Treasure hunt with volatile pheromones

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Abstract

In the treasure hunt problem, a team of mobile agents need to locate a single treasure that is hidden in their environment. We consider the problem in the discrete setting of an oriented infinite rectangular grid, where agents are modeled as synchronous identical deterministic time-limited finite-state automata, originating at a rate of one agent per round from the origin. Agents perish τ rounds after their creation, where $\tau \geq 1$ is a parameter of the model. An algorithm solves the treasure hunt problem if every grid position at distance τ or less from the origin is visited by at least one agent. Agents may communicate only by leaving indistinguishable traces (pheromone) on the nodes of the grid, which can be sensed by agents in adjacent nodes and thus modify their behavior. The novelty of our approach is that, in contrast to existing literature that uses permanent pheromone markers, we assume that pheromone traces evaporate over μ rounds from the moment they were placed on a node, where $\mu \geq 1$ is another parameter of the model. We look for uniform algorithms that solve the problem without knowledge of the parameter values, and we investigate the implications of this very weak communication mechanism to the treasure hunt problem. We show that, if pheromone persists for at least two rounds ($\mu \geq 2$), then there exists a treasure hunt algorithm for all values of agent lifetime. We also develop a more sophisticated algorithm that works for all values of μ, hence also for the fastest possible pheromone evaporation of $\mu = 1$, but only if agent lifetime is at least 16.

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1 Introduction

Treasure hunt is the fundamental problem of employing a team of searchers to locate a "treasure" that is hidden somewhere in their environment. It is one of the fundamental primitives in swarm robotics and a natural abstraction of foraging behavior of animals. Although various formulations of the problem exist at least since the 1960s, when Beck introduced the linear search problem [14], treasure hunt as a group search problem was first investigated from a distributed algorithms perspective by Feinerman et al. [41, 42, 43], under the name ANTS (Ants Nearby Treasure Search). In the ANTS problem, the search is performed by a team of randomized searchers, starting at the origin of an infinite 2-dimensional rectangular grid and having no means of communication once they start moving. Subsequent works considered stronger communication models, such as local communication by exchanging constant-size messages when two agents are located on the same node [40, 39, 25, 21, 54, 53], or communication by leaving permanent markers on grid nodes [56, 1, 2], that can be detected by other agents.

In this paper, we introduce a new model in which not only agents communicate indirectly, by dropping and sensing markers on nodes, but also these markers gradually evaporate and eventually disappear. This is directly inspired by the behavior of actual pheromone trails in nature. A common feature of the papers that we mentioned above is that the team of searchers is of constant size. However, with evaporating pheromones, we can no longer expect a constant-size team of constant-memory agents to explore all the grid nodes up to arbitrary distances.[1] Therefore, we propose a new model taking into account pheromone evaporation, in which a potentially infinite number of identical, synchronous, deterministic, time-limited finite-state automata are created at a rate of one agent per round at the origin of a 2-dimensional grid. Agents have a finite lifetime represented by the parameter τ, and the treasure is guaranteed to be within reach, i.e., at distance $\leq \tau$ from the origin. Pheromone evaporation is controlled by a parameter μ, which determines the number of rounds it takes for a pheromone marker to disappear from the system, assuming it is not refreshed in the meantime by a new pheromone drop on the same node. Agents can sense the presence or absence of pheromone in their neighboring nodes, and they can compare pheromone values, i.e., they know, for any pair of directions, which neighbor has the freshest pheromone. Agent memory cannot depend on the parameters τ, μ.

1.1 Related work

Searching is a well-studied family of problems in which a group consisting of one or multiple searchers (mobile agents) need to find a target placed at some unknown location. The search is typically concluded when the first searcher finds the target. Numerous books and research papers have been written on this subject, studying diverse models involving stationary or mobile targets, graphs or geometric terrains, different types of knowledge about the environment, one or many searchers, etc. [5, 6, 17, 23, 45, 47, 58].

Deterministic search on a line with a single robot was introduced in [14, 15]. In the original formulation of [14], a probability distribution of treasure placements is known to the agent. An optimal algorithm with competitive ratio 9, for an unknown probability distribution, is proposed in [15]. The problem is further generalized in [8, 35], by introducing a cost for turning, as well as a more general star topology. Further variants include searching for multiple targets [9], maximizing the searched area with a given time budget [10], and providing a hint to the searcher before it starts exploring [7].

Various maintenance and patrolling problems have also been formulated as linear group search problems, under requirements and assumptions such as perpetual exploration [27, 50, 26] or distinct searcher speeds [29, 50, 13]. The closely related evacuation problem on the line, in which the search is concluded when *all* searchers reach the target, has also been studied in a series of papers [11, 22, 12, 20]. See also [30] for a survey of group search and

[1] Indeed, intuitively, if they find themselves sufficiently far from each other, then they can no longer communicate because pheromone will evaporate before it can be sensed by another agent, whereas if they never find themselves more than a constant distance from each other, then their overall behavior can be described by a single finite automaton, which fails to explore a sufficiently large grid due to state repetition that forces it to explore at most a constant-width half-line (see, e.g., [39, Lemma 5]).

evacuation in different domains.

Searching with advice (hints) is studied under various assumptions in several papers. The size of advice that must be provided to a lone deterministic searcher in a polygonal terrain with polygonal obstacles, in order to locate the treasure at a cost linear in the length of a shortest path from the initial position of the agent to the treasure, is investigated in [59]. An algorithm that enables a deterministic agent to find an inert treasure in the Euclidean plane, taking advantage of hints about the general direction of the treasure, is given in [18]. In [51, 57], they explore the tradeoff between advice size and search cost in graphs. In trees, [19] explores the impact of different kinds of initial knowledge given to a lone searcher on the time cost of treasure hunt, and [16] considers treasure hunt with faulty hints.

The speedup in search time obtained by multiple independent random walkers has been studied for various graph families, such as expanders and random graphs [37, 4, 38, 48, 28]. Multiple searchers following Lévy walk processes, a type of random walk in which jump lengths are drawn from a power-law distribution, and for which there is significant empirical evidence that it models the movement patterns of various animal species [31], are investigated in [24]. A further abstraction of multiple independent randomized searchers is studied in [46], where a group of non-communicating agents need to find an adversarially placed treasure, hidden in one of an infinite set of boxes indexed by the natural numbers. In this Bayesian search setting, searchers have random access to the boxes. A game-theoretic perspective to the Bayesian search framework of [46] is given in [52].

The ANTS (Ants Nearby Treasure Search) problem was introduced in [41, 42, 43] as a natural abstraction of foraging behavior of ants around their nest. They explore the tradeoff between searcher memory and the speedup obtained by using multiple probabilistic searchers vs using a single searcher. Searchers may not communicate once they leave the nest. A variant of the ANTS problem in the geometric plane, with searchers that are susceptible to crash faults, is investigated in [3]. A notion of selection complexity, which measures how likely a given ANTS algorithm is to arise in nature, is introduced in [55], where they study the tradeoff between selection complexity and speedup in search time.

In follow-up work [40, 39, 25, 21, 54, 53] to the original ANTS formulation, searchers are modeled as finite state machines and can communicate outside the nest, when they are sufficiently close to each other, by exchanging messages of constant size. Under these assumptions, it is shown in [40] that the optimal search time can still be achieved by probabilistic finite state machines, matching the lower bound of [42]. The minimum number of searchers that can solve the ANTS problem, when they are controlled by randomized/deterministic finite/push-down automata, is investigated in [39, 25, 21]. A probabilistic fault-tolerant constant-memory algorithm is presented in [54], for the synchronous case. An algorithm that tolerates obstacles is presented in [53].

A different communication mechanism is considered in [56, 1, 2], where it is assumed that agents may communicate only by leaving permanent markers (pheromones) on nodes, which can be sensed later by other agents. Note that, although these papers use the word "pheromone" to describe the traces that agents leave on nodes, these are assumed permanent and do not evaporate. The usual term in the mobile agent literature to describe this type of movable or immovable marker that agents may choose to leave on nodes, and which can be detected later by other agents, is "token" or "pebble" [34].

1.2 Our contributions

We study the treasure hunt problem in the model that we outlined above, and which is developed in detail in Section 2. Thematically, our work is closest to the literature descending from the original ANTS problem formulation, and in particular to these papers that use pheromone (or tokens) as a means of communication [56, 1, 2]. The novelty of our approach is that we use *evaporating* pheromones as an agent communication mechanism. Indeed, in our model, a pheromone trace disappears μ rounds after it was dropped, unless it is refreshed by a new pheromone drop on the same node. Tokens that may disappear instantly from the system have actually been considered before in the literature, but only in the context of faults [44, 33, 32, 36].

To our knowledge, evaporating pheromone markers have never been considered before as a communication mechanism, from an algorithmic point of view. We study the impact of this weak agent communication model on the treasure hunt problem.

Our first result is a treasure hunt algorithm that works for all $\tau \geq 1$, assuming that the pheromone markers persist for at least two rounds ($\mu \geq 2$). This algorithm is optimal in terms of search time, number of pheromone drops, and number of agents used. Intuitively, the algorithm dispatches agents to the North and to the South of the origin by means of pheromone patterns around the nest. An agent knows when to leave the vertical axis in order to explore a horizontal half-line by detecting pheromone markers that were dropped by previous agents when they left the vertical axis. Because of the North-South dispatching at the origin, successive agents on the same side of the origin are at distance 2 from each other, therefore it is crucial that $\mu \geq 2$ for an agent to be able to detect pheromone that was dropped by the previous agent.

Our second result is a more complex algorithm that works for all $\mu \geq 1$. This algorithm is also based on a North-South dispatching of agents at the origin. The challenge here is that, since pheromone may be detectable for only one round after being dropped, we can no longer use the same approach as in the first algorithm. We resolve this by introducing *differentiation* of agent roles as a result of observing different pheromone patterns as they walk along the vertical axis. Now, some agents become *signaling* agents that stop moving at key positions and start dropping pheromone according to a predetermined pattern, whereas other agents become *explorers* that are dispatched to different horizontal half-lines according to these signals. This algorithm works for all $\tau \geq 16$.

Both algorithms are deterministic and uniform, i.e., they do not assume knowledge of the values of the parameters τ, μ. They solve the problem for *all* parameter values in the specified ranges, and the required memory per agent is constant.

2 Model and problem setting

The environment in which the agents operate is an infinite two-dimensional rectangular grid graph, equipped with a Cartesian coordinate system. Each node of the grid is identified by a pair of integer coordinates $(x, y) \in \mathbb{Z}^2$. The node $(0, 0)$ is called *nest*, as newly created agents appear at $(0, 0)$. We assume that the grid is oriented, with the four outgoing edges from each node receiving globally consistent distinct local port labels from $\{N, E, S, W\}$. Each node stores a nonnegative integer that represents the amount of pheromone present on that node. This value is decremented by 1 at each round and a value of zero represents the absence of pheromone.

Given natural numbers a, b, we use the notation $a \dot{-} b$ for proper subtraction: $a \dot{-} b = \max(a - b, 0)$. Moreover, if x is a nonnegative integer, we use $\mathbb{B}_x$ for the set of nodes at distance at most x from the nest, and $\mathbb{L}_x$ for the set of nodes at distance exactly x from the nest. We have $|\mathbb{B}_x| = 2x^2 + 2x + 1$ and $|\mathbb{L}_x| = 4x$.

2.1 Agent model

Agents are modeled as identical copies of a deterministic finite-state machine (FSM). An agent can move from node to node along the edges of the grid graph, and it may decide to drop pheromone before or after each move (but not both on the origin and on the destination node). It computes its next move based on the relative pheromone values of the neighboring nodes. More precisely, the agent does not have access to the actual stored pheromone values, but it can detect the presence or absence of pheromone in any direction, as well as whether one direction has equal, less, or more pheromone than another.

▶ **Definition 1** (Agents). *An* agent *is a finite-state machine* $\mathcal{A} = (Q, q_0, \mathsf{In}, \mathsf{Out}, \delta)$ *where:*

- Q *is a finite set of states and* $q_0 \in Q$ *is the initial state.*
- In *is the input alphabet. A symbol of* In *encodes the presence or absence of pheromone in the four cardinal directions, as well as the result of the comparison of pheromone levels for any pair of directions. This is clearly a finite amount of information, hence* In *is a finite set.*
- $\mathsf{Out} = \{N, S, E, W, \perp\} \times \{\text{before}, \text{after}, \perp\}$ *is the output alphabet, where the first element of an output symbol is the local port label through which the agent will exit the current node (*$\perp$ *for no movement), and the second element indicates whether pheromone will be dropped before or after the move (*$\perp$ *for no pheromone drop).*
- $\delta : Q \times \mathsf{In} \to Q \times \mathsf{Out}$ *is the transition function.*

▶ Note 2. By definition, an agent does not perceive other agents that may be present on the same node or on neighboring nodes. Moreover, an agent does not perceive and therefore cannot compare the pheromone level of its current node to those of neighboring nodes.

2.2 Model parameters

Agents have limited life, which is a parameter of the model and is represented by a positive integer τ. An agent "dies" upon having performed τ state transitions, meaning that it essentially disappears from the system.[2] We will call this parameter *lifetime*.

We also assume that every node has a maximum amount of pheromone that it can store, which is a second parameter of the model and is represented by a positive integer μ, which we will call *pheromone duration*. Whenever any number of agents decide to drop pheromone on a node at the same time, the pheromone value of that node is updated to μ. If an agent drops pheromone on some node, the pheromone value of that node will decrease from μ to 0 over the following μ rounds (assuming it is not refreshed in the meantime).

2.3 Execution

Given a protocol $\mathcal{A}$ (FSM) and an assignment of values to the parameters (τ, μ), the execution of the system proceeds deterministically in synchronous rounds.

▶ **Definition 3** (Execution). *The* execution of an FSM $\mathcal{A}$ for parameter values (τ, μ) *is an infinite sequence of system configurations defined as follows: In the initial configuration, there are no agents and no pheromone present on the grid. In each round i ($i \geq 1$), the*

[2] Perhaps less fatally, we may assume that after τ transitions, an agent is so tired that it cannot continue executing the protocol before returning to the nest for a brief nap. It may re-emerge from the nest at a later round without retaining its state.

next configuration is obtained from the current configuration by synchronously executing the following steps in the given order:

1. *A new agent (copy of $\mathcal{A}$) is created on node* $(0,0)$*, in the initial state* q_0*.*
2. *All agents read their inputs.*
3. *At each node, the quantity of pheromone is decreased by* 1*, if not already zero (pheromone evaporation).*
4. *All agents compute their transition function based on the input from step 2 and change their state accordingly.*
5. *All agents that computed in step 4 a pheromone drop action "*before*" drop pheromone on their current nodes. For each node on which at least one agent drops pheromone, the pheromone quantity of that node is updated to* μ*.*
6. *All agent moves (as computed in step 4) are executed.*
7. *All agents that computed in step 4 a pheromone drop action "*after*" drop pheromone on their current nodes. For each node on which at least one agent drops pheromone, the pheromone quantity of that node is updated to* μ*.*
8. *If this is round* $i \geq \tau$*, the agent that was created at the beginning of round* $i - \tau + 1$ *"dies" as it has now performed* τ *state transitions.*

▶ Note 4. As agents are anonymous and deterministic, and because pheromone does not accumulate higher than μ on a single node, if two (or more) agents ever find themselves at the same node and in the same state, then they will effectively continue moving as one agent from that point on. In particular, agents do not appear simultaneously at the nest, but they are created at a rate of one agent per round.

▶ **Definition 5.** *For* $i \geq 1$*, we denote by* A_i *the agent that is created at the beginning of round* i*.*

2.4 The treasure hunt problem

In the *treasure hunt* problem, a treasure is placed at an unknown location in the grid and the goal is for at least one agent to visit that node. In that case, we say that the agent *locates* the treasure. Locating the treasure for any (unknown) treasure location up to distance d from the nest is trivially equivalent to exploring all nodes up to distance d from the nest. We recast, then, the treasure hunt problem as an exploration problem:

▶ **Definition 6** (Treasure hunt problem)**.** *A given FSM* $\mathcal{A}$ solves the treasure hunt problem for the pair of parameters (τ, μ) *if, with lifetime* τ *and pheromone duration* μ*, every node at distance* τ *or less is visited by at least one agent. In this case, we will also say that the FSM is* correct for (τ, μ).

We will seek a uniform algorithm that solves the problem without knowledge of the model parameters, i.e., a single FSM that is correct for arbitrarily large values of τ and μ (ideally, for all $\tau \geq \tau_0$ and $\mu \geq \mu_0$, for some constants τ_0, μ_0).

For a given FSM, we will consider the following measures of efficiency as functions of τ and μ:

- *Completion time*: the number of rounds until the treasure is located.
- *Pheromone utilization*: the total number of times that any agent decides to drop pheromone at its destination node until the treasure is located.
- *Agent utilization*: the number of agents *effectively* used by the algorithm, i.e., the smallest r such that the algorithm remains correct even if the system stops creating new agents after round r.

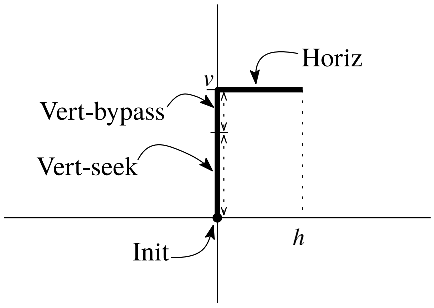

Figure 1 Illustration of the sequence of states for a typical agent with signature $[v; h]$ executing Algorithm 1.

