = x^2 + 5 \quad -(1) \; ; \qquad P(x) = -2 \; ; \; Q(x) = x^2 + 5$$

$$\frac{dy}{dx} + (-2)y = x^2 + 5$$

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

multiply eqn (1) by $e^{\int P(x)dx}$

$$y e^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)\, dx$$

$$y e^{-2x} = \int e^{-2x}(x^2 + 5)\, dx$$

$$ye^{-2x} = \int \underbrace{x^2}_{u}\,\underbrace{e^{-2x}}_{v}\,dx + 5\int e^{-2x}\,dx$$

$$= x^2\left[\frac{e^{-2x}}{-2}\right] - \int 2x\left[\frac{e^{-2x}}{-2}\right]dx + 5\left[\frac{e^{-2x}}{-2}\right]$$

∫(uv)dx=u∫v dx-∫{u' (∫vdx)} dx

$$= -\frac{x^2e^{-2x}}{2} + \int \underbrace{x}_{u}\,\underbrace{e^{-2x}}_{v}\,dx - \frac{5}{2}e^{-2x}$$

$$= -\frac{x^2e^{-2x}}{2} + x\left[\frac{e^{-2x}}{-2}\right] - \int 1\left[\frac{e^{-2x}}{-2}\right]dx - \frac{5}{2}e^{-2x}$$

$$= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx - \frac{5}{2}e^{-2x}$$

$$= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right] - \frac{5}{2}e^{-2x} + c$$

$$ye^{-2x} = -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{5e^{-2x}}{2} + c$$

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} - \frac{5}{2} + \frac{c}{e^{-2x}}$$

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{11}{4} + ce^{2x}$$

$\boxed{3}$ $(1+x)\frac{dy}{dx} - xy = x + x^2$

$$\frac{dy}{dx} - \frac{x}{1+x}y = \frac{x+x^2}{1+x} \qquad \left[\div \text{ by}(1+x)\right]$$

$$\frac{dy}{dx} - \frac{x}{1+x}\cdot y = \frac{x(1+x)}{(1+x)} = x \quad \text{—— (1)}$$

$$P(x) = -\frac{x}{1+x}, \quad Q(x) = x$$

$$e^{\int P(x)dx} = e^{-\int \frac{x}{1+x}dx} = e^{-\int \frac{1+x-1}{1+x}dx}$$

$$= e^{-\int\left(1 - \frac{1}{1+x}\right)dx}$$

$$e^{\ln x} = x$$

$$= e^{-x + \ln(1+x)}$$

$$= e^{-x}\underbrace{e^{\ln(1+x)}}$$

$$= e^{-x}\underbrace{(1+x)} = (1+x)e^{-x}$$

multiply eqn (1) by I.F.

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)\, dx$$

$$y(1+x)e^{-x} = \int (1+x)e^{-x}(x)\, dx$$

$\int(uv)dx=u\int v\, dx-\int\{u'\, (\int vdx)\}\, dx$

$$= \int xe^{-x}dx + \int \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v} dx$$

$$= \int xe^{-x}dx + x^2\left[\frac{e^{-x}}{-1}\right] - \int\left(2x\left[\frac{e^{-x}}{-1}\right]\right)dx$$

$$= \int xe^{-x}dx - x^2e^{-x} + \int 2xe^{-x}dx$$

$$= \int 3xe^{-x}dx - x^2e^{-x}$$

$$= 3\int \underbrace{x}_{u}\,\underbrace{e^{-x}}_{v}\,dx - x^2e^{-x}$$

$$= 3\left[x\left[\frac{e^{-x}}{-1}\right] - \int 1\left[\frac{e^{-x}}{-1}\right]dx\right] - x^2e^{-x}$$

$$= 3\left[-xe^{-x} + \int e^{-x}dx\right] - x^2e^{-x}$$

$$y(1+x)e^{-x} = 3\left[-xe^{-x} + \left(\frac{e^{-x}}{-1}\right)\right] - x^2e^{-x} + c$$

