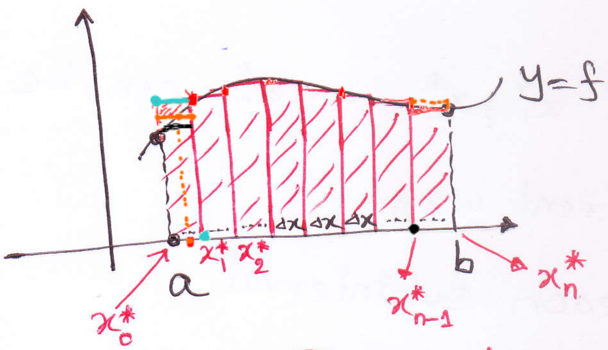
Integration Using Riemann Sums — Week 1

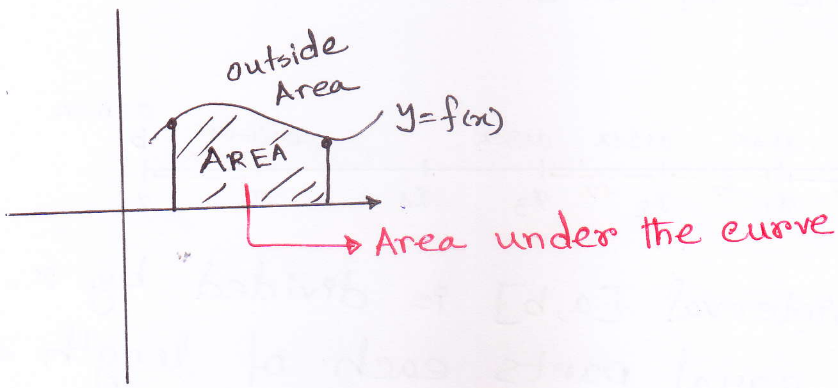
A function f is said to be integrable on a finite closed interval $[a,b]$ if the limit exists and hence denoted as

$$\text{Area} = \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x}_{\text{Riemann Sum}} = \underbrace{\int_{a\,(\text{lower limit})}^{b\,(\text{upper limit})} \underbrace{f(x)\,dx}_{\text{integrand}}}_{\text{Riemann integral}}$$

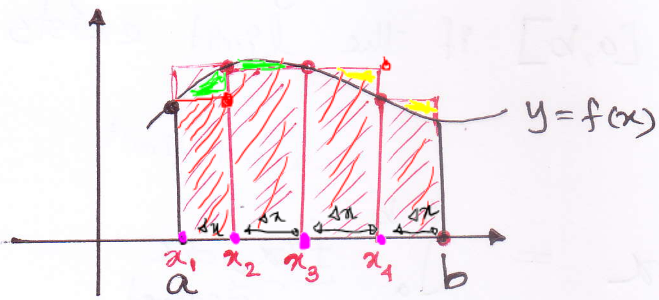
$$\underbrace{\Delta x}_{\text{subintervals}} = \frac{b-a}{n}$$



n-many rectangles



Left end point approximation

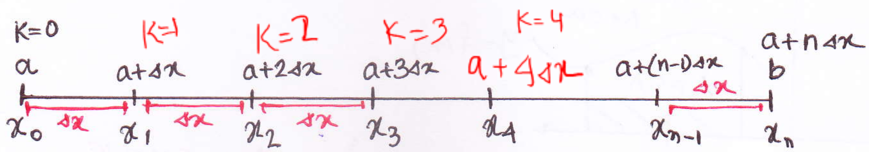


$$x_K^* = x_{K-1} = a+(K-1)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{K=1}^{n} \underbrace{f(x_K^*)}_{\text{height}} \underbrace{\Delta x}_{\text{weidth}} \quad \text{—— } ⊛ \text{ From previous page}$$

In the eqn ⊛, the values $x_1^*, x_2^*, \ldots, x_n^*$ can be chosen into three different ways

- → Left end point of each subinterval
- → Right end point of each subinterval
- → Midpoint of each subinterval

