Question 1 1: Write-up: 2 Dimensional divergence-less flow can be written in terms of a stream-function. The spatial derivatives of the stream-function gives us the components of the velocity.

$$\vec{v} = -\vec{\nabla} \times \vec{\psi}$$

with

$$\vec{\psi} = (0, 0, \psi)$$

The plots of the stream function tells us the path of the parcel for a steady state flow. Since there is no vertical velocity $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$, and the horizontal flow pattern remains the same vertically. So, there is no horizontal vorticity either.

$$\vec{\zeta}_x = \vec{\zeta}_y = 0$$

a) For a function

$$\psi = \psi(x, y)$$

The change in ψ is the same as the rate of change of ψ in the x direction times the change in x direction plus the rate of change of ψ in the y direction times the change in y direction

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Where

$$v = -\frac{\partial \psi}{\partial x}$$

and

$$u = \frac{\partial \psi}{\partial y}$$

For a constant function, or an isoline, where through which the function is constant,

$$\psi = C$$

the derivatives are

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$$

So, change in ψ

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

That is,

$$d\psi = v dx - u dy = 0$$

$$\frac{u}{v} = \frac{dx}{dy}$$

Or

$$\frac{u}{dx} = \frac{v}{dy}$$

This is the same thing as finding the solution to an exact differential:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and finding $F(x, y)$ for which

$$P = \frac{\partial F}{\partial x}; Q = \frac{\partial F}{\partial y}; P dx + Q dy = F$$

with $P = v$, $Q = u$, $F = \psi$ and $y' = \frac{v}{u}$

b) The sketch is plotted via MATLAB

$$\psi = -Ux + A\sin[k(x - ct)]$$

Set t = 0

$$\psi = -Ux + A\sin[kx]$$

Table 1: $\psi = -Ux + A\sin[kx]$

X	Y	ψ	X	Y	ψ
0	-2	2U	π/k	-2	2U + A
0	-1	U	π/k	-1	U + A
0	0	0	π/k	0	0 + A
0	1	-U	π/k	1	-U + A
0	2	-2U	π/k	2	-2U + A

c)

$$(x(t=0), y(t=0)) = 0$$

$$v = \frac{\partial - Uy + A\sin[k(x-ct)]}{\partial x} = Ak\cos[k(Ut-ct)]$$

$$u = \frac{\partial Uy - A\sin[k(x-ct)]}{\partial y} = U$$

$$x = \int U dt = Ut + C$$

For a particle that passes through the origin at t=0,

$$x(t=0) = 0 = U * (0) + C \Rightarrow C = 0$$

So,

$$x = Ut$$

$$y = \int v dt = Ak \int \cos[k(Ut-ct)]dt = \frac{A}{U-c}\sin[k(x-ct)] + D$$

$$y(t=0) = 0 = \frac{A}{U-c}\sin[k((0)-c(0))] + D \to D = 0$$

$$y = \frac{Ak}{U-c}\sin[k(x-c(t))]$$

$$(x, y) = (Ut, \frac{A}{U-c}\sin[k(U-c)t])$$

As c approaches U, $(c-U) \to 0$ So,

$$y = \frac{A}{U-c}\sin[k(U-c)t] = \frac{Ak(k(U-c)t)}{U-c} = Akt$$

d) Plots added after a few pages

e)

$$u = U$$

$$v = Ak\cos[k(x-ct)]$$

$$\nabla \cdot \vec{u} = \frac{\partial U}{\partial x} + \frac{\partial v}{\partial y} = 0 + Ak\frac{\partial \cos[k(x-ct)]}{\partial y} = 0$$

$$\underline{\nabla \cdot \vec{u} = 0}$$

$$\frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial z} = 0, \frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0$$

So, the horizontal components of vorticity are zero.