3 A treasure hunt algorithm for $\tau \geq 1$ and $\mu \geq 2$

We propose a deterministic and uniform algorithm that solves the treasure hunt problem for all combinations of parameters (τ, μ) with $\tau \geq 1$ and $\mu \geq 2$. We give a compact representation of the algorithm as a hybrid state transition diagram in Appendix A, and the full pseudocode in Section 3.1.

Before we give an informal description of the algorithm, we define the notion of *agent signature*:

▶ **Definition 7** (Agent signature). *Let $v, h \in \mathbb{Z}$ with $|v| + |h| = \tau$. We say that an agent has* signature $[v; h]$ *if it starts (from the nest) by moving $|v|$ steps to the North (resp. South), up to node $(0, v)$, if $v \geq 0$ (resp. $v < 0$), followed by $|h|$ steps to the East (resp. West), up to node (h, v), if $h \geq 0$ (resp. $h < 0$).*

The algorithm creates agents of all possible signatures $[v; h]$, thus ensuring correctness by visiting all nodes at distance $\leq \tau$ from the nest. Each agent drops pheromone once upon leaving the nest on its first move, and once more if and when it leaves the vertical axis. Figure 1 shows the sequence of states of a typical agent executing the algorithm.

The first two agents use pheromone information to the East and to the West of the nest to take signatures $[0, -\tau]$ and $[0, \tau]$ (state Init, lines 2-5). Subsequently created agents use pheromone information to the North and to the South of the nest to alternate between the two vertical directions: If there is more pheromone to the North of the nest then they start moving South, otherwise they start moving North (state Init, lines 7-9).

A northbound agent (southbound agents behave symmetrically) starts moving to the North in state Vert-seek. In this state, it checks horizontally adjacent nodes for the presence of pheromone previously dropped by agents leaving the vertical axis. Once it finds such pheromone traces, it switches to state Vert-bypass and keeps moving to the North until it reaches the first node $(0, v)$ whose East and West neighbors do not *both* have pheromone.

At that point, if no horizontal neighbor has pheromone then it turns East, taking signature $[v, \tau - v]$, whereas if only the East neighbor has pheromone then it turns West, taking signature $[v, v - \tau]$ (state Vert-bypass). Once it leaves the vertical axis, an agent keeps moving horizontally until the end of its lifetime in state Horiz.

3.1 Pseudocode

We give the transition function executed by each agent during step 4 of each round (cf. Definition 3) in Algorithm 1. We denote by φ_x, for $x \in \{N, E, S, W\}$, the pheromone value of the neighboring node in direction x. These represent the input to the FSM. In accordance with Definition 1, the pheromone values are never used directly but only as part of comparisons to each other and to the value 0. The output of the FSM is composed of the pair of values $(\mathsf{dir}, \mathsf{drop})$ at the end of the transition function computation.

3.2 Correctness

▶ **Theorem 8.** *Algorithm 1 correctly solves the treasure hunt problem for all combinations of parameters (τ, μ) with $\tau \geq 1$ and $\mu \geq 2$.*

The complete proof of Theorem 8 is available in Appendix B. The proof is based on the following simple properties of Algorithm 1: (Propositions 19, 20, 24, and 28 in Appendix B)

- Whenever an agent switches to state Horiz it moves horizontally (East or West) and drops pheromone. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until the end of its lifetime.
- Whenever an agent switches to state Vert-seek it moves vertically (North or South) and drops pheromone. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until one of the following happens: it reaches the end of its lifetime, or it switches to state Vert-bypass moving in the same direction as before, or it switches to state Horiz moving West.
- Whenever an agent switches to state Vert-bypass it moves vertically, and it drops pheromone only if it switches from state Init to Vert-bypass. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until one of the following happens: it reaches the end of its lifetime, or it switches to state Horiz.
- During its first transition, every agent switches to one of the states Horiz, Vert-seek, or Vert-bypass.

Based on these, we conclude that every agent has a signature as per Definition 7. The rest of the proof is devoted to showing that the first 4τ agents pick up distinct signatures, and thus they explore all nodes at distance τ or less from the nest. This is accomplished by a series of lemmas, where we show first (as part of Lemma 33 in Appendix B) that agents A_1 and A_2 have signatures $[0; \tau]$ and $[0; -\tau]$, respectively, and that subsequent agents are alternately dispatched to the North and to the South half-planes. Then, the following two technical lemmas describe completely the state transitions of agents on the vertical axis:

▶ **Lemma 9.** *For all odd i with $3 \leq i \leq 4\tau - 1$, and for all y with $1 \leq y \leq \lceil \frac{i-1}{4} \rceil$, A_i is at node $(0, y)$ at the beginning of round $i + y$ and:*

1. *It senses pheromone $\mu \dot{-} (i - 4y)$ to the East and $\mu \dot{-} (i - 2 - 4y)$ to the West.*
2. *If $y = 1$, it is in state* Vert-seek *if $i - 3 \geq \mu$, otherwise it is in state* Vert-bypass.
3. *If $y \geq 2$, it is in state* Vert-seek *if $i - 4y + 2 \geq \mu$, otherwise it is in state* Vert-bypass.

▶ **Lemma 10.** *For all even i with $4 \leq i \leq 4\tau$, and for all y with $1 \leq y \leq \lceil \frac{i-2}{4} \rceil$, A_i is at node $(0, -y)$ at the beginning of round $i + y$ and:*

1. *It senses pheromone $\mu \dot{-} (i - 4y - 1)$ to the East and $\mu \dot{-} (i - 3 - 4y)$ to the West.*
2. *If $y = 1$, it is in state* Vert-seek *if $i - 4 \geq \mu$, otherwise it is in state* Vert-bypass.
3. *If $y \geq 2$, it is in state* Vert-seek *if $i - 4y + 1 \geq \mu$, otherwise it is in state* Vert-bypass.

From Lemmas 9 and 10, we deduce the signatures of the first 4τ agents and conclude the proof of Theorem 8.

Algorithm 1 A treasure hunt algorithm for $\tau \geq 1$, $\mu \geq 2$.

```
Variables
state ∈ {Init, Vert-seek, Vert-bypass, Horiz}          ▷ Initial value: Init
dir ∈ {N, E, S, W, ⊥}                                  ▷ Initial value: ⊥
drop ∈ {before, after, ⊥}                              ▷ Initial value: ⊥

Transition function
if state = Init then
    if φ_N = φ_W = φ_S = φ_E = 0 then
        state ← Horiz; dir ← E; drop ← after
    else if φ_E > φ_W then
        state ← Horiz; dir ← W; drop ← after
    else
        state ← Vert-bypass if φ_W > 0 else Vert-seek
        dir ← S if φ_N > φ_S else N
        drop ← after
    end if
else if state = Vert-seek then
    if φ_W = φ_E = 0 then
        state ← Vert-seek; drop ← ⊥                    ▷ keep searching
    else
        Interpret-Signals
    end if
else if state = Vert-bypass then
    if φ_W = φ_E = 0 then
        state ← Horiz; dir ← E; drop ← after
    else
        Interpret-Signals
    end if
else if state = Horiz then
    drop ← ⊥
end if

procedure Interpret-Signals
    if φ_W = 0 and φ_E > 0 then
        state ← Horiz; dir ← W; drop ← after
    else if φ_W > 0 then
        state ← Vert-bypass; drop ← ⊥
    end if
end procedure
```

3.3 Complexity

Recall the definitions of $\mathbb{B}_x$ (ball of radius x around the nest) and $\mathbb{L}_x$ (layer of nodes at distance x from the nest) from Section 2.

▶ **Theorem 11.** *If the treasure is located at distance at most d, where $1 \leq d \leq \tau$, then Algorithm 1 locates the treasure in time at most $5d - 1$.*

Proof. By Lemmas 33 and 41, agents $A_1, \dots, A_{4d}$ have all possible signatures with vertical component at most d (in absolute value). Moreover, by Corollary 30, each agent A_i reaches distance d from the nest in round $i + d - 1$. It follows that, by the time agent A_{4d} reaches distance d from the nest, hence by round $5d - 1$, all nodes at distance d or less from the nest have been explored. ◀

▶ **Theorem 12.** *If the treasure is located at distance $d = \tau$, then any treasure hunt algorithm needs at least $5\tau - 1$ rounds to locate the treasure in the worst case.*

Proof. A given agent can explore at most one node at distance τ within its lifetime. Since $\mathbb{L}_\tau$ contains 4τ nodes, a correct algorithm must create at least 4τ agents, the last of which reaches distance τ in round $4\tau + \tau - 1 = 5\tau - 1$. It follows that, in the worst case, the treasure cannot be located before round $5\tau - 1$. ◀

▶ **Theorem 13.** *Let $\mathcal{A}$ be any treasure hunt algorithm that is correct for a pair of parameters (τ, μ). For every $d \leq \tau$, $\mathcal{A}$ needs at least $\sqrt{5}d$ rounds to explore all nodes up to distance d.*

Proof. Fix a $d \leq \tau$ and let T be the first round at the end of which $\mathcal{A}$ explores all nodes up to distance d. Clearly, $T \geq d$ because otherwise no agent can reach any node at distance d. We also assume that $T < \sqrt{5}d$, and we will show a contradiction.

Consider $\mathbb{B}_x$, for $x \leq d$ to be determined below. Among the agents $A_1, \dots, A_T$, those with $i \geq T - x + 1$ have moved at most x times by the end of round T, therefore they are unable to explore any node outside of $\mathbb{B}_x$. For every $i \leq T - x$, agent A_i moves at most $T - i + 1$ times by the end of round T, and it needs at least x moves before it can exit $\mathbb{B}_x$. Therefore, A_i explores at most $T - i + 1 - x$ nodes outside of $\mathbb{B}_x$. Summing over all agents with $i \leq T - x$ and taking also into account $\mathbb{B}_x$ itself, we conclude that, by the end of round T, algorithm $\mathcal{A}$ can explore at most

$$|\mathbb{B}_x| + \sum_{i=1}^{T-x} \big(T - i + 1 - x\big) = 2x^2 + 2x + 1 + \frac{(T-x)(T-x+1)}{2}$$

nodes. The above expression is minimized for $x = \frac{T}{5} - \frac{3}{10} < d$, whence we obtain that $\mathcal{A}$ explores at most

$$\frac{2T^2}{5} + \frac{4T}{5} + \frac{31}{40}$$

nodes. By definition of T, at that round $\mathcal{A}$ has explored at least $\mathbb{B}_d$, therefore:

$$\frac{2T^2}{5} + \frac{4T}{5} + \frac{31}{40} \geq 2d^2 + 2d + 1$$

whence it follows that $T > \sqrt{5}d$, a contradiction. ◀

▶ **Theorem 14.** *Algorithm 1 effectively uses 4τ agents, and that is optimal.*

Proof. By Lemmas 33 and 41, agents $A_1, \dots, A_{4\tau}$ have all possible signatures with vertical component at most τ (in absolute value). Moreover, by Lemmas 33, 34, and 40, every agent A_i only senses pheromone left by some earlier agent A_j, $j < i$. It follows that, even if no agents are generated after round 4τ, the above agents $A_1, \dots, A_{4\tau}$ will still perform the same trajectories and explore all nodes up to distance τ. Therefore, the effective agent utilization of Algorithm 1 is 4τ. This is optimal because there exist 4τ nodes at distance τ, and an agent can only visit at most one node at distance τ during its lifetime. ◀

▶ **Theorem 15.** *The pheromone utilization of Algorithm 1 is at most $O(d)$, and this is asymptotically optimal.*

Proof. By Theorem 11, the treasure is located in time $O(d)$, and each of the $O(d)$ agents that are created until then drops pheromone at most 2 times: once when it leaves the nest, and once if and when it leaves the vertical axis. Hence, the pheromone utilization of Algorithm 1 is $O(d)$.

To prove optimality, consider a treasure hunt algorithm $\mathcal{A}$ that uses asymptotically less than d pheromone, i.e., its pheromone utilization is bounded by some function $f(d)$ such that $\lim_{d\to\infty} \frac{f(d)}{d} = 0$. Let N be the number of states of the FSM $\mathcal{A}$.

By our assumption on $f(d)$, for every $\varepsilon > 0$ there exists a d_ε such that for all $d > d_\varepsilon$, $f(d) < \varepsilon \cdot d$. Let us fix, then, a $d_0 > N + 1$ such that $f(d_0) < \frac{d_0}{N+1}$. Moreover, it is well known and has been observed several times in the literature (see, e.g., [39, Lemma 5]) that a deterministic FSM that moves in a grid and does not interact with its environment can explore at most a constant-width band, infinite in one direction. Let W be the constant that bounds the number of nodes of any particular layer that are visited by such an agent.

Now, consider the execution of $\mathcal{A}$ in a system with parameters (τ, μ), where $\tau \geq WNd_0+1$ and the treasure is located at distance d_0. The number of layers on which at least one agent drops pheromone is clearly bounded by the pheromone utilization of $\mathcal{A}$, and hence by $f(d_0) < \frac{d_0}{N+1}$. It follows that there exists at least one layer $d_1 \leq d_0 - (N + 1)$, such that no agent drops pheromone on any of the layers $d_1, d_1 + 1, \dots, d_1 + N$. Therefore, any agent that arrives at layer d_0 is already repeating a sequence of states during which it drops no pheromone.

Consider, now, the execution of $\mathcal{A}$ in the same system but with the treasure placed at distance $d^\star = WNd_0 + 1$. As agents do not perceive the presence of treasure, they will behave as in the previous case. In particular, even though there is an infinite number of agents coming out of layer d_0, their trajectories are contained in at most most $4d_0 \cdot N$ distinct bands of constant width W, infinite in one direction. This is because the trajectory of an agent that is coming out of d_0 is completely determined by the node from which it exits layer d_0 and the state in which it leaves the layer.

It follows that algorithm $\mathcal{A}$ explores at most $4WNd_0$ nodes of layer $d^\star$, but layer $d^\star$ contains $4d^\star > 4WNd_0$ nodes. Therefore, the adversary can place the treasure at a node that will not be explored by $\mathcal{A}$. ◀

4 A treasure hunt algorithm for $\mu \geq 1$ and $\tau \geq 16$

Similarly to Algorithm 1, the algorithm that we present in this section creates agents of all possible signatures $[v; h]$, as per Definition 7.

The main difficulty here is that the dropped pheromone can evaporate in one round only, in the case of $\mu = 1$. To explore a grid up to an unknown distance τ, where τ is also the lifetime of an agent, every node at distance τ has to be visited by at least one agent

(Definition 6). This agent cannot stop even for one round and has to follow a shortest path to the node at distance τ. At the same time, agents have to be sent alternatively exploring each half of the grid (north and south, in our case), and so the shortest time interval between two following agents (moving to the positions to explore) is two rounds. This makes it difficult to solve the problem with pheromone evaporating in one round. It disappears too fast to provide any information to the next arriving agent.

In order to overcome this challenge, we use in Algorithm 2 two types of agents: *signaling* agents, that stop moving at key positions and start dropping pheromone according to some predetermined pattern, and *explorer* agents, that read these patterns on their way to the extreme grid positions without stopping even for a single round. Since signaling consumes rounds from the lifetime of signaling agents, these agents must stop at a sufficient distance away from the extreme positions, to still have enough lifetime to signal the required pattern. This distance is expressed by the parameter s of our algorithm. This also has an impact on the minimum agent lifetime that is required for the algorithm to operate correctly, as the furthest signaling agents must have enough lifetime to reach their signaling positions and complete the required pattern. Algorithm 2 works for all values of μ, but only for $\tau \geq 16$.

Binary word notations: Let us define some finite binary word notations that we will use in order to present the algorithm. The empty word is denoted by ϵ and the length of a word w by $|w|$. For any word w and integer $j \in \{1, \ldots, |w|\}$, $w[j]$ denotes the j^{th} most significant bit of w. Let $\mathsf{shiftleft}(w)$ return a word obtained by removing the most significant bit ($w[1]$) from w.

We now present Algorithm 2 by refering to the pseudocode that we give in this section, and to the technical lemmas that are proved in Appendix C. Algorithm 2 uses a constant number of special states, as follows: Given a binary word w of length at most 9, $\mathsf{Pattern}(w)$ is the first of a sequence of $|w|$ states, during which the agent stays on the same node and drops (or not) pheromone according to the bit pattern w. $\mathsf{Forward}(s)$-$\mathsf{Explore}(E)$ (resp. $\mathsf{Forward}(s)$-$\mathsf{Explore}(W)$) is the first of a sequence of states during which the agent moves s steps forward (north or south, in the same direction as it was moving before entering this state) and then turns east (resp. west) and keeps moving in that direction until the end of its lifetime.

In the main part of the algorithm, agents leave the nest alternatively moving either north or south, on the vertical axis, until arriving s steps away from a non-explored yet line where they either stop for signaling (moving to state $\mathsf{Pattern}(w)$) or continue moving to reach this non-explored yet line to turn there either east (state $\mathsf{Forward}(s)$-$\mathsf{Explore}(E)$) or west (state $\mathsf{Forward}(s)$-$\mathsf{Explore}(W)$) for exploring each half of the line, s steps away. This is proven in Lemma 54.