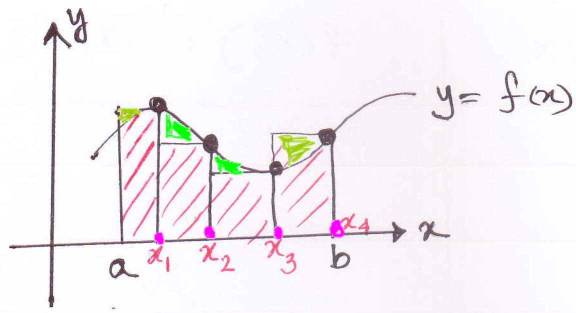


The subinterval $[a,b]$ is divided by $x_1, x_2, \ldots, x_{n-1}$ into n equal parts each of length $\Delta x = \frac{b-a}{n}$

and $x_0 = a$, $x_n = b$

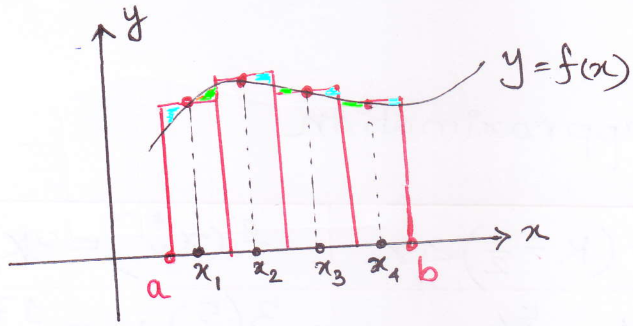
$$x_K^* = a + K\Delta x \;;\quad K = 0, 1, 2, 3, 4, \ldots, n$$

Right end point approximation



$$\boxed{x_k^* = x_k = a + k\Delta x}$$

Midpoint approximation



$$x_k^* = \frac{1}{2}(x_{k-1} + x_k)$$

average of left & right end pt approximation

$$= \frac{1}{2}\Big[\underbrace{a + (k-1)\Delta x}_{\text{Left}} + \underbrace{a + k\Delta x}_{\text{Right}}\Big]$$

$$= \frac{1}{2}(2a + 2k\Delta x - \Delta x)$$

$$= a + \frac{1}{2}(2k-1)\Delta x$$

$$= a + \left(k - \frac{1}{2}\right)\Delta x$$

Mid $= \dfrac{\text{Right} + \text{left}}{2}$

$= \dfrac{k + (k-1)}{2}$

$= k - \dfrac{1}{2}$

$$\boxed{x_k^* = a + \left(k - \tfrac{1}{2}\right)\Delta x}$$

Example [1] $f(x) = 3x + 1$, $a = 2$, $b = 6$, $n = 4$

Left end point approximation

$\Delta x = \dfrac{b-a}{n} = \dfrac{6-2}{4} = \dfrac{4}{4} = 1$

k	$x_k^* = a + (k-1)\Delta x$	$f(x_k^*) = 3x_k^* + 1$
1	$2 + (1-1)1 = 2$	$3(2)+1 = 7$
2	$2 + (2-1)1 = 3$	$3(3)+1 = 10$
3	$2 + (3-1)1 = 4$	13
4	$2 + (4-1)1 = 5$	16

$$\sum_{k=1}^{4} f(x_k^*) = 46$$

$$A = \sum_{k=1}^{4} f(x_k^*)\Delta x = 46 \times 1 = 46$$

[3]

Right end point approximation

K	$x_K^* = a + K\Delta x$	$f(x_K^*) = 3x_K^* + 1$
1	$2+1(1) = 3$	$3(3)+1 = 10$
2	$2+2(1) = 4$	$3(4)+1 = 13$
3	$2+3(1) = 5$	$3(5)+1 = 16$
4	$2+4(1) = 6$	$3(6)+1 = 19$

$\Delta x = 1$

found in previous exercise

$$\sum_{K=1}^{4} f(x_K^*) = 58$$

$$A = \sum_{K=1}^{4} f(x_K^*)\Delta x = 58 \times 1 = 58$$

Mid-point approximation

K	$x_K^* = a + (K - \frac{1}{2})\Delta x$	$f(x_K^*) = 3x_K^* + 1$
1	$2+(1-\frac{1}{2})1 = 5/2$	$3(\frac{5}{2})+1 = 17/2$
2	$2+(2-\frac{1}{2})1 = 7/2$	$3(7/2)+1 = 23/2$
3	$2+(3-\frac{1}{2})1 = 9/2$	$3(9/2)+1 = 29/2$
4	$2+(4-\frac{1}{2})1 = 11/2$	$3(11/2)+1 = 35/2$

