MAT120 Week 6

TRIPLE INTEGRAL IN SPHERICAL COORDINATE

$$\underset{G}{\iiint} f(\rho,\theta,\varphi)\, dV = \underset{\substack{\text{appropriate}\\ \text{limits}}}{\iiint} f(\rho,\theta,\varphi)\rho^2 \underbrace{\sin\varphi}\, d\rho\, d\theta\, d\varphi$$

$G \to$ Solid

ρ (rho) $\to$ constant that represents a sphere centered at the origin.

Eqn of sphere centered at the origin

$$x^2+y^2+z^2=\rho^2$$

$\theta \to$ constant, represents a half plane (height) [z-axis represents height].

$\varphi \to$ constant that represents a right circular cone with its' vertex at the origin and its' line of symmetry along the z-axis for $\varphi = \frac{\pi}{2}$ and in the $\underbrace{xy\text{-plane}}_{z=0}$ if $\varphi = \frac{\pi}{2}$.

A right circular cone is a circular cone whose altitude intersects the plane of the circle at the circle's center.

The height of an object or a point in relation to sea level or ground level is known as altitude.

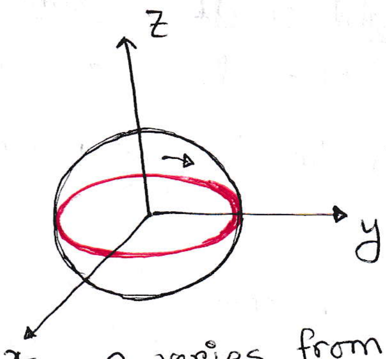
Relation

$$\underbrace{(\rho, \theta, \varphi)}_{\text{spherical coordinate}} \rightarrow \underbrace{(x, y, z)}_{\text{cartesian coordinate}}$$

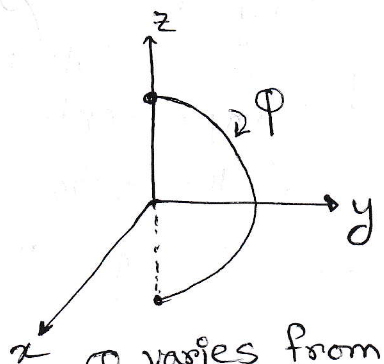
$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

$\therefore x^2 + y^2 + z^2 = \rho^2 \rightarrow$ Eqn of sphere centered at the origin

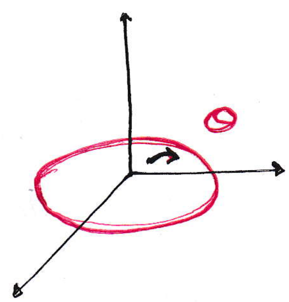
$\therefore \rho = \sqrt{x^2 + y^2 + z^2}$



ρ varies from 0 to ρ_0



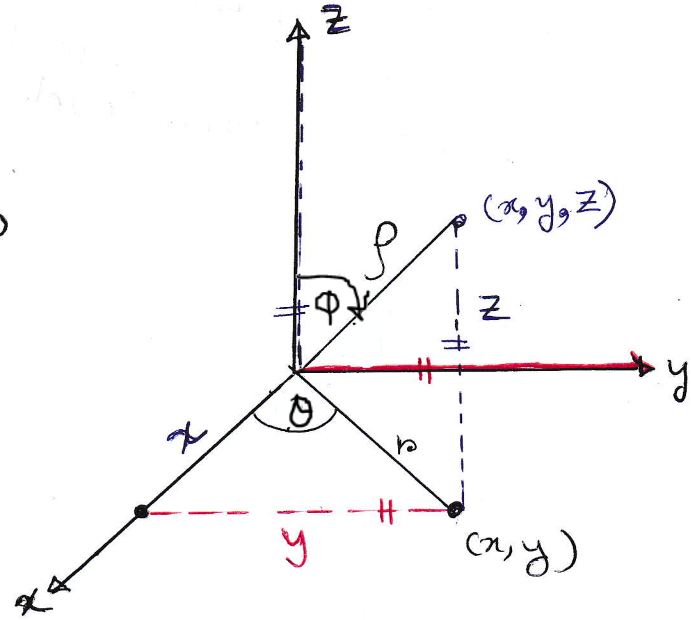
φ varies from 0 to π (half circle)



θ varies from 0 to 2π (full circle)