$$\frac{1}{e^{-x}} = e^{x}$$

$$y(1+x) = 3\left[-x-1\right] - x^2 + ce^{+x} \quad [\div \text{ by } e^{-x}]$$

$$= -3x - 3 - x^2 + ce^{+x}$$

$$y = -\frac{1}{1+x}\left(x^2+3x+3 - ce^{+x}\right)$$

$\boxed{4}$ $y\,dx - 4(x+y^6)\,dy = 0$

$\because \dfrac{dy}{dx} + P(x)y = Q(x)$

Similarly

$\dfrac{dx}{dy} + P(y)x = Q(y)$

$$\frac{dx}{dy} - \frac{4(x+y^6)}{y} = 0 \quad (\div\, y\,dy)$$

$$\frac{4(x+y^6)}{y} = \frac{4x+4y^6}{y} = \frac{4x}{y} + 4y^5$$

$$\frac{dx}{dy} - \frac{4}{y}\cdot x - 4y^5 = 0$$

$$\frac{dx}{dy} \underbrace{- \frac{4}{y}}_{P(y)}\cdot x = \overbrace{4y^5}^{Q(y)} \quad \text{①}$$

$P(y) = -\dfrac{4}{y}$, $Q(y) = 4y^5$

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = e^{-\ln y^4} = e^{\ln\left(\frac{1}{y^4}\right)} = \frac{1}{y^4}$$

multiply eqn ① by I.F.

$$xe^{\int P(y)dy} = \int e^{\int P(y)dy}\,Q(y)\,dy$$

$$x\cdot\frac{1}{y^4} = \int \frac{1}{y^4}\cdot 4y^5 dy = \int 4y\,dy$$

$$x \cdot \frac{1}{y^4} = 4\frac{y^2}{2} + c$$

$$= 2y^2 + c$$

$$\frac{x}{y^4} = 2y^2 + c$$

$$x = 2y^2 \cdot y^4 + cy^4$$

$$x = 2y^6 + cy^4$$

$\boxed{5}$ $\cos x \frac{dy}{dx} + (\sin x) y = 1$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x} \qquad [\div \text{ by } \cos x]$$

$$\frac{dy}{dx} + \tan x \cdot y = \sec x \quad \text{—} (1)$$

$P(x) = \tan x$; $Q(x) = \sec x$

$$e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$$

multiply eqn (1) by I.F.

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)dx$$

$$y \sec x = \int \sec x \cdot \sec x \, dx$$

$$= \int \sec^2 x \, dx$$

$$y \sec x = \tan x + c$$

$$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$$y = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} + C \cdot \frac{1}{\sec x}$$

$$= \frac{\sin x}{\cos x} \times \frac{\cos x}{1} + c \cdot \cos x$$

$$y = \sin x + c \cos x$$

$\boxed{6}$ Solve the initial value problem

$$y' + \tan x \cdot y = \cos^2 x \;;\; \underbrace{y(0) = -1}_{x=0,\; y=-1}$$

$$\frac{dy}{dx} + \tan x \cdot y = \cos^2 x \quad — (1) \qquad P(x) = \tan x,\; Q(x) = \cos^2 x$$

$$IF = e^{\int P(x)dx} = e^{\int \tan x\, dx} = e^{\ln|\sec x|} = \sec x$$

multiply eqn (1) by I.F.

$$y e^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)\, dx$$

$$y \sec x = \int \sec x \cos^2 x\, dx$$

$$= \int \frac{1}{\cos x} \cdot \cos^2 x\, dx$$

$$= \int \cos x\, dx$$

$$y \sec x = \sin x + c \quad — (2)$$

$$\sec x = \frac{1}{\cos x}$$

Substitute $x=0$, $y=-1$ into (2)

$$(-1)\sec(0) = \sin(0) + C$$

$$-1 = 0 + C$$

$$C = -1$$

substitute $C=-1$ into (2)

$$y\sec x = \sin x - 1$$

$$y = \frac{\sin x}{\sec x} - \frac{1}{\sec x}$$

$$= \frac{\sin x}{\frac{1}{\cos x}} - \cos x$$

$$= \sin x \cos x - \cos x$$

$$y = \cos x(\sin x - 1)$$ ■