Box and Tiao Transformation for estimating distributed lag models

Consider a transfer function noise model

$$y_t = v(B)x_t + e_t, \quad (1)$$

where

$$v(B) = v_0 + v_1 B + \cdots + v_s B^s$$

and the error term, and e_t, follows an $ARMA\ (p, q)$ model

$$\phi(B)e_t = \theta(B)a_t, \quad (2)$$

where a_t is white noise satisfying

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

and

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

Eqn. (1) may be expressed as

$$y_t = v(B)x_t + \frac{\theta(B)}{\phi(B)}a_t. \quad (2)$$

Rearranging eqn. (2) and multiplying $\phi(B)/\theta(B)$ on both sides of eqn. (2), we have

$$\frac{\phi(B)}{\theta(B)}y_t = v(B)\frac{\phi(B)}{\theta(B)}x_t + a_t. \quad (3)$$

Consider the transformed variables suggested by Box and Tiao (1975). Eqn. (3) becomes

$$\tilde{y}_t = v(B)\tilde{x}_t + a_t, \quad (4)$$

where $\tilde{y}_t = \phi(B) \cdot \theta^{-1}(B)y_t = \tilde{y}_t$, and $\tilde{x}_t = \phi(B) \cdot \theta^{-1}(B)x_t$.

Rearranging eqn. (4), we have

$$\tilde{y}_t = \sum_{j=0}^{s} v_j \tilde{x}_{t-j} + a_t. \quad (4)$$

Since the error term in eqn. (4) is white noise, we could fit it using the least squares regression.

STEPS OF THE ESTIMATION PROCEDURE

The steps of the estimation procedure may be summarized as follows:

1. Run the OLS regression on

$$y_t = \sum_{j=1}^{s} v_j x_{t-j} + e_t. \qquad (5)$$

   and collect the residuals $\{\hat{e}_t\}$;
2. Indentify an $ARMA$ model for $\{\hat{e}_t\}$ collected in step 1;
3. Apply Box and Tiao transformation using the model identified in step 2 to filter $\{y_t\}$ and $\{x_t\}$ for all t;
4. Run the OLS regression of eqn. (5) using the transformed variables obtained in Step 3;
5. Check whether the regression residuals on Step 4 are serially uncorrelated.
   i. If not, repeat Step 2 to 4;
   ii. If yes, the model estimation is complete.