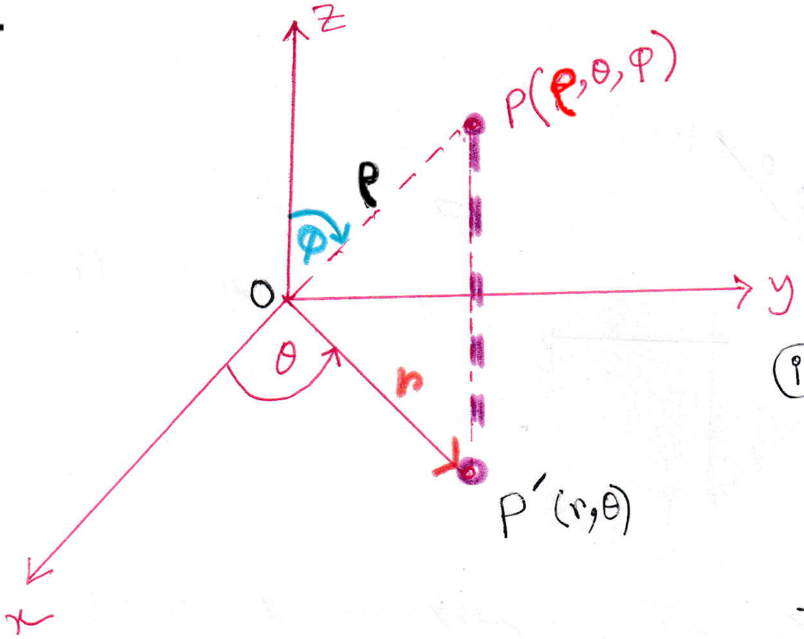
$\rho \rightarrow$ radius of sphere in 3D-plane $\}$ $\rho \geq 0$

$r \rightarrow$ radius of circle in 2D-plane $\}$ $r \geq 0$



Notes from MAT 110

Spherical Coordinate



Eqn of sphere, center (0,0,0)
$[x^2+y^2+z^2=\rho^2]$

(i) $\rho \to$ rho
$\rho = |\vec{OP}| \to$ radius of sphere

$\rho \geq 0$

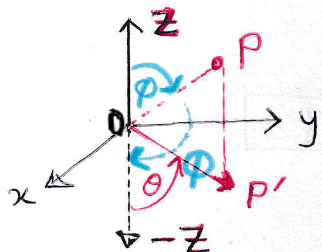
(ii) $\theta \to$ angle from x axis to the projection of the point P on xy-plane
$\to$ Vertical projection of P on xy-plane

(iii) $\to$ Let's call the projecting point on xy-plane P' from the point P.

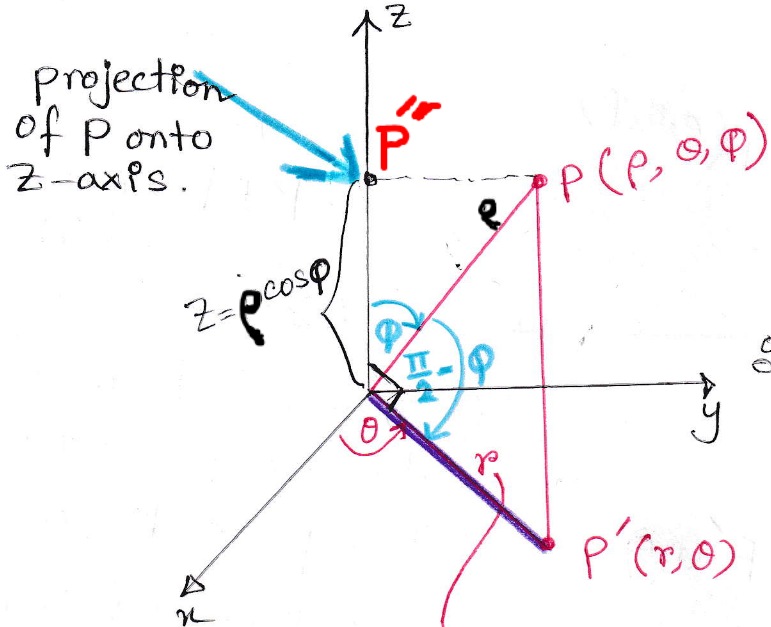
$P' \to (r,\theta)$ $\quad r = d\{(0,0), P'\}$

$\theta \in (0, 2\pi)$

(iv) $\varphi \to$ angle between OP to z-axis.
It starts to rotate from z-axis towards x-y plane. $\therefore \varphi \in (0, \pi/2)$
Then it rotates from xy-plane towards the $-$ve side of z-axis. $\therefore \varphi \in (\pi/2, \pi)$
Hence the complete turn of φ is $(0, \pi)$.



Notes from MAT 110



$z \perp$ on xy-plane

$\therefore$ z axis to the line is $90°$.