Such a signaling, for exploring each next line at distance h, is achieved by using three agents. One is placed s lines before, and at one cell east from the vertical axis, i.e. at $(1, h-s)$ if heading north (resp. $(1, -(h-s))$, if heading south). The second one is also s lines before, but at one cell west from the vertical axis, i.e. at $(-1, h-s)$ (resp. $(-1, -(h-s))$). The third agent, is at $(0, h-s+1)$ (resp. $(0, -(h-s+1))$) . A newly arrived agent (at $(0, h-s)$ (resp. $(0, -(h-s))$)) senses the pheromone dropped by these three agents and performs actions according to the parsing of the sensed pattern. This part of the algorithm is controlled mainly by the INTERPRET-SIGNALS-PHASE2() procedure (see Alg. 2).

Operating in this way, with agents "jumping" each time s steps vertically, for exploring a line there, leaves at least the s first horizontal lines of each half of the grid unexplored. Hence, we need a special procedure for exploring these lines. For that, up to horizontal lines at

Algorithm 2 A treasure hunt algorithm for $\tau \geq 16, \mu \geq 1$ and $s = 6$

Variables

state $\in$ {Init, Vert-seek, Vert-bypass, Horiz, Pattern(w), Forward(k)-Explore(E) Forward(k)-Explore(W)}, $w \in \{0,1\}^9, k \in [0, s]$ ▷ Initial value: Init

dir $\in \{N, E, S, W, \perp\}$ ▷ Initial value: $\perp$

drop $\in$ {before, after, $\perp$} ▷ Initial value: $\perp$

moves $\in [0, s+1]$ ▷ Initial value: 0

Transition function

```
if state = Init then
    if φ_N = φ_W = φ_S = φ_E = 0 then
        state ← Pattern(11001); dir ← E; drop ← after            ▷ Start signaling E
    else if φ_N = φ_W = φ_S = 0 and φ_E > 0 then
        state ← Pattern(111); dir ← W; drop ← after              ▷ Start signaling W
    else if φ_N = φ_S = 0 and φ_W = φ_E > 0 then
        state ← Pattern(01); dir ← N; drop ← after               ▷ Start signaling N
    else if φ_S = 0 and φ_N = φ_W = φ_E > 0 then
        state ← Horiz; dir ← E; drop ← ⊥                         ▷ Explore E
    else if φ_S = 0 and φ_N = φ_E < φ_W then
        state ← Horiz; dir ← W; drop ← ⊥                         ▷ Explore W
    else if φ_S = 0 and φ_N = φ_W > φ_E then
        state ← Vert-bypass; dir ← N; drop ← after               ▷ Go signaling E on line (0, 1)
    else if φ_S = 0 and φ_N = φ_E > φ_W then
        state ← Vert-bypass; dir ← S; drop ← after               ▷ Go signaling E on line (0, −1)
    else if φ_S > φ_N and φ_S > φ_E and φ_S > φ_W then
        state ← Vert-seek; dir ← N; drop ← after                 ▷ Go signaling W on line (0, 1)
    else if φ_N > φ_S and φ_N > φ_E and φ_N > φ_W then
        state ← Vert-seek; dir ← S; drop ← after                 ▷ Go signaling W on line (0, −1)
    end if
else if state = Vert-seek then
    if φ_dir = φ_E = φ_W = 0 then
        drop ← ⊥                                                 ▷ keep searching
    else if moves < s + 1 then
        state ← Vert-bypass; INTERPRET-SIGNALS-PHASE1
    else
        state ← Vert-bypass; INTERPRET-SIGNALS-PHASE2
    end if
else if state = Vert-bypass then
    if moves < s + 1 then
        INTERPRET-SIGNALS-PHASE1
    else
        INTERPRET-SIGNALS-PHASE2
    end if
else if state = Horiz then
    drop ← ⊥
else if state = Forward(k)-Explore(d) then
    if k > 1 then
        state ← Forward(k − 1)-Explore(d)
    else
        state ← Horiz; dir ← d
    end if
    drop ← ⊥
```

▶ Algorithm 2 (continued)

else if state = Pattern(w) **then**
    dir $\leftarrow \perp$
    if $|w| > 1 \wedge w[1] = 1$ **then** ▷ $w[1]$ returns the first bit of the binary word w
        drop $\leftarrow$ after
    else
        drop $\leftarrow \perp$
    end if
    if $w \neq \epsilon$ **then**
        state $\leftarrow$ Pattern(shiftleft(w)) ▷ shiftleft(w) removes the first bit of w
    end if
end if
if dir $\neq \perp \wedge$ moves $< s + 1$ **then**
    moves $\leftarrow$ moves $+ 1$
end if
procedure INTERPRET-SIGNALS-PHASE1
    if $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ **then**
        state $\leftarrow$ Pattern($1\,01\,01\,01$); dir $\leftarrow E$; drop $\leftarrow \perp$ ▷ Start signaling E
    else if $\varphi_{\mathsf{dir}} = \varphi_W = 0$ **and** $\varphi_E > 0$ **then**
        state $\leftarrow$ Pattern($1\,01\,00\,01\,01$); dir $\leftarrow W$; drop $\leftarrow \perp$ ▷ Start signaling W
    else if $\varphi_{\mathsf{dir}} = 0$ **and** $\varphi_E = \varphi_W > 0$ **then**
        state $\leftarrow$ Pattern($1\,01\,00\,01\,01$); drop $\leftarrow \perp$ ▷ Start signaling N or S
    else if $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W > 0$ **then**
        state $\leftarrow$ Horiz; dir $\leftarrow E$; drop $\leftarrow \perp$ ▷ Explore E
    else if $\varphi_{\mathsf{dir}} = \varphi_E > \varphi_W$ **then**
        state $\leftarrow$ Horiz; dir $\leftarrow W$; drop $\leftarrow \perp$ ▷ Explore W
    else if $\varphi_{\mathsf{dir}} = \varphi_E < \varphi_W$ **then**
        state $\leftarrow$ Forward(s)-Explore(E); drop $\leftarrow \perp$ ▷ Move forward s steps and explore E
    else if $\varphi_{\mathsf{dir}} = \varphi_W > \varphi_E$ **then**
        state $\leftarrow$ Forward(s)-Explore(W); drop $\leftarrow \perp$ ▷ Move forward s steps and explore W
    else if $\varphi_{\mathsf{dir}} > \varphi_E$ **and** $\varphi_{\mathsf{dir}} > \varphi_W$ **then**
        drop $\leftarrow \perp$ ▷ Continue to bypass pheromone traces
    end if
end procedure
procedure INTERPRET-SIGNALS-PHASE2
    if $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ **then**
        state $\leftarrow$ Pattern($1\ 01\ 01$); dir $\leftarrow E$; drop $\leftarrow \perp$ ▷ Start signaling E
    else if $\varphi_{\mathsf{dir}} = \varphi_W = 0$ **and** $\varphi_E > 0$ **then**
        state $\leftarrow$ Pattern($1\,01\,01$); dir $\leftarrow W$; drop $\leftarrow \perp$ ▷ Start signaling W
    else if $\varphi_{\mathsf{dir}} = 0$ **and** $\varphi_E = \varphi_W > 0$ **then**
        state $\leftarrow$ Pattern($1\,01\,01$); drop $\leftarrow \perp$ ▷ Start signaling N or S
    else if $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W > 0$ **then**
        state $\leftarrow$ Forward(s)-Explore(E); drop $\leftarrow \perp$ ▷ Move forward s steps and explore E
    else if $\varphi_{\mathsf{dir}} = \varphi_W > \varphi_E$ **then**
        state $\leftarrow$ Forward(s)-Explore(W); drop $\leftarrow \perp$ ▷ Move forward s steps and explore W
    else if $\varphi_{\mathsf{dir}} > \varphi_E$ **and** $\varphi_{\mathsf{dir}} > \varphi_W$ **then**
        drop $\leftarrow \perp$ ▷ Continue to bypass pheromone traces
    end if
end procedure

distance s, the signaling agents stay longer for guiding some of the incoming agents to explore these lines (some other agents are still guided to "jump" for exploring the lines s steps further). This part of the algorithm is controlled mainly by the Interpret-Signals-phase1() procedure (see Alg. 2) and proven in Lemma 52.

▶ Remark 16. The technical analysis shows that the algorithm works with $s = 6$. We give a short intuition for this value. As briefly explained above, the point is that signaling patterns cannot be established too far from the nest, because agents do not have enough remaining lifetime to complete the pattern. As such, the exploration of horizontal lines that are far from the nest must be signaled by patterns that are set up closer to the nest. In fact, s depends on the longest such signaling pattern, dropped by a signaling agent (at a distance further than s from the nest). This in turn establishes the closest position of such agent to the grid extremity (at distance τ), where it can complete the signaling before it dies. In our algorithm, to explore lines after distance s, only 6 rounds are used by a signaling agent, which explains why $s = 6$. We actually need to encode 6 actions (3 for signaling agents and 3 for exploring). This requires 12 rounds of signaling due to the N/S dispatching at the nest, but we can get away with s being only 6 because the agents arrive at different times. Still, signaling agents have to stay alive there only for 6 rounds each.

Regarding the minimal τ which is 16, it is due to the transition from operation in the s first lines to the next ones. During this transition, signaling agents should have enough remaining lifetime to reach line s and to signal the required pattern (in these lines, the signaling pattern of each agent requires 10 rounds; there are $3 + 5$ actions to signal here). So 10 rounds for signaling and 6 rounds to reach the line at distance 6 gives $\tau \geq 16$ rounds.

Let us detail now the operation of the algorithm during the first rounds intended to explore the x-axis (this differs from the exploration of other lines). Agents start at the nest in state Init. Each of the first three agents are placed respectively east, west and north to the nest and start signaling according to the predetermined pattern (lines 3, 5 and 7, Alg. 2). This signaling instructs the 4th agent (A_4) to explore the east half of the x-axis (line 9) and the 5th agent (A_5), to explore the remaining (west) half of the x-axis (line 11). This is proven in Lemma 46. The next four agents are instructed to move to lines $(0, 1)$ and $(0, -1)$ (lines 13 - 19), two agents on each line, to stop on the East and West from the vertical axis (cells $(1, 1), (-1, 1), (1, -1), (-1, -1)$). This is for instructing to explore lines $(0, 1), (0, -1), (0, s + 1)$ and $(0, -s - 1)$ (as explained in the previous paragraph). This is proven in Lemmas 47 and 48.

Notice that starting from round 8, every even round, an agent in Vert-seek leaves the nest to the North, and every odd round, an agent in Vert-seek leaves the nest to the South (Lemma 48). This alternation allows to explore both the north and the south halves of the grid, without knowing its size.

States Vert-seek and Vert-bypass are used in a similar way as in the previous algorithm, to overcome the difficulty caused by the pheromone traces left from previous drops in case of $\mu > 1$. An agent has to bypass (in state Vert-bypass) these traces (lines 74 and 89) until arriving to a line with either no pheromone or with "fresh" pheromones, just dropped in the previous round (treated in all other lines of the Interpret-Signals-phase1() and Interpret-Signals-phase2() procedures). Starting with the 8th agent, agents leave the nest in state Vert-seek and move vertically in this state until sensing some dropped pheromone, moving then to Vert-bypass (lines 22 - 27).

▶ **Theorem 17.** *Algorithm 2 solves the treasure hunt problem for $\mu \geq 1, \tau \geq 16$ and $s = 6$ in $11\tau - 6s + 2$ rounds, using $10\tau - 6s + 3$ agents and $28\tau + O(s) + 8$ pheromone drops.*

The complete proof of Theorem 17 is available in Appendix C.

5 Concluding remarks

We have presented the first algorithms for the treasure hunt problem under the weak communication mechanism of evaporating pheromone markers. In Algorithm 1, the assumption that pheromone lasts for at least two rounds ($\mu \geq 2$) leads to a fairly simple algorithm design with very few states. By contrast, Algorithm 2 is significantly more complicated, as it needs to be able to handle both an extremely fast evaporation rate ($\mu = 1$) and larger values of μ.

Algorithm 2 covers all values of the evaporation parameter $\mu \geq 1$, but it requires a lifetime of $\tau \geq 16$. It would be interesting to determine the smallest τ_0 such that there exists a treasure hunt algorithm that works for all $\mu \geq 1$ and for all $\tau \geq \tau_0$. With ad-hoc arguments, it can be seen that $\tau_0 > 2$. However, it is far from obvious how to generalize these arguments to larger values of τ_0. On the other hand, there may be room to improve the upper bound of $\tau_0 \leq 16$, with some fine-tuning of the signaling patterns.

Another interesting direction for future work is improving on the complexities of Algorithm 2, or studying tradeoffs between completion time, pheromone utilization, and agent utilization. Since both of our algorithms use only a constant number of pheromone drops per agent, one idea would be to increase the frequency of pheromone drops. It seems that this would not help to reduce agent utilization or the completion time. Indeed, the limiting factor in Algorithm 2 seems to be not the amount of pheromone that is dropped or that might be dropped, but indeed the number of grid positions that are available in order to set up an efficient pattern, i.e., a pattern that resides in the neighborhood of the main axis so that it can be immediately sensed by agents.

The assumption of detecting pheromones only in adjacent nodes to the agent, although natural, could be relaxed. However, if the sensing range is increased even to 2 while maintaining the principle that the agent can pinpoint exactly the position of the pheromone and compare pheromone levels between all nodes in its 2-neighborhood, then Algorithm 1 resolves the problem for all values of the parameters. Indeed, the only reason why Algorithm 1 fails for $\mu = 1$ is that, due to the North-South dispatching at the nest, agents are dispatched into the same half-plane every two rounds, and therefore any pheromone dropped by an agent evaporates before the next agent can sense it. Consequently, in order to study a meaningful problem with an increased sensing range, some loss of information would have to be introduced at distance 2 or more.

Our algorithms are quite far from modeling natural ant foraging patterns. Indeed, depending on species, ants in nature tend to employ a wide range of communication methods, including multiple types of pheromone of various degrees of volatility, repellent pheromones, contact, or sounds [49]. However, our proposed solutions are more appropriate for artificial agent systems, where the parameter τ might correspond to agents with limited energy, and evaporating markers could be useful to prevent area pollution. The appropriate parameter values will depend on the specific application.

As a general remark, we believe that the communication model of evaporating pheromone markers is inherently interesting and we would like to study other agent coordination problems in this model. Orthogonally, one may consider less predictable evaporation mechanisms, such as evaporation governed by a random process, or controlled by an adversary.

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A State transition diagram of Algorithm 1

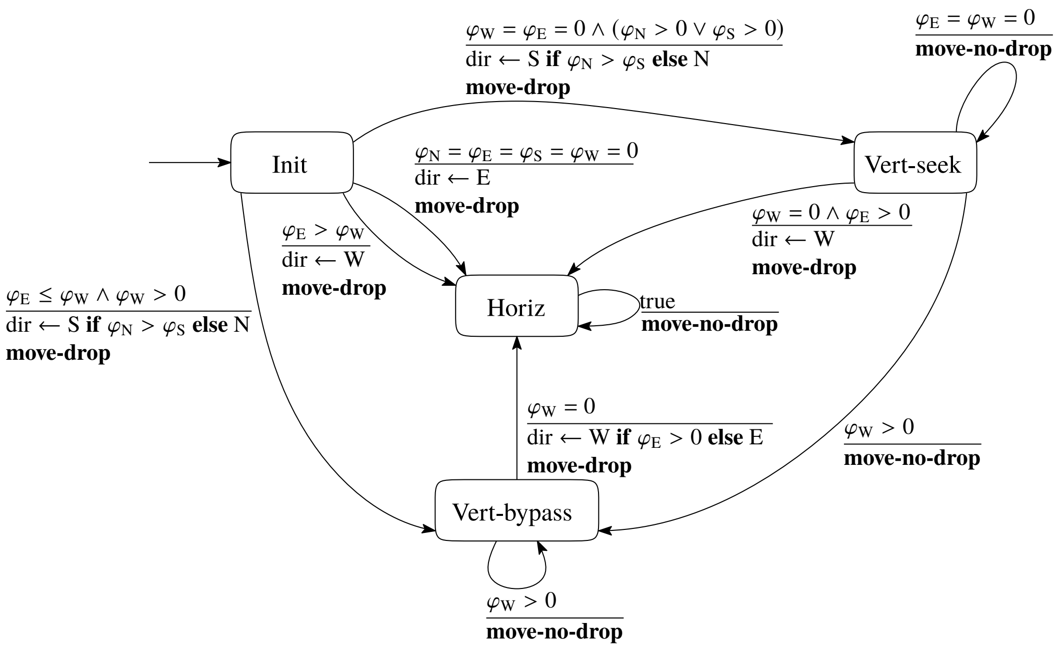

Figure 2 A hybrid state transition diagram representing Algorithm 1. On each transition, the guard condition is given above the horizontal line. The actions that are executed if the transition is triggered are given below the horizontal line. The values φ_x, for $x \in \{N, E, S, W\}$, represent the pheromone values in neighboring nodes at the beginning of the round. dir $\in \{N, E, S, W\}$ is a variable whose value persists between transitions. The statement **move-drop** instructs the agent to move in the direction indicated by the variable dir, dropping pheromone on the destination node. The statement **move-no-drop** instructs the agent to move in the direction indicated by the variable dir, without dropping pheromone on the destination node. Exactly one guarded transition is enabled from each state.

B Proof of Theorem 8 from Section 3

In this section, we show the correctness of Algorithm 1 under the assumption that $\tau \geq 1$ and $\mu \geq 2$. We start by introducing some notation.