$$\vec{\zeta} \cdot \hat{z} = \hat{z}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

$$= \hat{z}(\frac{\partial Ak\cos[k(x-ct)]}{\partial x} - \frac{\partial U}{\partial y})$$

$$\boxed{\vec{\zeta} = \hat{z}(-Ak^2\sin[k(x-ct)])}$$

Q2: Definitions:

Write-up:

The momentum equations in spherical coordinates have similar form to the ones in Cartesian coordinates. We can find the definition of the divergence operator in spherical coordinates by using the divergence theorem. The procedure for finding the 3 momentum equations in spherical coordinates is the same as the way for Cartesian, via force balances. However there are a bunch of $\cos(\phi)$ factors that need some extra attention.

$$\int_V u dV = \int_r \int_\phi \int_\lambda u r^2 \cos(\phi) d\phi d\lambda dr$$

$$M = \int_r^{r+\delta r} \int_\lambda^{\lambda+\delta\lambda} \int_\phi^{\phi+\delta\phi} \rho r^2 \cos(\phi) d\phi d\lambda dr$$

$$\vec{p} = \int_r^{r+\delta r} \int_\lambda^{\lambda+\delta\lambda} \int_\phi^{\phi+\delta\phi} \rho \vec{u} r^2 \cos(\phi) d\phi d\lambda dr$$

$$x = r\cos(\phi)\cos(\lambda)$$

$$y = r\cos(\phi)\sin(\lambda)$$

$$z = r\sin(\phi)$$

$$r^2 = x^2 + y^2 + z^2$$

For water in the ocean, if there are no currents, the only stress felt is the normal stress, i.e. the pressure, otherwise we also have wind stress. We will be using the stress on the faces from different directions for force balance.

Table 2: $\tau_{ij} = \delta_{ij} P - \mu \frac{\partial v_j}{\partial x_i}$

r	ϕ	λ
$\tau_{rr} = +P - \mu \frac{\partial v_r}{\partial r}$	$\tau_{r\phi} = -\mu \frac{\partial v_\phi}{\partial r}$	$\tau_{r\lambda} = -\mu \frac{\partial v_\lambda}{\partial r}$
$\tau_{\lambda r} = -\mu \frac{\partial v_r}{\partial \lambda}$	$\tau_{\lambda\phi} = -\mu \frac{\partial v_\phi}{\partial \lambda}$	$\tau_{\lambda\lambda} = P - \mu \frac{\partial v_\lambda}{\partial \lambda}$
$\tau_{\phi r} = -\mu \frac{\partial v_r}{\partial \phi}$	$\tau_{\phi\phi} = P - \mu \frac{\partial v_\phi}{\partial \phi}$	$\tau_{\phi\lambda} = -\mu \frac{\partial v_\lambda}{\partial \phi}$

The areas of the different faces will be useful in finding the net flow of momentum in a direction and in turn the net flow through the volume element.

Table 3: Areas of the different faces i $\equiv$ in,o $\equiv$ out

in	out
$Ai = \int_\phi^{\phi+\delta\phi} \int_r^{r+\delta r} r dr d\phi$	$Ao = \int_\phi^{\phi+\delta\phi} \int_r^{r+\delta r} r dr d\phi$
$Bi = \int_\lambda^{\lambda+\delta\lambda} \int_r^{r+\delta r} r\cos(\phi) dr d\lambda$	$Bo = \int_\lambda^{\lambda+\delta\lambda} \int_r^{r+\delta r} r\cos(\phi+\delta\phi) dr d\lambda$
$Ci = \int_\lambda^{\lambda+\delta\lambda} \int_\phi^{\phi+\delta\phi} r^2 \cos(\phi) d\phi d\lambda$	$Co = \int_\lambda^{\lambda+\delta\lambda} \int_\phi^{\phi+\delta\phi} r^2 \cos(\phi) d\phi d\lambda$

