MODELING (DE)

Chapter 3.1 Linear Models

Text: Differential Equations with Boundary-Value Problems

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→ Growth and Decay
- → Population
- → Half life

→ Newton's Law of Cooling/Warming

MAT 120

Ch 3.1 LINEAR EQN of GROWTH & DECAY

Linear DE: $\frac{dy}{dx} + P(x)y = f(x)$ $\left.\begin{array}{l} x = t \\ y = P \end{array}\right\}$ week 7 recall

Linear DE of population function w.r.t. time

$$\frac{dP}{dt} + P(t)\,P = f(t)$$

population

function of t → time

$P(t_0) = P_0$ → initial population at initial time

The population of a community is known to increase/decrease at a rate proportional to the number of people present at time t.

$$\frac{dP}{dt} \propto P$$

$\Rightarrow \frac{dP}{dt} = kP$ → k is a constant of proportionality serves as a model for diverse phenomena involving either growth (↑) or decay (↓).

compare

$\Rightarrow \frac{dP}{dt} - kP = 0$

$\frac{dP}{dt} + P(t)\,P = f(t)$

$\left.\begin{array}{l} P(t) = -k \\ f(t) = 0 \end{array}\right\}$

$\Rightarrow e^{-kt}\left(\frac{dP}{dt} - kP\right) = e^{-kt}(0)$

(multiply by I.F.)

$$\begin{aligned} I.F. &= e^{\int P(t)dt} \\ &= e^{\int -k\,dt} \\ &= e^{-kt} \end{aligned}$$

$\Rightarrow \int \left[\underbrace{e^{-kt}\left(\frac{dP}{dt} - kP\right)}\right] dt = \int 0 \, dt$

Recall from week 7 $ye^{\int P(x)dx}$

$\Rightarrow Pe^{-kt} = c$ Refer to Ch 2.3 general sol. of Linear DE

$\therefore P = Ce^{kt}$

$\boxed{P(t) = Ce^{kt}}$ (a) → standard Linear eqn of growth & decay

$\left\{\begin{array}{l} \text{in pop. function} \\ y = P \\ x = t \end{array}\right\}$

Initially $t_0 = 0$, P_0 = initial Population

$P(t_0) = P(0) = Ce^{k(0)}$

$\Rightarrow P_0 = C$

Now Substitute $C = P_0$ into (a)

$\boxed{P(t) = P_0 e^{kt}}$

→ initial Population function.

Growth & Decay

Examples

$\frac{dP}{dt} \propto P$

1 The population of a community is known to increase $\frac{dP}{dt} = kP$ at a rate $\frac{dP}{dt}$ proportional to the number of people present at time "t". If the population has doubled in 5 years, how long will it take to triple? to $\underbrace{\text{quadruple}}_{\text{4 times larger}}$?

Eqn of growth:

$P(t) = Ce^{kt}$ —— (i)

initially $t = 0$ [∵ The countdown of time starts at '0']

∴ substitute into (i) we have

$P(0) = Ce^{k(0)} \Rightarrow P_0 = Ce^0 \Rightarrow C = P_0$ → initial population

Substitute C into (i)

$P(t) = P_0 e^{kt}$ —— (ii)

Given: Population doubled in 5 years → (given) (The population has doubled in 5 years)

$\underbrace{t = 5, \quad P(5) = 2P_0}_{\text{Substitute in (ii)}}$

$P(5) = P_0 e^{k5}$

$\Rightarrow 2\cancel{P_0} = \cancel{P_0} e^{5k}$

$\Rightarrow e^{5k} = 2$

$\Rightarrow \ln e^{5k} = \ln 2$

$\Rightarrow 5k = \ln 2$

$\Rightarrow \boxed{k = \frac{\ln 2}{5}}$

$\ln e^{5k} = 5k \ln e = 5k(1) = 5k$

$\boxed{e^{\ln x} = x}$

{How long will it take to triple?

$\Rightarrow t = ?$ when $P(t) = 3P_0$

substitute $P(t)$ into eqn (ii)

$P(t) = P_0 e^{kt}$ —— (ii)

$3\cancel{P_0} = \cancel{P_0} e^{\frac{\ln 2}{5} t}$ ∵ $k = \frac{\ln 2}{5}$

$e^{\frac{\ln 2}{5} t} = 3 \Rightarrow \ln e^{\frac{\ln 2}{5} t} = \ln 3$

$\Rightarrow \frac{\ln 2}{5} t = \ln 3$

$\Rightarrow t = \frac{5 \ln 3}{\ln 2}$

$= 7.92481$

≈ 8 years

$\frac{\ln 2}{5} t \ln e = \frac{\ln 2}{5} t$

{How long will it take to quadruple?