▶ **Definition 18.** *For i, r satisfying $1 \leq i \leq r < i + \tau$, let $\mathsf{state}_i(r)$, $\mathsf{dir}_i(r)$, and $\mathsf{drop}_i(r)$ be the values of variables* state, dir, *and* drop, *respectively, of agent A_i at the end of the execution of its transition function during round r. For convenience, let $\mathsf{state}_i(i-1) = \mathsf{Init}$.*

The output symbol $(\mathsf{dir}_i(r), \mathsf{drop}_i(r))$ encodes the actions of agent i in round r, as described in steps 5-7 of Definition 3.

In Propositions 19, 20, 24, and 28 below, we prove some basic properties of the four states: Horiz, Vert-seek, Vert-bypass, and Init, respectively. Proposition 19 states that whenever an agent switches to state Horiz it moves horizontally (East or West) and drops pheromone. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until the end of its lifetime.

Proposition 20 states that whenever an agent switches to state Vert-seek it moves vertically (North or South) and drops pheromone. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until one of the following happens: it reaches the end of its lifetime, or it switches to state Vert-bypass moving in the same direction as before, or it switches to state Horiz moving West.

Proposition 24 states that whenever an agent switches to state Vert-bypass it moves vertically, and it drops pheromone only if it switches from state Init to Vert-bypass. Subsequently, it keeps moving in the same direction in the same state without dropping pheromone until one of the following happens: it reaches the end of its lifetime, or it switches to state Horiz.

Finally, Proposition 28 states that during its first transition, every agent switches to one of the states Horiz, Vert-seek, or Vert-bypass.

▶ **Proposition 19.** *Let $r \geq i$ be such that $\mathsf{state}_i(r) = \mathsf{Horiz} \neq \mathsf{state}_i(r-1)$. Then $\mathsf{dir}_i(r) \in \{E, W\}$, $\mathsf{drop}_i(r) = \mathsf{after}$, and, for all $r' \in (r, i+\tau)$, we have $\mathsf{state}_i(r') = \mathsf{Horiz}$, $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$, and $\mathsf{drop}_i(r') = \perp$.*

Proof. Because $\mathsf{state}_i(r-1) \neq \mathsf{Horiz}$, $\mathsf{state}_i(r)$ must have been assigned the value Horiz in line 3, 5, 19, or 28. In any case, this is followed by an assignment to dir of a value from $\{E, W\}$ and by an assignment drop $\leftarrow$ after. Therefore $\mathsf{dir}_i(r) \in \{E, W\}$ and $\mathsf{drop}_i(r) = \mathsf{after}$. The claim follows by induction on r'. Assuming that $\mathsf{state}_i(r') = \mathsf{Horiz}$ and $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$ for a particular $r' \geq r$, in the following round agent A_i executes the conditional in lines 23-24, which sets drop $\leftarrow \perp$ and does not modify the other variables. Therefore, $\mathsf{state}_i(r'+1) = \mathsf{state}_i(r') = \mathsf{Horiz}$, $\mathsf{dir}_i(r'+1) = \mathsf{dir}_i(r') = \mathsf{dir}_i(r)$, and $\mathsf{drop}_i(r'+1) = \perp$. ◀

▶ **Proposition 20.** *Let $r \geq i$ be such that $\mathsf{state}_i(r) = \mathsf{Vert\text{-}seek} \neq \mathsf{state}_i(r-1)$. Then $\mathsf{dir}_i(r) \in \{N, S\}$, $\mathsf{drop}_i(r) = \mathsf{after}$, and there exists a unique $t \in (r, i+\tau]$ such that for all $r' \in (r, t)$ we have $\mathsf{state}_i(r') = \mathsf{Vert\text{-}seek}$, $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$, and $\mathsf{drop}_i(r') = \perp$, and one of the following holds:*

1. $\mathsf{state}_i(t) = \mathsf{Horiz}$ *and* $\mathsf{dir}_i(t) = W$.
2. $\mathsf{state}_i(t) = \mathsf{Vert\text{-}bypass}$ *and* $\mathsf{dir}_i(t) = \mathsf{dir}_i(r)$.
3. $t = i + \tau$.

Proof. Because $\mathsf{state}_i(r-1) \neq \mathsf{Vert\text{-}seek}$, $\mathsf{state}_i(r)$ must have been assigned the value Vert-seek in line 7. This is followed by an assignment to dir of a value from $\{N, S\}$ and by an assignment drop $\leftarrow$ after, therefore $\mathsf{dir}_i(r) \in \{N, S\}$ and $\mathsf{drop}_i(r) = \mathsf{after}$.

Let $t^\star$ be the largest t such that for all $r' \in [r, t)$, $\mathsf{state}_i(r') = \mathsf{Vert\text{-}seek}$ and $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$. It must be that $r + 1 \le t^\star \le i + \tau$, since $\mathsf{state}_i(r) = \mathsf{Vert\text{-}seek}$ and $\mathsf{state}_i(r')$, $\mathsf{dir}_i(r')$ are undefined for $r' \ge i + \tau$.

By definition of $t^\star$, in order to conclude the proof, we need to show that for all $r' \in (r, t^\star)$ we have $\mathsf{drop}_i(r') = \perp$, that $t^\star$ satisfies one of the properties 1-3 in the statement of the Proposition, and that $t^\star$ is unique. We prove these in Claims 21, 22, and 23 below.

▷ Claim 21. For all $r' \in (r, t^\star)$, $\mathsf{drop}_i(r') = \perp$.

Proof. By definition of $t^\star$, we have $\mathsf{state}_i(r') = \mathsf{state}_i(r' - 1) = \mathsf{Vert\text{-}seek}$ for all $r' \in (r, t^\star)$. It follows that agent A_i executes line 13 in round r', therefore $\mathsf{drop}_i(r') = \perp$. ◀

▷ Claim 22. If $t^\star < i + \tau$, then either $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$ and $\mathsf{dir}_i(t^\star) = W$, or $\mathsf{state}_i(t^\star) = \mathsf{Vert\text{-}bypass}$ and $\mathsf{dir}_i(t^\star) = \mathsf{dir}_i(r)$.

Proof. By definition of $t^\star$, if $t^\star < i + \tau$, then we must have $\mathsf{state}_i(t^\star) \neq \mathsf{state}_i(t^\star - 1) = \mathsf{Vert\text{-}seek}$, or $\mathsf{dir}_i(t^\star) \neq \mathsf{dir}_i(t^\star - 1) = \mathsf{dir}_i(r)$.

In the first case, during round $t^\star$, A_i must have executed procedure Interpret-Signals in line 15, therefore either the conditional in lines 27-28, or the one in lines 29-30. Lines 27-28 result in $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$ and $\mathsf{dir}_i(t^\star) = W$. Lines 29-30 result in $\mathsf{state}_i(t^\star) = \mathsf{Vert\text{-}bypass}$ and $\mathsf{dir}_i(t^\star) = \mathsf{dir}_i(t^\star - 1) = \mathsf{dir}_i(r)$.

In the second case, during round $t^\star$, A_i must have executed procedure Interpret-Signals in line 15, and within the procedure the conditional in lines 27-28, which results in $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$ and $\mathsf{dir}_i(t^\star) = W$. ◀

▷ Claim 23. $t^\star$ is the unique $t \in (r, i + \tau]$ that satisfies the statement of the proposition.

Proof. For any $t' \in (r, t^\star)$ we have $t' < t^\star \le i + \tau$ and $\mathsf{state}_i(t') = \mathsf{Vert\text{-}seek}$ by definition of $t^\star$. Therefore, t' cannot satisfy any of the properties 1-3.

For any $t' \in (t^\star, i + \tau]$, the interval (r, t') contains $t^\star$. Moreover, since $t^\star < i + \tau$ in this case, by Claim 22 we have $\mathsf{state}_i(t^\star) \neq \mathsf{Vert\text{-}seek}$. Therefore, it is not true that for all $r' \in (r, t')$ we have $\mathsf{state}_i(r') = \mathsf{Vert\text{-}seek}$. ◀

This concludes the proof of Proposition 20. ◀

▶ **Proposition 24.** *Let $r \ge i$ be such that $\mathsf{state}_i(r) = \mathsf{Vert\text{-}bypass} \neq \mathsf{state}_i(r - 1)$. Then $\mathsf{dir}_i(r) \in \{N, S\}$, $\mathsf{drop}_i(r) = \mathsf{after}$ if and only if $\mathsf{state}_i(r - 1) = \mathsf{Init}$, and there exists a unique $t \in (r, i + \tau]$ such that for all $r' \in (r, t)$ we have $\mathsf{state}_i(r') = \mathsf{Vert\text{-}bypass}$, $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$, and $\mathsf{drop}_i(r') = \perp$, and one of the following holds:*

1. $\mathsf{state}_i(t) = \mathsf{Horiz}$.
2. $t = i + \tau$.

Proof. Because $\mathsf{state}_i(r-1) \neq \mathsf{Vert\text{-}bypass}$, $\mathsf{state}_i(r)$ must have been assigned the value $\mathsf{Vert\text{-}bypass}$ in line 7 or in line 30 (following a call of Interpret-Signals from line 15). If it was in line 7, then $\mathsf{state}_i(r - 1) = \mathsf{Init}$ and it was followed by an ssignment to dir of a value from $\{N, S\}$ and an assignment $\mathsf{drop} \leftarrow \mathsf{after}$. If it was in line 30 following a call of Interpret-Signals from line 15, then $\mathsf{state}_i(r - 1) = \mathsf{Vert\text{-}seek}$ and dir is not changed, therefore $\mathsf{dir}_i(r) = \mathsf{dir}_i(r - 1) \in \{N, S\}$ by Proposition 20. Moreover, line 30 contains an assignment $\mathsf{drop} \leftarrow \perp$, therefore $\mathsf{drop}_i(r) = \perp$.

Let $t^\star$ be the largest t such that for all $r' \in [r, t)$, $\mathsf{state}_i(r') = \mathsf{Vert\text{-}bypass}$ and $\mathsf{dir}_i(r') = \mathsf{dir}_i(r)$. It must be that $r + 1 \le t^\star \le i + \tau$, since $\mathsf{state}_i(r) = \mathsf{Vert\text{-}bypass}$ and $\mathsf{state}_i(r')$, $\mathsf{dir}_i(r')$ are undefined for $r' \ge i + \tau$.

By definition of $t^\star$, in order to conclude the proof, we need to show that for all $r' \in (r, t^\star)$ we have $\mathsf{drop}_i(r') = \bot$, that $t^\star$ satisfies one of the properties 1-2 in the statement of the Proposition, and that $t^\star$ is unique. We prove these in Claims 25, 26, and 27 below.

▷ Claim 25. For all $r' \in (r, t^\star)$, $\mathsf{drop}_i(r') = \bot$.

Proof. By definition of $t^\star$, we have $\mathsf{state}_i(r') = \mathsf{state}_i(r'-1) = \mathsf{Vert\text{-}bypass}$ for all $r' \in (r, t^\star)$. It follows that agent A_i executes line 30 following a call of Interpret-Signals from line 21 in round r', therefore $\mathsf{drop}_i(r') = \bot$. ◀

▷ Claim 26. If $t^\star < i + \tau$, then $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$.

Proof. By definition of $t^\star$, if $t^\star < i + \tau$, then we must have $\mathsf{state}_i(t^\star) \neq \mathsf{state}_i(t^\star - 1) = \mathsf{Vert\text{-}bypass}$, or $\mathsf{dir}_i(t^\star) \neq \mathsf{dir}_i(t^\star - 1) = \mathsf{dir}_i(r)$.

In both cases, in order to change either state or direction, A_i must have executed during round $t^\star$ line 19 or line 28 following a call of procedure Interpret-Signals in line 21. Both possibilities result in $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$. ◀

▷ Claim 27. $t^\star$ is the unique $t \in (r, i + \tau]$ that satisfies the statement of the proposition.

Proof. For any $t' \in (r, t^\star)$ we have $t' < t^\star \leq i + \tau$ and $\mathsf{state}_i(t') = \mathsf{Vert\text{-}bypass}$ by definition of $t^\star$. Therefore, t' cannot satisfy any of the properties 1-2.

For any $t' \in (t^\star, i + \tau]$, the interval (r, t') contains $t^\star$. Moreover, since $t^\star < i + \tau$ in this case, by Claim 26 we have $\mathsf{state}_i(t^\star) = \mathsf{Horiz}$. Therefore, it is not true that for all $r' \in (r, t')$ we have $\mathsf{state}_i(r') = \mathsf{Vert\text{-}bypass}$. ◀

This concludes the proof of Proposition 24. ◀

▶ **Proposition 28.** *For all $i \geq 1$, $\mathsf{state}_i(i) \in \{\mathsf{Horiz}, \mathsf{Vert\text{-}seek}, \mathsf{Vert\text{-}bypass}\}$.*

Proof. In its first round, agent A_i starts in state Init and it enters one of the conditionals in lines 2-10. This results in $\mathsf{state}_i(i) \in \{\mathsf{Horiz}, \mathsf{Vert\text{-}seek}, \mathsf{Vert\text{-}bypass}\}$. ◀

By Propositions 19, 20, 24, and 28, it follows immediately that the trajectory of every agent A_i is of one of the following types:

- τ horizontal moves in the same direction $\mathsf{dir}_i(i) \in \{W, E\}$, dropping pheromone only on the first horizontal move.
- For some $t^\star < \tau$, $t^\star$ vertical moves in the same direction $\mathsf{dir}_i(i) \in \{N, S\}$, dropping pheromone only on the first vertical move, followed by $\tau - t^\star$ horizontal moves in the same direction $\mathsf{dir}_i(t^\star) \in \{E, W\}$, dropping pheromone only on its first horizontal move.
- τ vertical moves in the same direction $\mathsf{dir}_i(i) \in \{N, S\}$, dropping pheromone only on the first vertical move.

We have, therefore, the following two corollaries:

▶ **Corollary 29.** *Every agent A_i, $i \geq 1$, has a signature $[v_i; h_i]$ and it drops pheromone only on its first vertical move (if $|v_i| > 0$) and on its first horizontal move (if $|h_i| > 0$).*

▶ **Corollary 30.** *The distance of every agent from the nest strictly increases at each round.*

▶ **Definition 31.** *For $i \geq 1$ and $x, y \in \mathbb{Z}$, let $\varphi(x, y, i)$ denote the amount of pheromone at node (x, y) at the end of round i. For convenience, let $\varphi(x, y, 0) = 0$.*

In particular for the nodes around the nest, for $i \geq 0$, let $\varphi_N(i) = \varphi(0, 1, i)$, $\varphi_E(i) = \varphi(1, 0, i)$, $\varphi_S(i) = \varphi(0, -1, i)$, and $\varphi_W(i) = \varphi(0, -1, i)$. Let $\Phi_i = \big(\varphi_N(i), \varphi_E(i), \varphi_S(i), \varphi_W(i)\big)$.

▶ **Lemma 32.** *For all $i \geq 1$, $\varphi_{\mathsf{dir}_i(i)}(i) = \mu$. Moreover, for all $d \in \{N, E, S, W\} \setminus \{\mathsf{dir}_i(i)\}$, $\varphi_d(i) = \varphi_d(i-1) \dot{-} 1$.*

Proof. By Corollary 29, every agent drops pheromone on its first move. Therefore, $\varphi_{\mathsf{dir}_i(i)}(i) = \mu$. Moreover, since all agents other than A_i that are present in the system in round i have started their signature trajectory strictly before round i, and because the distance of an agent from the nest strictly increases at each round (Corollary 30), it follows that they cannot arrive at a node neighboring the nest in round i. The pheromone on neighbors of the nest in any direction $d \neq \mathsf{dir}_i(i)$ is, therefore, subject only to evaporation: $\varphi_d(i) = \varphi_d(i-1) \dot{-} 1$. ◀

▶ **Lemma 33.** *All of the following hold:*

- *A_1 has signature $[0; \tau]$ and $\Phi_1 = (0, \mu, 0, 0)$.*
- *A_2 has signature $[0; -\tau]$ and $\Phi_2 = (0, \mu \dot{-} 1, 0, \mu)$.*
- *$\mathsf{dir}_3(3) = N$, $\mathsf{state}_3(3) = \mathsf{Vert\text{-}bypass}$, and $\Phi_3 = (\mu, \mu \dot{-} 2, 0, \mu \dot{-} 1)$.*
- *For all even $i \geq 4$, $\mathsf{dir}_i(i) = S$ and $\Phi_i = (\mu \dot{-} 1, \mu \dot{-} (i-1), \mu, \mu \dot{-} (i-2))$.*
- *For all odd $i \geq 5$, $\mathsf{dir}_i(i) = N$ and $\Phi_i = (\mu, \mu \dot{-} (i-1), \mu \dot{-} 1, \mu \dot{-} (i-2))$.*

Proof. We have $\Phi_0 = (0, 0, 0, 0)$, hence in round 1 agent A_1 executes line 3. This results in $\mathsf{state}_1(1) = \mathsf{Horiz}$ and $\mathsf{dir}_1(1) = E$. The signature of A_1 is $[0; \tau]$. By Lemma 32, $\Phi_1 = (0, \mu, 0, 0)$.

In round 2, agent A_2 senses pheromone $\varphi_E(1) = \mu > 0 = \varphi_W(1)$, so it executes line 5. This results in $\mathsf{state}_2(2) = \mathsf{Horiz}$ and $\mathsf{dir}_2(2) = W$. Its signature is $[0; -\tau]$. By Lemma 32, $\Phi_2 = (0, \mu \dot{-} 1, 0, \mu)$.

