4 Solving Ordinary Differential Equations

The goal is to solve

$$y'(t) = \frac{dy}{dt} = f(t, y), \; a \leq t \leq b, \; y(a) = \alpha$$

where $y = [y_1(t), y_2(t), \ldots, y_K(t)]$.

Euler's Explicit Method

We first discretive the domain $[t_0, T]$, into $n + 1$ points $t_i = t_0 + ih$ where $h = \frac{T - t0}{n}$, then you replace $y'(t)$ by $\frac{y(t+h)-y(t)}{h}$ and you define

$$y(t_{i+1}) = y(t_i) + hf(t_i, y_i)$$

which can be simplified by denoting $y^i = y(t_i)$ and gives you

$$\frac{y^{i+1} - y^i}{h} = f(t_i, y^i) \Longrightarrow y^{i+1} = y^i + hf(t_i, y^i)$$

Runge Kutta of Order 4

We first discretive the domain $[t_0, T]$, into $n+1$ points $t_i = t_0 + ih$ where $h = \frac{T-t_0}{n}$, then

$$\begin{aligned}
k_1 &= hf(t_i, y_i) \\
k_2 &= hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}) \\
k_3 &= hf(t_i + \frac{h}{2}, y_i + \frac{k_2}{2}) \\
k_3 &= hf(t_i + h, y_i + k_3) \\
y(t_{i+1}) &= y(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

which can be rewritten by denoting $y^i = y(t_i)$ as

$$\begin{aligned}
k_1 &= hf(t_i, y^i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right) \\
k_4 &= hf\left(t_i + h, y_i + k_3\right) \\
y^{i+1} &= y^i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$