In spherical coordinates, the divergence operator is defined as

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{\cos(\phi)} \left(\frac{1}{r} \frac{\partial F_\lambda}{\partial \lambda} + \frac{1}{r} \frac{\partial F_\phi \cos(\phi)}{\partial \phi} + \frac{\cos(\phi)}{r^2} \frac{\partial F_r r^2}{\partial r} \right)$$

And the velocities are

$$(u, v, w) = (r\cos(\phi)\frac{D\lambda}{Dt}, r\frac{D\phi}{Dt}, \frac{Dr}{Dt})$$

So the material derivative is,

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \frac{u}{r\cos(\phi)}\frac{\partial A}{\partial \lambda} + \frac{v}{r}\frac{\partial A}{\partial \phi} + w\frac{\partial A}{\partial r}$$

The divergence operator in spherical coordinates can be derived using the divergence theorem:

$$\int\int \vec{F}\cdot d\vec{S} = \int\int\int \vec{\nabla}\cdot\vec{F}\cdot d\vec{V}$$

$$\int\int \vec{F}\cdot d\vec{A}_o - \int\int \vec{F}\cdot d\vec{A}_i + \int\int \vec{F}\cdot d\vec{B}_o - \int\int \vec{F}\cdot d\vec{B}_i + \int\int \vec{F}\cdot d\vec{C}_o - \int\int \vec{F}\cdot d\vec{C}_i$$

$$= \int\int\int \vec{\nabla}\cdot\vec{F}\cdot r^2\cos(\phi)d\phi d\lambda dr$$

$$\begin{aligned}
&-\int_{\phi}^{\phi+\delta\phi}\int_{r}^{r+\delta r}\vec{F}rdrd\phi + \int_{\phi}^{\phi+\delta\phi}\int_{r}^{r+\delta r} rdrd\phi \\
&\quad -\int_{\lambda}^{\lambda+\delta\lambda}\int_{r}^{r+\delta r}\vec{F}r\cos(\phi)drd\lambda = \int_{\lambda}^{\lambda+\delta\lambda}\int_{r}^{r+\delta r}\vec{F}r\cos(\phi+\delta\phi)drd\lambda \\
&\quad\quad -\int_{\lambda}^{\lambda+\delta\lambda}\int_{\phi}^{\phi+\delta\phi}\vec{F}r^2\cos(\phi)d\phi d\lambda + \int_{\lambda}^{\lambda+\delta\lambda}\int_{\phi}^{\phi+\delta\phi}\vec{F}r^2\cos(\phi)d\phi d\lambda
\end{aligned} \tag{1}$$

Taylor expanding, canceling terms, and taking the limit of small

$$= \frac{\partial\vec{F}}{\partial\lambda}rd\lambda drd\phi + \frac{\partial\vec{F}\cos(\phi)}{\partial\phi}rd\phi drd\lambda + \frac{\partial\vec{F}r^2}{\partial r}\cos(\phi)drd\phi d\lambda$$

Plugging it back into the divergence theorem

$$= \left(\frac{\partial\vec{F}}{\partial\lambda}r + \frac{\partial\vec{F}\cos(\phi)}{\partial\phi}r + \frac{\partial\vec{F}r^2}{\partial r}\cos(\phi)\right)drd\phi d\lambda = \vec{\nabla}\cdot\vec{F}\cdot r^2\cos(\phi)d\phi d\lambda dr$$

$$\vec{\nabla}\cdot\vec{F} = +\frac{1}{r^2\cos(\phi)}\left(\frac{\partial\vec{F}}{\partial\lambda}r + \frac{\partial\vec{F}\cos(\phi)}{\partial\phi}r + \frac{\partial\vec{F}r^2}{\partial r}\cos(\phi)\right)$$

$$\vec{\nabla}\cdot\vec{F} = +\frac{1}{r\cos(\phi)}\frac{\partial\vec{F}}{\partial\lambda} + \frac{1}{r\cos(\phi)}\frac{\partial\vec{F}\cos(\phi)}{\partial\phi} + \frac{1}{r^2}\frac{\partial\vec{F}r^2}{\partial r}$$

