Reference Book
Anton's Calculus
10th Ed.

Chapter 14.6

POLAR COORDINATE IN CYLINDRICAL COORDINATES

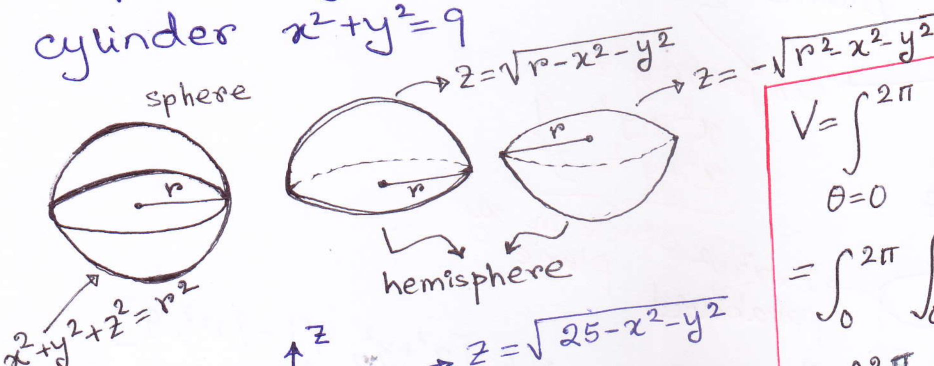
$$\iiint_G f(r,\theta,z)\,\underbrace{dV}_{dz\,dA \,=\, dz\,r\,dr\,d\theta} = \iiint f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

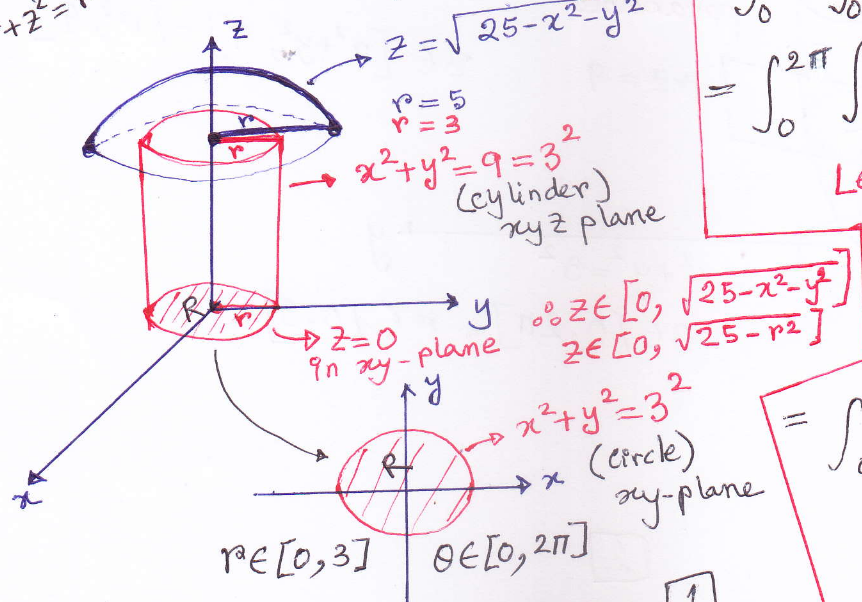
$$= \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1(\theta)}^{r_2(\theta)} \int_{z=g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

$G \rightarrow$ Solid

$f(r,\theta,z)$ will be considered "1" if it is not provided in the problem.

Example $\boxed{1}$: Use cylindrical coordinates to find the volume of the solid G bounded above by the hemisphere $z=\sqrt{25-x^2-y^2}$, below by the xy plane & lateraly (imaginatively) (across tangentially) by the cylinder $x^2+y^2=9$





$r\in[0,3]$ $\theta\in[0,2\pi]$

$$V=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 r\,[z]_0^{\sqrt{25-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 r\sqrt{25-r^2}\,dr\,d\theta$$

Let $u=25-r^2$
$du=-2r\,dr$
$-\frac{1}{2}du=r\,dr$
$r=0 \rightarrow u=25$
$r=3 \rightarrow u=16$

$$=\int_0^{2\pi}\int_{25}^{16}\sqrt{u}\left(-\frac{1}{2}du\right)d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_{25}^{16} u^{1/2}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_{16}^{25} u^{1/2}\,du\,d\theta$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{u^{\cancel{1/2}+1}}{\frac{1}{2}+1}\right]_{16}^{25} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{u^{3/2}}{3/2}\right]_{16}^{25} d\theta$$

$$= \frac{1}{2}\times\frac{2}{3}\int_0^{2\pi}\left[25^{3/2} - 16^{3/2}\right] d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[125 - 64\right] d\theta$$