$\to t = ?$ when $\underbrace{P(t) = 4P_0}$ substitute into (ii)

$$P(t) = P_0 e^{kt} \quad \text{— (ii)}$$

$$4\cancel{P_0} = P_0 e^{kt} = \cancel{P_0} e^{\frac{\ln 2}{5} t}$$

$$e^{\frac{\ln 2}{5} t} = 4 \Rightarrow \ln e^{\frac{\ln 2}{5} t} = \ln 4$$

$$\Rightarrow \frac{\ln 2}{5} t = \ln 4 \Rightarrow t = \frac{5 \ln 4}{\ln 2} \approx 10 \text{ years}$$

$\boxed{2}$ Suppose it is known that the population of the community in Problem $\boxed{1}$ is 10,000 after 3 years. What was the initial population? What will be the population in 10 years?

$P(t)$ $t = 3$ After 3 years

Given $P(3) = 10{,}000$

Find $P_0 = ?$, $P(10) = ?$

Population eqn of Growth: $P(t) = P_0 e^{kt}$ — (1) from Problem $\boxed{1}$

$$\therefore P(3) = P_0 e^{k3} \qquad \because K = \frac{\ln 2}{5} \text{ from Problem 1}$$

$$\Rightarrow 10{,}000 = P_0 e^{3\left[\frac{\ln 2}{5}\right]}$$

$$= P_0 e^{\frac{3}{5} \ln 2}$$

$$= P_0 e^{\ln 2^{3/5}} \qquad e^{\ln x} = x$$

$$= P_0 2^{3/5}$$

$$P_0 = \frac{10{,}000}{2^{3/5}} = 6597.54 \approx 6597$$

Substitute $t = 10$ into (1):

$$P(10) = P_0 e^{k(10)}$$

$$= 6597.54 e^{\left[\frac{\ln 2}{5}\right](10)}$$

$$= 6597.54 e^{\frac{10}{5} \ln 2} = 6597.54 e^{\ln 2^2} = 6597.54 e^{\ln 4}$$

$$= 6597.54(4)$$

$$= 26390.2$$

$$\approx 26390$$

3 The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?

$\frac{dp}{dt} \propto P$ $\qquad 15\% = \frac{15}{100} = 0.15$

$\frac{dP}{dt} = KP$; $P_0 = 500$

$P(10) = 500 + \frac{15}{100} \cdot (500) = 575$

∴ Population is ↑

Find: $P(30) = ?$

skip

Solving $\frac{dP}{dt} - KP = 0$

we get $P(t) = Ce^{Kt}$ —(i)

initially $t=0$, $P(0) = P_0 \Rightarrow C = P_0$.

$P(t) = P_0 e^{kt}$

$P(30) = 500e^{\frac{1}{10}\ln\left(\frac{23}{20}\right)30}$

Substitute C into (i)

Standard form of initial function

$\boxed{P(t) = P_0 e^{Kt}}$

$P(t) = 500e^{Kt}$ ∵ $P_0 = 500$ —(ii)

∴ $P(10) = 500e^{K10}$

$575 = 500e^{10K}$

$e^{10K} = \frac{575}{500} = \frac{23}{20}$

$\ln e^{10K} = \ln\frac{23}{20}$

$10K = \ln\frac{23}{20}$

$$\boxed{K = \frac{1}{10}\ln\frac{23}{20}}$$

Again, in eqn (ii)

$P(t) = 500e^{Kt}$

if $t = 30$ then

$P(30) = 500e^{\left[\frac{1}{10}\ln\left(\frac{23}{20}\right)\right]30}$

$= 500e^{\frac{30}{10}\ln\frac{23}{20}}$

$= 500e^{3\ln\frac{23}{20}}$

$= 500e^{\ln\left(\frac{23}{20}\right)^3}$

$= 500\left(\frac{23}{20}\right)^3$

$= 760.438$

≈ 760

$\boxed{4}$ $\boxed{a}$ A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium $\rightarrow P_0$ has disintegrated.

$\boxed{b}$ Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining. Compare

$\boxed{a}$ Let $A(t)$ denote the amount of plutonium remaining at time t.

As The ~~linear~~ function of Growth/Decay, the solution of the initial value problem:

$\frac{dP}{dt} = kP$

$$\frac{dA}{dt} = kA \; ; \; A(0) = A_0 \rightarrow t=0 \text{ initial time}$$

is $A(t) = A_0 e^{kt}$ — (i)

$A(15) =$ 0.043% of the initial amount A_0 of plutonium has disintegrated

which is $(100 - 0.043)\% = 99.957\%$ of A_0 remains

$\therefore A(15) = 0.99957 A_0$

$\cancel{A_0} e^{K15} = 0.99957 \cancel{A_0}$

[(i) $\Rightarrow A(t) = A_0 e^{kt}$
$\Rightarrow A(15) = A_0 e^{K15}$]

