Transfer Function Noise model and Intervention Analysis

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Transfer function noise model

Aka ARMAX or dynamic regression

Optional reading: Chapter 14 of Wei (2005)

Transfer function noise (TFN) model

- A TFN model is a time series regression that predict values of a dependent variable based on both the current and lagged values of one or more explanatory variables.
- A distributed lag model in statistics and econometrics*
- E.g. sales and advertisement are the example of the dependent variable and the input or explanatory variable in a TFN model

TFN Model

- Mathematical form (causal single-input, single-output system)

$$y_t = v(B)x_t + n_t$$

where $v(B) = \sum_{j=0}^{\infty} v_j B^j$, and x_t and n_t are independent.

- The coefficients $v_0, v_1, \cdots$ are referred to as the impulse response function of the system.* (Distributed lag models require impulse response functions of the same sign.)

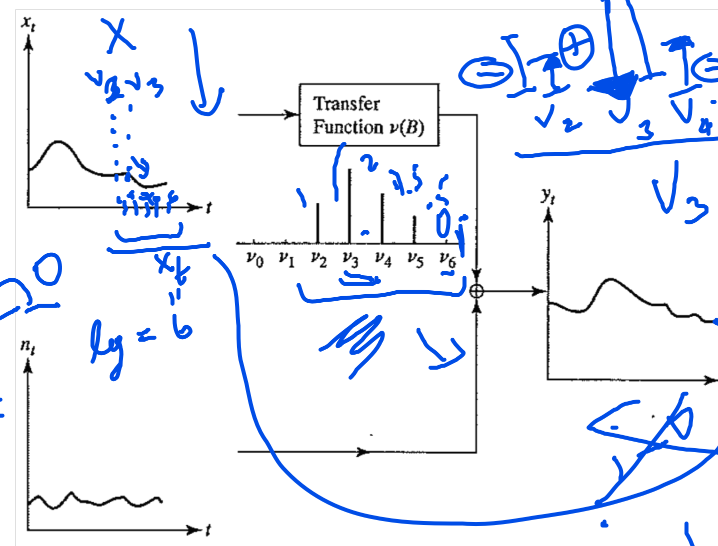


FIGURE 14.1 Dynamic transfer function system.

Transfer function noise model

- TFN model: $y_t = v(B)x_t + n_t$
- For the above equation to be *meaningful*, the impulse responses must be absolutely summable, i.e., $\sum_{j=0}^{\infty}|v_j| < \infty$.
  - In this case, the system is said to be stable.
- The value $g = \sum_{j=0}^{\infty} v_j$ is called the stead-state gain
  - It represents the impact on Y when X_{t-j} are held constant over time.

$v_j x$

$v_1 + v_2 x + v_3 x + v_4 + v_5 x$

$0 \quad 1 + 2 + 1.5 \quad .5$

$g = 1 + 2 + 1.5 + .5 = 5.$

$g = 1 - 2 + 3 - 6 = -4.$

Model with infinite number of parameters

1. Unstructured estimation
2. Structured estimation/approximation
   ❖Finite distributed lag model, e.g. Almon distributed lag model

$$v_j = \sum_{j=0}^{n} a_j i^j ,$$

where $i = 0, \dots, k$ and $n < k$.

   ❖Rational (infinite) distributed lag model, e.g. Koyck distributed lag model

parameters

j^n

V_j is among of effect

$i, i-1, i-2$

V_j = constant

$V_j = V_j(i, i-1, i-2, i-3)$

$y_i = \beta_0 + \beta_1 X_i + \epsilon_i$

Koyck distributed lag model

y_t

β_0

$\sum^n$

$$y_t = \alpha + \sum_{i=0}^{\infty} \beta\lambda^i x_{t-i} + \sum_{i=0}^{\infty} \lambda^i \xi_{t-i}\,.$$

$i \neq 0$

- That is, we have $v_i = \beta\lambda^i$. Suppose that $|\lambda| < 1$
- We can approximate the Koyck distributed lag model using the following ARX model

$$y_t = a + \lambda y_{t-1} + \beta x_t + \xi_t.$$

n

$i = 1$

$\boxed{\lambda^i = 0 \quad \forall i \neq 1}$

$\infty \quad \sum^n \text{ vs } \sum^\infty$

Rational distributed lag model

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + n_t$$

- Jorgenson (1966, Econometrica) proves that $v(B) = \sum_{i=0}^{\infty} v_i B^i$

can be approximated by a ratio of two polynomials

$$v(B) = \frac{\delta_0 + \delta_1 B + \cdots + \delta_r B^r}{1 - \vartheta_1 B - \cdots - \vartheta_s B^s} = \frac{\delta(B)}{\vartheta(B)}.$$

- Using the rational distributed lag function, we cam approximate y_t as

$$y_t = \frac{\delta(B)}{\vartheta(B)} x_t + n_t.$$

where we allows the error term ε_t to follow a stationary ARMA process. The above equation satisfies the form of a transfer function noise model.

