



A general class of zero-or-one inflated beta regression models

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ABSTRACT

This paper proposes a general class of regression models for continuous proportions when the data contain zeros or ones. The proposed class of models assumes that the response variable has a mixed continuous–discrete distribution with probability mass at zero or one. The beta distribution is used to describe the continuous component of the model, since its density has a wide range of different shapes depending on the values of the two parameters that index the distribution. We use a suitable parameterization of the beta law in terms of its mean and a precision parameter. The parameters of the mixture distribution are modeled as functions of regression parameters. We provide inference, diagnostic, and model selection tools for this class of models. A practical application that employs real data is presented.

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1. Introduction

Statistical modeling of continuous proportions has received close attention in the past few years. Some examples of proportions measured on a continuous scale include the fraction of income contributed to a retirement fund, the proportion of weekly hours spent on work-related activities, the fraction of household income spent on food, the percentage of ammonia escaping unconverted from an oxidation plant, etc. Usual regression models, such as normal linear or nonlinear regression models, are not suitable for such situations. Different strategies have been proposed for modeling continuous proportions that are perceived to be related to other variables. Regression models that assume a beta distribution for the response variable are of particular interest.

It is well known that the beta distribution is very flexible for modeling limited range data, since its density has different shapes depending on the values of the two parameters that index the distribution: left-skewed, right-skewed, “U”, “J”, inverted “J”, and uniform (see Johnson et al., 1995, Section 25.1). Beta regression models have been studied by Kieschnick and McCullough (2003), Ferrari and Cribari-Neto (2004), Espinheira et al. (2008a,b), Paolino (2001), Smithson and Verkuilen (2006), Korhonen et al. (2007), Simas et al. (2010) and Ferrari and Pinheiro (2010), among others.

Oftentimes, proportions data include a non-negligible number of zeros and/or ones. When this is the case, the beta distribution does not provide a satisfactory description of the data, since it does not allow a positive probability for any particular point in the interval $[0, 1]$. A mixed continuous–discrete distribution might be a better choice. This approach has been considered by Ospina and Ferrari (2010), who used the beta law to define the continuous component of the distribution.

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The discrete component is defined by a Bernoulli or a degenerate distribution at zero or at one. The proposed distributions are usually referred to as zero-and-one inflated beta distributions (mixture of a beta and a Bernoulli distribution) and zero-inflated beta distributions or one-inflated beta distributions (mixture of a beta and a degenerate distribution at zero or at one, depending on the case).

This paper is concerned with a general class of regression models for modeling continuous proportions when zeros or ones appear in the data. Such a class of models is tailored for the situation where only one of the extremes (zero or one) is present in the dataset. We shall consider a mixture of a beta distribution and a degenerate distribution at a fixed known point c , where $c \in \{0, 1\}$. The beta distribution is conveniently parameterized in terms of its mean and a precision parameter. We shall allow the mean and the precision parameter of the beta distribution and the probability of a point mass at c to be related to linear or non-linear predictors through smooth link functions. Inference, diagnostic, and selection tools for the proposed class of models will be presented. Closely related to our work are the papers by Hoff (2007) and Cook et al. (2008). Our model, however, is more general and convenient than those proposed by these authors. It relaxes linearity assumptions, allows all the parameters of the underlying distribution to be modeled as functions of unknown parameters and covariates, and uses a suitable parameterization of the beta law. Unlike their papers, our article offers a comprehensive framework for the statistical analysis of continuous data observed on the standard unit interval with a point mass at one of its extremes.

The paper unfolds as follows. Section 2 presents a general class of zero-or-one inflated beta regression models. Section 3 discusses maximum likelihood estimation. Section 4 is devoted to diagnostic measures. Section 5 contains an application using real data and concluding remarks are given in Section 6. Some technical details are collected in two appendices.

2. Zero-or-one inflated beta regression models

The beta distribution with parameters μ and ϕ ($0 < \mu < 1$ and $\phi > 0$), denoted by $\mathcal{B}(\mu, \phi)$, has the density function

$$f(y; \mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y^{\mu\phi-1} (1-y)^{(1-\mu)\phi-1}, \quad y \in (0, 1), \quad (2.1)$$

where $\Gamma(\cdot)$ is the gamma function. The parameterization employed in (2.1) is not the usual one, but it is suitable for modeling purposes.

If $y \sim \mathcal{B}(\mu, \phi)$, then $E(y) = \mu$ and $\text{Var}(y) = \mu(1-\mu)/(\phi+1)$. Hence, μ is the distribution mean and ϕ plays the role of a precision parameter, in the sense that, for a fixed μ , the larger the value of ϕ , the smaller the variance of y . Since the beta distribution is very flexible, allowing a wide range of different forms, it is an attractive choice for modeling continuous proportions. A possible shortcoming is that it is not appropriate for modeling datasets that contain observations at the extremes (either zero or one). Our focus is on the case where only one of the extremes appears in the data—a situation often found in empirical research. It is then natural to model the data using a mixture of two distributions: a beta distribution and a degenerate distribution in a known value c , where c equals zero or one, depending on the case. Under this approach, we assume that the probability density function of the response variable y with respect to the measure generated by the mixture¹ is given by

$$\text{bi}_c(y; \alpha, \mu, \phi) = \begin{cases} \alpha, & \text{if } y = c, \\ (1-\alpha)f(y; \mu, \phi), & \text{if } y \in (0, 1), \end{cases} \quad (2.2)$$

where $f(y; \mu, \phi)$ is the beta density (2.1). Note that α is the probability mass at c and represents the probability of observing zero ($c = 0$) or one ($c = 1$). If $c = 0$, density (2.2) is called a zero-inflated beta distribution, and if $c = 1$, the density is called a one-inflated beta distribution.

The mean of y and its variance can be written as

$$\begin{aligned} E(y) &= \alpha c + (1-\alpha)\mu, \\ \text{Var}(y) &= (1-\alpha) \frac{\mu(1-\mu)}{\phi+1} + \alpha(1-\alpha)(c-\mu)^2. \end{aligned}$$

Note that $E(y)$ is the weighted average of the mean of the degenerate distribution at c and the mean of the beta distribution $\mathcal{B}(\mu, \phi)$ with weights α and $1-\alpha$. Also, $E(y | y \in (0, 1)) = \mu$ and $\text{Var}(y | y \in (0, 1)) = \mu(1-\mu)/(\phi+1)$. Other properties of this distribution can be found in Ospina and Ferrari (2010).

A general class of zero-or-one inflated beta regression models can be defined as follows. Let y_1, \dots, y_n be independent random variables such that each y_t , for $t = 1, \dots, n$, has probability density function (2.2) with parameters $\alpha = \alpha_t$, $\mu = \mu_t$,

¹ The probability measure P , corresponding to this distribution, defined over the measure space $((0, 1) \cup \{c\}, \mathfrak{B})$, where \mathfrak{B} is the class of all Borelian subsets of $(0, 1) \cup \{c\}$, is such that $P \ll \lambda + \delta_c$, with λ representing the Lebesgue measure, and δ_c is such that $\delta_c(A) = 1$ if $c \in A$ and $\delta_c(A) = 0$ if $c \notin A$ and $A \in \mathfrak{B}$.

and $\phi = \phi_t$. We assume that α_t , μ_t , and ϕ_t are defined as

$$\begin{aligned} h_1(\alpha_t) &= \eta_{1t} = f_1(v_t, \rho), \\ h_2(\mu_t) &= \eta_{2t} = f_2(x_t, \beta), \\ h_3(\phi_t) &= \eta_{3t} = f_3(z_t, \gamma), \end{aligned} \quad (2.3)$$

where $\rho = (\rho_1, \dots, \rho_p)^\top$, $\beta = (\beta_1, \dots, \beta_k)^\top$ and $\gamma = (\gamma_1, \dots, \gamma_m)^\top$ are vectors of unknown regression parameters; $(p+k+m < n)$, $\eta_1 = (\eta_{11}, \dots, \eta_{1n})^\top$, $\eta_2 = (\eta_{21}, \dots, \eta_{2n})^\top$, and $\eta_3 = (\eta_{31}, \dots, \eta_{3n})^\top$ are predictor vectors; $f_1(\cdot, \cdot)$, $f_2(\cdot, \cdot)$, and $f_3(\cdot, \cdot)$ are linear or nonlinear twice continuously differentiable functions in the second argument, such that the derivative matrices $\mathcal{V} = \partial \eta_1 / \partial \rho$, $\mathcal{X} = \partial \eta_2 / \partial \beta$, and $\mathcal{Z} = \partial \eta_3 / \partial \gamma$ have ranks p , k and m , respectively, for all ρ , β , and γ .

Here $v_t = (v_{t1}, \dots, v_{tp'})^\top$, $x_t = (x_{t1}, \dots, x_{tk'})^\top$, and $z_t = (z_{t1}, \dots, z_{tm'})^\top$ are observations on $p' + k' + m'$ known covariates. We also assume that the link functions $h_1 : (0, 1) \rightarrow \mathbb{R}$, $h_2 : (0, 1) \rightarrow \mathbb{R}$, and $h_3 : (0, +\infty) \rightarrow \mathbb{R}$ are strictly monotonic and twice differentiable. Various different link functions may be used. For μ and α , one may choose $h_2(\mu) = \log\{\mu/(1-\mu)\}$ (logit link); $h_2(\mu) = \Phi^{-1}(\mu)$ (probit link), where $\Phi(\cdot)$ denotes the standard normal distribution function; $h_3(\mu) = \log\{-\log(1-\mu)\}$ (complementary log-log link); $h_2(\mu) = -\log\{-\log(\mu)\}$ (log-log link), among others. Possible specifications for ϕ are $h_3(\phi) = \log \phi$ (log link) and $h_3(\phi) = \sqrt{\phi}$ (square-root link).

Model (2.2)–(2.3) has a number of interesting features. The variance of y_t is a function of $(\alpha_t, \mu_t, \phi_t)$ and, as a consequence, of the covariates values. Hence, non-constant response variances are naturally accommodated by the model. Also, the role that the covariates and the parameters play in the model is clear. For example, suppose $c = 0$. In this case, v_t and ρ affect $\Pr(y_t = 0)$, x_t and β control $E(y_t \mid y_t \in (0, 1))$, and z_t and γ influence the precision of the conditional distribution of y_t , given that $y_t \in (0, 1)$. This feature can be very useful for modeling purposes. For instance, if the response is the individual mobile communications expenditure proportion (MCEP), model (2.2)–(2.3) allows the researcher to take into account that some individuals do not spend at all on MCEP and to separately assess the effects of the heterogeneity among consumers and non-consumers of mobile communications (in this connection, see Yoo, 2004).