In round 3, agent A_3 senses pheromone $\varphi_E(2) = \mu \dot{-} 1 < \mu = \varphi_W(2)$, where the inequality holds because $\mu \geq 2$, and $\varphi_N(2) = \varphi_S(2) = 0$, so it executes lines 7-9, hence $\mathsf{dir}_3(3) = N$ and $\mathsf{state}_3(3) = \mathsf{Vert\text{-}bypass}$. By Lemma 32, $\Phi_3 = (\mu, \mu \dot{-} 2, 0, \mu \dot{-} 1)$.

We prove the rest of the claim by induction on i. For even $i \geq 4$, either $i = 4$ and thus in round i agent A_i senses pheromone $\varphi_E(3) = \mu \dot{-} 2 < \mu \dot{-} 1 = \varphi_W(3)$ and $\varphi_N(3) = \mu > 0 = \varphi_S(3)$, or $i \geq 6$ and thus in round i agent A_i senses pheromone $\varphi_E(i-1) = \mu \dot{-} (i-2) \leq \mu \dot{-} (i-3) = \varphi_W(i-1)$ and $\varphi_N(i-1) = \mu > \mu \dot{-} 1 = \varphi_S(i-1)$ by the inductive hypothesis, where the last inequality holds because $\mu \geq 2$. In both cases, it executes lines 7-9 and $\mathsf{dir}_i(i) = S$. By Lemma 32, $\Phi_i = (\mu \dot{-} 1, \mu \dot{-} (i-1), \mu, \mu \dot{-} (i-2))$.

For odd $i \geq 5$, in round i agent A_i senses pheromone $\varphi_E(i-1) = \mu \dot{-} (i-2) \leq \mu \dot{-} (i-3) = \varphi_W(i-1)$ and $\varphi_N(i-1) = \mu \dot{-} 1 < \mu = \varphi_S(i-1)$ by the inductive hypothesis, so it executes lines 7-9 and $\mathsf{dir}_i(i) = N$. By Lemma 32, $\Phi_i = (\mu, \mu \dot{-} (i-1), \mu \dot{-} 1, \mu \dot{-} (i-2))$. ◀

▶ **Lemma 34.** *For every odd $i \geq 3$ and $r \geq i+1$, if agent A_i is on the positive part of the y-axis at the beginning of round r, then it may only sense pheromone previously dropped by some agent A_j with odd j, $3 \leq j \leq i$.*

Proof. Agents A_j with even j, $j \geq 4$, remain in the South half-plane in view of Corollary 30 and because their first step is to the South (Lemma 33), so any pheromone that they may drop cannot be sensed by A_i.

Agents A_j with odd j, $j > i$, are always at distance at least 2 behind agent A_i by Corollary 30, so A_i cannot sense any pheromone that they may drop.

Finally, agents A_1 and A_2 stay on the x-axis by Lemma 33, so A_i may only sense pheromone dropped by those agents during round i. ◀

▶ Remark 35. In the following lemma and its proof, we will say that an agent A_i is at a certain node at the beginning of a certain round $j > i$, when it would be more correct to say that agent A_i arrives at that node during step 6 of round $j-1$. The two wordings are equivalent except for the case $j = i + \tau$, because, by definition of the model, A_i is no longer present in the system in round $i + \tau$. Slightly abusing terminology, we will use the former wording even for the case $j = i + \tau$. Similarly, we will say that A_i is in state s at the beginning of round j, if it switches to state s during step 4 of round $j - 1$. Finally, we will say that A_i senses a certain amount of pheromone in a certain direction at the beginning of round j, if that amount of pheromone is present in that direction after step 7 of round $j - 1$.

▶ **Lemma 9.** *For all odd i with $3 \leq i \leq 4\tau - 1$, and for all y with $1 \leq y \leq \lceil \frac{i-1}{4} \rceil$, A_i is at node $(0, y)$ at the beginning of round $i + y$ and:*

1. *It senses pheromone $\mu \dot{-} (i - 4y)$ to the East and $\mu \dot{-} (i - 2 - 4y)$ to the West.*
2. *If $y = 1$, it is in state* Vert-seek *if $i - 3 \geq \mu$, otherwise it is in state* Vert-bypass*.*
3. *If $y \geq 2$, it is in state* Vert-seek *if $i - 4y + 2 \geq \mu$, otherwise it is in state* Vert-bypass*.*

Proof. The proof is by induction on odd i. For $i = 3$, the statement boils down to the following Claim.

▷ Claim 36. A_3 is at node $(0, 1)$ in state Vert-bypass at the beginning of round 4 and senses no pheromone to the East or to the West.

Proof. By Lemma 33, the first step of A_3 is to the North and it switches to state Vert-bypass. Moreover, By Lemma 34, A_3 senses no pheromone to the East or to the West at the beginning of round 4. ◀

Now, given an odd $i^\star$, $3 \leq i^\star \leq 4\tau - 3$, we assume that the statement holds for all odd i, $3 \leq i \leq i^\star$. We prove it for the case $i = i^\star + 2$, which we rewrite below as Claim 37. The remainder of the proof of Lemma 9 is devoted to the proof of Claim 37.

▷ Claim 37. For all y with $1 \leq y \leq \lceil \frac{i^\star+1}{4} \rceil$, $A_{i^\star+2}$ is at node $(0, y)$ at the beginning of round $i^\star + 2 + y$ and:

1. It senses pheromone $\mu \dot{-} (i^\star + 2 - 4y)$ to the East and $\mu \dot{-} (i^\star - 4y)$ to the West.
2. If $y = 1$, it is in state Vert-seek if $i^\star - 1 \geq \mu$, otherwise it is in state Vert-bypass.
3. It $y \geq 2$, it is in state Vert-seek if $i^\star - 4y + 4 \geq \mu$, otherwise it is in state Vert-bypass.

We prove Claim 37 by induction on y. We prove the base case $y = 1$ as Claim 38 below.

▷ Claim 38. $A_{i^\star+2}$ is at node $(0, 1)$ at the beginning of round $i^\star + 3$ and:

1. It senses pheromone $\mu \dot{-} (i^\star - 2)$ to the East and $\mu \dot{-} (i^\star - 4)$ to the West.
2. It is in state Vert-seek if $i^\star - 1 \geq \mu$, otherwise it is in state Vert-bypass.

Proof. By Lemma 33, the first step of $A_{i^\star+2}$ in round $i^\star + 2$ is to the North, therefore $A_{i^\star+2}$ is at node $(0, 1)$ at the beginning of the next round $i^\star + 3$.

At the beginning of round $i^\star + 2$, when it first appears at the nest in state Init, it senses pheromone $\Phi_{i^\star+1}$ around it. By Lemma 33 and because $i^\star \geq 3$ is odd, hence $i^\star + 1 \geq 4$ is even, we have $\Phi_{i^\star+1} = (\mu \dot{-} 1, \mu \dot{-} i^\star, \mu, \mu \dot{-} (i^\star - 1))$. Therefore, during its first transition, it executes line 7 and switches to state Vert-seek if $\mu \dot{-} (i^\star - 1) = 0$, or equivalently $i^\star - 1 \geq \mu$. Otherwise, it switches to state Vert-bypass.

It remains to show property 1 of the Claim. By inspection of Algorithm 1, an agent that is in one of the states {Vert-seek, Vert-bypass} can only switch direction to the East if it is in state Vert-bypass and senses no pheromone to the East or to the West (lines 18-19). Therefore, by the inductive hypothesis for agents A_i with odd i, $3 \leq i \leq i^\star$, agent A_i can only turn East from node $(0, 1)$ if $i - 3 \leq \mu - 1$ and $\mu \dot{-} (i - 4) = \mu \dot{-} (i - 6) = 0$. It can be easily verified that $i = 3$ satisfies these conditions, whereas for any $i \geq 5$ the conditions can only be satisfied if $\mu \geq i-2$ and $\mu \leq i-4$, which is impossible. We conclude that only A_3 drops pheromone at node $(1,1)$, and it does so during round 4, so the pheromone level at node $(1,1)$ at the beginning of round 5 is μ. It follows that agent $A_{i^\star+2}$, which is at node $(0,1)$ at the beginning of round $i^\star+3$, senses a pheromone level of $\mu \dot{-} (i^\star+3-5) = \mu \dot{-} (i^\star-2)$ to the East.

Similarly, by inspection of Algorithm 1, an agent that is in one of the states {Vert-seek, Vert-bypass} can only switch direction to the West if it senses no pheromone to the West and strictly positive pheromone to the East (lines 27-28). By the inductive hypothesis for agents A_i with odd i, $3 \leq i \leq i^\star$, agent A_i can only turn West from node $(0,1)$ if $\mu \dot{-} (i-4) > 0$ and $\mu \dot{-} (i-6) = 0$. As can be easily verified, $i=3$ does not satisfy the first condition, $i=5$ satisfies both, and for any $i \geq 7$ the conditions can only be satisfied if $i-4 \leq \mu-1 \Leftrightarrow \mu \geq i-3$ and $\mu \leq i-6$, which is impossible. We conclude that if $i^\star=3$, then no previous agent has dropped pheromone at $(-1,1)$ and hence $A_{i^\star+2}$ senses a pheromone level of $0 = \mu \dot{-} (i^\star-4)$ to the West. If $i^\star \geq 5$, then only A_5 has previously dropped pheromone at $(-1,1)$, and it did so during round 6, so the pheromone level at node $(-1,1)$ at the beginning of round 7 is μ. It follows that agent $A_{i^\star+2}$, which is at node $(0,1)$ at the beginning of round $i^\star+3$, senses a pheromone level of $\mu \dot{-} (i^\star+3-7) = \mu \dot{-} (i^\star-4)$ to the West. ◀

For the inductive step, we assume that Claim 37 holds for some $y^\star$, $1 \leq y^\star \leq \left\lceil \frac{i^\star+1}{4} \right\rceil - 1$, and we prove it for the case $y = y^\star+1$, which we rewrite below as Claim 39 for convenience.

▷ Claim 39. $A_{i^\star+2}$ is at node $(0, y^\star+1)$ at the beginning of round $i^\star+3+y^\star$ and:

1. It senses pheromone $\mu \dot{-} (i^\star-2-4y^\star)$ to the East and $\mu \dot{-} (i^\star-4y^\star-4)$ to the West.
2. It is in state Vert-seek if $i^\star-4y^\star \geq \mu$, otherwise it is in state Vert-bypass.

Proof. We first show that $A_{i^\star+2}$ is at node $(0, y^\star+1)$ at the beginning of round $i^\star+3+y^\star$. By the inductive hypothesis, $A_{i^\star+2}$ is at node $(0, y^\star)$ at the beginning of round $i^\star+2+y^\star$ and it senses pheromone $\mu \dot{-} (i^\star+2-4y^\star)$ to the East and $\mu \dot{-} (i^\star-4y^\star)$ to the West. We distinguish two cases:

If $A_{i^\star+2}$ is in state Vert-seek at the beginning of round $i^\star+2+y^\star$, then, to show that it advances one step to the North, it suffices to show that if no pheromone is present to the West, then no pheromone is present to the East either. Note that, by assumption, $y^\star \leq \left\lceil \frac{i^\star+1}{4} \right\rceil - 1$, therefore $4y^\star \leq i^\star$. Therefore, if no pheromone is present to the West, it must mean that $\mu \dot{-} (i^\star-4y^\star) = 0 \Rightarrow i^\star-4y^\star \geq \mu \Rightarrow i^\star+2-4y^\star \geq \mu \Rightarrow \mu \dot{-} (i^\star+2-4y) = 0$, which proves that no pheromone is present to the East either.

If $A_{i^\star+2}$ is in state Vert-bypass at the beginning of round $i^\star+2+y^\star$, then to show that it advances one step to the North it suffices to show that it senses strictly positive pheromone to the West, or $\mu \dot{-} (i^\star-4y^\star) > 0 \Leftrightarrow 0 \leq i^\star-4y^\star \leq \mu-1$. The first inequality holds because $y^\star \leq \left\lceil \frac{i^\star+1}{4} \right\rceil - 1$ by assumption. For the second inequality, note that by the inductive hypothesis (Claim 37 holds for $y^\star$), since $A_{i^\star+2}$ is in state Vert-bypass at the beginning of round $i^\star+2+y^\star$, we must have either $y^\star=1$ and $i^\star-1 \leq \mu-1$, or $y^\star \geq 2$ and $i^\star-4y^\star+4 \leq \mu-1$. In both cases, $i^\star-4y^\star$ is at most $\mu-4$, therefore the second inequality is also satisfied.

Next, we show property 2. Assuming $i^\star-4y^\star \leq \mu-1$, and taking also into account the fact that $4y^\star \leq i^\star$, we must have $\mu \dot{-} (i^\star-4y^\star) > 0$. By Claim 37 for $y=y^\star$ (inductive hypothesis), $A_{i^\star+2}$ is at node $(0, y^\star)$ at the beginning of round $i^\star+y^\star+2$ and it senses pheromone $\mu \dot{-} (i^\star-4y^\star)$ to the West, which we just showed is strictly positive. Therefore, since it was in one of the states {Vert-seek, Vert-bypass}, its next state is Vert-bypass by lines 29-30.

On the other hand, assuming $i^\star - 4y^\star \geq \mu$, we have $\mu \dot{-} (i^\star - 4y^\star) = \mu \dot{-} (i^\star + 2 - 4y^\star) = 0$. Hence, $A_{i^\star+2}$ senses no pheromone to the East or to the West at the beginning of round $i^\star + y^\star + 2$. To show that its next state is Vert-seek, it suffices to show that its state at the beginning of round $i^\star + y^\star + 2$ is Vert-seek. This holds by the inductive hypothesis, since if $y^\star = 1$ our assumption $i^\star - 4y^\star \geq \mu$ becomes $i^\star - 4 \geq \mu \Rightarrow i^\star - 1 \geq \mu$, and if $y^\star \geq 2$ our assumption $i^\star - 4y^\star \geq \mu$ implies $i^\star - 4y^\star + 4 \geq \mu$.

Finally, we show property 1. By the inductive hypothesis for agents A_i with odd i, $3 \leq i \leq i^\star$, agent A_i goes up the y-axis as far as node $(0, \lceil \frac{i-1}{4} \rceil)$, where it senses no pheromone to the West. To the East, it senses $\mu - 1$ if $i = 4\lambda + 1$ for some $\lambda \in \mathbb{N}$, or 0 if $i = 4\lambda + 3$ for some $\lambda \in \mathbb{N}$.

By inspection of Algorithm 1, it follows that if $i = 4\lambda + 1$, then agent A_i always turns West at node $(0, \lceil \frac{i-1}{4} \rceil)$. If $i = 4\lambda + 3$, A_i turns East at node $(0, \lceil \frac{i-1}{4} \rceil)$ only if it is in state Vert-bypass, which is equivalent, by the inductive hypothesis, to $\lceil \frac{i-1}{4} \rceil = 1$ and $i - 3 \leq \mu - 1$, or $\lceil \frac{i-1}{4} \rceil \geq 2$ and $i - 4\lceil \frac{i-1}{4} \rceil + 2 \leq \mu - 1$. It can be easily verified that the first condition is satisfied for $i = 3$, while the second condition is satisfied for all $i = 4\lambda + 3$ with $\lambda \geq 1$. We conclude, then, that an agent A_i with $i = 4\lambda + 3$ always turns East at node $(0, \lceil \frac{i-1}{4} \rceil)$.

By the above analysis, it follows that the only agent that may turn East at node $(0, y^\star+1)$ is agent $A_{4y^\star+3}$ in round $5y^\star+4$. Thus, if $i^\star+2 \geq 4y^\star+5$, agent $A_{i^\star+2}$, which is at node $(0, y^\star+1)$ at the beginning of round $i^\star + y^\star + 3$, senses pheromone $\mu \dot{-} (i^\star + y^\star + 3 - (5y^\star + 5)) = \mu \dot{-} (i^\star - 4y^\star - 2)$ to the East. If $i^\star+2 \leq 4y^\star+3$, agent $A_{i^\star+2}$ senses no pheromone to the East because it does not arrive after agent $A_{4y^\star+3}$. And indeed, in this case $\mu \dot{-} (i^\star - 4y^\star - 2) = 0$, because $i^\star - 4y^\star - 2 \leq 1 - 2 = -1$.

Moreover, the only agent that may turn West at node $(0, y^\star + 1)$ is agent $A_{4y^\star+5}$ in round $5y^\star + 6$. Thus, if $i^\star + 2 \geq 4y^\star + 7$, agent $A_{i^\star+2}$, which is at node $(0, y^\star + 1)$ at the beginning of round $i^\star+y^\star+3$, senses pheromone $\mu \dot{-} (i^\star+y^\star+3-(5y^\star+7)) = \mu \dot{-} (i^\star-4y^\star-4)$ to the West. If $i^\star + 2 \leq 4y^\star + 5$, agent $A_{i^\star+2}$ senses no pheromone to the West because it does not arrive after agent $A_{4y^\star+5}$. And indeed, in this case $\mu \dot{-} (i^\star - 4y^\star - 4) = 0$, because $i^\star - 4y^\star - 4 \leq 3 - 4 = -1$. ◀

This concludes the proof of Claim 37, and hence the proof of Lemma 9. ◀

With completely symmetric arguments, which we omit here, we can prove Lemmas 40 and 10 below, which are analogous to Lemmas 34 and 9 for the South half-plane.

