1 Taylor Series

Preliminaries

The idea is to find an polynomial that approximate a function $f(x)$.

$$f(x) = \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x--a)^3 + \cdots + c_n(x-a)^n}_{\text{Polynomial of degree n denoted } P_x(x)} + \underbrace{R_n(x)}_{\text{the ramainder}}$$

If we remove the remainder we have the polynomial $P_n(x)$ that approximate $f(x)$, we will use $\simeq$

$$f(x) \simeq P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x--a)^3 + \cdots + c_n(x-a)^n$$

How to find the values $c_0, c_2, \ldots, c_n$? We will chose:

- c_0 such that $P_n(a) = f(a)$: which gives $c_0 = f(a)$
- c_1 such that $P_n'(a) = f'(a)$ which gives $c_1 = f'(a)$
- c_2 such that $P_n''(a) = f''(a)$ which gives $c_1 = \frac{f''(a)}{2}$
- $\ldots$
- c_n such that $P_n^{(n)}(a) = f^{(n)}(a)$ which gives $c_n = \frac{f^{(n)}(a)}{n!}$

Taylor Polynomial

The Taylor polynomial $P_n(x)$ of degree n of f at the point a is:

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples:

1. Talor Polynomial of e^x centered at 0 (this means $a = 0$) of degree n.

$$e^0 = 1,\ (e^x)'\big|_{x=0} = 1,\ (e^x)''\big|_{x=0} = 1,\ (e^x)'''\big|_{x=0} = 1, \ldots$$

   Therefore,

$$e^x \simeq P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^2}{n!}.$$

2. Talor Polynomial of $\ln(1+x)$ centered at 0 of degree 3:

$$\ln(x)\big|_{x=0} = 0,\ \big(\ln(1+x)\big)'\big|_{x=0} = \frac{1}{(1+x)}\big|_{x=0} = 1,\ \big(\ln(1+x)\big)''\big|_{x=0} = \left(\frac{1}{(1+x)}\right)'\big|_{x=0} = -1,$$

$$(\ln(1+x)'''\big|_{x=0} = 2$$

Therefore,

$$\ln(1+x) \simeq P_3(x) = 0 + 1 \cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 = x - fracx^2 2 + \frac{x^3}{3}$$

As an exercise compute the following

1. Talor Polynomial of $\sin(x)$ centered at 0 of degree n.
2. Talor Polynomial of $\frac{1}{x}$ centered at 1 of degreee n.

Error Estimate

How accurate is the Taylor polynomial compare to the original function $f(x)$? For that we need to measure the error:

$$Error(x) = \left|f(x) - P_n(x)|\right| = \left|R(x)\right| \text{ that is the absolute value of the remainder}$$

Taylor bound (Remainder)

Let $P_n(x)$ be the Taylor polyomial of degree n of f at the point a, then the error is bounded by:

$$|R_n(x)| = |f(x) - P_n(x)| \leq \left| f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!} \right|$$

where $f^{(n+1)}(c)$ is the maximum value of $f^{(n+1)}$ between x and a.

Example:
Error between $f(x) = x^2\cos(x)$ and its Taylor Polynomial of degree 2 centered at 0 when $x \in [-\pi, \pi]$.

To compute the error let us first compute

$$f^{(3)}(x) = -6x\cos(x) + (-6 + x^2)\sin(x)$$

This implites that for $-\pi \leq x \leq \pi$:

$$|f^{(3)}(x)| \leq |-6x||\cos(x)| + (|-6| + |x^2|)|\sin(x)| \leq 6\pi + 6 + \pi^2$$

Thefore

$$|f(x) - P_2(x)| \leq (6\pi + 6 + \pi^2)\frac{|x|^3}{3!} \leq (6\pi + 6 + \pi^2)\frac{\pi^3}{6} = \text{Error}$$