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Calculating Logs in Your Head

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1 Introduction

In Biosatistics, one often works with rates. Often, these models are fit on the log scale. Also, calculating logarithms are important in a wide range of problems from epidemic growth, long term changes in prices, and in other cases where one is compounding rates. The logarithm function basically turns multiplications into additions. In order to really get the handle of of this function it is very helpful to actually know how to calculate logarthms in one's head. That is, one needs to understand what values are large and what values are not large. If the log rate is 10 is that small, big, or really huge? Of course, one can simply raise e to the power 10 in a calculator, but it is better if one has an internal feel for what this means. To do this calculation, it turns out not to be that hard. One basically needs to remember the logarithm values for 2, 3, and 5. With these three numbers and remembering some basic rules of logarithms, one can get a fairly good approximation of the logarithm function.

The calculation scheme that I used below was somewhat inspired by a rule of thumb used by population scientist and financial investments. It is sometimes referred to as the "rule of 70"[1]. That basic question is, if a population grows by $x\%$ a year, how many years will it take to double? The answer is that it will take $70/x$ years. When thinking about this second question, I concocted the general rule which I present below.

I do find it useful to know how to do these calculations in my head because then I can think about the problem at hand without constantly accessing a black box to make the conversions. Furthermore, besides calculating these values "in your head", it is useful to know these techniques even if they requires writing some numbers on a piece of scrap paper. This allows one to get a feel for the calculations and to understand the order of magnitude of the calculations. (See the short story, "Feeling of Power" by Isaac Asimov[2], for another interesting example about forgetting how to do mental arithmetic.)

Note 1: Here I am talking about the natural logarithm function. That is, the logarithm to the base e. Most disciplines write this as ln. I am a statistician and our usually convention is to write this as log. However, for the general reader, I will use the usual convention and use the term $\ln(x)$ which means $\log_e(x)$. If one wants to get $\log_{10}$ then one can divide all the values here by $\ln(10)$ which is about 2.3. Similarly, if one wants $\log_2$, then one can divide all the numbers here by $\ln(2)$ which is about 0.69.

[1]See `https://en.wikipedia.org/wiki/Rule_of_72` downloaded on February 18, 2016. Note that they state that the earliest known reference is to 1494 in Venice and they suggest that back then it was a well known rule.

[2]Asimov, I, "Feeling of Power", first appeared in magazine **IF**, Quinn Publishing, February 1958. Also available on line at: http://downlode.org/Etext/power.html accessed on February 18, 2016.

70

110 1.

160 3

2 The algorithm

It is best to first start with calculating values between 10 and 100. Then, the algorithm can be expanded to the more general case.

2.1 For numbers between 10 and 100

Here is the basic algorithm for numbers which are about between 10 and 100. If the number of interest is around either boundary, then you could use this algorithm or the more general formula. This is an algorithm to use in your head, so consider these more as guidelines and not as hard rules.

So, suppose you want to know the log of a number A. That is for the equation $e^x = A$, you wish to solve for x for a know value of A.

Basic Algrorithm 20 years to double.

Memorize Memorize the approximate values for $\ln(2)$, $\ln(3)$, and $\ln(5)$. If you can perhaps remember them to two decimal places, which would be: $(0.69, 1.10, 1.61)$. That is, $\ln(2) \approx 0.69$, etc. If that is too much, then use the one decimal place approximations: $(0.7, 1.1, 1.6)$.

110 / r%×100

Pick a target number T: Find a number T which is close to A and is a factor of only the numbers 2, 3, and 5. That is,

$$T = 2^a 3^b 5^c.$$

Calculate the log: Let R equal $A - T$. Then the $\ln(A)$ can be calculated as:

$$log(A) \approx a * (0.69) + b * (1.10) + c * (1.61) + \frac{R}{T}.$$

Proof. First note the following:

$$\begin{aligned} A &= T\left(1 + \frac{A-T}{T}\right) \\ &= 2^a 3^b 5^c \left(1 + \frac{R}{T}\right). \end{aligned}$$

Now, remember the basic Taylor series approximation for natural logarithms. That is, for $\epsilon \in (-1, 1)$, $\ln(1+\epsilon) \approx \epsilon$. Taking logarithms, we then have:

$$\begin{aligned} \ln(A) &\approx a * \ln(2) + b * \ln(3) + c * \ln(5) + \frac{R}{T} \\ &\approx a * (0.69) + b * (1.10) + c * (1.61) + \frac{R}{T} \end{aligned}$$

