COMP 3400

Assignment 2 Solutions

Question 1

Quesition 1a

```
1 expUp :: Int -> Int -> int
2 expUp x y = h_exp x (-y) 1
3
4 h_exp :: Int -> Int -> Int -> Int
5 h_exp x 0 ans = ans
6 h_exp x y ans = h_exp x (y+1) (ans*x)
```

Question 1b

The iteration invariant is:

$$\texttt{h\_exp x y ans} = \frac{\texttt{ans}}{x^y}.$$

LHS(5) = h_exp x 0 ans = $\frac{\texttt{ans}}{x^0}$ = ans = RHS(5).

LHS(6) = h_exp x y ans = $\frac{\texttt{ans}}{x^y}$ = $\frac{\texttt{ans}\times x}{x^{y+1}}$ = h_exp x (y+1) (ans*x) = RHS(6)

Question 1c

expUp x y = h_exp x (-y) 1 = $\frac{1}{x^{-y}}$ = x^y.

Question 1d

BV(expUp) = y or BV(h_exp) = -y

Question 2

```
iter 0 f = id
iter n f = iter (n-1) f . f
```

Question 3

Question 3a

`foo` returns the last element of its second input provided it is nonempty, otherwise `foo` returns its first input.

Question 3b

```
foo :: a -> [a] -> a
foo x [] = x
foo _ xs = last xs
```

Question 4

```
data Nat = Zero | Succ Nat deriving Show

emb :: Nat -> Int
emb Zero     = 0
emb (Succ n) = 1 + emb n

plus :: Nat -> Nat -> Nat
plus n Zero     = n
plus n (Succ m) = plus (Succ n) m

times :: Nat -> Nat -> Nat
times _ Zero = Zero
times n (Succ m) = plus n $ times n m
```

Let `m::Nat` be arbitrary and `n::Nat`, then

$$P(\texttt{n}) \iff \texttt{emb(times m n) = (emb m)} * \texttt{(emb n)}.$$

Base case:

$$
\begin{aligned}
&\texttt{emb (times m Zero)} & \\
&\quad = \texttt{emb Zero} & \text{by line (12)} \\
&\quad = \texttt{0} & \text{by line (4)} \\
&\quad = \texttt{(emb m)} * \texttt{0} & \text{by definition of } (*) \\
&\quad = \texttt{(emb m)} * \texttt{(emb Zero)} & \text{by line (4)}
\end{aligned}
$$

Thus $P(\texttt{Zero})$.

Induction hypothesis: Presume $P(\texttt{n})$ is true.

Induction:
We presume `emb (plus m n) = emb m + emb n`.[1]

$$
\begin{aligned}
&\texttt{emb (times m (Succ n))} & \\
&\quad = \texttt{emb (plus m \$ times m n)} & \text{by line (13)} \\
&\quad = \texttt{(emb m) + (emb \$ times m n)} & \text{by presumption} \\
&\quad = \texttt{(emb m) + (emb m)} * \texttt{(emb n)} & \text{by induction hypothesis} \\
&\quad = \texttt{(emb m)} * \texttt{(1 + emb n)} & \text{by distributivity of } (*) \\
&\quad = \texttt{(emb m)} * \texttt{(emb \$ Succ n)} & \text{by line (5)}
\end{aligned}
$$

Thus $P(\texttt{succ n})$.

The PMI implies that $P(\texttt{n})$ is valid for arbitrary `n::Nat`. $\square$

[1] Because the instructor allowed it.