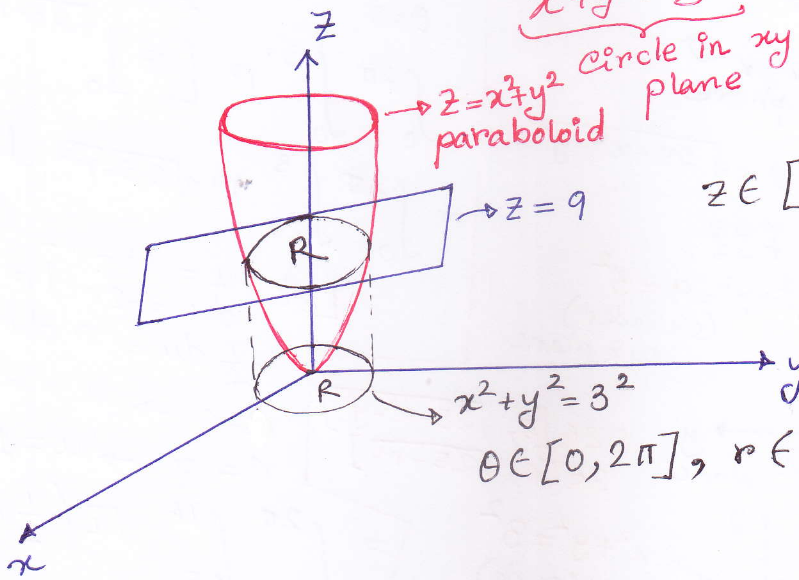
$$= \frac{1}{3}\times 61\,\left[\theta\right]_0^{2\pi} = \frac{122\pi}{3}$$

Example $\boxed{2}$ The solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$. Find the volume of the solid.

Comparing "z"

$x^2 + y^2 = 9$

$\underbrace{x^2 + y^2 = 3^2}_{\text{circle in xy plane}}$



$z \in [x^2+y^2, 9] = [r^2, 9]$

We know $x^2+y^2 = r^2$

$\theta \in [0, 2\pi]$, $r \in [0, 3]$

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{z=r^2}^{9} r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{3} r \int_{r^2}^{9} dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{3} r \left[z \right]_{r^2}^{9} dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{3} r \left[9 - r^2 \right] dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{3} \left[9r - r^3 \right] dr \, d\theta$$

$$= \int_{0}^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_{0}^{3} d\theta$$

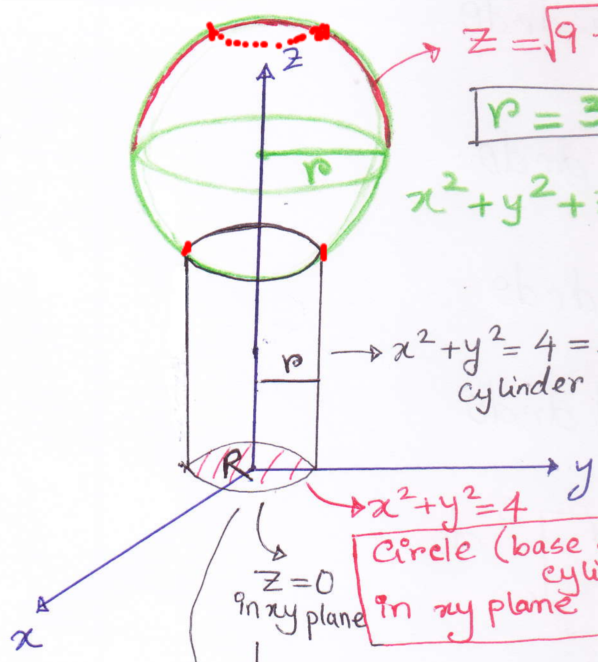
$$= \int_{0}^{2\pi} \left[\frac{81}{2} - \frac{81}{4} \right] d\theta$$

$$= \frac{81}{4} \int_{0}^{2\pi} d\theta$$

$$= \frac{81}{4} \left[\theta \right]_{0}^{2\pi}$$

$$= \frac{81\pi}{2}.$$

Example $\boxed{3}$: Find the volume of the solid that is bounded above by the sphere $\underbrace{x^2+y^2+z^2=9}_{z=\pm\sqrt{9-x^2-y^2}}$ $\rightarrow r=3$ and inside the cylinder $x^2+y^2=4$. $\rightarrow r=2$



$\rightarrow z=\sqrt{9-x^2-y^2}$ (upper hemisphere)

$\boxed{r=3}$ sphere

$x^2+y^2+z^2=9$ (sphere)

$\rightarrow x^2+y^2=4=2^2$ $\boxed{r=2}$ cylinder

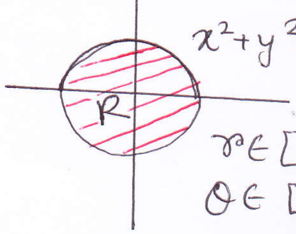
cylinder in xyz plane

$\rightarrow x^2+y^2=4$

Circle (base of the cylinder in xy plane)

∵ The radius of the sphere is larger than the radius of the cylinder, hence the sphere won't be able to go inside the cylinder completely but partially.

∴ While $x^2+y^2+z^2=9$ & $x^2+y^2=4$ intersect, we receive $r=2$.