$\therefore 0.99957 A_0 = A_0 e^{K15}$

$\ln e^{15K} = \ln 0.99957$

$15K = \ln 0.9957$

$K = \frac{1}{15} \ln 0.9957$

$= -0.00002867$

$\therefore A(t) = A_0 e^{kt}$

$\Rightarrow \underbrace{A(t) = A_0 e^{-0.00002867t}}$

Initial function of growth for Plutonium

$\boxed{b}$ Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2} A_0$

From (a) we have:

$$A(t) = A_0 e^{-0.00002867t}$$

$$\frac{1}{2} \cancel{A_0} = \cancel{A_0} e^{-0.00002867t} \qquad \because A(t) = \frac{1}{2} A_0$$

$$\therefore e^{-0.00002867t} = \frac{1}{2}$$

$$\frac{1}{e^{0.00002867t}} = \frac{1}{2}$$

$$e^{0.00002867t} = 2$$

$$\ln e^{0.00002867t} = \ln 2$$

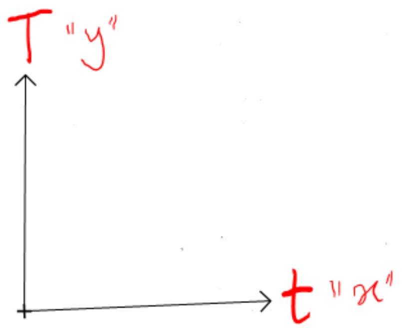
$$0.00002867t = \ln 2$$

$$t = \frac{\ln 2}{0.00002867} = 24.180 \approx 24 \text{ years}$$

Newton's Law of Cooling / Warming

It is given by

$$\frac{dT}{dt} = k\,(T - T_m)$$



k – constant

T(t) – temperature of the object with respect to time

time is non-negative

T_m – ambient temperature (It is the air temperature of any environment where computers and related equipment are kept)

OR (Room temperature)

Example:

When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to a room temperature of $\underbrace{70°F}_{T_m}$?

$T(0) = 300°F$

$T(3) = 200°F$ (3 minutes later)

$T_m = 70°F$

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T-70) \quad \to \quad dT = k(T-70)\,dt$$

$$\frac{dT}{T-70} = k\,dt$$

$$\int \frac{dT}{T-70} = \int k\,dt$$

$$\ln|T-70| = kt + c_1$$

$$\log_e|T-70| = kt + c_1$$

$$\{\, T - 70 = e^{kt+c_1} \qquad \left\{\begin{array}{l}\log_e x = y\\ x = e^y\end{array}\right.$$

$$T = 70 + e^{kt}e^{c_1}$$

$$T = 70 + e^{kt}C_2 \qquad \left[\text{Relabel constant } e^{c_1} = C_2\right]$$

$$T(t) = 70 + c_2e^{kt}$$

$$T(0) = 70 + C_2e^0$$

$$300 = 70 + C_2$$

$$\therefore C_2 = 230$$

$$\therefore \boxed{T(t) = 70 + 230e^{kt}} \; \text{— (i)}$$

Now substitute $t = 3$ ∵ Given $T(3) = 200°F$

$$T(3) = 70 + 230e^{k(3)}$$

$$200 = 70 + 230e^{3k}$$

$$230e^{3k} = 130$$

$$e^{3k} = \frac{130}{230} = \frac{13}{23}$$

$$\ln e^{3k} = \ln\frac{13}{23}$$

$$3k = \ln\frac{13}{23}$$

$$k = \frac{1}{3}\ln\frac{13}{23} = -0.19018$$

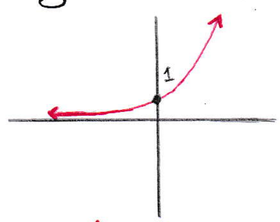
$t = ?$ when $T(t) = 70°$

$$T(t) = 70 + 230e^{-0.19018t}$$

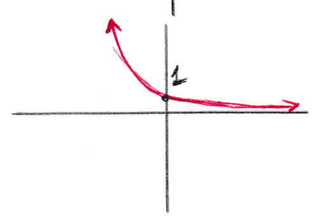
↳ There is no finite solution to $T(t) = 70$ ∵ $\lim_{t\to\infty} T(t) = 70$.

Yet infinitely we expect the cake to reach the room temperature which is 70°F after a reasonably long period of time ∴ Consider $T(t) = 71$ ∵ 71° is close to 70°. Note that the room temperature is always changing, hence it won't be always 70° (fixed).

$y = e^x$



$y = e^{-x}$



We have: from eqn (i), where $K = -0.19018$

$$T(t) = 70 + 230e^{-0.19018t}$$

$$71 = 70 + 230e^{-0.19018t}$$

$$1 = 230e^{-0.19018t}$$

$$e^{-0.19018t} = \frac{1}{230}$$

$$\ln e^{-0.19018t} = \ln\left(\frac{1}{230}\right)$$

$$-0.19018t = -5.438079$$

$$t = 28.59 \approx 29 \text{ minutes}$$

$t = 0$

$$T(0) = 70 + 230e^{-0.19018(0)}$$

$$= 70 + 230e^{0}$$

$$= 300$$