$\Delta x = 1$

$$\sum_{K=1}^{4} f(x_K^*) = \frac{104}{2} = 52$$

$$A = \sum_{K=1}^{4} f(x_K^*)\Delta x = 52 \times 1 = 52$$

Alternatively: Mid pt approximation = Average of left & right end pt approximation

$$= \frac{1}{2}(46+58) = 52$$

Theorem

If the function f is continuous on [a,b], [n is unknown or extremely large] then the net signed area A between $y = f(x)$ and interval [a,b] is defined by $A = \lim_{n\to\infty} \sum_{K=1}^{n} f(x_K^*)\Delta x$

$$\lim_{n\to\infty} \frac{1}{n^{n}} \sum_{K=1}^{n} K^{n-1} = \frac{1}{n}$$

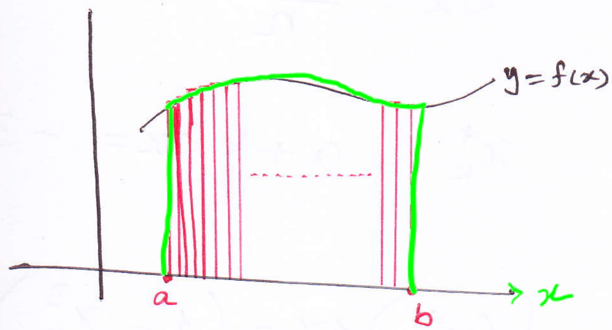
where

a) $\lim_{n\to\infty} \frac{1}{n} \sum_{K=1}^{n} 1 = 1 = \frac{1}{1}$ $\to K^0$

b) $\lim_{n\to\infty} \frac{1}{n^2} \sum_{K=1}^{n} K = 1/2$

c) $\lim_{n\to\infty} \frac{1}{n^3} \sum_{K=1}^{n} K^2 = 1/3$

d) $\lim_{n\to\infty} \frac{1}{n^4} \sum_{K=1}^{n} K^3 = \frac{1}{4}$



$\lim_{n\to\infty} \frac{1}{n^6} \sum_{K=1}^{n} K^5 = ? = \frac{1}{6}$

→ We can see the pattern of the theorem above. Anything that does not match with the above pattern will produce '0' as a result.

<u>Ex</u> $\lim_{n\to\infty} \frac{1}{n^3} \sum_{K=1}^{n} K = 0$ (False)

Ex Find the area under the curve $y = 1 - x^3$ over the interval $[-3, -1]$

[We can use left end, right end or midpoint approximation.

$\because [-3, -1] = [a, b]$

width $\rightarrow$ $\Delta x = \frac{b-a}{n} = \frac{-1-(-3)}{n} = \frac{2}{n}$

input to find height $\rightarrow$ $x_k^* = a + k\Delta x = -3 + k\left(\frac{2}{n}\right) = -3 + \frac{2k}{n}$ using right end point approximation

height $\rightarrow$ $f(x_k^*)\Delta x = (1 - x_k^3)\frac{2}{n}$ width

$= \left[1 - \left(-3 + \frac{2k}{n}\right)^3\right]\frac{2}{n}$

$= \left[1 - \left(-27 + 3(-3)^2\left(\frac{2k}{n}\right) + 3(-3)\left(\frac{2k}{n}\right)^2 + \left(\frac{2k}{n}\right)^3\right)\right]\frac{2}{n}$

$= \left[1 - \left(-27 + \frac{54k}{n} - \frac{36k^2}{n} + \frac{8k^3}{n^3}\right)\right]\frac{2}{n}$

$= \left(1 + 27 - \frac{54k}{n} + \frac{36k^2}{n^2} - \frac{8k^3}{n^3}\right)\frac{2}{n}$

$f(x_k^*)\Delta x = \frac{56}{n} - \frac{108k}{n^2} + \frac{72k^2}{n^3} - \frac{16k^3}{n^4}$

$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}(56) - \frac{1}{n^2}k(108) + \frac{1}{n^3}k^2(72) - \frac{1}{n^4}k^3(16)\right]$

$= (1)56 - \left(\frac{1}{2}\right)108 + \left(\frac{1}{3}\right)72 - \left(\frac{1}{4}\right)16 = 22$

Area