$r\in[0,2]$

$\theta\in[0,2\pi]$

$z\in[0,\sqrt{9-x^2-y^2}]=[0,\sqrt{9-r^2}]$

$$V=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2} r\int_0^{\sqrt{9-r^2}} dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2} r\,\big[z\big]_0^{\sqrt{9-r^2}}\,dr\,d\theta$$

Let $9-r^2=u$

$-2r\,dr=du$

$r\,dr=-\frac{1}{2}du$

$r=0\rightarrow u=9$

$r=2\rightarrow u=5$

$$=\int_0^{2\pi}\int_0^{2} r\left(\sqrt{9-r^2}\right)dr\,d\theta$$

$$=\int_0^{2\pi}\int_9^{5}\sqrt{u}\left(-\frac{1}{2}du\right)d\theta$$

$$=-\frac{1}{2}\int_0^{2\pi}\int_9^{5}u^{1/2}\,du\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_5^{9}u^{1/2}\,du\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[\frac{u^{3/2}}{3/2} \right]_5^9 d\theta$$

$$= \frac{1}{2} \times \frac{2}{3} \int_0^{2\pi} \left[9^{3/2} - 5^{3/2} \right] d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[27 - 5\sqrt{5} \right] d\theta$$

$$= \frac{27 - 5\sqrt{5}}{3} \left[\theta \right]_0^{2\pi}$$

$$= \frac{2\pi}{3} \left(27 - 5\sqrt{5} \right)$$

Example: $\boxed{4}$ Find the volume of the solid that is bounded by the cylinder $y = x^2$ and by the plane $\underbrace{y + z = 4}$ and $z = 0$.

$z = 4 - y$
$= 4 - \cancel{x}^2$
$\because y = x^2$ given

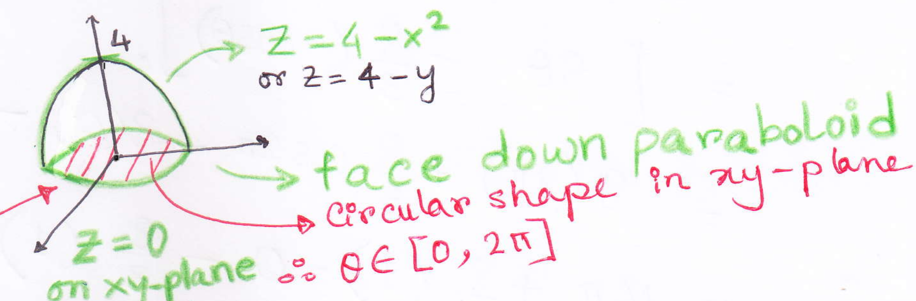
$y = x^2$; $y = 4 - z$

$x^2 = 4 - z$

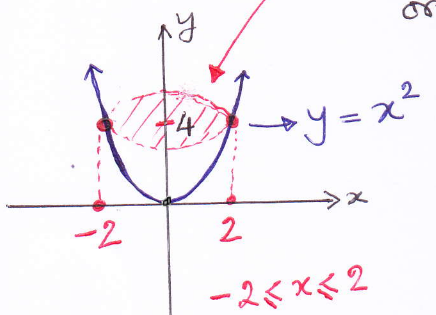
$z = 4 - x^2$

$\int_{z=0}^{4-y}$ or $\int_0^{4 - r\sin\theta}$

$\because y = r\sin\theta$



$z \in [0, 4 - x^2]$
or $z \in [0, 4 - y] = [0, 4 - r\sin\theta]$



$-2 \leq x \leq 2$
$\Rightarrow 0 \leq r \leq 2$
$r \in [0, 2]$

Given $y + z = 4$
$z = 4 - y = 4 - x^2$ $[\because y = x^2]$
$\therefore z = 4 - x^2$
But $z = 0$ in xy plane
in xy-plane $0 = 4 - x^2 \Rightarrow \boxed{x = \pm 2}$
Also in xy-plane $0 = 4 - y \Rightarrow y = 4$
Note $(-2)^2 = 4$, $(2)^2 = 4$
As given $x^2 = y$

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{4-r\sin\theta} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r \left[z \right]_0^{4-r\sin\theta} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r \left[4 - r\sin\theta \right] dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[4r - r^2 \sin\theta \right] dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \sin\theta \left(\frac{r^3}{3} \right) \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[2(2)^2 - \frac{\sin\theta}{3} (2)^3 - 2(0)^2 + \frac{\sin\theta}{3} (0)^3 \right] d\theta$$

$$= \int_0^{2\pi} \left(8 - \frac{8\sin\theta}{3} \right) d\theta$$

$$= \left[8\theta - \frac{8}{3}(-\cos\theta) \right]_0^{2\pi}$$

$$= 8(2\pi) + \frac{8}{3} \cos 2\pi - 8(0) - \frac{8}{3} \cos(0)$$

$$= 16\pi + \frac{8}{3}(1) - 0 - \frac{8}{3}(1)$$

$$= 16\pi.$$