Model building process

- The procedure of building the single input TFN model includes
    1. Preliminary identification of the impulse response coefficients ν_i*'s*;
    2. Specification of the noise term ε_t; n_t
    3. Specification of the transfer function using a rational polynomial in *B if necessary*;
    4. Estimation of the TFN model specified in Step 2 and 3;
    5. Model diagnostic checks.
- See the supplement materials for Model diagnostic checks and estimation using the Box and Tiao approach.
- In practice, we may model the multiple inputs TFN model using vector autoregression.

Preliminary identification (prewhitening)

- Suppose that x follows an $ARMA$ model

$$\phi_x(B)x_t = \theta_x(B)\alpha_t, \qquad \alpha_t \sim NID(0, \sigma_\alpha^2).$$

- Apply the operator $\phi_x(B)/\theta_x(B)$ on both sides of the above equation

$$\tau_t = \frac{\phi_x(B)}{\theta_x(B)} y_t = \nu(B) \underbrace{\frac{\phi_x(B)}{\theta_x(B)} x_t}_{\alpha_t} + \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t = \nu(B)\alpha_t + n_t, \qquad (*)$$

where $\tau_t = \frac{\phi_x(B)}{\theta_x(B)} Y_t$ and $n_t = \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t$

- By design, $\{n_t\}$ is independent of $\{\alpha_t\}$.

Preliminary identification (prewhitening)

- Multiplying both sides of eqn. $(*)$ by α_{t-j} for $j \geq 0$, we have

$$\tau_t \alpha_{t-j} = \nu(B)\alpha_t \alpha_{t-j} + n_t \alpha_{t-j}.$$

- Taking expectation, we have

$$cov(\tau_t, \alpha_{t-j}) = \nu_j \cdot var(\alpha_{t-j}).$$

- By definition

$$\nu_j = \frac{cov(\tau_t, \alpha_{t-j})}{var(\alpha_t)} = corr(\tau_t, \alpha_{t-j}) \cdot \frac{se(\tau_t)}{se(\alpha_t)}.$$

- Thus, we can test the statistical significance of ν_j by examining the statistical significance of $corr(\tau_t, \alpha_{t-j})$.

Intervention analysis

Dummy variables in dynamic regression

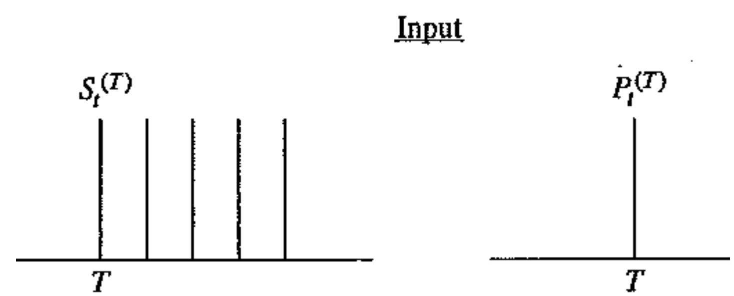
Reading: Chapter 10 of Wei (2005)

What do intervention analysis study

- Given that a known intervention occurs at time T
  1. Is there any evidence of a change in the time series (such as the increase of mean level)?
  2. If so, by how much?

Two common types of intervention variables

- Step function:
$$S_t^{(T)} = \begin{cases} 0, & t < T \\ 1, & t \geq T \end{cases}$$
- Pulse function:
$$P_t^{(T)} = \begin{cases} 0, & t \neq T \\ 1, & t = T \end{cases}$$
- Relation between step and pulse functions:
$$P_t^{(T)} = S_t^{(T)} - S_{t-1}^{(T)} = (1 - B)S_t^{(T)}$$



Possible responses to intervention

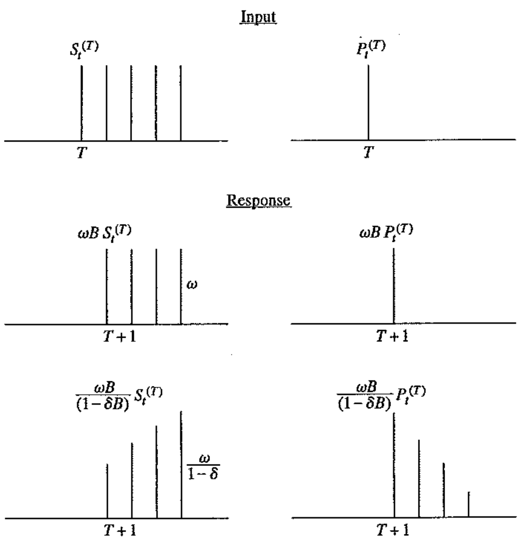


FIGURE 10.1 Responses to step and pulse inputs.