Special cases. Model (2.2)–(2.3) embodies two general classes of models: the *zero-inflated beta regression model* ($c = 0$) and the *one-inflated beta regression model* ($c = 1$), the first of which is suitable when the data include zeros and the second, when ones appear in the dataset. Each of them leads to a corresponding linear model when the predictors are linear functions of the parameters. In this case, we have $p = p'$, $k = k'$, $m = m'$, $h_1(\alpha_t) = v_t^\top \rho$, $h_2(\mu_t) = x_t^\top \beta$ and $h_3(\phi_t) = z_t^\top \gamma$. Here, $\mathcal{V} = (v_1^\top, \dots, v_n^\top)^\top$, $\mathcal{X} = (x_1^\top, \dots, x_n^\top)^\top$, and $\mathcal{Z} = (z_1^\top, \dots, z_n^\top)^\top$. Also, the nonlinear beta regression model (Simas et al., 2010; Ferrari and Pinheiro, 2010) is a limiting case of our model obtained by setting $\alpha_t = \alpha \rightarrow 0$. If, in addition, the predictor for μ_t is linear and ϕ_t is constant through the observations, we arrive at the beta regression model defined by Ferrari and Cribari-Neto (2004).

3. Likelihood inference

The likelihood function for $\theta = (\rho^\top, \beta^\top, \gamma^\top)^\top$ based on a sample of n independent observations is

$$L(\theta) = \prod_{t=1}^n \text{bi}_c(y_t; \alpha_t, \mu_t, \phi) = L_1(\rho)L_2(\beta, \gamma), \quad (3.1)$$

where

$$\begin{aligned} L_1(\rho) &= \prod_{t=1}^n \alpha_t^{11_{[c]}(y_t)} (1 - \alpha_t)^{1-11_{[c]}(y_t)}, \\ L_2(\beta, \gamma) &= \prod_{t: y_t \in (0, 1)} f(y_t; \mu_t, \phi_t), \end{aligned}$$

with $11_A(y_t)$ being an indicator function that equals 1 if $y_t \in A$, and 0, if $y_t \notin A$. Here, $\alpha_t = h_1^{-1}(\eta_{1t})$, $\mu_t = h_2^{-1}(\eta_{2t})$, and $\phi_t = h_3^{-1}(\eta_{3t})$, as defined in (2.3), are functions of ρ , β , and γ , respectively. Notice that the likelihood function $L(\theta)$ factorizes in two terms, the first of which depends only on ρ (discrete component), and the second, only on (β, γ) (continuous component). Hence, the parameters are separable (Pace and Salvan, 1997, p. 128) and the maximum likelihood inference for (β, γ) can be performed separately from that for ρ , as if the value of ρ were known, and vice-versa.

The log-likelihood function is given by

$$\ell(\theta) = \ell_1(\rho) + \ell_2(\beta, \gamma) = \sum_{t=1}^n \ell_t(\alpha_t) + \sum_{t: y_t \in (0, 1)} \ell_t(\mu_t, \phi_t), \quad (3.2)$$

where

$$\ell_t(\alpha_t) = 11_{[c]}(y_t) \log \alpha_t + (1 - 11_{[c]}(y_t)) \log(1 - \alpha_t), \quad (3.3)$$

$$\ell_t(\mu_t, \phi_t) = \log \Gamma(\phi_t) - \log \Gamma(\mu_t \phi_t) - \log \Gamma((1 - \mu_t) \phi_t) + (\mu_t \phi_t - 1) y_t^* + (\phi_t - 2) y_t^\dagger, \quad (3.4)$$

$y_t^* = \log\{y_t/(1 - y_t)\}$ and $y_t^\dagger = \log(1 - y_t)$ if $y_t \in (0, 1)$, and $y_t^* = 0$ and $y_t^\dagger = 0$ otherwise. We have the following conditional moments of y_t^* and y_t^\dagger :

$$\begin{aligned}\mu_t^* &= E(y_t^* | y_t \in (0, 1)) = \psi(\mu_t \phi_t) - \psi((1 - \mu_t) \phi_t), \\ \mu_t^\dagger &= E(y_t^\dagger | y_t \in (0, 1)) = \psi((1 - \mu_t) \phi_t) - \psi(\phi_t), \\ v_t^* &= \text{Var}(y_t^* | y_t \in (0, 1)) = \psi'(\mu_t \phi_t) - \psi'((1 - \mu_t) \phi_t), \\ v_t^\dagger &= \text{Var}(y_t^\dagger | y_t \in (0, 1)) = \psi'((1 - \mu_t) \phi_t) - \psi'(\phi_t), \\ c^{*\dagger} &= \text{Cov}(y_t^*, y_t^\dagger | y_t \in (0, 1)) = -\psi'((1 - \mu_t) \phi_t),\end{aligned}\quad (3.5)$$

with $\psi(\cdot)$ denoting the digamma function.² Notice that $\ell_1(\rho)$ represents the log-likelihood function of a regression model for binary responses, in which the success probability for the t th observation is $\alpha_t = h_1^{-1}(\eta_{1t})$ (McCullagh and Nelder, 1989, Section 4.4.1). On the other hand, $\ell_2(\beta, \gamma)$ is the log-likelihood function for (β, γ) in a nonlinear beta regression model based on the observations that fall in the interval $(0, 1)$. Hence, the maximum likelihood (ML) estimation for this model can be accomplished by separately fitting a binomial regression model to the indicator variables $1_{\{c\}}(y_t)$, for $t = 1, \dots, n$, and a nonlinear beta regression model to the observations $y_t \in (0, 1)$, for $t = 1, \dots, n$.

The score function, obtained by the differentiation of the log-likelihood function with respect to the unknown parameters (see (A.1)–(A.3); Appendix A), is given by

$$U(\theta) = (U_\rho(\rho)^\top, U_\beta(\beta, \gamma)^\top, U_\gamma(\beta, \gamma)^\top)^\top,$$

where

$$\begin{aligned}U_\rho(\rho) &= \mathbf{v}^\top \mathcal{A} D \mathcal{A}^* (\mathbf{y}^c - \alpha), \\ U_\beta(\beta, \gamma) &= \mathbf{X}^\top (I_n - Y^c) T \Phi (\mathbf{y}^* - \mu^*), \\ U_\gamma(\beta, \gamma) &= \mathbf{Z}^\top (I_n - Y^c) H [\mathcal{M} (\mathbf{y}^* - \mu^*) + (\mathbf{y}^\dagger - \mu^\dagger)].\end{aligned}\quad (3.6)$$

Here, $\mathbf{y}^* = (y_1^*, \dots, y_n^*)^\top$, $\mathbf{y}^\dagger = (y_1^\dagger, \dots, y_n^\dagger)^\top$, $\mathbf{y}^c = (1_{\{c\}}(y_1), \dots, 1_{\{c\}}(y_n))^\top$, $\mu^* = (\mu_1^*, \dots, \mu_n^*)^\top$, $\mu^\dagger = (\mu_1^\dagger, \dots, \mu_n^\dagger)^\top$, and $\alpha = (\alpha_1, \dots, \alpha_n)^\top$ are n -vectors and $\mathcal{M} = \text{diag}(\mu_1, \dots, \mu_n)$, $\mathcal{A} = \text{diag}(1/\alpha_1, \dots, 1/\alpha_n)$, $\mathcal{A}^* = \text{diag}(1/(1 - \alpha_1), \dots, 1/(1 - \alpha_n))$, $D = \text{diag}(1/h'_1(\alpha_1), \dots, 1/h'_1(\alpha_n))$, $\Phi = \text{diag}(\phi_1, \dots, \phi_n)$, $T = \text{diag}(1/h'_2(\mu_1), \dots, 1/h'_2(\mu_n))$, $H = \text{diag}(1/h'_3(\phi_1), \dots, 1/h'_3(\phi_n))$, and $Y^c = \text{diag}(1_{\{c\}}(y_1), \dots, 1_{\{c\}}(y_n))$ are $n \times n$ diagonal matrices. Also, I_n represents the $n \times n$ identity matrix. The maximum likelihood estimators (MLEs) of ρ and $(\beta^\top, \gamma^\top)^\top$ are obtained as the solutions of the nonlinear systems $U_\rho(\rho) = 0$ and $(U_\beta(\beta, \gamma)^\top, U_\gamma(\beta, \gamma)^\top)^\top = 0$. There are no closed form expressions for these estimators, and their computations should be performed numerically using a nonlinear optimization algorithm, e.g., some form of Newton's method (Newton–Raphson, Fisher's scoring, BHHH, etc.) or a quasi-Newton algorithm such as BFGS. For more details on nonlinear optimization, see Press et al. (1992).

An iterative algorithm for maximum likelihood estimation is presented in Appendix B. An alternative way for numerically finding the maximum likelihood estimates of the parameters is provided by the GAMLSS framework implemented in R; see below. An advantage of the estimation procedure described in Appendix B is that it is expressed as an iterative reweighted least squares, enabling us to develop specific diagnostic tools for our general class of zero-or-one inflated beta regression models. The diagnostics measures proposed in Section 4 are not explored in GAMLSS, and represent a central contribution of the present paper.

From the observed information matrix given in (A.4), it can be shown that the Fisher information matrix is

$$K(\theta) = \begin{pmatrix} K_{\rho\rho} & 0 & 0 \\ 0 & K_{\beta\beta} & K_{\beta\gamma} \\ 0 & K_{\gamma\beta} & K_{\gamma\gamma} \end{pmatrix}, \quad (3.7)$$

where $K_{\rho\rho} = \mathbf{v}^\top \mathbf{W}_1 \mathbf{v}$, $K_{\beta\beta} = \mathbf{X}^\top \mathbf{W}_2 \mathbf{X}$, $K_{\gamma\beta} = K_{\beta\gamma}^\top = \mathbf{X}^\top \mathbf{W}_3 \mathbf{Z}$, $K_{\gamma\gamma} = \mathbf{Z}^\top \mathbf{W}_4 \mathbf{Z}$ with $\mathbf{W}_1 = (\mathcal{A}^* + \mathcal{A}) D^2$, $\mathbf{W}_2 = \Phi T \{V^* \mathcal{A}^{*-1}\} T \Phi$, $\mathbf{W}_3 = T \{\Phi (\mathcal{M} V^* + C) \mathcal{A}^{*-1}\} H$, and $\mathbf{W}_4 = H \{(\mathcal{M}^2 V^* + 2 \mathcal{M} C + V^\dagger) \mathcal{A}^{*-1}\} H$. Notice that $K_{\rho\beta} = K_{\beta\rho}^\top = 0$ and $K_{\rho\gamma} = K_{\gamma\rho}^\top = 0$, thus indicating that the parameters γ and $(\beta^\top, \gamma^\top)^\top$ are globally orthogonal (Cox and Reid, 1987) and their MLEs, $\hat{\rho}$ and $(\hat{\beta}^\top, \hat{\gamma}^\top)^\top$, are asymptotically independent.

