7.8 Infinite Integrals (Improper Integrals)

Week 1

Anton's Calculus 10th Ed.

Consider the followings:

$$\int_c^{\infty} (ax+b)\,dx \;;\; \int_{-\infty}^{c} (ax+b)\,dx$$

$$\int_{-\infty}^{\infty} (ax+b)\,dx \;;\; \int_a^3 \frac{1}{x-3}\,dx$$

An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinitely at one or more points in the range of integration.

$$\int f(x)\,dx = F(x)$$

Range of integration

integrand

$F(x)$ →
- Converge (If the solution exists)
- Diverge (If the solution does not exist)

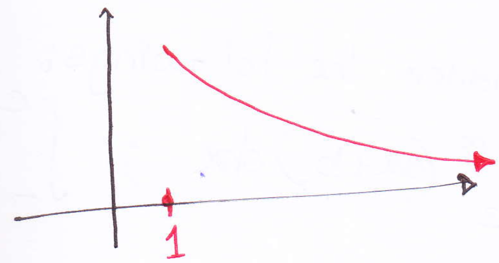
It cannot be computed using normal integral.

So we introduce limit.

Consider:

$$\int_1^{\infty} x^{-2}\, dx$$

$y = \dfrac{1}{x^2}$, $x \in [1, \infty)$



It can be computed by replacing infinite limits with finite values:

$$\int_1^{\infty} x^{-2}\, dx = \lim_{l \to \infty} \int_1^{l} x^{-2}\, dx = \lim_{l \to +\infty} \left[\frac{x^{-2+1}}{-2+1} \right]_1^l$$

$$= \lim_{l \to +\infty} \left[\frac{-1}{x} \right]_1^l$$

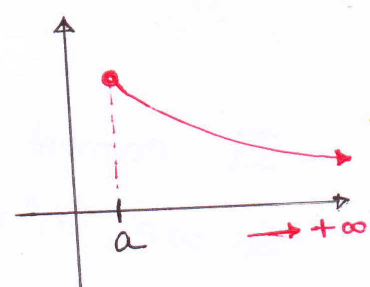
$$= \lim_{l \to +\infty} \left[-\frac{1}{l} + 1 \right]$$

$$= -\frac{1}{\infty} + 1 = 0 + 1 = 1$$

Observe few cases:

1(a) The improper integral of 'f' over the interval $[a, +\infty)$ is defined as:

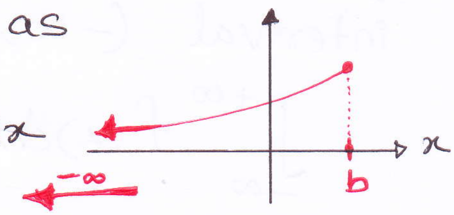
$$\int_a^{+\infty} f(x)\, dx = \lim_{l \to +\infty} \int_a^{l} f(x)\, dx$$



The integral is said to converge if the limit exists and diverge if it does not.

<u>1(b)</u> The improper integral of f over the integral $(-\infty, b]$ is defined as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{K \to -\infty} \int_{K}^{b} f(x)\,dx$$



The Integral is said to converge if the limit exists and diverge it it does not.

$\boxed{\text{Example}}$ $\displaystyle\int_{4}^{+\infty} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$

$$= \lim_{l \to +\infty} \int_{4}^{l} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{l \to +\infty} \Big[\ln(x-1) - \ln(x+1)\Big]_{4}^{l}$$

$$= \lim_{l \to +\infty} \left[\ln\left(\frac{x-1}{x+1}\right)\right]_{4}^{l}$$

$$= \lim_{l \to +\infty} \left[\ln\left(\frac{l-1}{l+1}\right) - \ln\left(\frac{3}{5}\right)\right]$$

$$= \lim_{l \to +\infty} \ln\frac{l-1}{l+1} - \lim_{l \to +\infty} \ln\left(\frac{3}{5}\right)$$

