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(i) We have $\text{logit}(\pi(x_i)) = \alpha + \beta x_i$: Find the log likelihood function.

Let $x_0 = 0 \to n_0$ trials, y_0 successes
$= 1 \to x_1 n_1$ trials, y_1 successes
$\pi_0 = \pi(x_0) = n_0 / y_0$ & $\pi(x_1) = y_1/n_1 = \pi_1$

These are 2 indep binomials w/
$y_0 \sim Bin(n_0, \pi_0)$
$y_1 \sim Bin(n_1, \pi_1)$

If we had N independant binomials their joint pdf would be equal to

$$(1)\ f(x_1, \dots, x_N) = \prod_{i=1}^{N} f(x_i) = \prod_{i=1}^{N} (\pi(x_i))^{y_i} (1-\pi(x_i))^{n_i - y_i}$$

where $N = 2$.

$$= \prod_{i=1}^{N} \frac{(\pi(x_i))^{y_i}}{(1-\pi(x_i))^{y_i}} (1-\pi(x_i))^{n_i} = \prod_{i=1}^{N} \left(\frac{\pi(x_i)}{1-\pi(x_i)}\right)^{y_i} (1-\pi(x_i))^{n_i}$$

$$= \left[\prod_{i=1}^{N} \left(\frac{\pi(x_i)}{1-\pi(x_i)}\right)^{y_i}\right] \underbrace{\left[\prod_{i=1}^{N} (1-\pi(x_i))^{n_i}\right]}_{A} = \prod_{i=1}^{N} \exp \log \left\{\left(\frac{\pi(x_i)}{1-\pi(x_i)}\right)^{y_i}\right\} A$$

$$= \left[\prod_{i=1}^{N} \exp\left(y_i \log\left(\frac{\pi(x_i)}{1-\pi(x_i)}\right)\right)\right] \left[\prod_{i=1}^{N} (1-\pi(x_i))^{n_i}\right]$$

Recall $\log\left(\frac{\pi(x_i)}{1-\pi(x_i)}\right) = \alpha + \beta x_i$

$$= \left\{\prod_{i=1}^{N} \exp(y_i(\alpha + \beta x_i))\right\} \left[\prod_{i=1}^{N} (1-\pi(x_i))^{n_i}\right]$$ Taking the log gives

$$= \left[\sum_{i=1}^{N} y_i(\alpha + \beta x_i)\right] + \left[\sum_{i=1}^{N} n_i \log(1-\pi(x_i))\right], \quad 1-\pi(x_i) = 1 - \frac{e^{\alpha + x_i\beta}}{1+e^{\alpha+\beta x_i}} = \frac{1}{1+e^{\alpha+\beta x_i}}$$

$$\therefore\ \boxed{l = \log(L(\beta,\alpha)) = \left[\sum_{i=1}^{N=2} y_i(\alpha+\beta x_i)\right] + \left[\sum_{i=1}^{N=2} -n_i \log(1+e^{\alpha+\beta x_i})\right]}$$

This is the log likelihood f^n of $\text{logit}(\pi(x_i)) = \alpha+\beta x_i$ with 2 independant binomial outcomes: y_0 & y_1.

(ii) Find likelihood equations:

$$\frac{\partial l}{\partial \beta} = \left[\left(\sum_{i=1}^{N=2} y_i x_i\right)\right] - \left[\sum_{i=1}^{N=2} n_i \frac{\beta e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}\right] = 0$$

$$\Rightarrow \boxed{\sum_{i=1}^{2} y_i x_i = \beta \sum_{i=1}^{2} n_i \pi(x_i)} \quad (*) \text{ as } \pi(x_i) = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$$

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{N=2} y_i - \left[\sum_{i=1}^{2} n_i \frac{\alpha e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}\right] = 0$$

$$\Rightarrow \sum_{i=1}^{2} y_i = \alpha \sum_{i=1}^{2} \frac{n_i e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$$

$$\Rightarrow \boxed{\sum_{i=1}^{2} y_i = \alpha \sum_{i=1}^{2} n_i \pi(x_i)} \quad (**) \text{ as } \pi(x_i) = \frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$$

(iii) Want to show that $\hat{\beta}$ is sample log odds.

$$\therefore\ \hat{\beta} = \log\left[\frac{\hat{\pi}(x_1)/(1-\hat{\pi}(x_1))}{\hat{\pi}(x_0)/(1-\hat{\pi}(x_0))}\right] \text{ where } \hat{\pi}(x_i) = y_i/n_i$$

$$\therefore\ y_i = \hat{\pi}(x_i)\, n_i$$

Note that $\text{logit}(\pi(x_i)) = \alpha + \beta x_i \Rightarrow \text{logit}(\hat{\pi}(x_i)) = \hat{\alpha} + \hat{\beta} x_i$.

$$\Rightarrow \text{logit}(\hat{\pi}(x_1)) = \hat{\alpha} + \hat{\beta} x_1 \quad \& \quad \text{logit}(\hat{\pi}(x_0)) = \hat{\alpha} + \hat{\beta} x_0.$$

as $x_0 = 0$ & $x_1 = 1$ we get

$$\text{logit}(\hat{\pi}(x_1)) = \hat{\alpha} + \hat{\beta} \quad \& \quad \text{logit}(\hat{\pi}(x_0)) = \hat{\alpha}$$

$$\Rightarrow \hat{\beta} = \text{logit}(\hat{\pi}(x_1)) - \hat{\alpha} = \text{logit}(\hat{\pi}(x_1)) - \text{logit}(\hat{\pi}(x_0))$$

$$\Rightarrow \hat{\beta} = \log\left(\frac{\hat{\pi}(x_1)}{1-\hat{\pi}(x_1)}\right) - \log\left(\frac{\hat{\pi}(x_0)}{1-\hat{\pi}(x_0)}\right)$$

$$\boxed{\hat{\beta} = \log\left[\frac{\hat{\pi}(x_1) / 1-\hat{\pi}(x_1)}{\hat{\pi}(x_0) / 1-\hat{\pi}(x_0)}\right]}$$

Hence $\hat{\beta}$ is the sample log odds