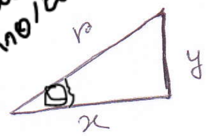
Ch 7.4

Integration by Trigonometric substitution

$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \sin\theta/\cos\theta$



$x = r\cos\theta$
$y = r\sin\theta$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \begin{cases} 1 - \cos^2\theta = \sin^2\theta \\ 1 - \sin^2\theta = \cos^2\theta \end{cases}$

$1 + \tan^2\theta = \sec^2\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x} \Rightarrow \cot\theta = \frac{x}{y}$

$\sin\theta = \frac{y}{r} \Rightarrow \operatorname{cosec}\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r} \Rightarrow \sec\theta = \frac{r}{x}$

$\sqrt{16 - x^2} = \sqrt{16\left(1 - \frac{1}{16}x^2\right)} = 4\sqrt{1 - \frac{1}{16}x^2}$

$\boxed{3}$ $\displaystyle\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

$= \displaystyle\int \frac{x^2}{4\sqrt{1 - \frac{1}{16}x^2}}\,dx$

$= \displaystyle\frac{1}{4}\int \frac{x^2}{\sqrt{1 - \left(\frac{x}{4}\right)^2}}\,dx$

let

$\star\ \frac{x^2}{16} = \sin^2\theta$

$\Rightarrow x/4 = \sin\theta$

$\therefore \frac{1}{4}dx = \cos\theta\, d\theta$

$dx = 4\cos\theta d\theta$

$\because \frac{x}{4} = \sin\theta$

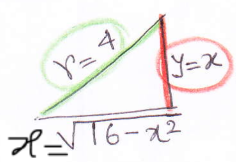
$\therefore \theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\therefore x^2 = 16\sin^2\theta$

$\therefore \sin\theta = \frac{x}{4} \xrightarrow{} \frac{y}{r}$ ($x \to y$, $4 \to r$)

$\therefore \cos\theta = \frac{x}{r} = \frac{\sqrt{16 - x^2}}{4}$

$= \displaystyle\frac{1}{4}\int \frac{16\sin^2\theta\ 4\cos\theta\, d\theta}{\sqrt{1 - \sin^2\theta}}$



$x = \sqrt{r^2 - y^2}$
$= \sqrt{16 - x^2}$

$\therefore r^2 = x^2 + y^2$
$x = \sqrt{r^2 - y^2}$

$= \displaystyle\int \frac{16\sin^2\theta\cos\theta}{\cos\theta}\,d\theta$

$= 16\displaystyle\int \sin^2\theta\, d\theta$

$= 16\displaystyle\int \frac{1}{2}(1 - \cos 2\theta)\,d\theta = 8\int(1 - \cos 2\theta)\,d\theta$

$= 8\left[\theta - \frac{\sin 2\theta}{2}\right] + C$

$= 8\left[\sin^{-1}\frac{x}{4} - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$

$\cos\theta = \frac{x}{r}$

$= 8\left[\sin^{-1}\frac{x}{4} - \sin\theta\cos\theta\right] + C$

$= 8\left[\sin^{-1}\frac{x}{4} - \frac{x}{4}\cdot\frac{\sqrt{16 - x^2}}{4}\right] + C$

$= 8\sin^{-1}\frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2} + C$

(5) $\int \frac{dx}{(4+4x^2)^2}$ $\qquad$ $1+\tan^2\theta = \sec^2\theta$

$= \int \frac{dx}{\left[4(1+x^2)\right]^2}$ $\qquad$ $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

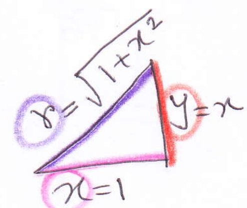
$= \frac{1}{16} \int \frac{dx}{(1+x^2)^2}$

$= \frac{1}{16} \int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^2}$ $\qquad$ $x = \tan\theta$

$\theta = \tan^{-1} x$ $\qquad$ $\tan\theta = \frac{y}{x}$

$= \frac{1}{16} \int \frac{\sec^2\theta}{\sec^4\theta} \, d\theta$ $\qquad$ $\therefore \tan\theta = \frac{y}{x}$

