Week 3

Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$m>0,\ n>0$

$\sqrt{1-x^3}$

$= (1-x^3)^{1/2}$ simplify

Exercise:

2 iii) $\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$

$x^3 = z$

$3x^2\,dx = dz$

$dx = \frac{1}{3x^2}dz$

$= \frac{1}{3z^{2/3}}dz$

$x^3 = z$

$x = z^{1/3}$

$\therefore x^2 = z^{2/3}$

$= \int_0^1 (1-x^3)^{-1/2}dx$

$= \int_0^1 \frac{1}{3} z^{-2/3}(1-z)^{-1/2}\,dz$

$-2/3 \rightarrow m-1$, $-1/2 \rightarrow n-1$

limits

$x=0 \rightarrow z=0$

$x=1 \rightarrow z=1$

$= \frac{1}{3}\int_0^1 z^{\frac{1}{3}-1}(1-z)^{\frac{1}{2}-1}\,dz$

$\rightarrow m-1 = -2/3$

$m = 1-2/3 = 1/3$

$\rightarrow n-1 = -1/2$

$n = \frac{1}{2}$

$= \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right)$

$$\sqrt{a^4(1-y^4)} = a^2\sqrt{1-y^4}$$

Let

2vi) $\displaystyle\int_0^a y^7\sqrt{a^4-y^4}\,dy$

a^4y^4 factor out a^4

let $y^4 = a^4 z^4$

$y = az$

$dy = a\,dz$

$$= \int_0^1 a^7 z^7 \sqrt{a^4 - a^4 z^4}\; a\,dz$$

limits

$y = 0 \Rightarrow az = 0$

$z = 0$

$y = a \Rightarrow az = a$

$z = 1$

$$= a^8 \int z^7 a^2 \sqrt{1-z^4}\,dz$$

$$= a^{10}\int_0^1 z^7 (1-z^4)^{1/2}\,dz$$

$$= a^{10}\int_0^1 z^4 \cdot \underline{z^3} (1-z^4)^{1/2}\,\underline{dz}$$

let

$z^4 = x$

$4z^3 dz = dx$

$z^3 dz = \frac{1}{4}dx$

$$= a^{10}\int_0^1 x\,\frac{1}{4}(1-x)^{1/2}\,dx$$

limits

$z = 0 \rightarrow x = 0$

$z = 1 \rightarrow x = 1$

$$= \frac{a^{10}}{4}\int_0^1 x^1 (1-x)^{1/2}\,dx$$

$\rightarrow n-1$

$$= \frac{a^{10}}{4}\int_0^1 x^{\overset{m-1}{2-1}} (1-x)^{\overset{n-1}{3/2-1}}\,dx$$

$n - 1 = \frac{1}{2}$

$n = \frac{1}{2} + 1 = \frac{3}{2}$

$$= \frac{a^{10}}{4}\,\beta\left(2, \tfrac{3}{2}\right)$$

Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \; ; \; n>0$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \; ; \; \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$m>0, n>0$

1 i) $\int_0^4 x^{3/2} (4 - x)^{5/2}\, dx$

$1 \qquad 4x$

$x = 4y$

$dx = 4dy$

limits

$x=0 \to y=0$

$x=4 \to y=1$

$$= \int_0^1 4^{3/2} y^{3/2} (4-4y)^{5/2}\, 4dy$$

$[4(1-y)]^{5/2}$

$$= 4^{3/2+1} \int_0^1 y^{3/2}\, 4^{5/2} (1-y)^{5/2}\, dy$$

$m-1 \qquad n-1$

$$= 4^{\frac{3}{2}+1+\frac{5}{2}} \int_0^1 y^{3/2} (1-y)^{5/2}\, dy$$

$m - 1 = 3/2$

$m = 3/2 + 1 = 5/2$

$$= 4^{\frac{10}{2}} \int_0^1 y^{5/2 - 1} (1-y)^{7/2 - 1}\, dy$$

$n - 1 = 5/2$

$n = 5/2 + 1 = 7/2$

$$= 4^5 \; \beta\left(\frac{5}{2}, 7/2\right)$$

$$= 1024 \frac{\Gamma(\frac{5}{2})\Gamma(7/2)}{\Gamma(\frac{5}{2}+\frac{7}{2})} = 1024 \frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)}.$$

