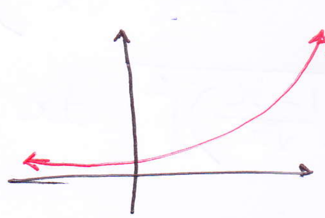
MAT 120 Week 4

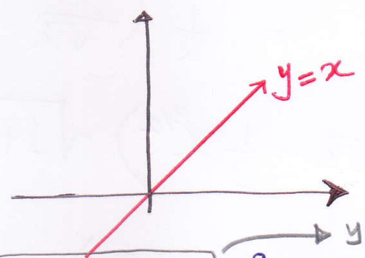
Ch 6.5
Anton's Calculus 10th Ed.

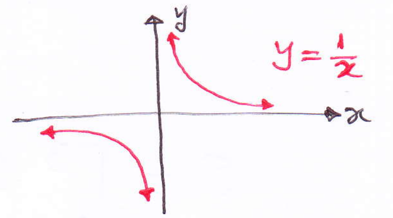
AREA of a SURFACE of REVOLUTION

A smooth function has continuous derivatives up to some desired order over some domain.
A function said to be smooth over a restricted interval I such as (a, b) or $[a, b]$.

$y = e^x$, $y = x$ etc are examples of smooth functions.





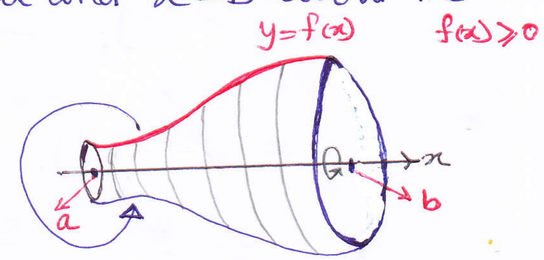


$y = \frac{1}{x}$ is a piecewise smooth function.
It is discontinuous at $x = 0$.

$y \geq 0$

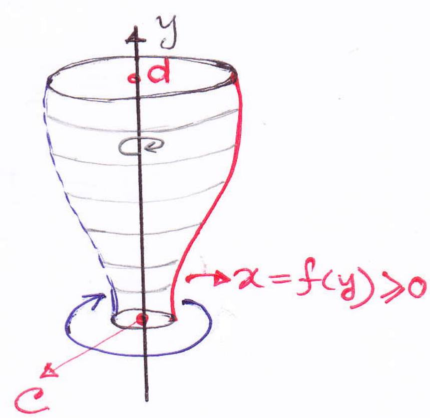
If f is a smooth $\boxed{\text{non-negative}}$ function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the $\boxed{x\text{-axis}}$ is defined as:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$



About the $\boxed{y\text{-axis}}$ is defined as:

$$S = \int_c^d 2\pi f(y)\sqrt{1 + [f'(y)]^2}\, dy$$



Find the area of the surface generated by revolving given curve about the x-axis:

$y=\sqrt{x}-\frac{1}{3}x^{3/2}$; $1\leq x\leq 3$

$S=2\pi f(x)\sqrt{1+[f'(x)]^2}$

$$f(x)=\sqrt{x}-\frac{1}{3}x^{3/2}$$

$$f'(x)=\frac{dy}{dx}=\frac{1}{2\sqrt{x}}-\frac{1}{3}\cdot\frac{3}{2}x^{1/2}$$

$$=\frac{1}{2\sqrt{x}}-\frac{1}{2}x^{1/2}$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)=\frac{1}{2}\left(\frac{1-x}{\sqrt{x}}\right)$$

$$S=\int_1^3 2\pi\left(\sqrt{x}-\frac{1}{3}x^{3/2}\right)\sqrt{1+\left[\frac{1}{2}\left(\frac{1-x}{\sqrt{x}}\right)\right]^2}\,dx$$

$$=2\pi\int_1^3\left(\sqrt{x}-\frac{1}{3}x^{3/2}\right)\sqrt{1+\frac{(1-x)^2}{4x}}\,dx$$

$$=2\pi\int_1^3\frac{3\sqrt{x}-x\sqrt{x}}{3}\sqrt{\frac{4x+1-2x+x^2}{4x}}\,dx$$

$$=2\pi\int_1^3\frac{\sqrt{x}\,(3-x)}{3}\sqrt{\frac{x^2+2x+1}{4x}}\,dx$$

$$=\frac{\cancel{2}\pi}{3}\int_1^3\cancel{\sqrt{x}}\,(3-x)\,\frac{\sqrt{(x+1)^2}}{\cancel{2}\cancel{\sqrt{x}}}\,dx$$

$$=\frac{\pi}{3}\int_1^3(3-x)(x+1)\,dx$$

$$=\frac{\pi}{3}\int_1^3\left(3x-x^2+3-x\right)dx$$

$$=\frac{\pi}{3}\int_1^3\left(2x-x^2+3\right)dx$$

$$= \frac{\pi}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3} + 3x\right]_1^3$$

$$= \frac{\pi}{3}\left[x^2 - \frac{x^3}{3} + 3x\right]_1^3$$

$$= \frac{\pi}{3}\left[3^2 - \frac{3^3}{3} + 3(3) - 1^2 + \frac{1^3}{3} - 3(1)\right]$$

$$= \frac{\pi}{3}\left[9 - 9 + 9 - 1 + \frac{1}{3} - 3\right]$$

$$= \frac{\pi}{3}\left[5 + \frac{1}{3}\right] = \frac{\pi}{3}\left[\frac{16}{3}\right] = \frac{16\pi}{9}$$

Find the area of the surface generated by revolving the given curve about the y-axis.

$x = 9y + 1$; $0 \leq y \leq 2$

$$\frac{dx}{dy} = 9$$

$$S = \int_0^2 2\pi f(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^2 (9y+1)\sqrt{1 + (9)^2}\, dy$$

$$= 2\pi \int_0^2 (9y+1)\sqrt{82}\, dy$$

$$= 2\sqrt{82}\,\pi \int_0^2 (9y+1)\, dy = 2\sqrt{82}\,\pi \left[\frac{9y^2}{2} + y\right]_0^2$$

$$= 2\sqrt{82}\,\pi \left[\frac{36}{2} + 2 - 0\right]$$

$$= 40\sqrt{82}\,\pi$$

$$x = \sqrt{9-y^2} \ ; \quad -2 \le y \le 2$$

$$f'(y) = \frac{dx}{dy} = \frac{1}{2\sqrt{9-y^2}}(-2y) = \frac{-y}{\sqrt{9-y^2}}$$

$$S = \int_{-2}^{2} 2\pi \sqrt{9-y^2}\sqrt{1+\left(\frac{-y}{\sqrt{9-y^2}}\right)^2}\, dy$$

$$= 2\pi \int_{-2}^{2} \sqrt{9-y^2}\sqrt{1+\frac{y^2}{9-y^2}}\, dy$$

$$= 2\pi \int_{-2}^{2} \sqrt{9-y^2}\sqrt{\frac{9-y^2+y^2}{9-y^2}}\, dy$$

$$= 2\pi \int_{-2}^{2} \cancel{\sqrt{9-y^2}}\ \frac{\sqrt{9} \to 3}{\cancel{\sqrt{9-y^2}}}\, dy$$

$$= 6\pi \int_{-2}^{2} dy$$

$$= 6\pi \left[y\right]_{-2}^{2} = 24\pi$$