The inverse of Fisher's information matrix is useful for computing asymptotic standard errors of MLEs. From (3.7) and a standard formula for the inverse of partitioned matrices (Rao, 1973, p. 33), we have

$$K(\theta)^{-1} = \begin{pmatrix} K^{\rho\rho} & 0 & 0 \\ 0 & K^{\beta\beta} & K^{\beta\gamma} \\ 0 & K^{\gamma\beta} & K^{\gamma\gamma} \end{pmatrix}, \quad (3.8)$$

² The digamma function is defined as $\psi(x) = d \log \Gamma(x) / dx$, $x > 0$.

where $K^{\rho\rho} = (\mathcal{V}^\top \mathbf{W}_1 \mathcal{V})^{-1}$, $K^{\beta\beta} = \{\mathcal{X}^\top (\mathbf{W}_2 - \mathbf{W}_3 \mathcal{Z} (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} \mathcal{Z}^\top \mathbf{W}_3) \mathcal{X}\}^{-1}$, $K^{\gamma\beta} = (K^{\beta\gamma})^\top = -(\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} \mathcal{Z}^\top \mathbf{W}_3 \mathcal{X}$, $(\mathcal{X}^\top (\mathbf{W}_2 - \mathbf{W}_3 \mathcal{Z} (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} \mathcal{Z}^\top \mathbf{W}_3) \mathcal{X})^{-1}$, and $K^{\gamma\gamma} = (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} + (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} \mathcal{Z}^\top \mathbf{W}_3 \mathcal{X} (\mathcal{X}^\top (\mathbf{W}_2 - \mathbf{W}_3 \mathcal{Z} (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1} \mathcal{Z}^\top \mathbf{W}_3) \mathcal{X})^{-1} \mathcal{X}^\top \mathbf{W}_3 \mathcal{Z} \times (\mathcal{Z}^\top \mathbf{W}_4 \mathcal{Z})^{-1}$.

Fitting the model using the GAMLSS implementation. The zero-or-one inflated beta distribution has been incorporated in the GAMLSS framework (Rigby and Stasinopoulos, 2005); see Ospina (2006). GAMLSS allows the flexible modeling of each of the three parameters that index the distribution using parametric terms involving linear or nonlinear predictors, smooth nonparametric terms (e.g., cubic splines or loess), and random effects. Maximum (penalized) likelihood estimation is approached through a Newton–Raphson or Fisher scoring algorithm with the backfitting algorithm for the additive components. Our approach consists of an application of the `gamlss` functions, which are fully documented in the `gamlss` package (Stasinopoulos and Rigby, 2007; see also <http://www.gamlss.org>). The structure of the `gamlss` functions is familiar to readers who are used to the R (or S-Plus) syntax (the `glm` function, in particular). The set of `gamlss` packages can be freely downloaded from the R library at <http://www.r-project.org/>.

Large sample inference. If the model specified by (2.2) and (2.3) is valid and the usual regularity conditions for maximum likelihood estimation are satisfied (Cox and Hinkley, 1974, p. 107), the MLEs of θ and $K(\theta)$, $\hat{\theta} = (\hat{\rho}^\top \hat{\beta}^\top, \hat{\gamma})^\top$ and $K(\hat{\theta})$, respectively, are consistent. Assuming that $I(\theta) = \lim_{n \rightarrow \infty} \{n^{-1} K(\theta)\}$ exists and is nonsingular, we have $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}_{p+k+m}(0, I(\theta)^{-1})$, where \xrightarrow{d} denotes convergence in distribution. Note that, if θ_l denotes the l th component of θ , it follows that

$$(\hat{\theta}_l - \theta_l) \{K(\hat{\theta})^{ll}\}^{-1/2} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $K(\hat{\theta})^{ll}$ is the l th diagonal element of $K(\hat{\theta})^{-1}$. A rigorous proof of the aforementioned asymptotic result can be obtained by extending arguments similar to those given in Fahrmeir and Kaufmann (1985).

Let $K(\hat{\rho})^{ss}$, $K(\hat{\beta})^{rr}$, and $K(\hat{\gamma})^{RR}$ be the estimated s th, r th and R th diagonal elements of $K^{\rho\rho}$, $K^{\beta\beta}$, and $K^{\gamma\gamma}$, respectively. The asymptotic variances of $\hat{\rho}_s$, $\hat{\beta}_r$, and $\hat{\gamma}_R$ are estimated by $K(\hat{\rho})^{ss}$, $K(\hat{\beta})^{rr}$, and $K(\hat{\gamma})^{RR}$, respectively. If $0 < \varsigma < 1/2$, and z_ς represents the ς th quantile of the $\mathcal{N}(0, 1)$ distribution, we have $\hat{\rho}_s \pm z_{1-\varsigma/2} (K(\hat{\rho})^{ss})^{1/2}$, $\hat{\beta}_r \pm z_{1-\varsigma/2} (K(\hat{\beta})^{rr})^{1/2}$, and $\hat{\gamma}_R \pm z_{1-\varsigma/2} (K(\hat{\gamma})^{RR})^{1/2}$ as the limits of asymptotic confidence intervals (ACI) for ρ_s , β_r , and γ_R , respectively, all with asymptotic coverage $100(1 - \varsigma)\%$. Additionally, an approximate $100(1 - \varsigma)\%$ the mean response for $\mu^* = E(y)$, the mean response for the given covariates v and x , can be computed as

$$(\hat{\mu}^* - z_{1-\varsigma/2} \text{s.e.}(\hat{\mu}^*), \hat{\mu}^* + z_{1-\varsigma/2} \text{s.e.}(\hat{\mu}^*)),$$

where $c = 0$ or $c = 1$, depending on the case; $\hat{\mu}^* = c\hat{\alpha} + (1 - \hat{\alpha})\hat{\mu}$, with $\hat{\alpha} = h_1^{-1}(\hat{\eta}_1)$; $\hat{\mu} = h_2^{-1}(\hat{\eta}_2)$; $\hat{\eta}_1 = f_1(v; \hat{\rho})$; $\hat{\eta}_2 = f_2(x; \hat{\beta})$;

$$\text{s.e.}(\hat{\mu}^*) = \sqrt{\{(c - \hat{\mu})/h'_1(\hat{\alpha})\}^2 v^\top \hat{K}^{\rho\rho} v + \{(1 - \hat{\alpha})/h'_2(\hat{\mu})\}^2 x^\top \hat{K}^{\beta\beta} x}.$$

Here, $\hat{K}^{\rho\rho}$ and $\hat{K}^{\beta\beta}$ respectively equal $K^{\rho\rho}$ and $K^{\beta\beta}$, evaluated at $\hat{\theta}$.

The likelihood ratio, Rao's score, and Wald's (W) statistics to test hypotheses on the parameters can be calculated from the log-likelihood function, the score vector, the Fisher information matrix, and its inverse given above. Their null distributions are usually unknown and the tests rely on asymptotic approximations. In large samples, a chi-squared distribution can be used as an approximation to the true null distributions. For testing the significance of the i th regression parameter that models μ , one can use the signed square root of Wald's statistic, $\hat{\beta}_i/\text{s.e.}(\hat{\beta}_i)$, where $\text{s.e.}(\hat{\beta}_i)$ is the asymptotic standard error of the MLE of β_i obtained from the inverse of Fisher's information matrix evaluated at the maximum likelihood estimates. The limiting null distribution of the test statistic is standard normal. Significance tests on the γ 's and ρ 's can be performed in a similar fashion.

Finite-sample performance of the MLEs. We now present the results of two Monte Carlo simulation experiments in order to investigate the finite-sample performance of the MLEs. The first experiment evaluates the impact of the magnitude of the probability of zero response on the MLEs. Here, the sample size is $n = 150$ and we focus on the zero-inflated beta regression model with $\text{logit}(\mu) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$, $\text{log}(\phi) = \gamma_0 + \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3$, and α being constant for all observations. The true parameter values were taken as $\beta_0 = -1$, $\beta_1 = 1$, $\beta_2 = -0.5$, $\beta_3 = 0.5$, $\gamma_0 = 2$, $\gamma_1 = 1$, $\gamma_2 = 0.5$, and $\gamma_3 = 0.5$. Here, we consider four different values for the probability of observing zero: $\alpha = 0.18$, $\alpha = 0.32$, $\alpha = 0.68$, and $\alpha = 0.82$.

The second experiment considers the zero-inflated beta regression model with $\text{logit}(\alpha) = \rho_0 + \rho_1 v_1 + \rho_2 v_2 + \rho_3 v_3$, $\text{logit}(\mu) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$, and $\text{log}(\phi) = \gamma_0 + \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3$. Here, we evaluate the performance of the MLEs when the sample size increases. The true values of the β 's and the γ 's are the same as in the first experiment. The true values of the ρ 's are $\rho_0 = -1$, $\rho_1 = 1$, $\rho_2 = -0.5$, and $\rho_3 = 0.5$. In this situation, the sample sizes considered are $n = 50$, 150 , and 300 .

The explanatory variables v_1 , x_1 , and z_1 were generated as independent draws from a standard normal distribution. The covariates v_2 , x_2 , and z_2 were generated from the Poisson distribution with unit mean, and v_3 , x_3 , and z_3 were generated from the binomial(0.2, 5) distribution. The total number of Monte Carlo replications was set at 5000 for each sample size. All simulations were carried out in R (Ihaka and Gentleman, 1996). Computations for fitting inflated beta regression models were performed using the `gamlss` package. The MLEs were obtained by maximizing the log-likelihood function using the

Table 1

Simulation results for the first experiment.

