MAT 120

INTRODUCTION to HIGHER ORDER DE

Inhomogeneous case

Non-homogeneous or Inhomogeneous DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x) \quad \text{—} (1)$$

$g(x) \neq 0$, otherwise eqn (1) will become Homogeneous DE

solve the complementary function: $y_c = 0$

$$\Rightarrow \underbrace{a_n y^{(n)} + a_n y^{(n-1)} + \dots + a_1 y' + a_0 y}_{y_c} = 0 \rightarrow \text{homogeneous DE with constant coefficient}$$

Then solve the particular solution: $y_p = g(x)$

Solution of $y = y_c + y_p$

Trial Particular solutions

$g(x)$	Form of y_p
1 (or any constant)	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+D
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
$x^2 e^{bx}$	$(Ax^2+Bx+C)e^{bx}$
$e^{3x}\sin 4x$	$e^{3x}(A\cos 4x + B\sin 4x)$
$5x^2 \sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$

This table shows the pattern of forming y_p

Exercise: Solve the following inhomogeneous DE

(1) $y'' + 3y' + 2y = 6$

Solution: For complimentary function:

$y'' + 3y' + 2y = 0$ [Let $y = e^{mx}$, $y' = me^{mx}$; $y'' = m^2 e^{mx}$]

$m^2 e^{mx} + 3me^{mx} + 2e^{mx} = 0$

$m^2 + 3m + 2 = 0$; $\div$ by e^{mx} & $e^{mx} \neq 0$

$m^2 + 2m + m + 2 = 0$

$(m+1)(m+2) = 0$

$m_1 = -1, \; m_2 = -2$

$\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$ $\quad y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

For particular

$\boxed{y_p = A}$ $\because$ $\boxed{g(x) = 6}$ $\longrightarrow$ 6 $\hookrightarrow$ A

$\therefore y_p' = 0$ & $y_p'' = 0$

Given $y'' + 3y' + 2y = 6$ $\longrightarrow$ in the question

Consider $y_p'' + 3y_p' + 2y_p = 6$

$0 + 3(0) + 2A = 6$

$A = 3$ $\quad \therefore \boxed{y_p = A = 3}$

$y = y_c + y_p$

$= c_1 e^{-x} + c_2 e^{-2x} + 3$

② $y'' - 10y' + 25y = 30x + 3$

Solution:

For complimentary function:

$y'' - 10y' + 25y = 0$

let $y = e^{mx}$

$m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

$(m-5)(m-5) = 0$

$m_1 = m_2 = 5 \longrightarrow$ repeating root $\quad y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$

$y_c = c_1 e^{5x} + c_2 x e^{5x}$

For particular solution:

$y_p = Ax + B \qquad \because g(x) = 30x + 3 \longrightarrow Ax + B$

$y_p' = A \;;\; y_p'' = 0$

Given: $y_p'' - 10y_p' + 25y_p = 30x + 3$ $\quad$ considering $y = y_p$

$0 - 10(A) + 25(Ax + B) = 30x + 3$

$\boxed{-10A} + \boxed{25Ax} + \boxed{25B} = \boxed{30x} + \boxed{3}$

Equating coefficients of like terms:

$\boxed{x}$

$25A = 30$

$A = \frac{6}{5}$

$\boxed{\text{Constant}}$

$-10A + 25B = 3$

$-10\left(\frac{6}{5}\right) + 25B = 3$

$-12 + 25B = 3$

$25B = 15 \Rightarrow B = \frac{3}{5}$

$\therefore \boxed{y_p = Ax + B = \frac{6}{5}A + \frac{3}{5}}$

$y = y_c + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}A + \frac{3}{5}$

③ $\frac{1}{4}y'' + y' + y = x^2 - 2x$

For complimentary function:

$\frac{1}{4}y'' + y' + y = 0$; let $y = e^{mx}$

$\Rightarrow \frac{1}{4}m^2 + m + 1 = 0$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m_1 = m_2 = -2$

$$\boxed{y_c = c_1 e^{-2x} + c_2 x e^{-2x}}$$

for particular solution:

$y_p = Ax^2 + Bx + C$ ∵ $g(x) = x^2 - 2x$

$y_p' = 2Ax + B$; $y_p'' = 2A$

Given $\frac{1}{4}y'' + y' + y = x^2 - 2x \Rightarrow \frac{1}{4}y_p'' + y_p' + y_p = x^2 - 2x$

$\frac{1}{4}(2A) + (2Ax + B) + (Ax^2 + Bx + C) = x^2 - 2x$

$\frac{1}{2}A + 2Ax + B + Ax^2 + Bx + C = x^2 - 2x$

Equating coefficients of like terms:

$\boxed{x^2}$

$A = 1$

$\boxed{x}$

$2A + B = -2$

$2(1) + B = -2$

$B = -4$

$\boxed{\text{Constant}}$

$\frac{1}{2}A + B + C = 0$

$\frac{1}{2}(1) + (-4) + C = 0$

$C = 4 - \frac{1}{2} = 7/2$

$y_p = Ax^2 + Bx + C$

$\therefore \boxed{y_p = x^2 - 4x + 7/2}$

$y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$.