$$= \lim_{l \to +\infty} \ln\frac{l+1-2}{l+1} - \lim_{l \to \infty} \ln(3/5)$$

$$= \lim_{l \to \infty} \ln\left(1 - \frac{2}{l+1}\right) - \ln\left(\frac{3}{5}\right)$$

$$= \ln\left(1 - \frac{2}{\infty+1}\right) - \ln(3/5)$$

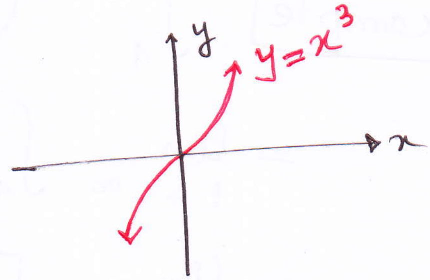
$$= \underbrace{\ln(1-0)}_{\ln 1 = 0} - \ln(3/5) = 0 - \ln(3/5) = \ln\left(\frac{5}{3}\right)$$

<u>Converges.</u>

2 The improper integral of f over the interval $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx}_{\text{piecewise continuos function from } (-\infty, c)\ \&\ (c, +\infty).}$$

where 'c' is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.



Example $\int_{-\infty}^{+\infty} x^3\,dx$

$$= \int_{-\infty}^{0} x^3\,dx + \int_{0}^{+\infty} x^3\,dx$$

$$= \lim_{K \to -\infty} \int_{K}^{0} x^3\,dx + \lim_{l \to +\infty} \int_{0}^{l} x^3\,dx$$

$$= \lim_{K \to -\infty} \left[\frac{x^4}{4}\right]_{K}^{0} + \lim_{l \to +\infty} \left[\frac{x^4}{4}\right]_{0}^{l}$$

$$= \lim_{K \to -\infty} \left[0 - \frac{K^4}{4}\right] + \lim_{l \to +\infty} \left[\frac{l^4}{4} - 0\right]$$

$$= -\frac{1}{4} \lim_{K \to -\infty} K^4 + \frac{1}{4} \lim_{l \to +\infty} l^4$$

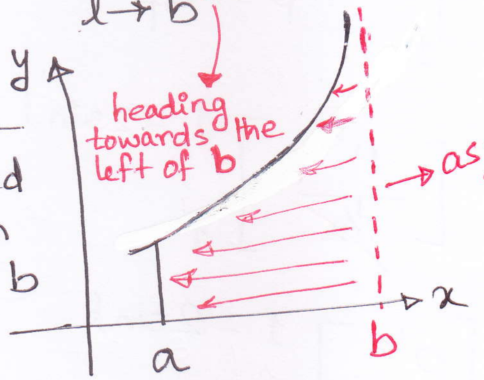
$$= \frac{1}{4} \left[\lim_{l \to +\infty} l^4 - \lim_{K \to -\infty} K^4\right]$$

$$= \frac{1}{4}\left[\infty - \infty\right] = \infty \quad \text{diverges.}$$

3. If f is continuous on the interval $[a,b]$ except for an infinite discontinuity at b, then the improper integral of f over the interval $[a,b]$ is defined as

$$\int_a^{b} f(x)\,dx = \lim_{l \to b^-} \int_a^{l} f(x)\,dx$$

The integral converges if the limit exists and diverges if it does not.