□

2.2 General Formula

Now consider the case for a number For A^* outside of the range 10 and 100. For such a number, one can multiply A^* by a power to get a number between 10 and 100. Therefore, find A and n such that:

$$A^* = A \times 10^n.$$

Now, note that $\ln(10) = \ln(2) + \ln(5)$. So, one can remember that $\ln(10)$ is about 2.30. Therefore, using the same basic algorithm from above, fine a target T around the value A and then:

$$\ln(A^*) \approx a * (0.69) + b * (1.1) + c * (1.61) + \frac{R}{T} + n * (2.30).$$

3 Precision

When doing these calculations, there is a certain amount of round off in the calculations. Below, we look at the level of approximations that one might consider. For quick calculations on the fly, perhaps only one decimal is good enough. If one wants to use a little scrape paper, then perhaps two decimals in sufficient. In any case, the below provides various levels of approximations for $\ln(2)$, $\ln(3)$, $\ln(5)$, $\ln(10)$, and $\ln(1+\epsilon)$. Note that for the below approximations, the accuracy decreases as one goes down the list.

For $\ln(2)$:

$$\begin{aligned}
\ln(2) &\approx .693315\ldots \\
&\approx .69 + (.01)(.31) \\
&\approx .69 + (.01)\left(\frac{1}{3}\right) \\
&\approx .69 \\
&\approx .70
\end{aligned}$$

For $\ln(3)$:

$$\begin{aligned}
\ln(3) &\approx 1.098612\ldots \\
&\approx 1.10 - (.01)(.14) \\
&\approx 1.10 - (.01)\left(\frac{1}{7}\right) \\
&\approx 1.10 - (.01)\left(\frac{1}{6}\right) \\
&\approx 1.10
\end{aligned}$$

For $\ln(5)$:

$$\ln(5) \approx 1.609438\ldots$$

$$
\begin{aligned}
&\approx 1.61-(.01)(.06)\\
&\approx 1.61-(.01)\left(\frac{1}{16}\right)\\
&\approx 1.61-(.01)\left(\frac{1}{12}\right)\\
&\approx 1.61\\
&\approx 1.6
\end{aligned}
$$

For ln(10):

$$
\begin{aligned}
\ln(10) &\approx 2.302585\ldots\\
&\approx 2.30-(.01)(.26)\\
&\approx 2.30+(.01)\left(\frac{1}{4}\right)\\
&\approx 2.30
\end{aligned}
$$

Also, for the Taylor series approximation, for $\epsilon \in (-1,1)$:

$$
\begin{aligned}
\ln(1+\epsilon) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\epsilon^n}{n}\\
&= \epsilon-\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}-\ldots\\
&\approx \epsilon
\end{aligned}
$$

From the above, if one wants to have accuracy in the second decimal place (that is for the 0.01 term), then one would need to keep track of the additional error. So, since $\ln(2) \approx 1.69 + (0.01)\left(\frac{1}{3}\right)$ then when one uses 0.69 there is an error of about $\frac{1}{3}$ in the second decimal place. So, if one has a target, T, which uses 2^3, then one could add an addition 0.01 to correct the second digit. Similarly, one would therefore remember that for ln(2), ln(3), ln(5), and ln(10), the error corrections in the second decimal of $\frac{1}{3}$, $\frac{-1}{6}$, $\frac{1}{12}$ and $\frac{1}{4}$ respectively.

For the error associated with the approximation of $\ln(1+\epsilon)$, this would depend on the target term. When the target is large and the remainder small, then one might not need to make any adjustment. One would rarely need to consider the ϵ^2 term. Also, note that if one is often looking for a second decimal place term. So, it is not really necessary to make too detailed a calculation of the $\frac{R}{T}$ term.

4 Examples

Here are some simple examples to show how the procedure works.

Example. Finding the ln(80). Since 80 is a 8×10, then let target be 80 which is $2^4 * 5$, so

$$\ln(80) = 4 * 0.69 + 1.61 \approx 2.76 + 1.61 = 4.37.$$

Note, that the true value is about: $\ln(80) = 4.382026\ldots$ If one wanted more precision one could consider using $\ln(2) \approx .69 + (.01)\frac{1}{3}$. This would result in adding $.01\left(\frac{4}{3}\right)$. This would give an answer closer in the second decimal place.

Okay, now let us try one where the number is not equal to the target. How about $\ln(97)$?

Example. Note that $96 = 8 * 12$, so let the target be 96 where $96 = 2^5 * 3$. So,

$$\ln(97) \approx 5 * 0.69 + 1.10 + \frac{1}{96} \approx 3.45 + 1.10 + 0.01 = 4.56.$$

The actual value is $\ln(97) = 4.574711\ldots$ If 2 decimal place accuracy is diserable, then including the $.01 * \left(\frac{5}{3}\right)$ term could be included.