$= \frac{1}{16} \int \frac{1}{\sec^2\theta} \, d\theta$ $\qquad$ We have $\tan\theta = \frac{x}{1}$ $\quad x \to y$, $1 \to x$



$= \frac{1}{16} \int \cos^2\theta \, d\theta$

$r = \sqrt{x^2+y^2}$

$= \sqrt{1+x^2}$

$= \frac{1}{16} \int \frac{1}{2}(1+\cos 2\theta) \, d\theta$

$= \frac{1}{32}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$ $\qquad$ $\sin\theta = \frac{y}{r} = \frac{x}{\sqrt{1+x^2}}$ $\qquad$ $\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{1+x^2}}$

$= \frac{1}{32}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$

$= \frac{1}{32}\left[\theta + \sin\theta\cos\theta\right] + C$

$= \frac{1}{32}\left[\tan^{-1}x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right] + C$

$= \frac{1}{32}\left[\tan^{-1}x + \frac{x}{1+x^2}\right] + C$

$1-\sin^2\theta = \cos^2\theta$

$1-\cos^2\theta = \sin\theta$

$1+\tan^2\theta = \sec^2\theta$ ✓

⑦ $\int \frac{\sqrt{x^2-9}}{x}\,dx$

$= \int \frac{\sqrt{9\left(\frac{x^2}{9}-1\right)}}{x}\,dx$

$= 3\int \frac{\sqrt{\left(\frac{x}{3}\right)^2-1}}{x}\,dx$

$= \cancel{3}\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{3}\,\cancel{\sec\theta}}\,(\cancel{3\sec\theta}\tan\theta)\,d\theta$

$= +3\int \tan\theta\,(\tan\theta)\,d\theta$

$= 3\int \tan^2\theta\,d\theta$

$= 3\int (\sec^2\theta-1)\,d\theta$

$= 3\left[\tan\theta-\theta\right]+C$

$= 3\left[\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right]+C$

$\sec^2\theta - 1 = \tan^2\theta$

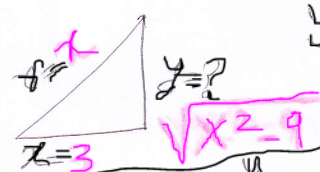
$\frac{x^2}{9} = \sec^2\theta \quad \left(\therefore \left(\frac{x}{3}\right)^2 = \sec^2\theta\right)$

$\Rightarrow \frac{x}{3} = \sec\theta \rightarrow \theta = \sec^{-1}\frac{x}{3}$

$\Rightarrow \frac{1}{3}dx = \sec\theta\tan\theta\,d\theta \qquad x = 3\sec\theta$

$\Rightarrow dx = 3\sec\theta\tan\theta\,d\theta \qquad \therefore \theta = \sec^{-1}\left(\frac{x}{3}\right)$

$\sec\theta = \frac{x}{3} = \frac{r}{x}$



$y = \sqrt{r^2-x^2}$

$= \sqrt{x^2-9}$

$\therefore \tan\theta = \frac{y}{x} = \frac{\sqrt{x^2-9}}{3}$

$\star$ $1-x^2 \longrightarrow 1-\sin^2\theta$ or $1-\cos^2\theta$

$\star\star$ $1+x^2 \longrightarrow 1+\tan^2\theta$ $\star\star\star$ $x^2-1=\sec^2\theta-1$

$\sec^2\theta-1=\tan^2\theta$

(23) $$\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta \, d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan\theta \, d\theta}{\sec\theta\sqrt{\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta} \, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos\theta \, d\theta$$

$$= \left[+\sin\theta\right]_{\pi/4}^{\pi/3}$$

$$= + \left[\sin\frac{\pi}{4} - \sin\frac{\pi}{3}\right]$$

$$= + \left[\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\right]$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