Estimator	$\alpha = 0.18$		$\alpha = 0.32$		$\alpha = 0.68$		$\alpha = 0.82$	
	Bias	$\sqrt{\text{MSE}}$	Bias	$\sqrt{\text{MSE}}$	Bias	$\sqrt{\text{MSE}}$	Bias	$\sqrt{\text{MSE}}$
$\hat{\alpha}$	0.00052	0.03149	0.00022	0.03778	0.00086	0.03802	−0.00207	0.03097
$\hat{\beta}_0$	−0.00095	0.05213	−0.00176	0.07724	0.00032	0.11219	−0.00226	0.19777
$\hat{\beta}_1$	0.00176	0.04006	0.00207	0.04101	0.00354	0.07070	0.00452	0.11554
$\hat{\beta}_2$	−0.00048	0.02711	−0.00076	0.04308	−0.00427	0.08532	−0.00328	0.11987
$\hat{\beta}_3$	0.00014	0.03764	0.00024	0.05145	0.00073	0.08256	0.00089	0.12046
$\hat{\gamma}_0$	0.01241	0.21359	0.02668	0.25779	0.02892	0.43969	0.06437	0.71228
$\hat{\gamma}_1$	0.03158	0.13205	0.03203	0.14619	0.07302	0.30639	0.20369	0.51323
$\hat{\gamma}_2$	0.03088	0.12514	0.04288	0.16232	0.09854	0.30874	0.23140	0.68394
$\hat{\gamma}_3$	0.02745	0.15964	0.01527	0.17183	0.07067	0.32288	0.19891	0.69026

Table 2

Simulation results for the second experiment.

Estimator	$n = 50$		$n = 150$		$n = 300$	
	Bias	$\sqrt{\text{MSE}}$	Bias	$\sqrt{\text{MSE}}$	Bias	$\sqrt{\text{MSE}}$
$\hat{\rho}_0$	−0.14554	0.81179	−0.02398	0.38136	−0.01263	0.26529
$\hat{\rho}_1$	−0.18009	0.56946	−0.04623	0.25131	−0.02501	0.17076
$\hat{\rho}_2$	−0.13892	0.66011	−0.03437	0.24178	−0.01803	0.16958
$\hat{\rho}_3$	0.09667	0.55895	0.02113	0.24240	0.01093	0.16877
$\hat{\beta}_0$	0.00588	0.16190	0.00171	0.07288	0.00055	0.05272
$\hat{\beta}_1$	0.00588	0.10917	0.00146	0.04882	0.00027	0.03313
$\hat{\beta}_2$	−0.00211	0.09761	−0.00131	0.05057	0.00018	0.02872
$\hat{\beta}_3$	0.00140	0.12356	0.00104	0.05264	0.00063	0.03049
$\hat{\gamma}_0$	0.12628	0.45696	0.04208	0.25820	0.02472	0.18738
$\hat{\gamma}_1$	0.04197	0.29273	0.03485	0.15767	0.00959	0.10359
$\hat{\gamma}_2$	0.09771	0.39828	0.02333	0.17590	0.01153	0.10208
$\hat{\gamma}_3$	0.04934	0.36516	0.02266	0.18366	0.00473	0.11669

RS algorithm (Rigby and Stasinopoulos, 2005). In order to analyze the results, we computed the bias and the root mean squared error ($\sqrt{\text{MSE}}$) of the estimates.

Table 1 summarizes the numerical results of the first experiment. We note that, for a fixed sample size, the bias and the $\sqrt{\text{MSE}}$ of the estimators of the continuous component (β 's and γ 's) increase with the expected proportion of zeros (α). This is to be expected, since the expected number of observations in (0, 1) decreases as α increases. Also, it is noteworthy that the bias and the $\sqrt{\text{MSE}}$ of MLEs corresponding to the dispersion covariates tend to be much more pronounced when compared with the MLEs of the parameters that model the mean response.

Table 2 presents simulation results for the second experiment. For the smallest sample size considered ($n = 50$), the estimation algorithm failed to converge in 1.3% of the samples. For large sample sizes, the algorithm converged for all the samples. The estimated biases of the MLEs of the ρ 's (parameters of the discrete component) are markedly high for small samples. This is not surprising; in standard logistic regression, the MLEs are considerably biased in small samples. For large samples, the bias of the ρ 's is negligible.

In the second experiment, the β 's and the γ 's are essentially estimated from the observations in (0, 1), which represent around 70% of the observations in our study. For all the sample sizes considered, the mean of the $\hat{\beta}$'s and $\hat{\gamma}$'s are close to the corresponding true values. Also, all the root mean square errors decrease when the sample size increases, as expected.

4. Diagnostics

Likelihood-based inference depends on parametric assumptions, and a severe misspecification of the model or the presence of outliers may impair its accuracy. We shall introduce some types of residuals for detecting departures from the postulated model and outlying observations. Additionally, we suggest some measures to assess goodness-of-fit.

Residuals. Residual analysis in the context of the zero-or-one inflated beta regression model (2.1)–(2.3) can be split into two parts. First, we focus on the residual analysis for the discrete and the continuous component of the model separately. For this purpose, we propose a standardized Pearson residual based on Fisher's scoring iterative algorithm for estimating ρ , β , and γ . Second, we define a randomized quantile residual as a global residual for the model, i.e., using information of the discrete and the continuous component simultaneously.

To study departures from the error assumptions as well as the presence of outlying observations of the discrete component for the zero-or-one inflated beta regression model, we define a version of the standardized Pearson

residual as

$$r_{pt}^D = \frac{11_{[c]}(y_t) - \hat{\alpha}_t}{\sqrt{\hat{\alpha}_t(1 - \hat{\alpha}_t)(1 - \mathbf{h}_{tt})}}, \quad t = 1, \dots, n, \quad (4.1)$$

where \mathbf{h}_{tt} is the t th diagonal element of the orthogonal projection matrix \mathbf{H} defined in (B.5) and “ $\hat{\cdot}$ ”, indicates evaluation at the MLEs. Note that r_{pt}^D is a version of the ordinary residual obtained in (B.4) that takes the leverage of the t th observation into account. Plots of the residuals against the covariates or the fitted values should exhibit no systematic trend.

The conditional distribution of y_t , given that $y_t \in (0, 1)$, is a beta distribution $\mathcal{B}(\mu_t, \phi_t)$. Ferrari and Cribari-Neto (2004) provided two different residuals (standardized and deviance) for the class of beta regression models. Espinheira et al. (2008b) proposed two new beta regression residuals and their numerical results favor one of the new residuals: the one that accounts for the leverages of the different observations. Here, we follow Espinheira et al. (2008b) to define a weighted version of the ordinary residual from Fisher’s scoring iterative algorithm for estimating β when γ is fixed (see Appendix B). Our proposed residual is defined as

$$r_{pt}^C = \frac{y_t - \hat{\mu}_t^*}{\sqrt{\hat{v}_t^*(1 - \hat{\alpha}_t)(1 - \hat{\mathbf{P}}_{tt})}}, \quad t : y_t \in (0, 1), \quad (4.2)$$

where $\hat{\mathbf{P}}_{tt}$ is the t th diagonal element of the projection matrix $\hat{\mathbf{P}}$ defined in (B.13). Note that r_{pt}^C is a version of the standardized Pearson residual obtained in (B.14) that takes the leverage of the t th observation into account.

To assess the overall adequacy of the zero-or-one inflated beta regression model to the data at hand, we propose the randomized quantile residual (Dunn and Smyth, 1996). It is a randomized version of Cox and Snell (1968) residual and given by

$$r_t^q = \Phi^{-1}(u_t), \quad t = 1, \dots, n, \quad (4.3)$$

where $\Phi(\cdot)$ denotes the standard normal distribution function, u_t is a uniform random variable on the interval $(a_t, b_t]$, with $a_t = \lim_{y \uparrow y_t} \text{Bl}_c(y_t; \hat{\alpha}_t, \hat{\mu}_t, \hat{\phi}_t)$ and $b_t = \text{Bl}_c(y_t; \hat{\alpha}_t, \hat{\mu}_t, \hat{\phi}_t)$. Here, $\text{Bl}_c(y_t; \alpha_t, \mu_t, \phi_t) = \alpha_t 11_{[c,1]}(y_t) + (1 - \alpha_t)F(y_t; \mu_t, \phi_t)$, where $F(\cdot; \mu_t, \phi_t)$ is the cumulative distribution function of the beta distribution $\mathcal{B}(\mu_t, \phi_t)$. In the zero-inflated beta regression model, u_t is a uniform random variable on $(0, \hat{\alpha}_t]$ if $y_t = 0$ and $u_t = \text{Bl}_0(y_t; \hat{\alpha}_t, \hat{\mu}_t, \hat{\phi}_t)$ if $y_t \in (0, 1)$. On the other hand, in the one-inflated beta regression model, u_t is a uniform random variable on $[\hat{\alpha}_t, 1)$ if $y_t = 1$ and $u_t = \text{Bl}_1(y_t; \hat{\alpha}_t, \hat{\mu}_t, \hat{\phi}_t)$ if $y_t \in (0, 1)$. Apart from sampling variability in $\hat{\alpha}_t$, $\hat{\mu}_t$, and $\hat{\phi}_t$ the r_t^q are exactly standard normal in $(a_t, b_t]$ and the randomized procedure is introduced in order to produce a continuous residual. The randomized quantile residuals can vary from one realization to another. In practice, it is useful to make at least four achievements.

A plot of these residuals against the index of the observations (t) should show no detectable pattern. A detectable trend in the plot of some residual against the predictors may be suggestive of link function misspecification. Also, normal probability plots with simulated envelopes are a helpful diagnostic tool (Atkinson, 1985). Simulation results not presented here indicated that the randomized quantile residuals perform well in detecting whether the distribution assumption is incorrect.

Global goodness-of-fit measure. A simple global goodness-of-fit measure is a pseudo R^2 , say R_p^2 , defined by the square of the sample correlation coefficient between the outcomes, y_1, \dots, y_n , and their corresponding predicted values, $\hat{\mu}_1^*, \dots, \hat{\mu}_n^*$, where $\hat{\mu}_t^* = \hat{E}(y_t) = c\hat{\alpha}_t + (1 - \hat{\alpha}_t)\hat{\mu}_t$. Other pseudo R^2 ’s are defined as $R_p^{2*} = 1 - \log \hat{L} / \log \hat{L}_0$ (McFadden, 1974) and $R_{LR}^2 = 1 - (\hat{L}_0 / \hat{L})^{2/n}$ (Cox and Snell, 1989, p. 208–209), where \hat{L}_0 and \hat{L} are the maximized likelihood functions of the null model and the fitted model, respectively. The ratio of the likelihoods or log-likelihoods may be regarded as measures of the improvement, over the model with only three parameters (α , μ , and ϕ), achieved by the model under investigation.