$r = \rho \sin \varphi$ → $x = r\cos\theta = \rho \sin\varphi \cos\theta$

→ $y = r\sin\theta = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$\boxed{\text{Rectangular to Spherical:}}$

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$\boxed{\text{Spherical to Rectangular}}$

$\rho^2 = r^2 + z^2$

$\rho^2 = x^2 + y^2 + z^2$

$\tan\theta = \dfrac{y}{x}$

$\cos\varphi = \dfrac{z}{\rho} = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$dV = dz\,dy\,dx$

$= dz\, r\, dr\, d\theta$

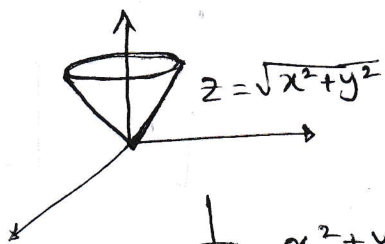
$= r\rho\, d\rho\, d\theta\, d\varphi$

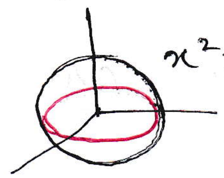
$= \rho\sin\varphi\, \rho\, d\rho\, d\theta\, d\varphi \quad \because r = \rho\sin\varphi$

$= \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$ ...

Examples

(1) Use spherical coordinate to find the volume of the solid G bounded above by the sphere $x^2+y^2+z^2=16$ and below by the cone $z=\sqrt{x^2+y^2}$.





$x^2+y^2+z^2=16=4^2$

$\therefore \rho = 4$

<u>Solution:</u> In the spherical coordinates:

the eqn $x^2+y^2+z^2=16$ is $\rho=4$ and

the eqn of the cone $z=\sqrt{x^2+y^2}$

$$\Rightarrow \rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta}$$

$$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\,\underbrace{(\cos^2\theta+\sin^2\theta)}_{\to 1}}$$

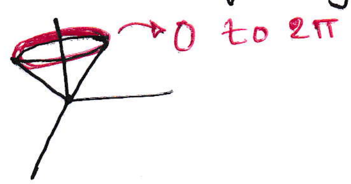
$$\rho\cos\varphi = \rho\sin\varphi$$

$$1 = \frac{\sin\varphi}{\cos\varphi} \Rightarrow \tan\varphi = 1 \Rightarrow \varphi = \tan^{-1} 1 \Rightarrow \varphi = \pi/4$$

$\therefore \rho \in [0, 4]$

$\varphi \in [0, \pi/4]$

$\theta \in [0, 2\pi]$ $\therefore$ the cone is given by $z=\sqrt{x^2+y^2}$



$$\text{Volume} = \iiint dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/4} \int_{\rho=0}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3}\right]_0^4 \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{64}{3} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi/4} d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\cos\frac{\pi}{4} - \cos 0\right] d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\frac{1}{\sqrt{2}} - 1\right] d\theta$$

$$= -\frac{64}{3}\left(\frac{1}{\sqrt{2}} - 1\right) \left[\theta\right]_0^{2\pi}$$

$$= -\frac{64}{3}\left(\frac{1}{\sqrt{2}} - 1\right)(2\pi)$$

$$= \frac{64\pi}{3}\left(2 - \sqrt{2}\right)$$

$\theta \in [0, 2\pi]$ $\rho \in [0, 4]$

(2) The solid bounded by the sphere $\rho = 4$ and below by the cone $\varphi = \frac{\pi}{3}$ $\varphi \in [0, \frac{\pi}{3}]$

Reading

Solution: $V = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/3} \int_{\rho=0}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\varphi \left[\frac{\rho^3}{3}\right]_0^4 d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\varphi \left(64/3\right) d\varphi \, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi/3} d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\cos\frac{\pi}{3} - \cos 0\right] d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\frac{1}{2} - 1\right] d\theta$$

$$= -\frac{64}{3} \left(-\frac{1}{2}\right) \left[\theta\right]_0^{2\pi}$$

$$= \frac{32}{3} \left[2\pi - 0\right]$$

$$= \frac{64\pi}{3}.$$

③ The solid enclosed by the sphere $x^2+y^2+z^2=4a^2$ $\theta \in [0, 2\pi]$ and the planes $z=0$ and $z=a$.

$z=0 \rightarrow$ xy-plane

<u>Solution:</u> In spherical coordinates the sphere and the plane $z=a$

Consider $x^2+y^2+z^2 = 4a^2 = (2a)^2$

$\therefore \rho = 2a \quad \therefore \rho \in [0, 2a]$

We know $z = \rho\cos\varphi$

$\therefore a = 2a\cos\varphi$

$\because z = a, \rho = 2a$

$$\frac{a}{2a} = \cos\varphi$$

$$\cos\varphi = \frac{1}{2}$$

$$\varphi = \cos^{-1}\frac{1}{2}$$

$$\varphi = \frac{\pi}{3}$$

$$\therefore \varphi \in \left[0, \frac{\pi}{3}\right]$$

$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/3}\int_{\rho=0}^{2a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^{2a} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{8a^3}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/3} d\theta$$

$$= -\frac{8a^3}{3}\int_0^{2\pi}\left[\cos\frac{\pi}{3} - \cos 0\right] d\theta$$

$$= -\frac{8a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}-1\right) d\theta$$

$$= -\frac{8a^3}{3}\left(-\frac{1}{2}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{4a^3}{3}\left[2\pi - 0\right]$$

$$= \frac{8\pi a^3}{3}.$$