Introduction to SARIMA and PAR/PMA model

- **A**utoregressive **I**ntegrated **M**oving **A**verage (ARIMA) Model

An $ARIMA(p,d,q)$ process is given by

$$\phi_p(B)(1-B)^d y_t = \theta_q(B)\varepsilon_t,$$

where $\phi(z)$ and $\theta(z)$ are polynomials of order p and q respectively and are given by

$$\phi_p(z) = 1 - \phi_1 z - \cdots - \phi_p z^p,$$

and

$$\theta_q(z) = 1 + \theta_1 z + \cdots + \theta_q z^q,$$

B is the backshift operator, and $\{\varepsilon_t\}$ is a white noise process with mean zero and variance σ^2. To ensure causality and invertibility, we need to have that $\phi_p(z)$ and $\theta_q(z)$ have no roots outside the unit circle, i.e. $|z| \leq 1$.

- **S**easonal **A**utoregressive **I**ntegrated **M**oving **A**verage (SARMA) Model

The $SARIMA(p,d,q)(P,D,Q)_s$ process is given by

$$\Phi_P(B^s)\phi_p(B)(1-B)^d(1-B^s)^D y_t = \Theta_Q(B^s)\theta_q(B)\varepsilon_t,$$

where s is the seasonal frequency, $\Phi_{\mathrm{P}}(z)$ and $\Theta_{\mathrm{Q}}(z)$ are polynomials of orders P and Q respectively, each containing no roots inside the unit circle with

$$\Phi_P(z^s) = 1 - \Phi_1 z^s - \Phi_2 z^{2s} - \cdots - \Phi_P z^{Ps},$$

and

$$\Theta_Q(z^s) = 1 + \Theta_1 z^s + \Theta_2 z^{2s} + \cdots + \Theta_Q z^{Qs}.$$

In practice, although it is case specific, it is not expected to have P, D, and Q greater than 1.

- Periodic Time Series Model

  – **P**eriodic **A**utoregressive **M**odel (PAR)

  Consider an observed time series $y_{s,n}$, where $s = 1, \dots, S$ denotes the season and $n = 1, \dots, N$ denotes the year. A periodic AR(p), or PAR(p), process has the form of

$$y_{s,n} = \mu_s + \phi_{1s} y_{s-1,n} + \cdots + \phi_{ps} y_{s-p,n} + \varepsilon_{s,n}, \qquad \{\varepsilon_{s,n}\} \sim NID(0, \sigma^2)$$

  where $y_{i,n} = y_{S+i,n-1}$ when $i \leq 0$, and NID denotes normal and independently distributed.

  Note that some $\phi_{is}, i = 1, \dots, p$ can take zero values, the order in the above PAR model is the maximum of all p_s, where p_s denotes the AR order per season s.

  ▪ Consider the example of a PAR(2) process

$$y_{s,n} = \phi_{1s} y_{s-1,n} + \phi_{2s} y_{s-2,n} + \varepsilon_{s,n}, s = 1,2,3,4.$$

$$y_{1,n} = \phi_{11} y_{\underbrace{0,n}_{4,n-1}} + \phi_{21} y_{\underbrace{-1,n}_{3,n-1}} + \varepsilon_{1,n}$$

$$y_{2,n} = \phi_{12} y_{1,n} + \phi_{22} y_{\underbrace{0,n}_{4,n-1}} + \varepsilon_{2,n}$$

$$y_{3,n} = \phi_{13} y_{2,n} + \phi_{23} y_{1,n} + \varepsilon_{3,n}$$

$$y_{4,n} = \phi_{14} y_{3,n} + \phi_{24} y_{2,n} + \varepsilon_{4,n}$$

  which can be written as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ -\phi_{23} & -\phi_{13} & 1 & 0 \\ 0 & -\phi_{24} & -\phi_{14} & 1 \end{bmatrix}}_{\Phi_0} \underbrace{\begin{bmatrix} y_{1,n} \\ y_{2,n} \\ y_{3,n} \\ y_{4,n} \end{bmatrix}}_{Y_n} = \underbrace{\begin{bmatrix} 0 & 0 & \phi_{21} & \phi_{11} \\ 0 & 0 & 0 & \phi_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\Phi_1} \underbrace{\begin{bmatrix} y_{1,n-1} \\ y_{2,n-1} \\ y_{3,n-1} \\ y_{4,n-1} \end{bmatrix}}_{Y_{n-1}} + \underbrace{\begin{bmatrix} \varepsilon_{1,n} \\ \varepsilon_{2,n} \\ \varepsilon_{3,n} \\ \varepsilon_{4,n} \end{bmatrix}}_{E_n}$$

  or

$$\Phi_0 Y_n = \Phi_1 Y_{n-1} + E_n$$

$$\rightarrow Y_n = \Phi_0^{-1} \Phi_1 Y_{n-1} + \Phi_0^{-1} E_n.$$

– **P**eriodic **M**oving **A**verage **M**odel (PMA)

A PMA model for y_t can be written as

$$y_t = \delta_s + \theta_{qs}(B)\varepsilon_t,$$

with

$$\theta_{qs} = 1 + \theta_{1s}B + \cdots + \theta_{qs}B^q,$$

where some $\theta_{is}, i = 1, \ldots, q$ can take zero values, the order in the above PMA model is the maximum of all q_s, where q_s denotes the MA order per season s. Naturally, we can combine the previous two periodic time series models in a PARMA(*p*,*q*) model.

- See Equation (20) of Lin (2017) for the example of PMA(2) model.

- <u>Autocovariance functions for a SARIMA model</u>

Consider the $SARIMA\ (0,1,1)(0,1,1)_{12}$ model

$$\underbrace{(1-B)(1-B^{12})y_t}_{w_t} = \underbrace{(1+\theta B)(1+\Theta B^{12})}_{1+\theta B+\Theta B^{12}+\theta\Theta B^{13}}\varepsilon_t.$$

For the process $\{w_t\}$, its autocovariance functions are calculated as

$$\gamma(0) = var(w_t) = \sigma^2\left(1+\theta^2+\Theta^2+\theta^2\Theta^2\right)$$

$$\gamma(1) = cov(w_t, w_{t-1}) = \sigma^2(\theta + \Theta\cdot\theta\Theta) = \theta\sigma^2(1+\Theta^2)$$

$$\gamma(2) = 0$$

...

$$\gamma(11) = 0$$

$$\gamma(12) = \sigma^2\theta(1+\theta^2)$$

$$\gamma(13) = \sigma^2\theta\Theta$$

$$\gamma(h) = 0, \qquad h > 13$$