Influence measures. A well-known measure of the influence of each observation on the regression parameter estimates is the likelihood displacement (Cook and Weisberg, 1982, Ch. 3). The likelihood displacement that results from removing the t th observation from the data is defined by

$$LD_t = \frac{2}{d} \{ \ell(\hat{\theta}) - \ell(\hat{\theta}_{(t)}) \},$$

where d is the dimension of θ and $\hat{\theta}_{(t)}$ is the MLE of θ obtained after removing the t th observation from the data. This definition does not consider that θ is actually split into two different types of parameters: the parameters of the discrete component and the parameters of the continuous component. Thus, it is more appropriate to consider the influence of the t th case on the estimation of ρ and (β, γ) separately. Therefore, we propose the following statistics:

$$LD_t^D = \frac{2}{p} \{ \ell_1(\hat{\rho}) - \ell_1(\hat{\rho}_{(t)}) \}, \quad t = 0, 1, \dots, n$$

$$LD_t^C = \frac{2}{k+m} \{ \ell_2(\hat{\beta}, \hat{\gamma}) - \ell_2(\hat{\beta}_{(t)}, \hat{\gamma}_{(t)}) \}, \quad t : y_t \in (0, 1).$$

Table 3

Sample summary statistics.

Variable	Mean	S.d.	Min.	1st qu.	Median	3rd qu.	Max.
<i>lnpop</i>	9.4042	1.3464	7.3920	8.3997	9.0620	10.0025	16.1606
<i>propurb</i>	0.7173	0.2044	0.2042	0.5887	0.7445	0.8862	1.0000
<i>propmen</i>	0.5062	0.0114	0.4766	0.4989	0.5070	0.5146	0.5370
<i>prop2029</i>	0.1660	0.0136	0.1322	0.1559	0.1661	0.1750	0.2042
<i>hdie</i>	0.8236	0.0587	0.6060	0.7955	0.8325	0.8640	0.9330

Simple approximations for LD_t^D and LD_t^C are given by

$$LD_t^D \approx c_{tt}^D = \frac{\mathbf{h}_{tt}}{p(1 - \mathbf{h}_{tt})} (r_{pt}^D)^2, \quad t = 0, 1, \dots, n$$

$$LD_t^C \approx c_{tt}^C = \frac{\mathbf{P}_{tt}}{(k + m)(1 - \mathbf{P}_{tt})} (r_{pt}^C)^2, \quad t : y_t \in (0, 1);$$

see Cook and Weisberg (1982, p. 191) and Wei (1998, p. 102). The approximations above are based on the iterative scheme for evaluating the MLE of ρ , β , and γ (Appendix B) and Taylor expansions of $\ell_1(\hat{\rho}_{(t)})$ and $\ell_2(\hat{\beta}_{(t)}, \hat{\gamma}_{(t)})$ around $\hat{\rho}$ and $(\hat{\beta}, \hat{\gamma})$, respectively. The quantities c_{tt}^D and c_{tt}^C can be useful to highlight influential cases. In practice, we recommend the plotting of c_{tt}^D and c_{tt}^C against t .

Model selection. Nested zero-or-one inflated beta regression models can be compared via the likelihood ratio test, using twice the difference between the maximized log-likelihoods of a full model and a restricted model whose covariates are a subset of the full model. Information criteria, such as the generalized Akaike information criterion (GAIC), can be used for comparing non-nested models. It is defined as

$$\text{GAIC} = \hat{D} + d\wp, \quad (4.4)$$

where $\hat{D} = -2\hat{\ell}$ is the global fitted deviance (Rigby and Stasinopoulos, 2005), $\hat{\ell}$ is the maximized log-likelihood and d is the dimension of θ . It is possible to interpret the first term on the right-hand side of (4.4) as a measure of the lack of fit and the second term as a “penalty” for adding d parameters. The model with the smallest GAIC is then selected. The Akaike information criterion AIC (Akaike, 1974), the Schwarz Bayesian criterion SBC (Schwarz, 1978), and the consistent Akaike information criterion (CAIC) are special cases of GAIC corresponding to $\wp = 2$, $\wp = \log(n)$, and $\wp = \log(n) + 1$, respectively.

Model misspecification. Possible sources of misspecification in zero-or-one inflated beta regressions include incorrectly chosen link functions, neglected nonlinearities and omitted variables. Ramsey’s RESET test (Ramsey, 1969), which was originally proposed for linear regression models, can be adapted to test various forms of misspecification in more general models. The idea behind Ramsey’s test is that when the model is correctly specified, powers of the fitted values or powers of the regressors should not provide additional contribution to the quality of the fit. The specifications of the regression structure of the model parameters are then augmented by additional terms that reduce to zero if the model is correctly specified. An extension of Ramsey’s test for a sub-class of the general class of zero-or-one inflated beta regression models is considered in a recent unpublished working paper by Pereira and Cribari-Neto (2011). It can be further extended to our general class of models.

5. An application

This section contains an application of the zero-inflated beta regression model to real data. We analyze data on the mortality in traffic accidents of 200 randomly selected Brazilian municipal districts of the southeast region in the year 2002. The data were extracted from the DATASUS database available at www.datasus.gov.br and IPEADATA database available at www.ipeadata.gov.br. The response variable y is the proportion of deaths caused by traffic accidents. The explanatory variables are the logarithm of the number of inhabitants of the municipality (*lnpop*), the proportion of the population living in the urban area (*propurb*), the proportion of men in the population (*propmen*), the proportion of residents aged between 20 and 29 years (*prop2029*), and the human development index of education of the municipal district (*hdie*). The main objective is to investigate the effect of the young population (*prop2029*) on the proportion of deaths caused by traffic accidents after controlling for potential confounding variables. We report the summary statistics on each of these variables in Table 3.

Fig. 1 presents a histogram and a box-plot of the response variable. The clump-at-zero (the bar with the dot above) in the histogram represents 39% of the data. We observe that the distribution of the data in $(0, 1)$ is asymmetric with an inverted “J” shape. Also, the box-plot reveals the presence of some outliers. Visual inspection suggests that a zero-inflated beta distribution may be a suitable model for the data.

We consider a zero-inflated beta regression model with the following specification:

$$\text{logit}(\alpha) = \rho_0 + \rho_1 \text{lnpop} + \rho_2 \text{propurb} + \rho_3 \text{propmen} + \rho_4 \text{prop2029} + \rho_5 \text{hdie},$$

$$\text{logit}(\mu) = \beta_0 + \beta_1 \text{lnpop} + \beta_2 \text{propurb} + \beta_3 \text{propmen} + \beta_4 \text{prop2029} + \beta_5 \text{hdie},$$

$$\log(\phi) = \gamma_0 + \gamma_1 \text{lnpop} + \gamma_2 \text{propurb} + \gamma_3 \text{propmen} + \gamma_4 \text{prop2029} + \gamma_5 \text{hdie}.$$

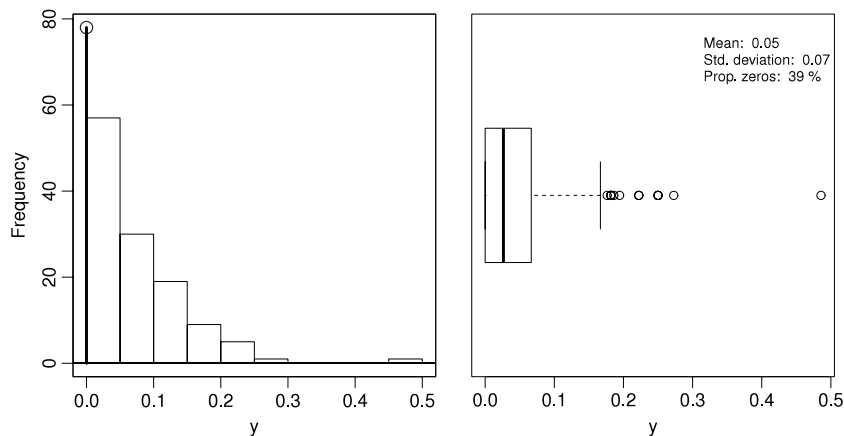


Fig. 1. Frequency histogram and box-plot of the observed proportions of deaths caused by traffic accidents.

Table 4

Parameter estimates (est.), standard errors (s.e.), and p -values (p).

α	Est.	S.e.	p
intercept	27.27	4.63	1.73e–08
lnpop	–1.17	0.27	2.07e–05
prop2029	–48.06	17.46	6.49e–03
hdei	–11.34	4.01	5.13e–03
μ	Estimate	S.e.	p
intercept	–4.72	1.20	1.11e–04
lnpop	–0.53	0.05	7.14e–17
prop2029	27.68	6.37	2.33e–05
hdei	3.10	1.44	3.28e–02
ϕ	Estimate	S.e.	p
intercept	9.46	3.25	4.08e–03
lnpop	0.47	0.10	8.54e–06
prop2029	–28.34	16.63	9.01e–02
hdei	–6.70	3.90	8.78e–02

Computations for fitting the model were carried out using the package `gamlss` (Stasinopoulos and Rigby, 2007) in the R software package (Ihaka and Gentleman, 1996) and the BEZI distribution implemented by Ospina (2006). The automatic procedure `stepAICall.B()` included in the `gamlss` package was used to perform model selection using the AIC. The following final model has been selected:

$$\begin{aligned}
 \text{logit}(\alpha) &= \rho_0 + \rho_1 \text{lnpop} + \rho_4 \text{prop2029} + \rho_5 \text{hdei}, \\
 \text{logit}(\mu) &= \beta_0 + \beta_1 \text{lnpop} + \beta_4 \text{prop2029} + \beta_5 \text{hdei}, \\
 \log(\phi) &= \gamma_0 + \gamma_1 \text{lnpop} + \gamma_4 \text{prop2029} + \gamma_5 \text{hdei}.
 \end{aligned} \tag{5.1}$$

In fact, the value of the likelihood ratio statistic for testing $H_0 : \theta = (\rho_2, \rho_3, \beta_2, \beta_3, \gamma_2, \gamma_3) = 0$ versus a two-side alternative is $\Lambda = 4.89$ (p -value = 0.56), i.e., H_0 is not rejected at the usual significance levels, and hence, *propurb* and *propmen* can be excluded from the model.

The parameter estimates and the corresponding standard errors for the final model are summarized in Table 4. The pseudo R^2 's are $R_p^{2*} = 0.597$ and $R_{LR}^2 = 0.584$, thus suggesting a reasonable fit of the model to the data. It is noteworthy that *prop2029*, which represents the proportion of the population aged between 20 and 29 years, has a positive effect on both the probability of observing at least one death due to traffic accidents ($1 - \alpha$) and the conditional mean proportion of deaths due to this cause (μ).

