A Non-Exact DE Made Exact

Considering $M(x,y)\,dx + N(x,y)\,dy = 0$ —— (1) & Say $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Then

- If $(M_y - N_x)/N$ $f(x)$ is a function of x alone, then an integrating factor for (1) is $e^{\int \frac{M_y - N_x}{N} dx}$

- If $(N_x - M_y)/M$ $\to f(y)$ is a function of y alone, then an integrating factor for (1) is $e^{\int \frac{N_x - M_y}{M} dy}$

Consider the following non-linear DE:

$$\underbrace{(xy)}_{M_y}dx + (2x^2+3y^2-20)dy = 0 \quad \text{—— (2)}$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \to N_x$

$M = xy$ $\qquad N = 2x^2+3y^2-20$

$M_y = \frac{\partial M}{\partial y} = x$ $\qquad N_x = \frac{\partial N}{\partial x} = 4x$

Eqn (2) is not an exact DE

<u>Consider:</u>

$$\frac{M_y - N_x}{N} = \frac{x-4x}{2x^2+3y^2-20} = \frac{-3x}{2x^2+3y^2-20} \to f(x,y)$$

$\to$ depends on x and y

$$\frac{N_x - M_y}{M} = \frac{4x-x}{xy} = \frac{3x}{xy} = \frac{3}{y} \to f(y)$$

$\to$ depends only on y

$\therefore$ The integrating factor: $e^{\int \frac{3}{y} dy} = e^{3\ln y} = e^{\ln y^3} = y^3.$

Multiply eqn (2) by integrating factor we have:

$$y^3(xy\,dx + (2x^2+3y^2-20)\,dy) = (0)\,y^3$$

$$\underbrace{xy^4}\,dx + (\underbrace{2x^2y^3+3y^5-20y^3})\,dy = 0 \quad \text{—— (3)}$$

$M = xy^4$ $\qquad$ $N = 2x^2y^3+3y^5-20y^3$

$\dfrac{\partial M}{\partial y} = 4xy^3$ $\qquad$ $\dfrac{\partial N}{\partial x} = 4xy^3$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence eqn (3) is an exact DE

$M = xy^4$

$\dfrac{\partial f}{\partial x} = M = xy^4$

$\int \dfrac{\partial f}{\partial x} = \int xy^4$

$\int \partial f = \int xy^4\,\partial x$

$f = \dfrac{x^2y^4}{2} + \phi(y)$ —— (4)

$\dfrac{\partial f}{\partial y} = \dfrac{4x^2y^3}{2} + \phi'(y)$ $\longrightarrow$ derive w.r.t. y

$N = 2x^2y^3 + \phi'(y)$

$\cancel{2x^2y^3} + 3y^5 - 20y^3 = \cancel{2x^2y^3} + \phi'(y)$

$\phi'(y) = 3y^5 - 20y^3$

$\int \phi'(y) = \int (3y^5 - 20y^3)\,dy$

$\phi(y) = \dfrac{3y^6}{6} - \dfrac{20y^4}{4} + c$

$= \frac{1}{2}y^6 - 5y^4 + C$

substitute $\phi(y)$ into (4)

$f(x,y) = \dfrac{x^2y^4}{2} + \frac{1}{2}y^6 - 5y^4 + C$