The required region is on the left of b

→ asymptote (It means the curve will not touch this line or exit this line)

Example $\displaystyle\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$

Observe $x \neq \pi/6$ $\because \sin\frac{\pi}{6} = \frac{1}{2}$

$$\therefore \sqrt{1-2\sin x} = \sqrt{1-2\sin\tfrac{\pi}{6}} = \sqrt{1-2\left(\tfrac{1}{2}\right)} = \sqrt{1-1} = 0.$$

$$\therefore f(x) = \frac{\cos x}{0} = \infty$$

$$= \lim_{l \to \frac{\pi}{6}^-} \int_0^{l} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$$

$$= \lim_{l \to \frac{\pi}{6}^-} \int_0^{\sin l} \frac{dz}{\sqrt{1-2z}}$$

Let $\sin x = z$
$\cos x\,dx = dz$

limits:
$x = 0 \to z = \sin 0 = 0$
$x = l \to z = \sin l$

$$= \lim_{l \to \frac{\pi}{6}^-} \int_0^{\sin l} \frac{dz}{\sqrt{1-2z}}$$

let $1-2z = p$

$-2dz = dp$

$dz = -\frac{1}{2} dp$

$$= \lim_{l \to \frac{\pi}{6}^-} \int_1^{1-2\sin l} -\frac{1}{2}\frac{dp}{\sqrt{p}}$$

limits

$z = 0 \rightarrow p = 1$

$z = \sin l \rightarrow p = 1 - 2\sin l$

$$= -\frac{1}{2} \lim_{l \to \frac{\pi}{6}^-} \int_1^{1-2\sin l} p^{-1/2}\, dp$$

$$= -\frac{1}{\cancel{2}} \lim_{l \to \frac{\pi}{6}^-} \left[\frac{p^{1/2}}{\cancel{1/2}}\right]_1^{1-2\sin l}$$

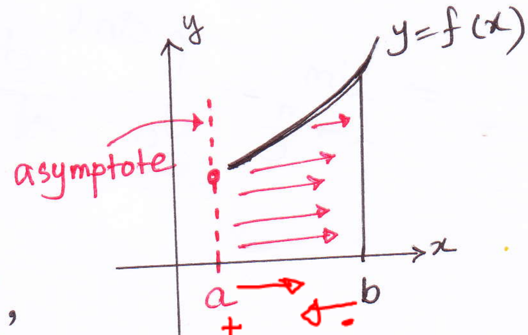
$$= -\lim_{l \to \frac{\pi}{6}^-} \left[p^{1/2}\right]_1^{1-2\sin l}$$

$$= -\lim_{l \to \frac{\pi}{6}^-} \left[\sqrt{1-2\sin l} - \sqrt{1}\right]$$

$$= -\sqrt{1-2\left(\tfrac{1}{2}\right)} + 1$$

$$= 1 \quad \text{Converges}$$

4 If f is continuous on the interval $[a,b]$, except for an infinite discontinuity at a, then the improper integral of 'f' over the interval $[a,b]$ is :



required region is on the right of 'a'

$$\int_a^b f(x)\,dx = \lim_{K \to a^+} \int_K^b f(x)\,dx$$

The integral converges if the limit exists & diverges if it does not.

$\boxed{\text{Example}}$ $\displaystyle\int_3^4 \frac{dx}{(x-3)^2}$ $x \neq 3$ otherwise $f(x)$ is undefined.