Fig. 2 shows the plot of randomized quantile residuals against the case number (Fig. 2(a)) and the quantile–quantile plot of the ordered randomized quantile residuals against the corresponding quantiles of a standard normal distribution with a 95% envelope based on 100 simulations (Fig. 2(b)). Fig. 2(a) singles out observation 138 as possibly atypical, and Fig. 2(b) indicates that it lies in the threshold of the envelope. Case 138 is, therefore, worthy of further investigation. Notice that Fig. 2(b) clearly shows that the distribution of the randomized quantile residuals is asymmetric, and the usual thresholds $(-2, 2)$ or $(-3, 3)$ should be used with care.

Diagnostic plots for the discrete component are given in Fig. 3(a)–(c). The plot of r_{pt}^D against the case number (Fig. 3(a)) suggests that the residuals appear to be randomly scattered around zero and that there is no atypical observation. The plot

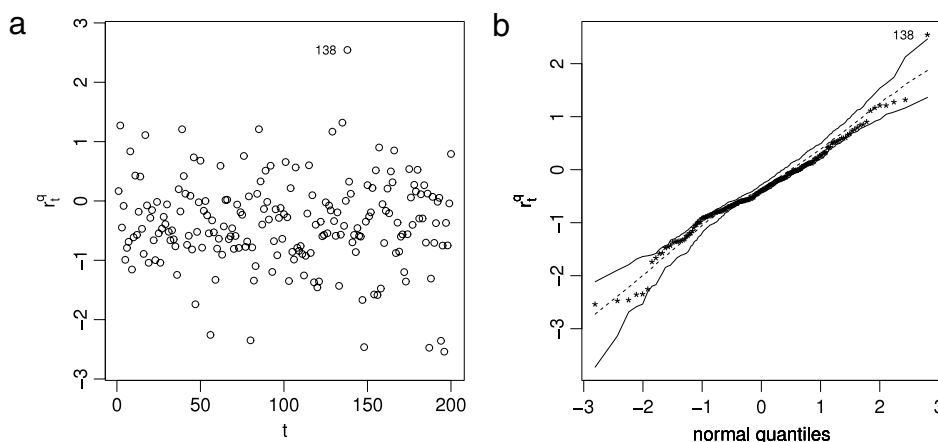


Fig. 2. Quantile residual plots.

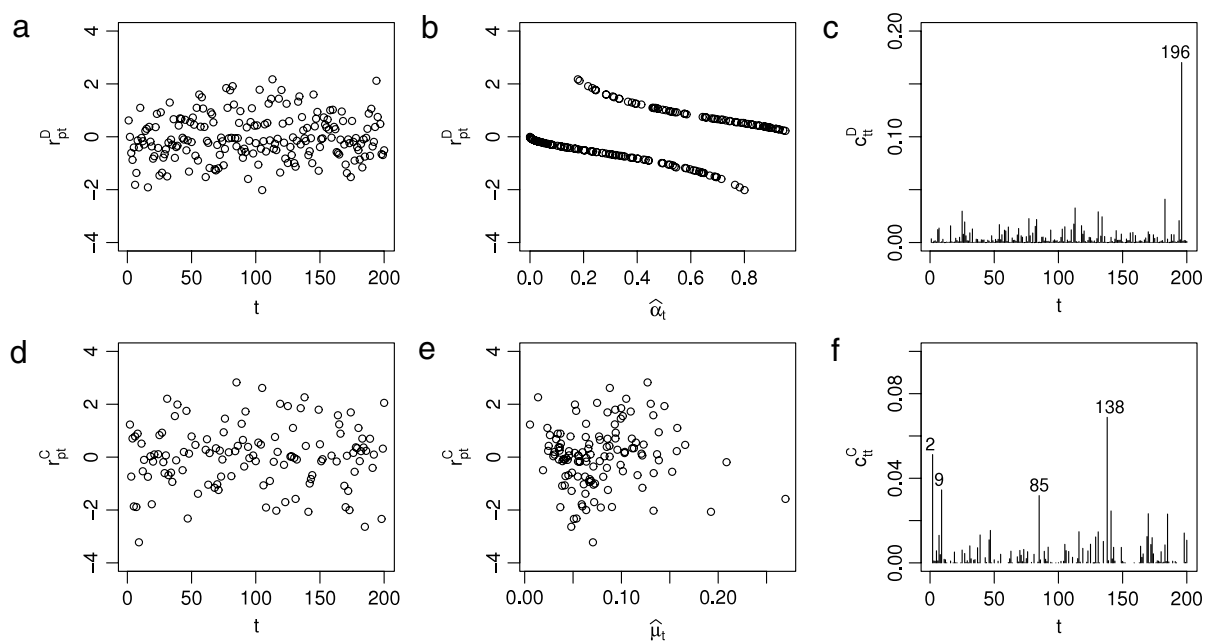


Fig. 3. Diagnostic plots for the discrete component ((a)–(c)) and the continuous component ((d)–(f)).

of r_{pt}^D against $\hat{\alpha}_t$ (Fig. 3(b)) is not suggestive of a lack of fit. The lower and upper bounds, corresponding to responses equal to zero and one, respectively, are typical of data with only two outcomes. Fig. 3(c) shows the plot of Cook statistics c_{tt}^D against the case number, from which observation **196** (an observation with the response variable equal to zero) is highlighted as influential.

Diagnostic plots for the continuous component are given in Fig. 3(d)–(f). Fig. 3(d) shows the plot of r_{pt}^D against case number and suggests that the residuals are randomly scattered around zero. No observation is distinguished as atypical. The plot of r_{pt}^D against $\hat{\mu}_t$ (Fig. 3(e)) is not suggestive of a lack of fit. Finally, the plot of c_{tt}^C against the case number indicates observations **2, 9, 85, and 138** as influential points for the continuous component.

We have reestimated the model after removing the following sets of observations: {138}, {196}, and {2, 9, 85, 138}. Our purpose is to investigate the impact of the exclusion of observations highlighted in the diagnostic plots on the inferences based on the estimated model. Table 5 gives relative changes in the parameter estimates and the standard errors. Observation 138 alone is clearly influential for the estimation of the continuous component of the model, while Observation 196 solely impacts the inference on the discrete component, as expected. The joint exclusion of cases 2, 9, 85, and 138 produces substantial changes in the parameter estimates of the continuous component and only slight changes in the estimated model for the discrete part. Our findings suggest that the proposed diagnostic tools are helpful for detecting atypical observations and influential cases and are able to indicate the component of the model, discrete or continuous, that is affected by each influential observation.

Table 5

Parameter estimates (est.), standard errors (s.e.), p -values (p), relative changes in estimates (rel. est.), and relative changes in standard errors (rel. s.e.) because of exclusions of observations (in percentage).

Obs.	Parameter	Est.	S.e	p	Rel. est.	Rel. s.e
138	ρ_0	27.26	4.63	0.00	0.04	−0.06
	ρ_1	−1.17	0.27	0.00	0.08	−0.17
	ρ_4	−48.06	17.46	0.01	−0.01	0.01
	ρ_5	−11.34	4.01	0.01	0.04	0.00
	β_0	−5.05	1.11	0.00	−6.96	7.06
	β_1	−0.57	0.06	0.00	−6.85	−1.48
	β_4	32.40	6.42	0.00	−17.07	−0.67
	β_5	2.93	1.40	0.04	5.54	2.64
	γ_0	10.66	3.00	0.00	−12.60	7.74
	γ_1	0.61	0.11	0.00	−27.20	−7.02
	γ_4	−43.46	17.26	0.01	−53.34	−3.78
	γ_5	−6.47	3.83	0.09	3.52	2.02
196	ρ_0	30.78	5.25	0.00	−12.86	−13.34
	ρ_1	−1.38	0.30	0.00	−18.15	−12.33
	ρ_4	−51.62	18.21	0.01	−7.41	−4.30
	ρ_5	−12.68	4.24	0.00	−11.80	−5.95
	β_0	−4.72	1.20	0.00	0.00	0.00
	β_1	−0.53	0.06	0.00	0.00	0.00
	β_4	27.68	6.38	0.00	0.00	0.00
	β_5	3.10	1.44	0.03	0.00	0.00
	γ_0	9.47	3.26	0.00	0.00	0.00
	γ_1	0.48	0.10	0.00	0.00	0.00
	γ_4	−28.34	16.64	0.09	0.00	0.00
	γ_5	−6.70	3.91	0.09	0.00	0.00
2, 9, 85, 138	ρ_0	26.97	4.62	0.00	1.11	0.17
	ρ_1	−1.16	0.27	0.00	1.17	0.24
	ρ_4	−47.28	17.45	0.01	1.63	0.06
	ρ_5	−11.27	4.00	0.01	0.67	0.09
	β_0	−3.98	1.11	0.00	15.69	6.93
	β_1	−0.58	0.05	0.00	−8.90	5.45
	β_4	29.71	5.87	0.00	−7.33	8.03
	β_5	2.29	1.36	0.09	26.14	5.94
	γ_0	7.48	3.35	0.03	20.96	−2.82
	γ_1	0.61	0.14	0.00	−27.75	−36.20
	γ_4	−29.35	16.95	0.08	−3.57	−1.86
	γ_5	−5.37	4.17	0.20	19.85	−6.70

An alternative strategy to model our data is the quasi-likelihood approach (Ramalho et al., 2011). No specific probability distribution is assumed for the data but, instead, a functional relation between the mean and the variance is specified in the form of a variance function. Also, the mean response is modeled through a regression structure. A clear disadvantage of the quasi-likelihood approach for our data is that it does not allow for the evaluation of the effect of the covariates in the probability of occurrence of deaths due to traffic accidents. We have fitted a quasi-likelihood model with specification for the mean given in (5.1) and variance function given by $V(\mu) = \mu(1 - \mu)$ to the traffic accidents data. This choice of the variance function is natural since it is in accordance with the relation between the mean and the variance for any beta distribution. Fig. 4 shows the plots of randomized quantile residuals against fitted values for our final zero-inflated regression model (Fig. 4(a)) and the quasi-likelihood model (Fig. 4(b)). Fig. 4(a) suggests that the residuals are randomly scattered around zero for the zero-inflated beta regression model, but a clear pattern is evident in Fig. 4(b), which corresponds to the quasi-likelihood fit. Our conclusion is that the zero-inflated beta regression model is a more appropriate model to the data at hand.

