6 Appendix A: Quaternion Algebra Reference

This appendix is a summary of quaternion definitions and algebra useful for orientation tracking. A quaternion $q = q_w + iq_x + jq_y + kq_z$ is defined by 4 coefficients: a scalar q_w and a vector part q_x, q_y, q_z. The fundamental quaternion units i, j, k are comparable to the imaginary part of complex numbers, but there are three of them for each quaternion. Each of the fundamental quaternion units is different, but the following relationships hold

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i \tag{25}$$

In general, there are two types of quaternions important for our application: quaternions representing a rotation and vector quaternions.

A valid rotation quaternion has unit length, i.e.

$$\|q\| = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} = 1 \tag{26}$$

and the quaterions q and $-q$ represent the same rotation.

A vector quaternion representing a 3D point or vector $\mathbf{u} = (u_x, u_y, u_z)$ can have an arbitrary length but its scalar part is always zero

$$q_{\mathbf{u}} = 0 + iu_x + ju_y + ku_z \tag{27}$$

The conjugate of a quaternion is

$$q^* = q_w - iq_x - jq_y - kq_z \tag{28}$$

and its inverse

$$q^{-1} = \frac{q^*}{\|q\|^2} \tag{29}$$

Similar to polynomials, two quaternion q and p are added as

$$q + p = (q_w + p_w) + i\,(q_x + p_x) + j\,(q_y + p_y) + k\,(q_z + p_z) \tag{30}$$

and multiplied as

$$\begin{aligned} q\,p &= (q_w + iq_x + jq_y + kq_z)\,(p_w + ip_x + jp_y + kp_z) && (31) \\ &= (q_w p_w - q_x p_x - q_y p_y - q_z p_z) \\ &\quad + i\,(q_w p_x + q_x p_w + q_y p_z - q_z p_y) \\ &\quad + j\,(q_w p_y - q_x p_z + q_y p_w + q_z p_x) \\ &\quad + k\,(q_w p_z + q_x p_y - q_y p_x + q_z p_w) \\ & && (32) \end{aligned}$$

Rotations with Quaternions A quaternion q can be converted to a 3×3 rotation matrix as

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2q_xq_y - 2q_wq_z & 2q_xq_z + 2q_wq_y \\ 2q_xq_y + 2q_wq_z & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2q_yq_z - 2q_wq_x \\ 2q_xq_z - 2q_wq_y & 2q_yq_z + 2q_wq_x & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{pmatrix} \tag{33}$$

Similarly, a 3×3 rotation matrix is converted to a quaterion as

$$q_w = \frac{\sqrt{1 + r_{11} + r_{22} + r_{33}}}{2}, \qquad q_x = \frac{r_{32} - r_{23}}{4q_w}, \qquad q_y = \frac{r_{13} - r_{31}}{4q_w}, \qquad q_z = \frac{r_{21} - r_{12}}{4q_w} \tag{34}$$