1(ii) $\int_0^b y^5 \sqrt{b^2-y^2}\, dy$

let $y = bz$

$dy = b\,dz$

limits

$y=0 \rightarrow z=0$

$y=b \rightarrow z=1$

$$= \int_0^1 b^5 z^5 \sqrt{b^2-b^2z^2}\; b\, dz$$

$$= b^6 \int_0^1 z^5\, b\sqrt{1-z^2}\, dz$$

$$= b^7 \int_0^1 z^5 (1-z^2)^{1/2} dz$$

$z^2 = x \rightarrow z^4 = x^2$

$2z\,dz = dx$

$z\,dz = \frac{1}{2} dx$

limits:

$z=0 \rightarrow x=0$

$z=1 \rightarrow x=1$

$$= b^7 \int_0^1 z \cdot z^4 (1-z^2)^{1/2} dz$$

$$= b^7 \int_0^1 \frac{1}{2} \cdot x^2 (1-x)^{1/2}\, dx$$

$$= \frac{b^7}{2} \int_0^1 x^{3-1} (1-x)^{3/2-1} dx$$

$$= \frac{b^7}{2} \beta\left(3, 3/2\right)$$

$$= \frac{b^7}{2} \frac{\Gamma(3) \cdot \Gamma(3/2)}{\Gamma(3+3/2)}$$

$$= \frac{b^7}{2} \frac{\Gamma(3) \cdot \Gamma(3/2)}{\Gamma(9/2)}$$

1 vii) $\int_0^\infty x^6 e^{-3x}\,dx$

$3x = z \longrightarrow \frac{z}{3} = x$

$3dx = dz$

$dx = \frac{1}{3}dz$

limits:

$x = 0 \rightarrow z = 0$

$x = \infty \rightarrow z = \infty$

$= \int_0^\infty \left(\frac{z}{3}\right)^6 e^{-z} \frac{1}{3}dz$

$= \frac{1}{3^7}\int_0^\infty e^{-z} z^{7-1}\,dz$

$= \frac{1}{3^7}\ \Gamma(7)$

1 ix) $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx$

$x^2 = z \longrightarrow x = \sqrt{z}$

$\therefore \sqrt{x} = z^{1/4}$

$2x\,dx = dz$

$dx = \frac{1}{2x}dz$

$= \frac{1}{2\sqrt{z}}dz$

limits

$x = 0 \rightarrow z = 0$

$x = \infty \rightarrow z = \infty$

$= \int_0^\infty z^{1/4} e^{-z} \frac{1}{2\sqrt{z}}dz$

$= \frac{1}{2}\int_0^\infty z^{\frac{1}{4}-\frac{1}{2}} e^{-z}\,dz$

$= \frac{1}{2}\int_0^\infty z^{-1/4} e^{-z}\,dz$ $\rightarrow n-1$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{\overset{n-1}{\frac{3}{4}-1}}\,dz$

$n - 1 = -\frac{1}{4}$

$n = -\frac{1}{4} + 1 = \frac{3}{4}$

$= \frac{1}{2}\ \Gamma(3/4)$

$$\to \frac{1}{\sqrt{x}\sqrt{\ln\frac{1}{x}}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{\ln(\frac{1}{x})}} = x^{-1/2} \frac{1}{\sqrt{\ln\frac{1}{x}}}$$

1 xi) $\displaystyle\int_0^1 \frac{1}{\sqrt{x \ln(\frac{1}{x})}}\, dx$

$\ln\left(\frac{1}{x}\right) = z$

$\ln 1 - \ln x = z$

$0 - \ln x = z$

$\ln x = -z$

$\log_e x = -z$

$x = e^{-z}$

$dx = -e^{-z} dz$

limits:

$x = 0 \to z = \infty \to \ln(\frac{1}{0}) = \ln\infty = \infty$

$x = 1 \to z = 0 \to \ln(\frac{1}{1}) = \ln 1 = 0$

$$= \int_{\infty}^{0} \frac{-e^{-z}\, dz}{\sqrt{e^{-z}\, z}}$$

$$= -\int_{\infty}^{0} \frac{e^{-z} dz}{e^{-z/2}\, z^{1/2}}$$

$\int_a^b f\, dx = -\int_b^a f\, dx$

$$= \int_0^{\infty} e^{-z-(-z/2)} z^{-1/2} dz$$

$$= \int_0^{\infty} e^{-z/2} z^{-1/2} dz \to \frac{z}{2} = y$$

$z = 2y$

$dz = 2dy$

limits

$z = 0 \to y = 0$

$z = \infty \to y = \infty$

$$= \int_0^{\infty} e^{-y} (2y)^{-1/2}\, 2\, dy$$

$$= 2^{-1/2+1} \int_0^{\infty} e^{-y} y^{-1/2} dy$$

$$= \sqrt{2} \int_0^{\infty} e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \sqrt{2}\, \Gamma\left(\frac{1}{2}\right)$$

1(xii) $\displaystyle\int_0^1 \left(1 - \frac{1}{x}\right)^{1/3} dx$

$$= \int_0^1 \left(\frac{x-1}{x}\right)^{1/3} dx = \int_0^1 -\left(\frac{1-x}{x}\right)^{1/3} dx$$

$$= -\int_0^1 x^{-1/3} (1-x)^{1/3} dx$$

⋮ continue

Formulas:

$\boxed{1}$ $\displaystyle\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.

OR

$\boxed{2}$ $\displaystyle\int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)\, dt = \beta(x,y)$.

3 iii) $\displaystyle\int_0^{\pi/6} \sin^2 6x \cos^4 3x\, dx$

$\sin^2 6x = (\sin 6x)^2 = (\sin 2\cdot 3x)^2 = (2\sin 3x \cos 3x)^2$

$\sin^2\theta$
$(\sin\theta)^2$

$$= \int_0^{\pi/6} (\sin 2\cdot 3x)^2 \cos^4 3x\, dx$$

$\sin^2 6x = (\sin 6x)^2 = (\sin 2\cdot 3x)^2$

$$= \int_0^{\pi/6} (2\sin 3x \cdot \cos 3x)^2 \cos^4 3x\, dx$$

$$= 4\int_0^{\pi/6} \sin^2 3x \cos^2 3x \cos^4 3x\, dx$$

$$= 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x\, dx$$

$3x = z$
$3\,dx = dz$
$dx = \frac{1}{3} dz$

$$= 4\cdot\frac{1}{3}\int_0^{\pi/2} \sin^{2} z \cos^{6} z\, dz$$

$2 \to p$, $6 \to q$

limits:
$x = 0 \to z = 0$
$x = \frac{\pi}{6} \to z = \pi/2$

$$= \frac{4}{3}\,\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$$

$$= \frac{4}{3}\,\frac{\Gamma(3/2)\Gamma(7/2)}{2\Gamma(5)} = \frac{2}{3}\,\frac{\Gamma(3/2)\Gamma(7/2)}{\Gamma(5)}$$

Alternative Method:

$$= \frac{2}{3}\int_0^{\pi/2} 2\sin^{2} z \cos^{6} z\, dz$$

$2x - 1 = 2$
$2x = 2 + 1 = 3$
$x = 3/2$

$2y - 1 = 6$
$2y = 7$
$y = 7/2$

$$= \frac{2}{3}\beta(x,y) = \frac{2}{3}\beta\left(\frac{3}{2}, \frac{7}{2}\right)$$

$\boxed{7}$

$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$

$\beta\left(\frac{3}{2},\frac{7}{2}\right) = \frac{\Gamma(3/2)\Gamma(7/2)}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\right)}$