Let RA_i and RA_o be the r-momentum flux through the A_i and A_o faces

$$RA_i = \int_{\phi}^{\phi+\delta\phi} \int_{r}^{r+\delta r} \tau^{\lambda r}(\phi, \lambda, r) r dr d\phi$$

$$RA_o = \int_{\phi}^{\phi+\delta\phi} \int_{r}^{r+\delta r} \tau^{\lambda r}(\phi, \lambda + d\lambda, r) r dr d\phi$$

Taylor expanding in λ and taking the limit of small dλ dϕ dr to drop the integrals:

$$RA_o \approx (\tau^{\lambda r}(\phi, \lambda, r) + \left.\frac{\partial \tau^{\lambda r}(\phi, \lambda, r)}{\partial \lambda}\right|_{\lambda} d\lambda) r dr d\phi$$

r-momentum flow through B:

$$RB_i = \int_{\lambda}^{\lambda+\delta\lambda} \int_{r}^{r+\delta r} \tau^{\phi r}(\phi, \lambda, r) r \cos(\phi) dr d\lambda$$

$$RB_o = \int_{\lambda}^{\lambda+\delta\lambda} \int_{r}^{r+\delta r} \tau^{\phi r}(\phi + \delta\phi, \lambda, r) r \cos(\phi + \delta\phi) dr d\lambda$$

$$RB_o \approx (\tau^{\phi r}(\phi, \lambda, r) r \cos(\phi) + \left.\frac{\partial \tau^{\phi r}(\phi, \lambda, r) r \cos(\phi)}{\partial \phi}\right|_{\phi} d\phi) dr d\lambda$$

r-momentum flow through C:

$$RC_i = \int_{\lambda}^{\lambda+\delta\lambda} \int_{\phi}^{\phi+\delta\phi} \tau^{rr}(\phi, \lambda, r) r^2 \cos(\phi) d\phi d\lambda$$

$$RC_o = \int_{\lambda}^{\lambda+\delta\lambda} \int_{\phi}^{\phi+\delta\phi} \tau^{rr}(\phi, \lambda, r + \delta r) r^2 \cos(\phi) d\phi d\lambda$$

$$RC_o \approx (\tau^{rr}(\phi, \lambda, r) r^2 + \left.\frac{\partial \tau^{rr}(\phi, \lambda, r) r^2}{\partial r}\right|_{r} dr) \cos(\phi) d\phi d\lambda$$

Momentum Balance in r-direction (flow in - flow out):

$$F_{rin} - F_{rout} = RA_i - RA_o + RB_i - RB_o + RC_i - RC_o$$

$$= \left(\tau^{\lambda r}(\phi, \lambda, r) - \tau^{\lambda r}(\phi, \lambda, r) - \left.\frac{\partial \tau^{\lambda r}(\phi, \lambda, r)}{\partial \lambda}\right|_{\lambda} d\lambda\right) r dr d\phi$$

$$\left(\tau^{\phi r}(\phi, \lambda, r) r \cos(\phi) - \tau^{\phi r}(\phi, \lambda, r) r \cos(\phi) - \left.\frac{\partial \tau^{\phi r}(\phi, \lambda, r) r \cos(\phi)}{\partial \phi}\right|_{\phi} d\phi\right) dr d\lambda$$

$$+ \left(\tau^{rr}(\phi, \lambda, r) - \tau^{rr}(\phi, \lambda, r) - \left.\frac{\partial \tau^{rr}(\phi, \lambda, r)}{\partial r}\right|_{r} dr\right) r^2 \cos(\phi) d\phi d\lambda$$

Simplifying,

$$= -\left(\frac{\partial \tau^{\lambda r}}{\partial \lambda} d\lambda\right) r dr d\phi - \left(\frac{\partial \tau^{\phi r} r \cos(\phi)}{\partial \phi} d\phi\right) dr d\lambda - \left(\frac{\partial \tau^{rr}}{\partial r} dr\right) r^2 \cos(\phi) d\phi d\lambda$$