④ $y'' + 3y = -48x^2e^{3x}$

For complimentary function:

$y'' + 3y = 0$

let $y = e^{mx}$

$m^2 + 3 = 0 \Rightarrow m^2 = -3$

$m = \pm\sqrt{-3} = \pm i\sqrt{3} = \underset{a}{0} \pm i\underset{b}{\sqrt{3}}$ $\quad m = a \pm ib$

$y_c = e^{0.x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

$\boxed{y_c = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x}$

For particular solution:

$y_p = (Ax^2 + Bx + C)e^{3x}$ $\quad \because g(x) = -48x^2 e^{3x}$

$Ax^2 + Bx + C$ $\quad Ae^{3x}$

combine for y_p

$y_p' = (Ax^2 + Bx + C)3e^{3x} + (2Ax + B)e^{3x}$

$y_p' = e^{3x}(3Ax^2 + 3Bx + 3C + 2Ax + B)$

$y_p'' = 3e^{3x}(3Ax^2 + 3Bx + 3C + 2Ax + B)$

$+ e^{3x}(6Ax + 3B + 2A)$

$= e^{3x}(9Ax^2 + 9Bx + 9C + 6Ax + 3B + 6Ax + 3B + 2A)$

$\therefore y_p'' = e^{3x}(9Ax^2 + 9Bx + 12Ax + 2A + 6B + 9C)$

Given: $y'' + 3y = -48x^2e^{3x} \Rightarrow y_p'' + 3y_p = -48x^2e^{3x}$

$e^{3x}(9Ax^2 + 12Ax + 9Bx + 2A + 6B + 9C) + 3e^{3x}(Ax^2 + Bx + C)$

$= -48x^2e^{3x}$

$9Ax^2 + 12Ax + 9Bx + 2A + 6B + 9C + 3Ax^2 + 3Bx + 3C = -48x^2$

$[\div \text{ by } e^{3x}]$

$\boxed{12Ax^2} + 12Ax + 12Bx + 2A + 6B + 12C = \boxed{-48x^2} + 0.x$

$+0$

Equating Coefficients of like terms:

$\boxed{x^2}$

$12A = -48$

$A = -4$

$\boxed{x}$

$12A + 12B = 0$

$A + B = 0$

$(-4) + B = 0$

$B = 4$

$\boxed{\text{Constant}}$

$2A + 6B + 12C = 0$

$A + 3B + 6C = 0$

$(-4) + 3(4) + 6C = 0$

$6C = -8$

$C = -\frac{4}{3}$

$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$\therefore \boxed{y_p = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}}$$

$$y = y_c + y_p = c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x + \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

(5) $y'' - 2y' + 5y = e^x \cos 2x$

For complimentary function

$y'' - 2y' + 5y = 0$

let $y = e^{mx}$

$m^2 - 2m + 5 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$m = 1 \pm 2i \rightarrow a = 1,\ b = 2$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$$\boxed{y_c = e^x(c_1 \cos 2x + c_2 \sin 2x)}$$

For particular solution:

$y_p = e^x(A\cos 2x + B\sin 2x)$ $\because$ $g(x) = \underbrace{e^x}_{Ae^x} \underbrace{\cos 2x}_{A\cos 2x + B\sin 2x}$

but $y_c = e^x(c_1 \cos 2x + c_2 \sin 2x)$

In that case y_p is the duplicate of y_c

$\therefore y_p = xe^x(A\cos 2x + B\sin 2x)$

$$y_p = xe^x(A\cos 2x + B\sin 2x)$$

$$y_p' = xe^x[-2A\sin 2x + 2B\cos 2x] + (xe^x + e^x)[A\cos 2x + B\sin 2x]$$

$$= xe^x[(-2A+B)\sin 2x + (2B+A)\cos 2x]$$
$$+ e^x[A\cos 2x + B\sin 2x]$$

$$y_p'' = xe^x[(-3A+4B)\cos 2x + (-3B-4A)\sin 2x]$$
$$+ e^x[(-4A+2B)\sin 2x + (4B+2A)\cos 2x]$$

Given $y'' - 2y' + 5y = e^x\cos 2x \Rightarrow y_p'' - 2y_p' + 5y_p = e^x\cos 2x$

$$\Rightarrow xe^x(-3A+4B)\cos 2x + xe^x(-3B-4A)\sin 2x + e^x(-4A+2B)\sin 2x$$
$$+ e^x(4B+2A)\cos 2x - 2xe^x(-2A+B)\sin 2x - 2xe^x(2B+A)\cos 2x$$
$$- 2e^x A\cos 2x - 2e^x B\sin 2x + 5xe^x A\cos 2x + 5xe^x B\sin 2x$$
$$= e^x\cos 2x$$

$\left(\frac{\circ}{\circ} \text{ by } e^x\right)$

$$\Rightarrow x(-3A+4B)\cos 2x + x(-3A-4B)\sin 2x + (-4A+2B)\sin 2x$$
$$+ (4B+2A)\cos 2x - 2x(-2A+B)\sin 2x - 2x(2B+A)\cos 2x$$
$$- 2A\cos 2x - 2B\sin 2x + 5xA\cos 2x + 5xB\sin 2x$$
$$= \cos 2x \quad \left[\frac{\circ}{\circ} \text{ by } e^x\right]$$

Equating coefficients of like terms:

$\boxed{x\cos 2x}$

$-3A+4B-2(2B+A)+5A=0$

$-3A+4B-4B-2A+5A=0$

$0=0$

$\boxed{x\sin 2x}$

$-3A-4B-2(-2A+B)+5B=0$

$-3A-4B+4A-2B+5B=0$

$A-B=0$

$\boxed{\cos 2x}$

$4B+2A-2A=1$

$4B=1$

$\boxed{B=1/4}$

$\boxed{\sin 2x}$

$-4A+2B-2B=0$

$-4A=0$

$\boxed{A=0}$

$$y_p = xe^x(A\cos 2x + B\sin 2x)$$
$$= xe^x\left(0 + \frac{1}{4}\sin 2x\right)$$
$$\therefore \boxed{y_p = \frac{1}{4}xe^x\sin 2x}$$

$$y = y_c + y_p$$
$$= e^x(C_1\cos 2x + C_2\sin 2x) + \frac{1}{4}xe^x\sin 2x$$