▶ **Lemma 40.** *For every even $i \geq 4$ and $r \geq i + 1$, if agent A_i is on the negative part of the y-axis at the beginning of round r, then it may only sense pheromone previously dropped by some agent A_j with even j, $4 \leq j \leq i$.*

▶ **Lemma 10.** *For all even i with $4 \leq i \leq 4\tau$, and for all y with $1 \leq y \leq \lceil \frac{i-2}{4} \rceil$, A_i is at node $(0, -y)$ at the beginning of round $i + y$ and:*
1. *It senses pheromone $\mu \dot{-} (i - 4y - 1)$ to the East and $\mu \dot{-} (i - 3 - 4y)$ to the West.*
2. *If $y = 1$, it is in state* Vert-seek *if $i - 4 \geq \mu$, otherwise it is in state* Vert-bypass.
3. *If $y \geq 2$, it is in state* Vert-seek *if $i - 4y + 1 \geq \mu$, otherwise it is in state* Vert-bypass.

▶ **Lemma 41.** *For every y, $1 \leq y \leq \tau - 1$:*
- *Agent A_{4y-1} has signature $[y; \tau - y]$.*
- *Agent A_{4y} has signature $[-y; \tau - y]$.*
- *Agent A_{4y+1} has signature $[y; -(\tau - y)]$.*
- *Agent A_{4y+2} has signature $[-y; -(\tau - y)]$.*

Mooreover:

- *Agent $A_{4\tau-1}$ has signature $[\tau; 0]$.*
- *Agent $A_{4\tau}$ has signature $[-\tau; 0]$.*

Proof. For y in the given range, we apply Lemma 9 or Lemma 10 as follows:

By Lemma 9 for $i = 4y - 1$, agent A_{4y-1} is at node $(0, \lceil \frac{(4y-1)-1}{4} \rceil) \equiv (0, y)$ at the beginning of round $5y - 1$ and senses pheromone $\mu \dot{-} (4y - 1 - 4y) = 0$ to the East and $\mu \dot{-} (4y - 1 - 2 - 4y) = 0$ to the West. Moreover, A_{4y-1} is in state Vert-bypass because if $y = 1$ we have $i - 3 = 4y - 1 - 3 = 0 \le \mu - 1$, and if $y \ge 2$ we have $i - 4y + 2 = 4y - 1 - 4y + 2 = 1 \le \mu - 1$. Therefore, it executes line 19, moving to the East and entering state Horiz. By Proposition 19, it will keep moving to the East until the end of its lifetime, hence its signature is $[y; \tau - y]$.

By Lemma 10 for $i = 4y$, agent A_{4y} is at node $(0, -\lceil \frac{4y-2}{4} \rceil) \equiv (0, -y)$ at the beginning of round $5y$ and senses pheromone $\mu \dot{-} (4y - 4y - 1) = 0$ to the East and $\mu \dot{-} (4y - 3 - 4y) = 0$ to the West. Moreover, A_{4y} is in state Vert-bypass because if $y = 1$ we have $i - 4 = 4y - 4 = 0 \le \mu - 1$, and if $y \ge 2$ we have $i - 4y + 1 = 4y - 4y + 1 = 1 \le \mu - 1$. Therefore, it executes line 19, moving to the East and entering state Horiz. By Proposition 19, it will keep moving to the East until the end of its lifetime, hence its signature is $[-y; \tau - y]$.

By Lemma 9 for $i = 4y + 1$, agent A_{4y+1} is at node $(0, \lceil \frac{(4y+1)-1}{4} \rceil) \equiv (0, y)$ at the beginning of round $5y + 1$ and senses pheromone $\mu \dot{-} (4y + 1 - 4y) = \mu - 1 > 0$ to the East and $\mu \dot{-} (4y + 1 - 2 - 4y) = 0$ to the West. Therefore, whether it is in state Vert-seek or Vert-bypass, it executes line 28, moving to the West and entering state Horiz. By Proposition 19, it will keep moving to the West until the end of its lifetime, hence its signature is $[y; -(\tau - y)]$.

By Lemma 10 for $i = 4y + 2$, agent A_{4y+2} is at node $(0, -\lceil \frac{4y+2-2}{4} \rceil) \equiv (0, -y)$ at the beginning of round $5y+2$ and senses pheromone $\mu \dot{-} (4y + 2 - 4y - 1) = \mu - 1 > 0$ to the East and $\mu \dot{-} (4y + 2 - 3 - 4y) = 0$ to the West. Therefore, whether it is in state Vert-seek or Vert-bypass, it executes line 28, moving to the West and entering state Horiz. By Proposition 19, it will keep moving to the West until the end of its lifetime, hence its signature is $[-y; -(\tau - y)]$.

For the last two items, we apply Lemma 9 for $i = 4\tau - 1$, and Lemma 10 for $i = 4\tau$.

By Lemma 9 for $i = 4\tau - 1$, agent $A_{4\tau-1}$ is at node $(0, \lceil \frac{(4\tau-1)-1}{4} \rceil) \equiv (0, \tau)$ at the beginning of round $5\tau - 1$. In view of Remark 35, this means that $A_{4\tau-1}$ arrives at node $(0, \tau)$ at the end of its lifetime, hence its signature is $[\tau; 0]$.

By Lemma 10 for $i = 4\tau$, agent $A_{4\tau}$ is at node $(0, -\lceil \frac{(4\tau)-2}{4} \rceil) \equiv (0, -\tau)$ at the beginning of round 5τ. In view of Remark 35, this means that $A_{4\tau}$ arrives at node $(0, -\tau)$ at the end of its lifetime, hence its signature is $[-\tau; 0]$. ◀

▶ **Theorem 8.** *Algorithm 1 correctly solves the treasure hunt problem for all combinations of parameters (τ, μ) with $\tau \ge 1$ and $\mu \ge 2$.*

Proof. By Lemmas 33 and 41, agents $A_1, \ldots, A_{4\tau}$ have all possible signatures for the given lifetime parameter τ. It follows that, by the end of the lifetime of agent $A_{4\tau}$, all nodes at distance τ or less from the nest are explored. ◀

▶ Remark 42. It may be instructive to pinpoint exactly why Algorithm 1 fails for $\mu = 1$. In the proof of Lemma 41, we need agents A_{4y+1} and A_{4y+2} to detect a strictly positive amount of pheromone to the East, at the point where they are supposed to turn West. With $\mu = 1$, they would detect a pheromone level of $\mu - 1 = 0$ to the East, which would break the algorithm. Intuitively, because of the alternating North-South dispatching of agents at the nest, the pheromone left by an agent in one half-plane needs to remain for at least *two* rounds so that it can be detected by the next agent that is sent to the same half-plane.

C Proof of Theorem 17 from Section 4

We prove next that Algorithm 2 solves the treasure hunt problem assuming that $\mu \geq 1$, $\tau \geq 16$ and $s = 6$. It does so in $11\tau - 6s + 2$ rounds, using $10\tau - 6s + 3$ agents and $28\tau + O(s) + 8$ pheromone drops. (Theorem 17).

In order to argue about the input of an agent at a node (i.e., the pheromone presence or absence in the four neighboring nodes and the comparison between the pheromone levels) we use the following definition of *pheromone scheme*.

▶ **Definition 43.** *Let* $\mathrm{PheroScheme}(x, y)$ *denote the* (absolute) pheromone scheme *(of the pheromone drops) at node* (x, y) *of the grid. It is defined as a binary word such that if* $\mathrm{PheroScheme}(x, y)[i] = 1$*, then at the beginning of round* i*, there is a quantity* μ *of the pheromone at* (x, y) *(it had to be dropped during round* $i - 1$*). Otherwise, if* $\mathrm{PheroScheme}(x, y)[i] = 0$*, the quantity of the pheromone is less than* μ*. i.e. no drop were done during the previous round.*

$\mathrm{PheroScheme}(x, y)[r_1, r_2]$ *denotes the pheromone scheme at node* (x, y) *for every round in the given range, from round* r_1 *up to round* r_2*. Notice that, in particular,* $\mathrm{PheroScheme}(0, 1)[1] = \mathrm{PheroScheme}(0, -1)[1] = \mathrm{PheroScheme}(1, 0)[1] = \mathrm{PheroScheme}(-1, 0)[1] = 0$.

An absolute pheromone scheme $\mathrm{PheroScheme}(x, y)$ *can be projected on the pheromone drops of a particular agent* A*. In this case, we say* $\mathrm{PheroScheme}(x, y)$ *of (or for)* A*. In particular, if no other agent (except* A*) drops pheromone in* (x, y) *during some interval of rounds* $[r_1, r_2]$*, then* $\mathrm{PheroScheme}(x, y)[r_1, r_2]$ *equals to* $\mathrm{PheroScheme}(x, y)[r_1, r_2]$ *of* A*.*

The observations below argue about the basic properties of the different types of states of an agent (defined by the value of variable state). States Init, Vert-seek, Vert-bypass and Horiz have similar properties as in Algorithm 1. These and the properties of the new states Pattern(w) and Forward(k)-Explore(E) can be easily verified from the pseudo-code.

▶ **Observation 44.**
At every next round after the change of state to Horiz*, an agent does not change its direction and drops no pheromone (lines 35 - 36).*
At every next round after the change of state to Pattern(w)*, an agent does not move (*dir $= \perp$*; line 44 - 45).*
During the next rounds after the change of state to either Forward(k)-Explore(E) *or* Forward(k)-Explore(W)*, an agent first does not change its direction during* k *rounds and then moves either east or west, respectively, and never drops pheromone (lines 37-43).*

▶ **Observation 45.** *Whenever in* Vert-seek*, an agent can move only to* Vert-bypass *(lines 25 and 27). Whenever in* Vert-bypass*, no change to* Vert-seek *is possible. After moving to* Vert-bypass*, an agent can either never change its direction or turn* only once *and only after moving to a new state:* Horiz*,* Pattern()*,* Forward()-Explore() *(by the code of* Interpret-Signals-phase1 *and of* Interpret-Signals-phase2*, and Observation 44).*

The next lemma proves that agents A_1, A_2 and A_3 move from the nest to the adjacent nodes where they stay forever in state Pattern() for signaling, as is precised by the given pheromone schemes (lines 2 - 7). Agents A_4 and A_5 read these schemes in round 4 and 5 respectively and move accordingly to explore the x-axis (lines 8 - 11).

▶ **Lemma 46.** *All the* x*-axis grid nodes at distance* $\leq \tau$ *are visited, by agents* A_4 *and* A_5 *moving east and west (resp.) starting from the nest (and being in state* Horiz*, while never dropping any pheromone).*

In their first round of life, agents A_1, A_2 and A_3 move each from the nest to the adjacent node, at the East, West and North, respectively, in state Pattern()*, where they (first drop pheromone and then) stay forever and contribute to the following pheromone schemes:*

$$\begin{aligned} &\text{rounds:}\ \ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \\ \text{PheroScheme}(1,0)[1,7] &= 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \\ \text{PheroScheme}(-1,0)[1,7] &= 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \\ \text{PheroScheme}(0,1)[1,6] &= 0\ 0\ 0\ 1\ 0\ 1 \end{aligned}$$

The overall number of pheromone drops by agents $A_1 - A_5$ (during their whole lifetime) is 10*.*

Proof. In round 1, agent A_1 senses no pheromone in the neighboring nodes and thus according to the condition in line 2, it will move east and drop there pheromone. It will then drop pheromone according to its state Pattern(11001), which will effect the pheromone quantity in node $(1, 0)$ starting from round 3. A_1 is the only agent which drops pheromone on the node $(0, 1)$ (this can be easily checked from the pseudo-code and observations 44 and 45). From all these claims, PheroScheme$(1, 0)[1, 7] = 0111001$.

In round 2, agent A_2 senses pheromone at the East (node $(1, 0)$), and no pheromone on the other cardinal directions. Only the condition at line 4 holds for A_2. Thus, it will move west and drop there pheromone. It will then drop pheromone according to its state Pattern(111), which will effect the pheromone quantity at node $(-1, 0)$ starting from round 4. A_2 is the only agent which drops pheromone on the node $(-1, 0)$ (this can be easily checked from the pseudo-code and observations 44 and 45). From all these claims, PheroScheme$(-1, 0)[1, 7] = 0011110$.

In round 3, according to the pheromone schemes PheroScheme$(1, 0)$ and PheroScheme$(-1, 0)$, agent A_3 senses pheromone at the East and west (nodes $(1, 0)$ and $(-1, 0)$), and no pheromone at two other cardinal directions. Only the condition at line 6 holds for A_3. Thus, it will move north and drop there pheromone. It will then drop pheromone according to its state Pattern(01), which will effect the pheromone quantity at node $(1, 0)$ starting from round 5. From all these claims, PheroScheme$(0, 1)[1, 4] = 0001$.

In round 4, according to the pheromone schemes PheroScheme$(0, 1)$, PheroScheme$(0, -1)$ and PheroScheme$(1, 0)$ up to round 4, agent A_4 senses equal quantity of pheromone at each of the East, west and north directions (there is μ pheromone on each of these nodes), and no pheromone at the South. This corresponds only to the condition at line 8. So, A_4 will move east and to a new state Horiz, without dropping any pheromone. From the latter, PheroScheme$(1, 0)[1, 5] = 00010$.

In round 5, according to the pheromone schemes PheroScheme$(1, 0)$, PheroScheme$(-1, 0)$ and PheroScheme$(0, 1)$ up to round 5, agent A_5 senses more pheromone at the West (there is μ pheromone) than at the East and at the North (there is $\mu - 1$ pheromone at each node), and still no pheromone at the South. This corresponds only to the condition at line 10. So, A_5 will move west and to a new state Horiz, without dropping any pheromone. From the latter, PheroScheme$(0, 1)[1, 6] = 000101$.

Whenever in state Horiz, the agent does not changes its direction (and drops no pheromone), by Observation 44. Thus, the x-axis is explored by agents A_4 and A_5 up to distance τ.

Agents A_4 and A_5 never drop any pheromone, while A_1, A_2 and A_3 drop 4, 4 and 2 times, respectively, according to the obtained pheromone schemes. This gives the stated 10 drops by agents $A_1 - A_5$. ◀

After agents A_4 and A_5, which are sent to explore the x-axis, next agents are sent to explore further lines (for signaling or actually exploring). According to the following lemmas

47 and 48, these agents leave the nest, at the end of their first round of lifetime, alternating either north or south, and either in state Vert-bypass (only for A_6 and A_7; Lemma 47) or in state Vert-seek (rest of the agents; Lemma 48). Then, Lemma 49 proves that each such agent keeps moving at each round in the direction adopted in the nest; without dropping any pheromone, but once, when exiting the nest.

▶ **Lemma 47.** *In round 6, agent A_6 moves to state* Vert-bypass *and goes north (while dropping pheromone there). A_6 is the first agent moving to this state from* Init *and going north. In round 7, agent A_7 moves to state* Vert-bypass *and goes south (while dropping pheromone there). A_7 is the first agent moving to this state from* Init *and going south. Moreover,*

$$\begin{aligned} \text{rounds: } & \ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \\ \text{PheroScheme}(1,0)[1,8] = & \ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \\ \text{PheroScheme}(-1,0)[1,8] = & \ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \\ \text{PheroScheme}(0,1)[1,8] = & \ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \\ \text{PheroScheme}(0,-1)[1,8] = & \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{aligned}$$

Proof. According to the pheromone schemes of Lemma 46, in round 6, A_6 in the nest (state Init) senses equal quantity of pheromone at the West and at the North (there is μ pheromone) and less pheromone on the East (there is $\mu - 1$) and still no pheromone at the South. This corresponds only to the condition at line 12. A_6 will move north in a new state Vert-bypass and drop pheromone there. That is why PheroScheme$(0,1)[1,7] =$ 0 0 0 1 0 1 1 (while Lemma 46 stated that PheroScheme$(0,1)[1,6] =$ 0 0 0 1 0 1). By Lemma 46 again, this is the first agent in state Vert-bypass.

According to the pheromone schemes up to round 7, in round 7, A_7 in the nest (state Init) senses equal quantity of pheromone at the East and at the North (there is μ pheromone) and less pheromone on the West (there is $\mu - 1$) and still no pheromone at the South. This corresponds only to the condition at line 14. A_7 will move south in a new state Vert-bypass and drop pheromone there. That is why PheroScheme$(0,1)[1,8] =$ 0 0 0 0 0 0 0 1. This is the very first agent in state Vert-bypass going south.

All the above implies the pheromone schemes stated by the lemma. ◀

▶ **Lemma 48.** *Starting from round 8, every even round, an agent in* Vert-seek *leaves the nest to the North (and drops there pheromone), and every odd round, an agent in* Vert-seek *leaves the nest to the South (and drops there pheromone). Before round 8, no agent ever moves to state* Vert-seek*.*

Proof. By lemmas 46 and 47, before round 8, no agent ever moves to state Vert-seek.

The rest of the lemma statement is proven by a simple induction on round numbers starting from round 8. In round 8, by pheromone schemes obtained in Lemma 47, A_8 in the nest (state Init) senses that in the South there is more pheromone (there is μ) than in any other direction. This corresponds only to the condition at line 16. A_8 will move north in a new state Vert-seek and drop pheromone there. In the next round 9 there will be μ pheromone north to the nest. Thus, condition at line 18 is the only condition satisfied for A_9 at the nest. It will move south in a new state Vert-seek and drop pheromone there. In the next round 10 there will be μ pheromone south to the nest.