let

$x = \sec\theta \longrightarrow \theta = \sec^{-1} x$

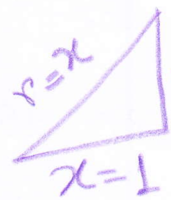
$x^2 = \sec^2\theta$

$dx = \sec\theta\tan\theta \, d\theta$

Limits $\begin{cases} x=\sqrt{2} \Rightarrow \theta = \sec^{-1}\sqrt{2} = \pi/4 \\ x = 2 \Rightarrow \theta = \sec^{-1} 2 = \pi/3 \end{cases}$

optional

$\sec\theta = \frac{x}{1}$ | $\sec\theta = \frac{r}{x}$

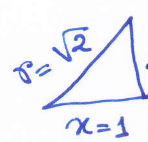


$y = \sqrt{r^2 - x^2}$
$= \sqrt{x^2-1}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{x^2-1}}{1}$
$= \sqrt{x^2-1}$

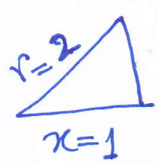
Limits

$\begin{cases} \sec^{-1}\sqrt{2} = \theta \\ \sec\theta = \sqrt{2} = \frac{r}{x} = \frac{\sqrt{2}}{1} \end{cases}$



$\therefore y = 1$ $\tan\theta = \frac{y}{x} = 1$
$\therefore \theta = 45^\circ$

$\begin{cases} \sec^{-1} 2 = \theta \\ \sec\theta = 2 = \frac{r}{x} = \frac{2}{1} \end{cases}$



$\cos\theta = \frac{1}{\sec\theta}$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\frac{1}{2}$
$= 60^\circ = \frac{\pi}{3}$

Examples

$\boxed{1}$ $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

$$= \int \frac{dx}{x^2\sqrt{4\left(1-\frac{x^2}{4}\right)}}$$

$$= \frac{1}{2}\int \frac{dx}{x^2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$\frac{x^2}{4} = \sin^2\theta$$

or $\frac{x}{2} = \sin\theta$

$$\frac{1}{2}dx = \cos\theta \, d\theta$$

$$dx = 2\cos\theta \, d\theta$$

$\therefore x = 2\sin\theta$

$x^2 = 4\sin^2\theta$

$\therefore \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$$= \frac{1}{2}\int \frac{2\cos\theta \, d\theta}{4\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta \, d\theta}{4\sin^2\theta\sqrt{\cos^2\theta}}$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta$$

Cosec = csc

$$= \frac{1}{4}\int \csc^2\theta \, d\theta$$

$$= \frac{1}{4}(-\cot\theta) + C$$

$$= -\frac{1}{4}\cot\theta + C$$

Given $\frac{x}{2} = \sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{2}$

$x=1 \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} = 30^\circ$

$x=\sqrt{2} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = 45^\circ$

Solve $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

$$= \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{4}\left[\frac{1}{\tan\theta}\right]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{4}\left[\frac{1}{\tan\frac{\pi}{4}} - \frac{1}{\tan\frac{\pi}{6}}\right]$$

$$= -\frac{1}{4}\left[\frac{1}{1} - \frac{1}{\frac{1}{\sqrt{3}}}\right]$$

$$= -\frac{1}{4}\left[1 - \sqrt{3}\right]$$

$$= \frac{\sqrt{3}-1}{4}$$

$\boxed{2}$ $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$

$$= \int \frac{x}{\underbrace{x^2-2\cdot x\cdot 2+2^2}+4}\,dx$$

$$= \int \frac{x}{(x-2)^2+4}\,dx$$

let $u = x-2$

$du = dx$

$\therefore x = u+2$

$$= \int \frac{u+2}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$

$u^2+4 = z$

$2u\,du = dz$

$u\,du = \frac{1}{2}dz$

$$= \int \frac{\frac{1}{2}dz}{z} + 2\int \frac{du}{4\left(\frac{u^2}{4}+1\right)}$$

$$= \frac{1}{2}\int \frac{1}{z}\,dz + \frac{1}{2}\int \frac{du}{\left(\frac{u}{2}\right)^2+1}$$

Recall $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

$$= \frac{1}{2}\ln z + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln[(x-2)^2+4] + \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$$