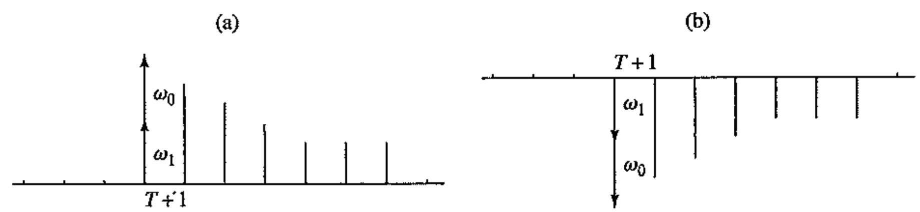


FIGURE 10.2 Response to combined inputs, $[\omega_0 B/(1 - \delta B)]P_t^{(T)} + \omega_1 B S_t^{(T)}$. (a) $\omega_0 > 0$ and $\omega_1 > 0$. (b) $\omega_0 < 0$ and $\omega_1 < 0$.

General model for intervention analysis

- For multiple intervention inputs, we have the following general class of models:

$$Z_t = \theta_0 + \sum_{j=1}^{k} \frac{w_j(B)B_j^b}{\delta_j(B)} I_{jt} + \frac{\theta(B)}{\psi(B)} a_t$$

where I_{jt} are intervention variables

$$\omega_j(B) = w_{j0} - w_{j1}B - \cdots - w_{s_j}B^{s_j}$$
$$\delta_j(B) = \delta_{j0} - \delta_{j1}B - \cdots - \delta_{r_j}B^{r_j}$$

Time series outliers (optional)

- Time series observations are sometimes influenced by interruptive events, such as strikes, outbreaks of war, or sudden political or economic crises.
- The consequence of these interruptive events create spurious observations that are inconsistent with the rest of the series. Such observations are usually referred to as outliers.

Additive and innovational outliers (optional)

- Z_t denotes the observed serie
- X_t is the outlier-free series satisfying a stationary and invertible ARMA model

$$\phi(B)X_t = \theta(B)a_t, \qquad a_t \sim NID(0, \sigma_a^2).$$

- Additive outlier (AO):

$$Z_t = \begin{cases} X_t & t \neq T \\ X_t + \omega & t = T \end{cases} = X_t + \omega P_t^{(T)} = \frac{\theta(B)}{\phi(B)} a_t + \omega P_t^{(T)}, \qquad (1)$$

- Innovational outlier (IO):

$$Z_t = X_t + \frac{\theta(B)}{\phi(B)} \omega P_t^{(T)} = \frac{\theta(B)}{\phi(B)} \left(a_t + \omega P_t^{(T)}\right), \qquad (2)$$

Hypothesis testing (optional)

H_0: Z_T is neither an AO nor an IO

H_1: Z_T is an AO

H_2: Z_T is an IO.

The likelihood ratio test statistics for AO and IO are

$$H_1 \text{ vs. } H_0: \qquad \lambda_{1,T} = \frac{\tau \hat{\omega}_{AT}}{\sigma_a}$$

and

$$H_2 \text{ vs. } H_0: \qquad \lambda_{2,T} = \frac{\hat{\omega}_{IT}}{\sigma_a}.$$

Under the null hypothesis H_0, both $\lambda_{1,T}$ and $\lambda_{2,T}$ are distributed as $N(0, 1)$.

Estimation when the timing is known (optional)

1. Define

$$e_t = \pi(B)Z_t,$$

where $\pi(B) = \phi(B)/\theta(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots$

2. From Equation (1) and (2), we have

$$\text{AO: } e_t = \omega\pi(B)P_t^{(T)} + a_t$$

$$\text{IO: } e_t = \omega P_t^{(T)} + a_t$$

AO model: $e_t = \omega\pi(B)P_t^{(T)} + a_t$ (optional)

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{T-1} \\ e_T \\ e_{T+1} \\ e_{T+2} \\ \vdots \\ e_n \end{bmatrix} = \omega \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\pi_1 \\ -\pi_2 \\ \vdots \\ -\pi_{n-T} \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{T-1} \\ a_T \\ a_{T+1} \\ a_{T+2} \\ \vdots \\ a_n \end{bmatrix}.$$

$$\hat{\omega}_{AT} = \frac{e_T - \sum_{j=1}^{n-T} \pi_j e_{T+j}}{\sum_{j=0}^{n-T} \pi_j^2}$$

$$\begin{aligned} \mathrm{Var}(\hat{\omega}_{AT}) &= \mathrm{Var}\left(\frac{\pi^*(F)e_T}{\tau^2}\right) \\ &= \frac{1}{\tau^4}\mathrm{Var}\left[\pi^*(F)a_T\right] \\ &= \frac{\sigma_a^2}{\tau^2}. \end{aligned}$$

$$\tau^2 = \sum_{j=0}^{n-T} \pi_j^2.$$

IO model: $e_t = \omega P_t^{(T)} + a_t$ (optional)

$$\text{IO:} \quad \hat{\omega}_{IT} = e_T$$

$$\begin{aligned}\operatorname{Var}(\hat{\omega}_{IT}) &= \operatorname{Var}(e_T) = \operatorname{Var}(\omega I_t^{(T)} + a_T) \\ &= \sigma_a^2.\end{aligned}$$