By a simple induction on even rounds (and the same arguments on the pseudo-code as above), it follows that if agent A_k in Vert-seek leaves the nest to the North in an even round k, it can do so only if the condition in line 16 holds. A_k drops pheromone on the North that is sensed by agent A_{k+1} in the next round. This, together with the fact that condition in line 16 holds in round k, the condition in line 18 holds for A_{k+1} in and odd round $k+1$. So it moves to Vert-seek and leaves the nest to the South dropping pheromone there. This implies, by similar arguments, that agent A_{k+2} in Vert-seek leaves the nest to the North in round $k+2$, and A_{k+3} to the South in round $k+3$. This completes the induction. ◀

▶ **Lemma 49.** *Any agent in* Vert-bypass *or in* Vert-seek *state keeps its direction (*dir *value) and keeps moving at every round according to the direction adopted in the nest, which is either north or south. Each such agent drops pheromone only after its first change to the state (*Vert-bypass *or* Vert-seek*), and never does this after.*

Proof. From lemmas 47 and 48, no agent is in state Vert-bypass or Vert-seek, before round 6. Then, in the nest, A_6 and A_7 move to state Vert-bypass moving north and south respectively, while dropping pheromone in the destination node. From Lemma 48, every agent exiting the nest starting from round 8 is in Vert-seek state, moving either north or south, while dropping pheromone in the destination node. Then, it does not change its direction and drops no pheromone, if it keeps the same state (line 23). It can change its state only to Vert-bypass in lines 25 and 27.

Vert-bypass state is treated only by the two procedures Interpret-Signals-phase1 and Interpret-Signals-phase2. In the procedures, the direction of the agent movement changes only together with the state change. If an agent stays in Vert-bypass, dir stays unchanged and no pheromone is dropped (lines 74 and 89), i.e, an agent continues its movements either north or south according to the direction adopted in the nest (without dropping any pheromone). ◀

Notice that by the analysis of pheromone drops until now (lemmas 46 and 49), agents $A_1 - A_5$ drop 10 times and then each agent exiting the nest drops pheromone once. It drops nothing afterword, if being still in either Vert-bypass or Vert-seek.

Next corollaries are obtained from the two previous lemmas and state that each agent in either Vert-bypass or Vert-seek state never stops moving in the same direction (when it is still in one of these states) and is followed by another such agent with a difference of two rounds.

▶ **Corollary 50.** *Any agent in either* Vert-bypass *or* Vert-seek *state and which is located in node $(0, y)$ at the beginning of a round, performed $|y|$ moves (during the first y rounds) on the y-axis of the grid, and* moves $= |y|$ *if* $|y| \leq s$ *and* moves $= s + 1$ *otherwise.*

▶ **Corollary 51.** *Let B be an agent in either* Vert-bypass *or* Vert-seek *state located in node $(0, y)$ at the beginning of round r, for $|y| \geq 1$. Then, at the beginning of round $r + 2$, there is agent B' in $(0, y)$ in either* Vert-bypass *or* Vert-seek *state, and there is no agent beforehand.*

In the next lemma, we focus only on the lines at distance 1 to s from the nest. For each such, not yet explored, line at distance y, three agents are used for signaling. Moving on the y-axis, they arrive to this line, node $(0, y)$, in either Vert-bypass or Vert-seek state, in round r. Those arriving in Vert-seek state, move first to Vert-bypass (line 25). Then, they act according to the Interpret-Signals-phase1 procedure (lines 25 and 31). One moves east (line 60), the second one (arriving in 2 rounds later; round $r + 2$) moves west (line 62) and the third one (in another 2 rounds; round $r + 4$) moves north, resp. south (line 64). In these locations the three agents stay forever, being in state Pattern(w) and dropping pheromone according to the binary word w (Pattern$(1\,01\,01\,01)$ for the first agent, Pattern$(1\,01\,00\,01\,01)$ for the second and third). We prove that these agents contribute to the following (absolute) pheromone schemes:

$$\begin{array}{rcccccccc} \textit{rounds:} & r+0 & 1\,2 & 3\,4 & 5\,6 & 7\,8 & 9\,10 & 11\,12 & 13\,14 \\ \text{PheroScheme}(1,y)[r,r+14] = & 0 & 0\,1 & 0\,1 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 0\,0 \\ \text{PheroScheme}(-1,y)[r,r+14] = & 0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,1 & 0\,1 & 0\,0 \\ \text{PheroScheme}(0,y+\frac{y}{|y|})[r,r+14] = & 0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,1 & 0\,1 \end{array}$$

Different three bit patterns formed by these schemes at each round allow agents to deduce the actions to perform. Every second round, another agent arrives at $(0, y)$. It senses pheromone in one of the neighboring nodes, while it is in Vert-bypass state and thus Interpret-Signals-phase1 is applied (by the same arguments as for the first three agents). In round $r + 6$, according to the pheromone schemes above and lines 59 - 60, the arriving agent turns east in state Horiz to explore the east half of the line. In round $r+8$, the arriving agent turns west to explore the west half of the line (lines 61 - 62). In round $r + 10$ (resp. $r + 12$), the arriving agent changes to Forward(s)-Explore(E) (resp. Forward(s)-Explore(W)) to explore the east (resp. west) half of the line, s steps farther (lines 69 - 72). Finally, in round $r + 14$, the agent continues moving in the same direction and arrives to the next unexplored line $y + \frac{y}{|y|}$, where the lemma conditions hold for it, as for agent B^1.

What concerns the complexities, for each line at distance 1 to s from the nest (north and south, i.e. $2s$ lines), the three signaling agents drop pheromone 12 times in overall. The exploring agents drop no pheromone.

▶ **Lemma 52.** *Let B^1 be an agent in state* Vert-bypass *located in node $(0, y)$ at round r such that $1 \le |y| \le s$ and $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ (no pheromone on the East, West and in front of the agent). Moreover, any agent located farther from the nest than B^1 at round r, never drops any pheromone. Then:*

1. *By round $r + 8$, there is an agent that starts moving horizontally (in state* Horiz*) to the East (resp. West) from node $(0, y)$.*
2. *By round $r + 12$, there is also an agent that moves to state* Forward(s)-Explore(E) *and an agent moving to* Forward(s)-Explore(W)*, both in node $(0, y)$.*
3. *In round $r + 15$, there is an agent in node $(0, y + \frac{y}{|y|})$ in state* Vert-bypass*, for which the conditions of the lemma also hold (as for B^1), i.e., $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ and any agent located farther (than node $(0, y + \frac{y}{|y|})$), never drops any pheromone.*
4. *Moreover, each of these agents, including B^1, enters node $(0, y)$, exactly $|y|$ rounds since exiting the nest.*
5. *The pheromone utilization (number of drops) during the time period $[r + 1, r + 15]$, on the nodes $(0, y), (1, y)$, $(-1, y)$ and $(0, y + \frac{y}{|y|})$ is 12 drops in overall. (These drops are done uniquely on nodes $(1, y)$, $(-1, y)$ and $(0, y + \frac{y}{|y|})$.)*

Proof. First, let us assume that the agents have enough lifetime to satisfy the lemma claims. We will prove here later that this is indeed the case. By Corollary 50 and the lemma assumption $1 \le |y| \le s$, in round r, B^1 either executes line 25 or line 31, i.e., Interpret-Signals-phase1. Notice that all the conditions in Interpret-Signals-phase1 are mutually exclusive (only one can hold at a time).

Then, since $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$, B^1 moves east in state Pattern(1 01 01 01) and drops no pheromone (line 60). Hence, there is still no pheromone at node $(1, y)$ up to round $r + 1$. Pheromone appears in this node starting from round $r + 2$, dropped by B^1 according to Pattern(1 01 01 01). Thus, PheroScheme$(1, y)[r, r + 8]$ of B^1 is 0 01 01 01 01. (In the following, we show that no other agent arrives at node $(1, y)$ before round $r+6$, and no other agent ever drops pheromone in this node.)

By Corollary 51, the next agent B^2 arrives at node $(0, y)$ in round $r+2$, in either Vert-bypass or Vert-seek state (and there is no any agent beforehand). B^2 senses pheromone on the East, and no pheromone on the West and on the North, according to the pheromone scheme for node $(1, y)$ obtained above. Thus, if it is in Vert-seek state, it moves to Vert-bypass in line 25 and calls Interpret-Signals-phase1. Otherwise, it also calls Interpret-Signals-phase1 in line 31. Inside the procedure, the condition in line 61 holds for B^2 and so it moves west in state Pattern(1 01 00 01 01), dropping no pheromone (line 62).

This proves that:

$$\begin{array}{rllllllllr} \textit{rounds:} & r+0 & 1\,2 & 3\,4 & 5\,6 & 7\,8 & 9\,10 & 11\,12 & 13\,14 & \\ \text{PheroScheme}(1, y)[r, r+14] = & 0 & 0\,1 & 0\,1 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & \text{for } B^1 \\ \text{PheroScheme}(-1, y)[r, r+14] = & 0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & \text{for } B^2 \\ \text{PheroScheme}(0, y+\frac{y}{|y|})[r, r+4] = & 0 & 0\,0 & 0 & & & & & & \end{array}$$

By Corollary 51 again, the next agent B^3 arrives at node $(0, y)$ in round $r+4$, in either Vert-bypass or Vert-seek state. According to the pheromone schemes obtained above, B^3 senses pheromone on the East (node $(1, y)$) and on the West (node $(-1, y)$) and no pheromone in node $(0, y+\frac{y}{|y|})$. Thus, if it is in Vert-seek state, it moves to Vert-bypass in line 25 and calls Interpret-Signals-phase1. Otherwise, it also calls Interpret-Signals-phase1 in line 31. Inside the procedure, the condition in line 63 holds for B^2 and so it moves forward (to node $(0, y+\frac{y}{|y|})$) in state Pattern(1 01 00 01 01), dropping no pheromone (line 64).

This proves that:

$$\begin{array}{rllllllllr} \textit{rounds:} & r+0 & 1\,2 & 3\,4 & 5\,6 & 7\,8 & 9\,10 & 11\,12 & 13\,14 & \\ \text{PheroScheme}(1, y)[r, r+14] = & 0 & 0\,1 & 0\,1 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & \text{for } B^1 \\ \text{PheroScheme}(-1, y)[r, r+14] = & 0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & \text{for } B^2 \\ \text{PheroScheme}(0, y+\frac{y}{|y|})[r, r+14] = & 0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,1 & 0\,1 & \text{for } B^3 \end{array}$$

We show below that these schemes are absolute (and not only projected on agents B^1, B^2 and B^3). As for now, they are absolute up to round $r+4$ by the arguments above, and up to (the beginning of) $r+6$ by Corollary 51.

Again by Corollary 51, in each of the rounds $r+6$, $r+8$, $r+10$, $r+12$, $r+14$, there is an agent at node $(0, y)$ either in Vert-bypass or in Vert-seek state (and no agents in between). Moreover, according to the pheromone schemes above, each two rounds starting from round $r+2$, the condition $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ in line 22 is false. Thus, if the agent arrived at $(0, y)$ is in Vert-seek state, it moves to Vert-bypass in line 25 and calls Interpret-Signals-phase1. Otherwise (it is in Vert-bypass state), it also calls Interpret-Signals-phase1 in line 31. We analyze below the actions of the agent inside this procedure in the aforementioned rounds:

- In round $r+6$, the condition in line 65 holds for the arriving agent B^4, so it moves east in state Horiz, dropping no pheromone (line 66). By Obs. 44, at every next round after, starting from node $(1, y)$, this agent does not change its direction (it continues moving east) and drops no pheromone.
- In round $r+8$, the condition in line 67 holds for the arriving agent B^5, so it moves west in state Horiz, dropping no pheromone (line 68). By Obs. 44, starting from node

$(-1, y)$, this agent does not change its direction (it continues moving west) and drops no pheromone. This and the previous item prove claim (1) of the lemma.

- In round $r+10$, the condition in line 69 holds for the arriving agent B^6, so it changes to state Forward(s)-Explore(E), moves in the original direction from node $(0, y)$, and drops no pheromone (line 70).
- In round $r+12$, the condition in line 71 holds for the arriving agent B^7, so it changes to state Forward(s)-Explore(W), moves in the original direction from node $(0, y)$, and drops no pheromone (line 72). This and the previous item prove claim (2) of the lemma.
- In round $r+14$, the condition in line 73 holds for the arriving agent B^8, so it does not change its state, neither the direction of movement, and drops no pheromone (line 74). In round $r+15$, the agent is in node $(0, y+\frac{y}{|y|})$, still in state Vert-bypass. This is the first agent arriving to this line that far from the nest in state Vert-bypass.
  By the lemma conditions, any agent located farther than B^1 at round r, never drops any pheromone. Earlier arrived agents B^1, B^2 and B^3 dropped pheromone in line $(0, y)$, and in $(0, y+\frac{y}{|y|})$ (for B^3). The four agents B^4 - B^7 that arrived after, moved to Horiz or Forward(s)-Explore(d) states, in which an agent never drops pheromone. Hence, any agent located farther than $(0, y+\frac{y}{|y|})$ in round $r+15$ (this can be an agent in Forward(s)-Explore(d) state), never drops any pheromone. In particular, $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ holds. This proves claim (3) of the lemma.

It follows now that the pheromone schemes above are indeed absolute. This is because the four agents B^4 - B^7 drop no pheromone, at least after arriving at $(0, y)$ and till the end of their lifetime, and the fifth agent B^8 arriving in round $r+14$ cannot change the schemes by this time.

We show now that claim (5) of the lemma holds. First, no two agents drop twice on the nodes $(1, y)$, $(-1, y)$ and $(0, y+\frac{y}{|y|})$, so the schemes reflect also the actual number of pheromone drops. In addition, by the above-mentioned arguments for agent B^8, no additional pheromone is dropped on these nodes by the end of $r+15$. As for node $(0, y)$, no drop is done there at least during $[r+1, r+15]$. Thus, the overall number of pheromone drops on $(0, y), (1, y)$, $(-1, y)$ and $(0, y+\frac{y}{|y|})$ is 12 during $[r+1, r+15]$. This proves claim (5).

Also notice that claim (4) of the lemma holds, by Corollary 50, since all the considered agents in this claim (B^1 and B^4 - B^8) are either in Vert-bypass or in Vert-seek state when entering node $(0, y)$.

Now let us show that $\tau \geq 16$ is indeed large enough to satisfy the lemma claims:
By the lemma claim (4), for every agent considered by the lemma, to arrive to node $(0, y)$, takes $|y| \leq s = 6$ rounds. Then, for agents B^4 - B^7 to perform the actions described by the lemma claims (1) and (2) and to make a move for the agent B^8 in claim (3) (from node $(0, y)$ to $(0, y+\frac{y}{|y|})$), agents B^1, B^2 and B^3 should keep alive up to the last pheromone drop they are instructed to do according to the binary word w in their Pattern(w) state. It is easy to see from the pheromone schemes obtained above that B^1 has to stay alive for at least $|y|+8$ rounds, B^2 and B^3 has to stay alive for at least $|y|+10 = 16 \leq \tau$. ◀

The following lemma proves that the lines at distance 1 to $2 \cdot s$ from the nest are visited. It also gives the time, agent and pheromone utilization complexities required for that. The proof is based on a simple induction proving Lemma 52 for every $1 \leq |y| \leq s$. For the basis, we show that Lemma 52 holds for $y = 1$ and A_6, and for $y = -1$ and A_7 (by lemmas 47 and 48). Then the induction step is obtained directly from the Lemma 52 itself, claim (3). Then, by the same claim (3), claim (2) of the following lemma is easily obtained too. We get also claim (1) below, using claims (1), (2) and (4) of Lemma 52. Claims (1) and (4) ensure that, for $1 \leq |y| \leq s$, every node on line $(0, y)$ at distance τ from the nest is visited (by agents in state Horiz). Claims (2) and (4) imply the same but for $s + 1 \leq |y| \leq 2 \cdot s$ (by agents in states Forward()-Explore()).

▶ **Lemma 53.**

1. *For every $1 \leq |y| \leq 2 \cdot s$, every node on line $(0, y)$ at distance $\leq \tau$ from the nest is visited by at least one agent, in $7 + 15 \cdot s - 2 + \tau - s$ rounds, and using $38s$ pheromone drops and $14s$ agents.*
2. *In round $6 + 15 \cdot s$, there is an agent B^{s+1} in node $(0, s+1)$ and in round $7 + 15 \cdot s$ an agent B^{-s-1} in node $(0, -s-1)$, both in state* Vert-bypass. *For each of them $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ and any agent located farther than B and B', never drops any pheromone. (B and B' satisfy Lemma 54 condition.)*

Proof. We use Lemma 52 and an induction on $1 \leq |y| \leq s$ to prove the current result. We should first show that there exist two agents satisfying Lemma 52 conditions for $|y| = 1$, i.e one agent for $y = 1$ in round 7, and another agent for $y = -1$ in round 8 (the base of induction). Assume that this is true (we show this later below). Then, claim (3) of the same lemma and a simple induction imply that there is an agent B^y on $(0, y)$ every 15 rounds satisfying the lemma conditions for every $1 \leq |y| \leq s$. Hence, Lemma 52 holds for every $1 \leq |y| \leq s$. We show that this implies the current lemma claim (1).

