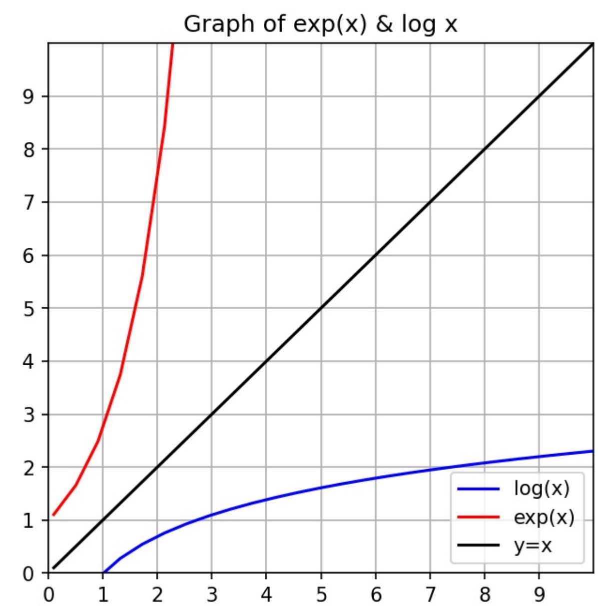
Notes for

$log\,(x)$

Note: While reading the notes, it is better to write the equations somewhere else so that you know which step might confuse you.

If we take a look on the graph of a natural $log(x)$ and e^x, we can see that both are just the same graph but is reflected by the line $y = x$.



In fact, natural $log(x)$ is the inverse of e^x. In other words, if

$$
\begin{aligned}
a^x &= y \\
log_a\left(a^x\right) &= log_a\,(y) \qquad (1)\\
x &= log_a\,(y) \qquad (2)
\end{aligned}
$$

Example:

$$1.\ \log_2 32 = 5 \ \rightarrow \ 32 = 2^5$$

$$2.\ \log_5\left(\frac{1}{25}\right) = -2 \ \rightarrow \ \frac{1}{25} = 5^{-2}$$

Thus, it is easier to remember how to deal with $log(x)$ function if we apply all the rule that we know for indices. i.e:

$$
\begin{gathered}
1.\ a^x * a^y = a^{x+y} \\
2.\ a^x / a^y = a^{x-y} \\
3.\ \left(a^x\right)^y = a^{xy} \\
4.\ a^x * b^x = (a*b)^x
\end{gathered}
$$

PROPERTY 1: Product Rule

from indices rule #1. Let $s = a^x$ and $t = a^y$, then

$$
\begin{gathered}
s * t = a^{x+y} \\
log_a(s*t) = log_a\left(a^{x+y}\right)
\end{gathered}
$$

here, note that from (1) and (2), we can make the right hand side to be

$$log_a(s*t) = x + y \tag{3}$$

but , since $s = a^x$ and $t = a^y$, we can derive the following

$$
\begin{gathered}
s = a^x \\
log_a(s) = log_a\left(a^x\right) \\
log_a(s) = x
\end{gathered}
$$

$$
\begin{gathered}
t = a^y \\
log_a(t) = log_a\left(a^y\right) \\
log_a(t) = y
\end{gathered}
$$

substituting both into RHS of (3),

$$log_a(s*t) = log_a(s) + log_a(t)$$

and we're done with our #1 property.

PROPERTY 2: Division Rule

from indices rule #2. Let $s = a^x$ and $t = a^y$, then

$$\frac{s}{t} = \frac{a^x}{a^y} = a^{x-y}$$

$$log_a\left(\frac{s}{t}\right) = log_a\left(a^{x-y}\right)$$

here, note that from (1) and (2), we can make the right hand side to be

$$log_a\left(\frac{s}{t}\right) = x - y \tag{4}$$

but , since $s = a^x$ and $t = a^y$, we can derive the following

$$\begin{aligned} s &= a^x \\ log_a(s) &= log_a\left(a^x\right) \\ log_a(s) &= x \end{aligned}$$

$$\begin{aligned} t &= a^y \\ log_a(t) &= log_a\left(a^y\right) \\ log_a(t) &= y \end{aligned}$$

substituting both into RHS of (4),

$$log_a\left(\frac{s}{t}\right) = log_a(s) - log_a(t)$$

and we're done with our #2 property.

PROPERTY 3: Power Rule

from indices rule #3, let $s = a^x$, then

$$s^y = \left(a^x\right)^y$$
$$s^y = a^{xy}$$

after take log_a for both side, we have

$$log_a\left(s^y\right) = log_a\left(a^{xy}\right)$$

and applying (1) and (2), we will get

$$log_a\left(s^y\right) = xy \tag{5}$$

now, what is x? we can find x because we define $s = a^x$. Thus

$$s = a^x$$
$$log_a(s) = log_a\left(a^x\right)$$
$$log_a(s) = x$$

substitute this into (5), we'll obtain

$$log_a\left(s^y\right) = log_a(s) * y$$

and we're done with #3 property.

PROPERTY 4 : Change Log Rule

let $s = a^x$, then apply a different log_b for both sides, we can have

$$s = a^x$$
$$log_b(s) = log_b\left(a^x\right)$$

here, we cannot use (1) and (2) because we use different base b (we use log_b not log_a)
but we can use the #3 property

$$log_b(s) = log_b\left(a^x\right)$$
$$log_b(s) = x * log_b(a)$$

if we divide both sides with $log_b(a)$, we obtain

$$log_b\,(s) \;=\; x * log_b\,(a)$$

$$\frac{log_b\,(s)}{log_b\,(a)} \;=\; \frac{x * log_b\,(a)}{log_b\,(a)}$$

$$\frac{log_b\,(s)}{log_b\,(a)} \;=\; x \tag{6}$$

again, can we find x? The answer is yes! since we already define $s \;=\; a^x$. So,

$$s \;=\; a^x$$
$$log_a\,(s) \;=\; x$$

substituting into (6), we're done with our #4 property.

$$\frac{log_b\,(s)}{log_b\,(a)} \;=\; log_a\,(s)$$