MAT 120
DIFFERENTIAL EQN (DE)

$\boxed{\text{Homogeneous Linear Eqn with constant coefficients}}$

Consider the n^{th} order DE below:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_2 y'' + a_1 y' + a_0 y = g(x) \quad \text{— (i)}$$

if $g(x) = 0$ then (i) is a homogeneous DE

if $g(x) \neq 0$ then (i) is a non-homogeneous DE

Consider

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0 \quad \text{— (ii)}$$

(homogeneous DE)

$a_i's$ are constant coefficients

$i = 0, 1, 2, 3, \dots, n$

Recall 1st order DE

$$a_1 y' + P(x) y = f(x) \quad \text{— (iii)}$$

if (iii) is homogeneous DE

then $a_1 y' + P(x) y = 0$

Replace $a_1 = a$ & $P(x) = b$

Now we have $ay' + by = 0$

$$\frac{ay'}{a} + \frac{b}{a} y = 0 \quad (\div \text{ by } a)$$

$$y' + \frac{b}{a} y = 0 \quad \text{— (iv)}$$

(iv) $\rightarrow y' + (m)y = 0$ $\left\{ \begin{array}{l} \text{let } \frac{b}{a} = m \\ \text{Relabel constant} \end{array} \right\}$

(iv) $y' + my = 0$

$\Rightarrow y' + P(x) y = Q(x)$ → standard form week 7

$\therefore m = P(x)$, $0 = Q(x)$

$I.F. = e^{\int P(x)\,dx}$

$= e^{\int m\,dx}$

$$\text{I.F.} \quad e^{\int P(x)dx} = e^{\int m\,dx} = e^{mx}$$

$$e^{mx}[y' + my] = e^{mx}(0) \quad \text{multiply (iv) with I.F.}$$

$$\int e^{mx}[y' + my] = \int 0\,dx \quad \text{intigrate both sides}$$

Ref to Week 7

$$ye^{mx} = c$$

$$y = ce^{-mx}$$

$$y = e^{mx} \rightarrow \text{Relabel constants: Consider } c = 1, \; \underset{\text{Replace}}{-m = +m}$$

$\boxed{y = e^{mx}}$

$\because$ we found $y = e^{mx}$

$\therefore$ $y' = me^{mx}$

$y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$

$\vdots$

$y^{(n-1)} = m^{n-1} e^{mx}$

$y^{(n)} = m^n e^{mx}$

Eqn (ii):

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$

Substitute $y, y', y'', \ldots, y^{(n)}$ into (ii)

$$a_n(m^n e^{mx}) + a_{n-1}(m^{n-1} e^{mx}) + \cdots + a_2(m^2 e^{mx}) + a_1(me^{mx}) + a_0 e^{mx} = 0 \quad \text{(v)}$$

eqn (v) may have distinct roots, repeating roots or complex root.

<u>Note</u> eqn (v) is known as <u>Auxiliary Equation</u>.

$\div$ (v) by e^{mx} & $e^{mx} \neq 0$

$\Rightarrow a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$

<u>Roots</u>

Consider the following quadratic eqns:

(a) $2m^2 - 5m - 3 = 0$

$\Rightarrow (2m+1)(m-3) = 0$

$\Rightarrow 2m+1 = 0 \; ; \; m-3 = 0$

$\Rightarrow m_1 = -\frac{1}{2}, \; m_2 = 3$

here the solutions m_1, m_2 are distinct roots

(b) $m^2 - 10m + 25 = 0 \Rightarrow m^2 - 2.m.5 + 5^2 = 0$ (completing sq)

$\Rightarrow (m-5)^2 = 0$

$\Rightarrow (m-5)(m-5) = 0$

$\Rightarrow m_1 = 5, \; m_2 = 5$

here the solutions m_1, m_2 are repeating roots

Considering (a)

$2m^2-5m-3=0$ —— Auxiliary eqn (a)

$m_1=-\frac{1}{2}, m_2=3$

∵ the solutions are unique

∴ the solution of Auxiliary eqn (a) can be written as

$y=c_1e^{m_1x}+c_2e^{m_2x}$

$=c_1e^{-\frac{1}{2}x}+c_2e^{3x}$

$=c_1f_1(x)+c_2f_2(x)$ [observe $f_1(x)$ & $f_2(x)$ are independent]