- *For $1 \leq |y| \leq s$:* By claims (1) and (4) of Lemma 52 and the fact that an agent in state Horiz moves at every round (lines 35 - 36), every node on line $(0, y)$ at distance τ from the nest is visited.
- *For $s + 1 \leq |y| \leq 2 \cdot s$:* By claims (2) and (4) of Lemma 52 and the fact that an agent in state Forward(k)-Explore(d) moves at every round (lines 37 - 43), every node at distance τ from the nest on every line $(0, y)$ is visited. Finally, by claim (2) of Lemma 52, 13 rounds (instead of 15, starting from round r in the lemma) are enough for sending an agent to explore line $(0, 2s)$ or $(0, -2s)$. Hence, the overall time complexity is $7 + 15 \cdot s - 2 + \tau - s$ rounds. This is the round when an agent visits node $(\tau - 2s, -2s)$.
  By claim (2) of Lemma 52, up to round $r + 12$, agents arriving at $(0, y)$ are used to explore lines at distance y and $y + s$ (resp. $y - s$, on the southern half of the grid) from the nest. By Corollary 51, during these rounds, 7 agents arrive (one agent every second round). Thus the overall agent utilization complexity (to explore these lines, in both halves of the grid) is $2 \cdot 7s$.
  To compute the pheromone utilization to explore all the lines at distance $1 \leq |y| \leq 2 \cdot s$, first note that every agent used for this, drops pheromone when exiting the nest (lemmas 47 and 48). To this we have to add the pheromone utilization of $2 \cdot 12s$ due to claim (5) of Lemma 52. This gives the claimed $38s = 2 \cdot 7s + 2 \cdot 12s$ drops.

In particular, Lemma 52 claim (3) holds for $|y| = s$ in round $6 + 15 \cdot s$ (by the calculations above) for agent B^{s+1} in node $(0, s + 1)$ and in round $7 + 15 \cdot s$ for agent B^{-s-1} in node $(0, -s - 1)$, as is required by the claim (2) of the current lemma.

It is left to prove Lemma 52 for $|y| = 1$:

By lemmas 47 and 48, the very first agent moving to the North (resp. South) from the nest in state Vert-bypass is A_6 (resp. A_7). By Lemma 2, any other agent exiting the nest before A_6 cannot visit lines $(y, 0)$ for $|y| > 1$. In fact, it can be easily checked from the pseudo-code (lines 2 - 12) that all agents exited the nest before round 6, either moved east or west in state Horiz, or moved east, west or north in state Pattern(). Thus, there is no pheromone and no agents in the grid, except for the nodes adjacent to the nest and on the x-axis. Because of that, in round 7 (resp. 8), all the conditions of Lemma 52 hold for A_6 (resp. A_7) while it is located in node $(0, 1)$ (resp. $(0, -1)$). ◀

Next, we address the rest of the lines, at distance (y) further than s from the nest. The exploration of the immediate line y being already accomplished, the algorithm here differs by the fact that the signaling agents (still three such agents) instruct the exploring agents only to change to either Forward(s)-Explore(E) or Forward(s)-Explore(W), and so only to explore the eastern or western half of the line which is s steps farther (claim (1) of the next lemma). See Interpret-Signals-phase2 that implements this part of the algorithm.

Thus, similarly to lines at distance 1 to s, for each further line (at distance y from $s+1$ to $\tau - s$), three agents are used for signaling. Moving on the y-axis, they arrive to the line, node $(0, y)$, in either Vert-bypass or Vert-seek state, in round r. Those arriving in Vert-seek state, move first to Vert-bypass (line 27). Then, they act according to the Interpret-Signals-phase2 procedure (lines 27 and 33). One moves east (line 79), the second one (arriving in 2 rounds later; round $r+2$) moves west (line 81) and the third one (in another 2 rounds; round $r+4$) moves north, resp. south (line 83). In these locations the three agents stay forever, being in state Pattern(1 01 01) and dropping pheromone according to the given binary word. We prove that these agents contribute to the following (absolute) pheromone schemes:

rounds: $r+0$	1 2	3 4	5 6	7 8	9 10
$\text{PheroScheme}(1, y)[r, r+14] = 0$	0 1	0 1	0 1	0 0	0 0
$\text{PheroScheme}(-1, y)[r, r+14] = 0$	0 0	0 1	0 1	0 1	0 0
$\text{PheroScheme}(0, y + \frac{y}{\lvert y\rvert})[r, r+14] = 0$	0 0	0 0	0 1	0 1	0 1

Every second round, another agent arrives at $(0, y)$. It senses pheromone in one of the neighboring nodes, while it is in Vert-bypass state and thus Interpret-Signals-phase2 is applied (by the same arguments as for the first three agents). In round $r+6$ (resp. $r+8$), the arriving agent changes to Forward(s)-Explore(E) (resp. Forward(s)-Explore(W)) to explore the east (resp. west) half of the line, s steps farther (lines 84 - 87). Finally, in round $r+10$, the agent continues moving in the same direction and arrives to the next line, at distance $y + \frac{y}{|y|}$, where the lemma conditions hold for it, as for agent B^1.

Concerning the complexities, for each line at distance y from $s+1$ to $\tau - s$, from the nest (north and south), the three signaling agents drop pheromone 9 times in overall. The exploring agents drop no pheromone.

▶ **Lemma 54.** *Let B^1 be an agent in state* Vert-bypass *located in node $(0, y)$ at round r such that $s+1 \le |y| \le \tau - s$ and $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ (no pheromone on the East, west and in front of the agent). Moreover, any agent located farther from the nest than B^1 at round r, never drops any pheromone. Then:*

1. *By round r+6, there is an agent that starts moving according to the state* Forward(s)-Explore(E) *and by round r+8, an agent that starts moving according to the state* Forward(s)-Explore(W)*, both from node $(0, y)$.*
2. *In round $r+11$, there is an agent in node $(0, y + \frac{y}{|y|})$ in state* Vert-bypass*, for which the conditions of the lemma also hold (as for B^1), i.e., $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ and any agent located farther (than node $(0, y + \frac{y}{|y|})$), never drops any pheromone.*
3. *Moreover, each of these agents, including B^1, enters node $(0, y)$, exactly $|y|$ rounds since exiting the nest.*

4. *The pheromone utilization (number of drops) during the time period* $[r+1, r+11]$*, on the nodes* $(0, y), (1, y)$*,* $(-1, y)$ *and* $(0, y + \frac{y}{|y|})$ *is 9 drops in overall. (These drops are done uniquely on nodes* $(1, y)$*,* $(-1, y)$ *and* $(0, y + \frac{y}{|y|})$*.)*

Proof. First, let us assume that the agents have enough lifetime to satisfy the lemma claims. We will prove here later that this is indeed the case. By Corollary 50, in round r, B^1 either executes line 27 or line 33, i.e., Interpret-Signals-phase2. Notice that all the conditions in Interpret-Signals-phase2 are mutually exclusive (only one can hold at a time).

Then, since $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$, B^1 moves east in state Pattern(1 01 01 01) and drops no pheromone (line 79). Hence, there is still no pheromone at node $(1, y)$ up to round $r+1$. Pheromone appears in this node starting from round $r+2$, dropped by B^1 according to Pattern(1 01 01). Thus, PheroScheme$(1, y)[r, r+8]$ of B^1 is 0 01 01 01. (In the following, we show that no other agent ever drops pheromone in this node.)

By Corollary 51, the next agent B^2 arrives at node $(0, y)$ in round $r+2$, in either Vert-bypass or Vert-seek state (and there is no any agent beforehand). B^2 senses pheromone on the East, and no pheromone on the West and on the North, according to the pheromone scheme for node $(1, y)$ obtained above. Thus, if it is in Vert-seek state, it moves to Vert-bypass in line 27 and calls Interpret-Signals-phase2. Otherwise (it is in state Vert-bypass), it also calls Interpret-Signals-phase2 in line 33. Inside the procedure, the condition in line 80 holds for B^2 and so it moves west in state Pattern(1 01 01), dropping no pheromone (line 81).

This proves that:

rounds:	$r+0$	1 2	3 4	5 6	7 8	9 10	
PheroScheme$(1, y)[r, r+14] =$	0	0 1	0 1	0 1	0 0	0 0	for B^1
PheroScheme$(-1, y)[r, r+14] =$	0	0 0	0 1	0 1	0 1	0 0	for B^2
PheroScheme$(0, y + \frac{y}{\lvert y\rvert})[r, r+4] =$	0	0 0	0				

By Corollary 51 again, the next agent B^3 arrives at node $(0, y)$ in round $r+4$, in either Vert-bypass or Vert-seek state. According to the pheromone schemes obtained above, B^3 senses pheromone on the East (node $(1, y)$) and on the West (node $(-1, y)$) and no pheromone in node $(0, y + \frac{y}{|y|})$. Thus, if it is in Vert-seek state, it moves to Vert-bypass in line 27 and calls Interpret-Signals-phase2. Otherwise (it is in state Vert-bypass), it also calls Interpret-Signals-phase2 in line 33. Inside the procedure, the condition in line 82 holds for B^2 and so it moves forward (to node $(0, y + \frac{y}{|y|})$) in state Pattern(1 01 01), dropping no pheromone (line 83).

This proves that:

rounds:	$r+0$	1 2	3 4	5 6	7 8	9 10	
PheroScheme$(1, y)[r, r+14] =$	0	0 1	0 1	0 1	0 0	0 0	for B^1
PheroScheme$(-1, y)[r, r+14] =$	0	0 0	0 1	0 1	0 1	0 0	for B^2
PheroScheme$(0, y + \frac{y}{\lvert y\rvert})[r, r+14] =$	0	0 0	0 0	0 1	0 1	0 1	for B^3

We show below that these schemes are absolute (and not only projected on agents B^1, B^2 and B^3). As for now, they are absolute up to round $r+4$ by the arguments above, and up to $r+6$ by Corollary 51.

Again by Corollary 51, in each of the rounds $r+6$, $r+8$ and $r+10$, there is an agent at node $(0, y)$ either in Vert-bypass or in Vert-seek state (and no agents in between). Moreover, according to the pheromone schemes above, each two rounds starting from round $r+4$, the condition $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ in line 22 is false. Thus, if the agent arrived at $(0, y)$ is in Vert-seek state, it moves to Vert-bypass in line 27 and calls Interpret-Signals-phase2. Otherwise (it is in Vert-bypass state), it also calls Interpret-Signals-phase2 in line 33. We analyze below the actions of the agent inside this procedure in the aforementioned rounds:

- In round $r+6$, the condition in line 84 holds for the arriving agent B^4, so it changes to state Forward(s)-Explore(E), moves in the original direction from node $(0, y)$, and drops no pheromone (line 70).
- In round $r+8$, the condition in line 86 holds for the arriving agent B^5, so it changes to state Forward(s)-Explore(W), moves in the original direction from node $(0, y)$, and drops no pheromone (line 87). This and the previous item prove claim (1) of the lemma.
- In round $r+10$, the condition in line 88 holds for the arriving agent B^6, so it does not change its state, neither the direction of movement, and drops no pheromone (line 89). In round $r+11$, the agent is in node $(0, y+\frac{y}{|y|})$, still in state Vert-bypass. This is the first agent arriving to this line that far from the nest in state Vert-bypass.
  By the lemma conditions, any agent located farther than B^1 at round r, never drops any pheromone. Later arrived agents B^1, B^2 and B^3 drop pheromone in line $(0, y)$. Agents B^4 and B^5 that arrived next moved to Horiz or Forward(s)-Explore(d) states in which an agent never drops pheromone. Hence, any agent located farther than $(0, y+\frac{y}{|y|})$ in round $r+11$ (this can be an agent in Forward(s)-Explore(d) state), never drops any pheromone. In particular, $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ holds. This proves claim (2) of the lemma.

It follows now that the pheromone schemes above are indeed absolute. This is because agents B^4 and B^5 drop no pheromone, at least after arriving at $(0, y)$ and till the end of their lifetime, and B^6 arriving in round $r+10$ cannot change the schemes by this time.

We show now that claim (4) of the lemma holds. First, no two agents drop twice on the nodes $(1, y)$, $(-1, y)$ and $(0, y+\frac{y}{|y|})$, so the schemes reflect also the actual number of pheromone drops. In addition, by the above-mentioned arguments for agent B^6, no additional pheromone is dropped on these nodes by the end of $r+11$. As for node $(0, y)$, no drop is done there at least during $[r+1, r+11]$. Thus, the overall number of pheromone drops on $(0, y), (1, y)$, $(-1, y)$ and $(0, y+\frac{y}{|y|})$ is 9 during $[r+1, r+11]$. This proves claim (4).

Also notice that claim (3) of the lemma holds, by Corollary 50, since all the considered agents in this claim (B^1, B^4, B^5 and B^6) are either in Vert-bypass or in Vert-seek state when entering node $(0, y)$.

Now let us show that $\tau \geq 16$ is indeed large enough to satisfy the lemma claims:
By the lemma claim (3), for every agent considered by the lemma, to arrive to node $(0, y)$, takes $|y|$ rounds. Then, for B^4, and B^5 to perform the actions described by the lemma claim (1) and to make a move for the agent in claim (2) (from node $(0, y)$ to $(0, y+\frac{y}{|y|})$), agents B^1, B^2 and B^3 should keep alive up to the last pheromone drop they are instructed to do according to the binary word w in their Pattern(w) state. It is easy to see from the pheromone schemes obtained above that each of them has to stay alive for at least $|y|+6 \leq \tau - s + 6 = \tau$ rounds. ◀

By a simple induction using Lemma 54, we prove the following lemma (in a way similar to the one used to prove Lemma 53).

▶ **Lemma 55.** *For every $2 \cdot s+1 \leq |y| \leq \tau$, every node on line $(0, y)$ at distance $\leq \tau$ from the nest is visited by at least one agent, in $7+15s+11(\tau-2s)-5+s$ rounds, using $10\tau-20s-2$ agents and dropping pheromone $28\tau-2+O(s)$ times.*

Proof. We use extensively Lemma 54. First, by Lemma 53, in round $6 + 15 \cdot s$, there is an agent B^{s+1} in node $(0, s+1)$ and in round $7 + 15 \cdot s$ an agent B^{-s-1} in node $(0, -s-1)$, both in state Vert-bypass. For each of them $\varphi_{\mathsf{dir}} = \varphi_E = \varphi_W = 0$ and any agent located farther than B and B', never drops any pheromone. B and B' satisfy Lemma 54 condition.

Then, claim (2) of the same lemma and a simple induction imply that there is an agent B^y every 15 rounds satisfying the lemma conditions for every $s + 1 \leq |y| \leq \tau - s$. Hence, Lemma 54 holds for every $s + 1 \leq |y| \leq \tau - s$.

By claim (1) of Lemma 54 an agent in state Forward(s)-Explore(d) at node $(0, y)$, for $s + 1 \leq |y| \leq \tau - s$, explores the line that is s steps farther. By claim (3) of Lemma 54 and the fact that an agent in state Forward(k)-Explore(d) moves at every round (lines 37 - 43), every node on the line $(0, y)$, for $2 \cdot s + 1 \leq |y| \leq \tau$, at distance τ from the nest is visited. Finally, by claim (1) of Lemma 54, 6 rounds (instead of 11) are enough for exploring line $-\tau$ (starting from the round satisfying Lemma 54 conditions for this line), because only one agent has to be sent to $(0, -\tau)$. Hence, the overall time complexity is $7 + 15s + 11(\tau - 2s) - 5 + s$ rounds. This is the round when an agent visits node $(0, -\tau)$.

By claim (1) of Lemma 54, up to round $r + 8$, agents arriving at $(0, y)$ are used to explore lines at distance $y + s$ (resp. $y - s$, on the southern half of the grid) from the nest. By Corollary 51, during these rounds, 5 agents arrive (one agent every second round). Thus the overall agent utilization complexity to explore lines at distance $2 \cdot s + 1 \leq |y| \leq \tau$, in both halves of the grid, is $2(5(\tau - 2s) - 1) = 10\tau - 20s - 2$ ($-1 \cdot 2$ is because each of the two lines at distance τ require only one agent to be explored).

To compute the pheromone utilization to explore all the lines at distance $2 \cdot s + 1 \leq |y| \leq \tau$, first note that every agent used for this drops pheromone when exiting the nest (Lemma 49). To this we have to add the pheromone utilization of $2 \cdot 9(\tau - 2s)$ due to claim (4) of Lemma 54. Then, there are additional possible drops in lines at distance $\tau - s + 1 \geq |y| \leq \tau$, by the agents entering these lines at most s rounds before the exploration completion of the same half of the grid. There are at most s such agents (in both halves of the grid), having at most s rounds to live. They either stay in state Vert-seek and thus drop no pheromone, or move to Vert-bypass, applying Interpret-Signals-phase2 and possibly dropping some constant number of pheromone before dying. Hence, the overall number of such drops is $O(s)$. This gives the claimed $28\tau - 2 + O(s) = (10\tau - 20s - 2) + 2 \cdot 9(\tau - 2s) + O(s)$ drops. ◀

The theorem below is obtained almost directly from Lemmas 46, 53 and 55.

▶ **Theorem 17.** *Algorithm 2 solves the treasure hunt problem for $\mu \geq 1, \tau \geq 16$ and $s = 6$ in $11\tau - 6s + 2$ rounds, using $10\tau - 6s + 3$ agents and $28\tau + O(s) + 8$ pheromone drops.*

Proof. The correctness of the algorithm follows directly from lemmas 46, 53 and 55. The completion time holds from Lemma 55.

The agent complexity is obtained by summing 5 agents $A_1 - A_5$, used for signaling and exploring the x-axis (Lemma 46), with $14s$ agents, used to explore lines at distance $1 \leq |y| \leq 2s$ (Lemma 53), with $10\tau - 20s - 2$ agents used to explore lines at distance $2 \cdot s + 1 \leq |y| \leq \tau$ (Lemma 55). This gives $10\tau - 6s + 3$ agents.

The pheromone complexity is obtained by summing 10 pheromones dropped by agents $A_1 - A_5$ (Lemma 46) with the pheromone utilization computed in Lemmas 53 and 55. This gives $28\tau + O(s) + 8$ pheromone drops. ◀