6. Concluding remarks

We developed a general class of zero-or-one inflated beta regression models that can be useful for practitioners when modeling response variables in the standard unit interval, such as rates or proportions, with the presence of zeros or ones. Explicit formulas for the score function, Fisher's information matrix, and its inverse are given. An iterative estimation procedure and its computational implementation are discussed. Interval estimation for different population quantities is presented. We also proposed a set of diagnostic tools that can be employed to identify departures from the postulated model and the atypical and influential observations. These tools include pseudo R^2 's, different residuals, influence measures, and model selection procedures. One particularly interesting feature of the proposed diagnostic analysis is that it allows the practitioner to separately identify influential cases on the discrete and the continuous components of the model. An application using real data was presented and discussed.

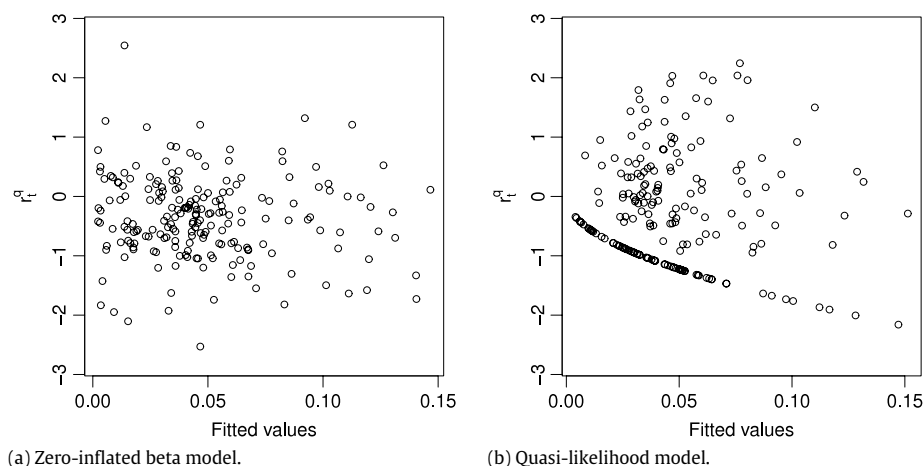


Fig. 4. Plots of randomized quantile residuals against fitted values.

As a final remark, we mention that a more general class of zero-or-one inflated beta regression models could allow for mixtures particularly within the continuous part of the model. Beta regression finite mixture models are considered in a recent paper by [Smithson et al. \(in press\)](#). Such models, however, do not allow for the presence of zeros or ones in the data. This is still an open, interesting research topic.

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Appendix A. Score vector and observed information matrix

Score vector. The elements of the score vector are given by

$$U_s = \frac{\partial \ell(\theta)}{\partial \rho_s} = \sum_{t=1}^n \frac{\partial \ell_t(\alpha_t)}{\partial \alpha_t} \frac{\partial \alpha_t}{\partial \eta_{1t}} \frac{\partial \eta_{1t}}{\partial \rho_s} = \sum_{t=1}^n \frac{11_{[c]}(y_t) - \alpha_t}{\alpha_t(1 - \alpha_t)} \frac{1}{h'_1(\alpha_t)} \frac{\partial \eta_{1t}}{\partial \rho_s}, \quad (\text{A.1})$$

$$U_r = \frac{\partial \ell(\theta)}{\partial \beta_r} = \sum_{t: y_t \in (0,1)} \frac{\partial \ell_t(\mu_t, \phi_t)}{\partial \mu_t} \frac{\partial \mu_t}{\partial \eta_{2t}} \frac{\partial \eta_{2t}}{\partial \beta_r} = \sum_{t: y_t \in (0,1)} \phi_t(y_t^* - \mu_t^*) \frac{1}{h'_2(\mu_t)} \frac{\partial \eta_{2t}}{\partial \beta_r}, \quad (\text{A.2})$$

$$U_R = \frac{\partial \ell(\theta)}{\partial \gamma_R} = \sum_{t: y_t \in (0,1)} \frac{\partial \ell_t(\mu_t, \phi_t)}{\partial \phi_t} \frac{\partial \phi_t}{\partial \eta_{3t}} \frac{\partial \eta_{3t}}{\partial \gamma_R} = \sum_{t: y_t \in (0,1)} [\mu_t(y_t^* - \mu_t^*) + (y_t^\dagger - \mu_t^\dagger)] \frac{1}{h'_3(\phi_t)} \frac{\partial \eta_{3t}}{\partial \gamma_R}, \quad (\text{A.3})$$

for $s = 1, \dots, p$, $r = 1, \dots, k$ and $R = 1, \dots, m$.

Observed information matrix. For $s, s' = 1, \dots, p$; $r, r' = 1, \dots, k$ and $R, R' = 1, \dots, m$, we have

$$\begin{aligned} J_{ss'} &= -\frac{\partial^2 \ell(\theta)}{\partial \rho_s \partial \rho_{s'}} = -\sum_{t=1}^n \left\{ \left(\frac{-11_{(0,1)}(y_t)}{(1 - \alpha_t)^2} - \frac{11_{[c]}(y_t)}{\alpha_t^2} \right) \left[\frac{1}{h'_1(\alpha_t)} \right]^2 \right. \\ &\quad + \left(\frac{11_{[c]}(y_t) - \alpha_t}{\alpha_t(1 - \alpha_t)} \right) \left[\frac{1}{h'_1(\alpha_t)} \right] \left[\frac{-h''_1(\alpha_t)}{(h'_1(\alpha_t))^2} \right] \frac{\partial \eta_{1t}}{\partial \rho_s} \frac{\partial \eta_{1t}}{\partial \rho_{s'}} \\ &\quad \left. - \sum_{t=1}^n \left\{ \frac{11_{[c]}(y_t) - \alpha_t}{\alpha_t(1 - \alpha_t)} \left[\frac{1}{h'_1(\alpha_t)} \right] \frac{\partial^2 \eta_{1t}}{\partial \rho_s \partial \rho_{s'}} \right\} \right\}, \\ J_{rr'} &= -\frac{\partial^2 \ell(\theta)}{\partial \beta_r \partial \beta_{r'}} = -\sum_{t: y_t \in (0,1)} \left\{ -\phi_t^2 v_t^* \frac{1}{h'_2(\mu_t)} + \phi_t(y_t^* - \mu_t^*) \left(\frac{-h''_2(\mu_t)}{(h'_2(\mu_t))^2} \right) \frac{1}{h'_2(\mu_t)} \frac{\partial \eta_{2t}}{\partial \beta_r} \frac{\partial \eta_{2t}}{\partial \beta_{r'}} \right\} \\ &\quad - \sum_{t: y_t \in (0,1)} \left\{ \phi_t(y_t^* - \mu_t^*) \frac{1}{(h'_2(\mu_t))} \frac{\partial^2 \eta_{2t}}{\partial \beta_r \partial \beta_{r'}} \right\}, \end{aligned}$$

$$J_{rR} = -\frac{\partial^2 \ell(\theta)}{\partial \beta_r \partial \gamma_R} = -\sum_{t: y_t \in (0,1)} \left\{ [y_t^* - \mu_t^* - \phi_t(\mu_t v_t^* + c_t^{*\dagger})] \frac{1}{(h_2'(\mu_t))} \frac{\partial \eta_{2t}}{\partial \beta_r} \frac{1}{(h_3'(\phi_t))} \frac{\partial \eta_{3t}}{\partial \gamma_R} \right\},$$

$$J_{RR'} = -\frac{\partial^2 \ell(\theta)}{\partial \gamma_R \partial \gamma_{R'}} = -\sum_{t: y_t \in (0,1)} \left\{ [-\mu_t^2 v_t^* - 2\mu_t c_t^{*\dagger} - v_t^\dagger] \frac{1}{(h_3'(\phi_t))} + [\mu_t(y_t^* - \mu_t^*) + (y_t^\dagger - \mu_t^\dagger)] \right. \\ \left. \times \left(\frac{-h_3''(\phi_t)}{(h_3'(\phi_t))^2} \right) \right\} \frac{1}{(h_3'(\phi_t))} \frac{\partial \eta_{3t}}{\partial \gamma_R} \frac{\partial \eta_{3t}}{\partial \gamma_{R'}} - \sum_{t: y_t \in (0,1)} \left\{ [\mu_t(y_t^* - \mu_t^*) + (y_t^\dagger - \mu_t^\dagger)] \frac{1}{(h_3'(\phi_t))} \frac{\partial^2 \eta_{3t}}{\partial \gamma_R \partial \gamma_{R'}} \right\}.$$

The observed Fisher information matrix can now be written as

$$J(\theta) = \begin{pmatrix} J_{\rho\rho} & 0 & 0 \\ 0 & J_{\beta\beta} & J_{\beta\gamma} \\ 0 & J_{\gamma\beta} & J_{\gamma\gamma} \end{pmatrix}, \quad (\text{A.4})$$

where

$$J_{\rho\rho} = \mathcal{V}^\top \{ (\mathcal{A}^{*2}(I_n - Y^c) + \mathcal{A}^2 Y^c) + \mathcal{A}^* \mathcal{A} (Y^c - \alpha^*) \mathcal{D}^* \mathcal{D} \} \mathcal{V}^\top - [(y^c - \alpha)^\top \mathcal{A} \mathcal{D}][\dot{\mathcal{V}}],$$

$$J_{\beta\beta} = \mathcal{X}^\top \{ \Phi T V^* + S T^2 (Y^* - \mathcal{M}^*) \} (I_n - Y^c) T \Phi \mathcal{X} + [(y^* - \mu^*)^\top (I_n - Y^c) T \Phi][\dot{\mathcal{X}}],$$

$$J_{\gamma\beta} = J_{\gamma\beta}^\top = -\mathcal{X}^\top \{ (Y^* - \mathcal{M}^*) - \Phi(\mathcal{M} V^* + C) \} (I_n - Y^c) T H \mathcal{Z},$$

$$J_{\gamma\gamma} = \mathcal{Z}^\top \{ H(\mathcal{M}^2 V^* + 2\mathcal{M} C + V^\dagger) + \{ \mathcal{M}(Y^* - \mathcal{M}^*) + (Y^\dagger - \mathcal{M}^\dagger) \} Q H^2 \} (I_n - Y^c) H \mathcal{Z} \\ + [((y^* - \mu^*)^\top \mathcal{M} + (y^\dagger - \mu^\dagger)^\top) (I_n - Y^c) H][\dot{\mathcal{Z}}],$$

with $\alpha^* = \text{diag}(\alpha_1, \dots, \alpha_n)$, $Y^* = \text{diag}(y_1^*, \dots, y_n^*)$, $Y^\dagger = \text{diag}(y_1^\dagger, \dots, y_n^\dagger)$, $\mathcal{M}^* = \text{diag}(\mu_1^*, \dots, \mu_n^*)$, $\mathcal{M}^\dagger = \text{diag}(\mu_1^\dagger, \dots, \mu_n^\dagger)$, and $Q = \text{diag}(h_3'(\phi_1), \dots, h_3'(\phi_n))$; $\dot{\mathcal{V}}$ is an $n \times p \times p$ array with faces $\mathcal{V}_t = \partial^2 \eta_{1t} / \partial \rho \partial \rho^\top$, $\dot{\mathcal{X}}$ is an $n \times k \times k$ array with faces $\mathcal{X}_t = \partial^2 \eta_{2t} / \partial \beta \partial \beta^\top$, and $\dot{\mathcal{Z}}$ is an $n \times m \times m$ array with faces $\mathcal{Z}_t = \partial^2 \eta_{3t} / \partial \gamma \partial \gamma^\top$, for $t = 1, \dots, n$. The column multiplication for three-dimensional arrays is indicated by the bracket operator $[\cdot][\cdot]$ as defined by Wei (1998, p. 188).