① ② ③
$\cos x, 2\cos x, \cos 2x$

①, ② similar

③ different

Considering ⓑ

$m^2-10m+25=0$ —— Auxiliary eqn ⓑ

$m_1=5, m_2=5$

∵ the solutions are repeating

∴ the solution of Auxiliary eqn ⓑ can be written as

Wrong

Consider: $y=c_1e^{m_1x}+c_2e^{m_2x}$

$=c_1e^{5x}+c_2e^{5x}$

$=c_1g_1(x)+c_2g_2(x)$

$g_1(x)$ & $g_2(x)$ are dependent

Correct

$y=c_1e^{m_1x}+c_2xe^{m_2x}$

$=c_1e^{5x}+c_2xe^{5x}$

$=c_1f_1(x)+c_2f_2(x)$ [observer $f_1(x)$ & $f_2(x)$ are now independent since we introduced x along with c_2, otherwise the solution will look like $\underbrace{c_1e^{5x}+c_2e^{5x}}_{\text{linearly dependent}}$

But we are looking forward to the independent set of solutions hence introduced x with c_2.

$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$

Going back to eqn (v)

→ if the roots are non repeating : $m_1 \neq m_2 \neq m_3 \cdots \neq m_n$

then solution of (v) will be:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

→ if the roots are repeating : $m_1 = m_2 = m_3 \cdots = m_n$

then the solution of (v) will be:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + c_3 x^2 e^{m_3 x} + c_4 x^3 e^{m_4 x} + \cdots + c_n x^{n-1} e^{m_n x}$$

→ if the roots are complex numbers:

$m = a \pm bi \Rightarrow m_1 = a + bi$
$\quad\quad\quad\quad\quad m_2 = a - bi$

then the solution of (v) will be:

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$\boxed{\text{Exercise}}$

Find the General solution of the given 2^{nd} order differential eqn:

(1) $3y'' + 2y' + y = 0$ —— (i)

substitute $y = e^{mx}$, $y' = m e^{mx}$, $y'' = m^2 e^{mx}$ into (i)

$$3m^2 e^{mx} + 2m e^{mx} + e^{mx} = 0 \quad (\div \text{ by } e^{mx})$$

$$3m^2 + 2m + 1 = 0$$

$$m = \frac{-2 \pm \sqrt{4-12}}{6}$$

$$= \frac{-2 \pm \sqrt{-8}}{6} = \frac{-2 \pm i2\sqrt{2}}{6} = \frac{-1 \pm i\sqrt{2}}{3} = -\frac{1}{3} \pm \frac{i\sqrt{2}}{3}$$

$a = -\frac{1}{3}$, $b = \frac{\sqrt{2}}{3}$

$$\therefore y = c_1 e^{-1/3 x} \cos\left(\frac{\sqrt{2}}{3}\right)x + c_2 e^{-1/3 x} \sin\left(\frac{\sqrt{2}}{3}\right)x$$

or $y = e^{-1/3 x}\left(c_1 \cos\frac{\sqrt{2}}{3}x + c_2 \sin\frac{\sqrt{2}}{3}x\right)$

(2) $y'' - 3y' + 2y = 0$ —(i)

substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$ into (i)

$m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$m^2 - 3m + 2 = 0$ ($\div$ by e^{mx})

$m^2 - 2m - m + 2 = 0$

$(m-2)(m-1) = 0$

$m_1 = 2$, $m_2 = 1$

$\therefore y = c_1 e^{2x} + c_2 e^{x}$

(3) $y'' + 8y' + 16y = 0$ —(i)

substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$ into (i)

$m^2 e^{mx} + 8me^{mx} + 16e^{mx} = 0$

$m^2 + 8m + 16 = 0$ ($\div$ by e^{mx})

$m^2 + 2 \cdot m \cdot 4 + 4^2 = 0$

$(m+4)^2 = 0$

$(m+4)(m+4) = 0 \Rightarrow m_1 = m_2 = -4$

$\therefore y = C_1 e^{-4x} + C_2 x e^{-4x}$

Find the General solution of higher order DE:

(4) $16 \frac{d^4y}{dx^4} + 24 \frac{d^2y}{dx^2} + 9y = 0$

$16y^{(4)} + 24y'' + 9y = 0$ —(i)

Substitute $y = e^{mx}$, $y'' = m^2 e^{mx}$, $y^{(4)} = m^4 e^{mx}$ into (i)

$16m^4 e^{mx} + 24m^2 e^{mx} + 9e^{mx} = 0$

$16m^4 + 24m^2 + 9 = 0$ ($\div$ by e^{mx})

$16m^4 + 24m^2 + 3^2 = 0 \Rightarrow (4m^2 + 3)^2 = 0$

$(4m^2)^2 + 2 \cdot 4m^2 \cdot 3 + 3^2 \Rightarrow (4m^2+3)(4m^2+3) = 0$

$$\Rightarrow (4m^2+3)(4m^2+3)=0$$

$4m^2+3=0$

$m_1=m_2=\pm\sqrt{\frac{-3}{4}}$

$=\pm i\frac{\sqrt{3}}{2}$ $a=0$, $b=\frac{\sqrt{3}}{2}$

$=0\pm i\sqrt{3}/2$

$4m^2+3=0$

$m_3=m_4=\pm\sqrt{\frac{-3}{4}}$

$a=0$, $b=\frac{\sqrt{3}}{2}$ $=\pm i\frac{\sqrt{3}}{2}$

$=0\pm i\sqrt{3}/2$

$4m^2+3=0$

$4m^2=-3$

$m^2=-\frac{3}{4}$

m_1, m_2

$m=\pm\sqrt{\frac{-3}{4}}$

$=\pm i\sqrt{\frac{3}{4}}$

$=\pm i\frac{\sqrt{3}}{2}$

$m_3 \to m_1$

$m_4 \to m_2$

m_3, m_4 are repeating roots of m_1, m_2

$\therefore y=\left[c_1e^{0x}\cos\left(\frac{\sqrt{3}}{2}\right)x+c_2e^{0.x}\sin\left(\frac{\sqrt{3}}{2}\right)x\right]+\left[c_3xe^{0.x}\cos\left(\frac{\sqrt{3}}{2}\right)x + c_4xe^{0.x}\sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

$=c_1\cos\left(\frac{\sqrt{3}}{2}\right)x+c_2\sin\left(\frac{\sqrt{3}}{2}\right)x+c_3x\cos\left(\frac{\sqrt{3}}{2}\right)x+c_4x\sin\left(\frac{\sqrt{3}}{2}\right)x$

Solve the given initial value & boundary value problem:

5) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0$; $y(1)=0$; $y'(1)=2$

$y''-4y'-5y=0$ — (i)

substitute $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$ into (i)

$m^2e^{mx}-4me^{mx}-5e^{mx}=0$ ($\div$ by e^{mx})

$m^2-4m-5=0$

$m^2-5m+m-5=0 \Rightarrow m_1=5, m_2=-1$

$(m-5)(m+1)=0$

$\therefore \boxed{y_{(x)}=c_1e^{5x}+c_2e^{-x}}$ — (ii)

$y(1)=c_1e^{5(1)}+c_2e^{-(1)}$

$y(1)=c_1e^5+c_2\frac{1}{e}$

$0=c_1e^5+c_2\frac{1}{e}$ — (a)

& $y'(x)=5c_1e^{5x}-c_2e^{-x}$

$y'(1)=5c_1e^{5(1)}-c_2e^{-(1)}$

$2=5c_1e^5-c_2\frac{1}{e}$ — (b)

(a) + (b) $\Rightarrow 6c_1e^5=2$

$c_1=\frac{1}{3e^5}$

substitute c_1 into (a)

$\frac{1}{3e^5}\cdot e^5+c_2\frac{1}{e}=0$

$\frac{1}{3}+c_2\frac{1}{e}=0$

$\frac{c_2}{e}=-\frac{1}{3} \Rightarrow c_2=-\frac{e}{3}$.

substitute c_1 & c_2 into (ii)

$y=\frac{1}{3e^5}\cdot e^{5x}-\frac{e}{3}\cdot e^{-x}$

$=\frac{1}{3}\left[e^{5(x-1)}-e^{(1-x)}\right]$