We want the accumulation of the r-component in the box, so we divide by the volume element $dV = r^2\cos(\phi)d\phi d\lambda dr$

$$\frac{F_{rin}-F_{rout}}{dV} = -(\frac{1}{r\cos(\phi)}\frac{\partial\tau^{\lambda r}(\phi,\lambda,r)}{\partial\lambda}+\frac{1}{r\cos(\phi)}\frac{\partial\tau^{\phi r}(\phi,\lambda,r)\cos(\phi)}{\partial\phi}+\frac{1}{r^2}\frac{\partial\tau^{rr}(\phi,\lambda,r)r^2}{\partial r})$$

Plug in the expressions for τ^{ij} into the above equation:

$$\frac{F_{rin}-F_{rout}}{dV} = -(\frac{1}{r\cos(\phi)}\frac{\partial-\mu\frac{\partial v_r}{\partial\lambda}}{\partial\lambda}+\frac{1}{r\cos(\phi)}\frac{\partial-\mu\frac{\partial v_r}{\partial\phi}\cos(\phi)}{\partial\phi}+\frac{1}{r^2}\frac{\partial(+P-\mu\frac{\partial v_r}{\partial r})r^2}{\partial r})$$

$$\frac{F_{rin}-F_{rout}}{dV} = (\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_r}{\partial\lambda}}{\partial\lambda}+\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_r}{\partial\phi}\cos(\phi)}{\partial\phi}+\frac{1}{r^2}\frac{\partial(\mu\frac{\partial v_r}{\partial r})r^2}{\partial r})-\frac{1}{r^2}\frac{\partial Pr^2}{\partial r}$$

This is $\frac{F_r}{V}$ where $\vec{F}$ is the force. The momentum balance in ϕ and λ is found using the same exact process, except the pressure, P, is at the respective face. ϕ-momentum balance:

$$\frac{F_{\phi in}-F_{\phi out}}{dV} = (\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_\phi}{\partial\lambda}}{\partial\lambda}+\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_\phi}{\partial\phi}\cos(\phi)}{\partial\phi}+\frac{1}{r^2}\frac{\partial(\mu\frac{\partial v_\phi}{\partial r})r^2}{\partial r})-\frac{1}{r\cos(\phi)}\frac{\partial P\cos(\phi)}{\partial\phi}$$

λ-momentum balance:

$$\frac{F_{\lambda in}-F_{\lambda out}}{dV} = (\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_\lambda}{\partial\lambda}}{\partial\lambda}+\frac{1}{r\cos(\phi)}\frac{\partial\mu\frac{\partial v_\lambda}{\partial\phi}\cos(\phi)}{\partial\phi}+\frac{1}{r^2}\frac{\partial(\mu\frac{\partial v_\lambda}{\partial r})r^2}{\partial r})-\frac{1}{r\cos(\phi)}\frac{\partial P}{\partial\lambda}$$

$$\frac{\vec{F}}{dV} = \frac{1}{dV}\frac{D\vec{p}}{Dt} = \frac{1}{dV}\frac{D(\rho\vec{u}dV)}{Dt} = \frac{D(\rho\vec{u})}{Dt}$$

$$\frac{D(\rho\vec{u})}{Dt} = -\nabla(\vec{P})+\mu\vec{\nabla}^2\vec{u}$$

$$\frac{D(\rho\vec{u})}{Dt} = \rho\frac{\partial\vec{u}}{\partial t}+\vec{u}\frac{\partial\rho}{\partial t}+\vec{u}(\vec{u}\cdot\vec{\nabla}(\rho))+\rho(\vec{u}\cdot\vec{\nabla}(\vec{u}))$$

But for seawater, we can assume the density to be approximately constant, so

$$\frac{D(\rho\vec{u})}{Dt} = \rho(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}(\vec{u})) = -\nabla(\vec{P})+\mu\vec{\nabla}^2\vec{u}$$