$$\downarrow$$

$$\frac{1}{(x-3)^2} = \frac{1}{(3-3)^2} = \frac{1}{0} = \infty$$

$$= \lim_{l \to 3^+} \int_l^4 \frac{dx}{(x-3)^2}$$

$$= \lim_{l \to 3^+} \int_{l-3}^{1} \frac{dz}{z^2}$$

let $x-3=z$
$dx = dz$

limits:
$x = l \to z = l-3$
$x = 4 \to z = 1$

$$= \lim_{l \to 3^+} \int_{l-3}^{1} z^{-2}\, dz$$

$$= \lim_{l \to 3^+} \left[\frac{z^{-1}}{-1}\right]_{l-3}^{1}$$

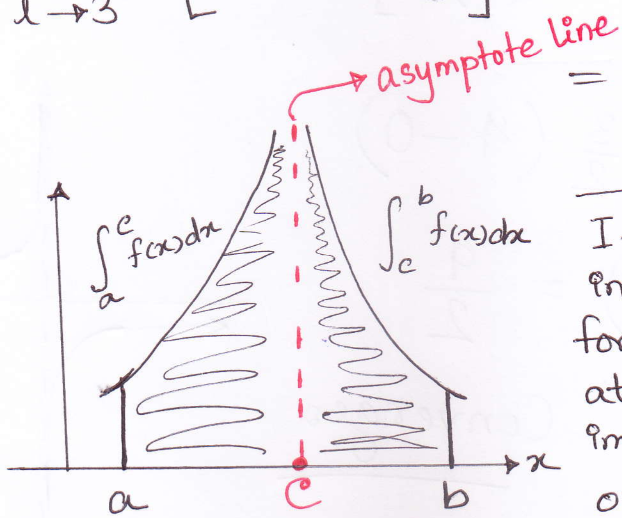
$$= -\lim_{l \to 3^+} \left[\frac{1}{z}\right]_{l-3}^{1}$$

$$= -\lim_{l \to 3^+} \left[1 - \frac{1}{l-3}\right] = -1 + \lim_{l \to 3^+} \frac{1}{l-3} = -1 + \frac{1}{3-3}$$

$$= -1 + \infty = \infty$$

diverges.

$\underline{\underline{5}}$



If f is continuous on the interval $[a,b]$, except for an infinite discontinuity at 'c' in (a,b), then the improper integral of 'f' over the interval $[a,b]$ is defined as:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The integral converges if both terms converge and diverges if either term diverges.

$\boxed{\text{Example}}$ $\int_{-1}^{8} x^{-1/3}\,dx$

$y = x^{-1/3} = \frac{1}{\sqrt[3]{x}}$

$\therefore x \neq 0$

Hence an infinite discontinuity at $x=0$

$$= \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{8} x^{-1/3}\,dx$$

$$= \lim_{l \to 0^-} \int_{-1}^{l} x^{-1/3}\,dx + \lim_{k \to 0^+} \int_{k}^{8} x^{-1/3}\,dx$$

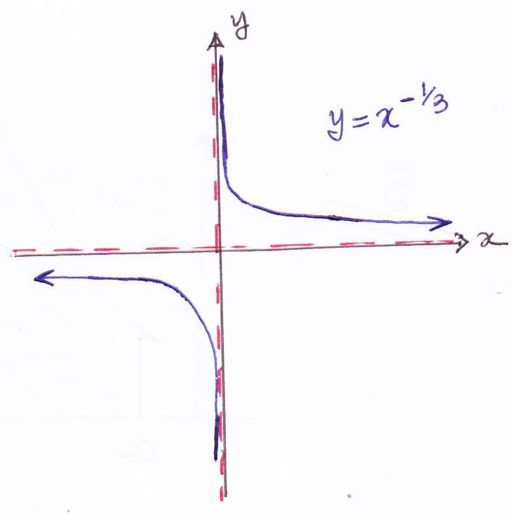
$$= \lim_{l \to 0^-} \left[\frac{x^{-1/3+1}}{-\frac{1}{3}+1}\right]_{-1}^{l} + \lim_{k \to 0^+} \left[\frac{x^{-1/3+1}}{-\frac{1}{3}+1}\right]_{k}^{8}$$

$$= \lim_{l \to 0^-} \left[\frac{x^{2/3}}{\frac{2}{3}}\right]_{-1}^{l} + \lim_{k \to 0^+} \left[\frac{x^{2/3}}{\frac{2}{3}}\right]_{k}^{8}$$

$$= \frac{3}{2} \lim_{l \to 0^-} \left[l^{2/3} - (-1)^{2/3}\right] + \frac{3}{2} \lim_{k \to 0^+} \left[8^{2/3} - k^{2/3}\right]$$

$$= \frac{3}{2}(0-1) + \frac{3}{2}(4-0)$$

$$= \frac{3}{2}(-1+4) = \frac{9}{2}$$



Converges