Example. An alternative way to get $\ln(97)$ is to use 100 as the target:

$$\ln(97) \approx 2 * 2.30 - 0.03 = 4.57.$$

This approximation is closer then the previous example. Note, in the above I used the approximation that $\ln(10) \approx 2.30$. Also note that $\frac{3}{97}$ is approximately $\frac{3}{100}$ which of course is 0.03. So, it might appear that calculating the term $\frac{R}{T}$ would take up a fair amount of head energy, in reality, one only needs to get one digit or so in the calculation.

As we can see in the last two examples, different approximations will get slightly different answers. However, the basic point is to get close enough when doing these head calculations.

Okay, now let us a try a big number, say $20,000$.

Example. The number $20,000$ can be represented by $2 * 10^4$. Therefore,

$$\ln(20000) \approx 0.69 + 4 * 2.30 = 9.89.$$

The actually answer is $\ln(20000) = 9.903488\ldots$ When calculating large numbers and approximating $\ln(10^n)$, one might want to track the extra $\frac{1}{4}$ to the second decimal. So, for every 10^4 one would add an extra value to the second decimal. That is, $\ln(10^n) \approx n * \left[2.30 + .01 * \left(\frac{1}{4}\right)\right]$. In the above calculation, that would account for most of the error in the last calculation. Adding the extra $\frac{1}{3}$ for the approximation of $\ln(2)$ would get the approximation to the third decimal place.

5 Time to Doubling

As mentioned in the introduction, the proposed methods of mentally calculating logarithms came to me while thinking about the time to doubling problem. That rule is sometimes called "the rule of 70". Basically, if one the annual growth is $x\%$ per year, then how many years does it take to double the original amount. The answer is that it takes approximately $70/x$ years.

Proof. Since the annual growth is $x\%$ per year, then the value at the end of the first year has increased by a factor of $\left(1+\frac{x}{100}\right)$ and after n years it has increased by a factor of $\left(1+\frac{x}{100}\right)^n$. So, the solution is to set this value to 2 and solve for n. To do this, first take (natural) logs of both sides. On one side use the Taylor approximation and on the other side note that $\ln(2)$ is approximately 0.70. Then, after some algebra, one gets the rule. In mathematical symbols:

$$\begin{aligned} 2 &= \left(1+\frac{x}{100}\right)^n \\ \ln(2) &= n * \ln\left(1+\frac{x}{100}\right) \\ 0.70 &\approx \frac{nx}{100} \\ \frac{70}{x} &\approx n \end{aligned}$$

□

In this proof of the rule of 70, the Taylor series approximation and the fact that $\ln(2)$ is about 0.70 play central roles. Also, from the above, it is now easy to develop rules for the time to tripling and other amounts. That is, if the annual growth rate is $x\%$, then in $110/x$ years the value will triple. Also, the increase will be 5 times in $160/x$ years and 10 times in $230/x$ years. Note that these additional "rules" are tightly connected to the approximations for $\ln(3)$, $\ln(5)$, and $\ln(10)$.

6 Discussion

First, a note on originality. I did develop this algorithm by myself, but the concepts are not that hard. So, it would not surprise me if this method is published somewhere else. The purpose of this note is to explain something which I believe is useful. When preparing this note, I did several searches on the internet to look at other methods. There are several comments on different websites which provide similar techniques. An interesting difference between this method and others that I have seen is that most work with logarithms to the base 10 and not with the natural logs. A curious benefit of working with the natural logs is that this method can take direct advantage of the Taylor series approximation. That is, this method can easily use the final correction based on $\ln(1+\epsilon) \approx \epsilon$. If one is calculating $\log_{10}$, then this final correction is $\log_{10}(1+\epsilon) \approx \epsilon/\ln(10)$. Therefore, these other methods that I have seen do not include this final correction term. By using the Taylor series approximation, one does not need to memorize too many value of the ln function since one can find some combination of 2, 3, and 5 which gets close enough when combined with the Taylor series approximation.

In practice, I can usually remember the one decimal point approximations. If there is a period of time where I'm working with logarithms a lot, then I start to remember the 2 decimal point approximations. I start to remember the rules with the additional fractions only when I find that I have a concentrated period of time working with logs. Also, as mentioned above, sometimes it is easier to move from doing all the calculations in my head to using a scratch pad for some of the calculations.

In any case, I do find it useful to have these rules available to me. True, now that scientific calculators are available as a free app for smart phones, it is easy to calculate a particular log value. Still, knowing the basic forms does help me think about the size of these numbers and does allow my mind to slowly think about some problems that I could not do when I needed to reference the "black box".

This was a fun little exercise and I hope some readers might find this useful.