Which gives us,

$$\rho(\frac{\partial u_r}{\partial t}+\vec{u}\cdot\vec{\nabla}(u_r)) = -\frac{1}{r^2}\frac{\partial Pr^2}{\partial r}+\mu(\frac{1}{r\cos(\phi)}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{r\cos(\phi)}\frac{\partial^2\cos(\phi)}{\partial\phi^2}+\frac{1}{r^2}\frac{\partial^2 r^2}{\partial r^2})u_r$$

For a = radius of the earth,

$$\frac{\delta r}{a} << 1$$

so,

$$r \approx a$$

Momentum Equations:

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \frac{u}{r\cos(\phi)}\frac{\partial A}{\partial \lambda} + \frac{v}{r}\frac{\partial A}{\partial \phi} + w\frac{\partial A}{\partial r}$$

$$\rho\frac{Du_r}{Dt} = \rho(\frac{\partial u_r}{\partial t} + \vec{u}\cdot\vec{\nabla}(u_r)) = -\frac{\partial P}{\partial r} + \vec{\nabla}^2 u_r$$

$$\rho\left(\frac{\partial u_r}{\partial t} + \frac{u}{r\cos(\phi)}\frac{\partial u_r}{\partial \lambda} + \frac{v}{r}\frac{\partial u_r}{\partial \phi} + w\frac{\partial u_r}{\partial r}\right) = -\frac{\partial P}{\partial r} + \mu(\frac{1}{a\cos(\phi)}\frac{\partial^2}{\partial \lambda^2} + \frac{1}{a\cos(\phi)}\frac{\partial^2\cos(\phi)}{\partial \phi^2} + \frac{\partial^2}{\partial r^2})u_r$$

$$\rho\frac{Du_\phi}{Dt} = \rho(\frac{\partial u_\phi}{\partial t} + \vec{u}\cdot\vec{\nabla}(u_\phi)) = -\frac{1}{a\cos(\phi)}\frac{\partial P\cos(\phi)}{\partial \phi} + \vec{\nabla}^2 u_\phi$$

$$\rho\left(\frac{\partial u_\phi}{\partial t} + \frac{u}{r\cos(\phi)}\frac{\partial u_\phi}{\partial \lambda} + \frac{v}{r}\frac{\partial u_\phi}{\partial \phi} + w\frac{\partial u_\phi}{\partial r}\right) = -\frac{1}{a\cos(\phi)}\frac{\partial P\cos(\phi)}{\partial \phi} + \mu(\frac{1}{r\cos(\phi)}\frac{\partial^2}{\partial \lambda^2} + \frac{1}{a\cos(\phi)}\frac{\partial^2 cos(\phi)}{\partial \phi^2} + \frac{\partial^2}{\partial r^2})u$$

$$\rho\frac{Du_\lambda}{Dt} = \rho(\frac{\partial u_\lambda}{\partial t} + \vec{u}\cdot\vec{\nabla}(u_\lambda)) = -\frac{1}{r\cos(\phi)}\frac{\partial P}{\partial \lambda} + \mu\vec{\nabla}^2 u_\lambda$$

$$\rho\left(\frac{\partial u_\lambda}{\partial t} + \frac{u}{r\cos(\phi)}\frac{\partial u_\lambda}{\partial \lambda} + \frac{v}{r}\frac{\partial u_\lambda}{\partial \phi} + w\frac{\partial u_\lambda}{\partial r}\right) = -\frac{1}{r\cos(\phi)}\frac{\partial P}{\partial \lambda} + \mu(\frac{1}{a\cos(\phi)}\frac{\partial^2}{\partial \lambda^2} + \frac{1}{a\cos(\phi)}\frac{\partial^2\cos(\phi)}{\partial \phi^2} + \frac{\partial^2}{\partial r^2})u_\lambda$$