Differential Equation (DE)

↳ An eqn involves derivative of a function

↳ A DE relates some function of one or more variables with its derivatives.

Initial Value Problems:

In initial value problem, we are given the value of function $\underbrace{f(x)}_{\hookrightarrow y}$ and its derivative $\underbrace{f'(x)}_{y'}$ at the same point (initial point), say at $x=0$ along with $f(0)=x_1$ and $f'(0)=x_2$

$y=f(x)$

$f'(\underset{\uparrow x}{0}) = \underset{\uparrow y}{3}$ $\qquad$ $f(\underset{x}{0}) = \underset{y}{2}$

Boundary Value Problem:

In boundary value problem, we are given the value of function $f(x)$, at two different points, say at

$f(a)=x_1$ and $f(b)=x_2$.

$f(\underset{\uparrow x}{2}) = \underset{\uparrow y}{3}$ $\qquad$ $(x,y)=(2,3)$

Separable Variables

Examples:

$\boxed{1}$ $x\frac{dy}{dx} = 4y \Rightarrow x\,dy = 4y\,dx$

$\int \frac{d}{dx} f(x)\,dx = f(x)$

$\int \frac{dy}{y} = \int \frac{4}{x}\,dx$

$\ln y = 4\ln x + c$

$\underbrace{e^{\ln x} = x}$

$\log_e y = \ln x^4 + c$

$a^{x+y} = a^x \cdot a^y$

$y = e^{\ln x^4 + c}$

$= e^{\ln x^4}\, \underline{e^c}$

$\begin{cases} e^c = ? \\ c = ? \end{cases}$

$= x^4 c$ Relabel the constant $e^c = c$

$y = c x^4$

$\boxed{2}$ $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

$= \frac{(y+1)^2}{x^2} = \frac{y^2 + 2y + 1}{x^2}$

$y \ln x\, dx = \frac{y^2 + 2y + 1}{x^2}\, dy$

$x^2 \ln x\, dx = \frac{y^2 + 2y + 1}{y}\, dy$

$\int uv\,dx = u\int v - \int\left(u' \int v\right)dx$

$\underline{\int x^2 \ln x\, dx} = \int\left(y + 2 + \frac{1}{y}\right)dy$

$\ln x \int x^2 dx - \int\left[\left\{\frac{d}{dx}\ln x\right\}\int x^2 dx\right]dx = \int\left(y + 2 + \frac{1}{y}\right)dy$

$\ln x = u$
$x^2 = v$

$\frac{x^3}{3}\ln x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx = \frac{y^2}{2} + 2y + \ln y$

$$\frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{y^2}{2} + 2y + \ln y$$

$$\frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C = \frac{y^2}{2} + 2y + \ln y$$

$$\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C = \frac{y^2}{2} + 2y + \ln y$$

$\boxed{3}$ $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

$$\sin 3x\,dx = -2y\cos^3 3x\,dy$$

$$\frac{\sin 3x}{\cos^3 3x}\,dx = -2y\,dy$$

$$\int \frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\cos^2 3x}\,dx = -2\int y\,dy$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\int \tan 3x \underbrace{\sec^2 3x\,dx} = -2\,\frac{y^2}{2}$$

Let $\tan 3x = z$

$\sec^2 3x(3)\,dx = dz$

$\sec^2 3x\,dx = \frac{1}{3}dz$

$$\frac{1}{3}\int z\,dz = -y^2$$

$$\frac{1}{3}\cdot\frac{z^2}{2} = -y^2$$

$$\frac{1}{6}\tan^2 3x = -y^2$$

$$y^2 = -\frac{1}{6}\tan^2 3x + C$$

$\boxed{4}$ $$\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$$

$$= \frac{xy-y+3x-3}{xy+4y-2x-8}$$

$$= \frac{y(x-1)+3(x-1)}{y(x+4)-2(x+4)}$$

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$$

$$\int \frac{y+3-5}{y+3}\,dy = \int \frac{x+4-5}{x+4}\,dx$$

$\frac{y+3}{y+3} - \frac{5}{y+3}$

$$\int \left(1-\frac{5}{y+3}\right)dy = \int\left(1-\frac{5}{x+4}\right)dx$$

$$y - 5\ln(y+3) = x - 5\ln(x+4) + c$$

$$y - x - c = \ln(y+3)^5 - \ln(x+4)^5$$

$$\ln\left[\frac{(y+3)^5}{(x+4)^5}\right] = y-x-c \Rightarrow \log_e\left(\frac{(y+3)^5}{(x+4)^5}\right) = y-x-c$$

$\log_e y = x$

$\Rightarrow y = e^x$

$$\frac{(y+3)^5}{(x+4)^5} = e^{y-x-c} = \frac{e^y}{e^x e^c}$$

$$\rightarrow e^y . e^{-x} . e^{-c}$$

$$\frac{e^x e^c}{(x+4)^5} = \frac{e^y}{(y+3)^5}$$

$$ce^x (x+4)^{-5} = e^y (y+3)^{-5}$$

Relabel the constant $e^c = c$

Boundary value problem

$\boxed{5}$ $\frac{dx}{dt} = 4(x^2+1)$; $\underbrace{x\left(\frac{\pi}{4}\right) = 1}_{t = \frac{\pi}{4},\ x = 1}$

$x(t) = 1$

$t = \frac{\pi}{4}$ $\therefore x = 1$

$$\int \frac{dx}{x^2+1} = \int 4\, dt$$

$$\tan^{-1} x = 4t + c \quad \text{—— } (1)$$

Substitute $t = \frac{\pi}{4}$, $x = 1$ into (1)

$$\tan^{-1} 1 = 4 \cdot \frac{\pi}{4} + c$$

$$\tan^{-1} \tan \frac{\pi}{4} = \pi + c$$

$$\frac{\pi}{4} = \pi + c$$

$$c = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

Substitute c into (1)

$$\tan^{-1} x = 4t - \frac{3\pi}{4}$$

$$x = \tan\left(4t - \frac{3\pi}{4}\right)$$

$\boxed{6}$

$$\frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \quad \underbrace{y(2) = 2}_{x=2,\ y=2}$$

$$\int \frac{1}{u^2-1}\,du = \frac{1}{2}\ln\frac{u-1}{u+1}$$

$$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$$

used formula

$$\frac{1}{2}\ln\frac{y-1}{y+1} = \frac{1}{2}\ln\frac{x-1}{x+1}$$

$$\ln\frac{y-1}{y+1} = \ln\frac{x-1}{x+1} + c$$

$$\log_e\left(\frac{y-1}{y+1}\right) = \ln\left(\frac{x-1}{x+1}\right) + c$$

$$\frac{y-1}{y+1} = e^{\ln\left(\frac{x-1}{x+1}\right)+c}$$

$$= e^{\ln\left(\frac{x-1}{x+1}\right)} e^{c}$$

$$\frac{y-1}{y+1} = \left(\frac{x-1}{x+1}\right)C \quad (1)$$

Relabel the constant $e^c = C$

Substitute $x = 2$, $y = 2$ into (1)

$$\frac{2-1}{2+1} = \frac{2-1}{2+1}C$$

$$\frac{1}{3} = \frac{1}{3}C$$

$$C = 1$$

Substitute $C = 1$ into (1)

$$\frac{y-1}{y+1} = \frac{x-1}{x+1}.$$