Appendix B. Iterative algorithm for maximum likelihood estimation

MLEs for ρ and $\vartheta = (\beta^\top, \phi)^\top$ can be obtained by using a re-weighted least-squares algorithm. For ρ , we have

$$\rho^{(m+1)} = (\mathcal{V}^{(m)\top} \mathbf{W}_1^{(m)} \mathcal{V}^{(m)})^{-1} \mathcal{V}^{(m)\top} \mathbf{W}_1^{(m)} \mathbf{y}_1^{(m)}, \quad m = 0, 1, \dots, \quad (\text{B.1})$$

where \mathbf{W}_1 is defined in (3.7) and

$$\mathbf{y}_1 = \mathcal{V} \rho + \mathbf{W}_1^{-1} \mathcal{A} \mathcal{D} \mathcal{A}^* (y^c - \alpha) \quad (\text{B.2})$$

is a local modified dependent variable. This cycle is repeated until convergence is achieved. Rewriting this equation as

$$\mathbf{y}_1 = \eta_1 - \tau_1 + \mathbf{W}_1^{-1} \mathcal{A} \mathcal{D} \mathcal{A}^* (y^c - \alpha), \quad (\text{B.3})$$

where $\eta_1 = \mathbf{f}_1(V; \rho)$ and $\tau_1 = \mathbf{f}_1(V; \rho) - \mathcal{V} \rho$, we can interpret (B.1) as an iterative process to fit a generalized linear model with design matrix \mathcal{V} , systematic part $h_1(\alpha_t) = \eta_{1t}$, t th diagonal element of the variance function $[\mathbf{W}_1^{-1} \mathcal{A} \mathcal{D} \mathcal{A}^*]_{tt}$, $t = 1, \dots, n$, and offset τ_1 . The offset quantity is subtracted, at each step, from the predictor η_1 . The iterative process (B.1) may be performed, for instance, in the R package by taking advantage of the library MASS (Venables and Ripley, 2002). This procedure allows us to extend the diagnostic results of ordinary regression models to the discrete component of the zero-or-one inflated beta regression model. Upon the converge of iterative process (B.1), we have $\hat{\rho} = (\hat{\mathcal{V}}^\top \mathbf{W}_1 \hat{\mathcal{V}})^{-1} \hat{\mathcal{V}}^\top \mathbf{z}$, where $\mathbf{z} = \hat{\mathcal{V}} \hat{\rho} + \hat{\mathbf{W}}_1^{-1} \hat{\mathcal{A}} \hat{\mathcal{D}} \hat{\mathcal{A}}^* (y^c - \hat{\alpha})$. The ordinary residual for this re-weighted regression is

$$r^* = \hat{\mathbf{W}}_1^{1/2} (\mathbf{z} - \hat{\eta}_1) = \hat{\mathbf{W}}_1^{-1/2} \hat{\mathcal{A}} \hat{\mathcal{D}} \hat{\mathcal{A}}^* (y^c - \hat{\alpha}).$$

Note that the t th element of r^* is

$$r_t^* = \frac{1l_{[c]} - \hat{\alpha}_t}{\sqrt{\hat{\alpha}_t(1 - \hat{\alpha}_t)}}. \quad (\text{B.4})$$

By writing $(\hat{\mathcal{V}}^\top \hat{\mathbf{W}}_1 \hat{\mathcal{V}}) \hat{\rho} = \hat{\mathcal{V}}^\top \mathbf{z}$, it is possible to obtain the approximations $E(r^*) \approx 0$ and $\text{Var}(r^*) \approx (I_n - \mathbf{H})$, where I_n is the $n \times n$ identity matrix and

$$\mathbf{H} = \mathbf{W}_1^{1/2} \mathcal{V} (\mathcal{V}^\top \mathbf{W}_1 \mathcal{V})^{-1} \mathcal{V}^\top \mathbf{W}_1^{1/2} \quad (\text{B.5})$$

is an orthogonal projection matrix onto the vector space spanned by the columns of \mathcal{V} . The geometric interpretation of \mathbf{H} as a projection matrix is discussed by Moolgavkar et al. (1984).

Now, let $\tilde{\mathbf{X}}$, $\tilde{\mathbf{T}}$, and $\tilde{\mathbf{W}}$ be the $(2n \times (k+n))$ and $(2n \times 2n)$, $(2n \times 2n)$ dimensional matrices

$$\tilde{\mathbf{X}} = \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}, \quad \tilde{\mathbf{T}} = \begin{pmatrix} (I_n - Y^c)T\Phi & \mathbf{0} \\ \mathbf{0} & (I_n - Y^c)H \end{pmatrix}, \quad \tilde{\mathbf{W}} = \begin{pmatrix} \mathbf{W}_2 & \mathbf{W}_3 \\ \mathbf{W}_3 & \mathbf{W}_4 \end{pmatrix}, \quad (\text{B.6})$$

respectively, and $\tilde{\mathbf{y}}^\top = ((y^* - \mu^*)^\top, [\mathcal{M}(y^* - \mu^*) + (y^\dagger - \mu^\dagger)]^\top)^\top$ be an $1 \times 2n$ auxiliary vector. The score vector corresponding to $\vartheta = (\beta^\top, \gamma^\top)^\top$ can be written as

$$U(\vartheta) = \tilde{\mathbf{X}}^\top \tilde{\mathbf{T}} \tilde{\mathbf{y}}. \quad (\text{B.7})$$

Fisher's information matrix for the parameter vector ϑ is given by

$$K(\vartheta) = \tilde{\mathbf{X}}^\top \tilde{\mathbf{W}} \tilde{\mathbf{X}}. \quad (\text{B.8})$$

Also, the iterative process for estimating ϑ takes the form

$$\vartheta^{(m+1)} = (\tilde{\mathbf{X}}^{(m)\top} \tilde{\mathbf{W}}^{(m)} \tilde{\mathbf{X}}^{(m)})^{-1} \tilde{\mathbf{X}}^{(m)\top} \tilde{\mathbf{W}}^{(m)} \tilde{\mathbf{y}}^{*(m)}, \quad (\text{B.9})$$

where $\tilde{\mathbf{y}}^{*(m)} = \tilde{\mathbf{X}} \vartheta + \tilde{\mathbf{W}}^{-1} \tilde{\mathbf{T}} \tilde{\mathbf{y}}$ and $m = 0, 1, 2, \dots$ are the iterations that are performed until convergence, which occurs when the distance between $\vartheta^{(m+1)}$ and $\vartheta^{(m)}$ becomes smaller than a given, small constant. The procedure is initialized by choosing suitable initial values for β and γ .

Assuming that γ is known, Fisher's scoring iterative scheme used to estimating β can be written as

$$\beta^{(m+1)} = (\mathbf{X}^{(m)\top} \mathbf{W}_2^{(m)} \mathbf{X}^{(m)})^{-1} \mathbf{X}^{(m)\top} \mathbf{W}_2^{(m)} \mathbf{y}_2^{(m)}, \quad (\text{B.10})$$

where \mathbf{W}_2 is defined in (3.7) and $\mathbf{y}_2 = \mathbf{X}\beta + \mathbf{W}_2^{-1}T\Phi(y^* - \mu^*)$, with $m = 0, 1, 2, \dots$. Upon convergence, we have

$$\hat{\beta} = (\hat{\mathbf{X}}^\top \hat{\mathbf{W}}_2 \hat{\mathbf{X}})^{-1} \hat{\mathbf{X}}^\top \tau, \quad (\text{B.11})$$

where $\tau = \hat{\mathbf{X}}\hat{\beta} + \hat{\mathbf{W}}_2^{-1}T\hat{\Phi}(y^* - \hat{\mu}^*)$. Then, $\hat{\beta}$ in (B.11) can be viewed as the least-squares estimate of β obtained by regressing τ on \mathbf{X} with weighting matrix \mathbf{W}_2 . The ordinary residual of this re-weighted least-squares regression is

$$r = (I_n - \hat{\mathbf{P}})\tau = \hat{\mathbf{W}}_2^{1/2}(\tau - \hat{\mathbf{X}}\hat{\beta}) = \hat{\mathbf{W}}_2^{-1/2}\hat{\Phi}T(y^* - \hat{\mu}^*). \quad (\text{B.12})$$

Here,

$$\hat{\mathbf{P}} = \hat{\mathbf{W}}_2^{1/2} \hat{\mathbf{X}} (\hat{\mathbf{X}}^\top \hat{\mathbf{W}}_2 \hat{\mathbf{X}})^{-1} \hat{\mathbf{X}}^\top \hat{\mathbf{W}}_2^{1/2} \quad (\text{B.13})$$

is a projection matrix. Note that if all the quantities are evaluated at the true parameter, $E(r) = 0$ and $\text{Var}(r) = (I_n - \mathbf{P})$. The \mathbf{P} matrix is similar to the leverage matrix in standard linear regression models, and hence, we refer to it as the generalized leverage matrix. It is possible to show that $I_n - \mathbf{P}$ is symmetric, idempotent, and spans the residual r -space. This implies that a small $1 - \mathbf{P}_{tt}$ indicates extreme points in the design space of the continuous component of the zero-or-one inflated beta regression model. Note that the t th element of r is

$$r_t = \frac{y^* - \hat{\mu}^*}{\sqrt{\hat{v}_t^*(1 - \hat{\alpha}_t)}}. \quad (\text{B.14})$$

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