Discrete Structures

(CKC111)

Week 11 & Week 12

Induction and Recursion

USM We lead

Recursive Definitions and Structural Induction

Section Summary

✓ Recursively Defined Functions

✓ Recursively Defined Sets and Structures

Introduction

- The object is defined in terms of itself. This process is called recursion.

- We can use recursion to define sequences, functions, and sets.

Example:

The sequence of powers of 2 is given by $a_n = 2^n$ for $n = 0, 1, 2, \ldots$.

<u>Recursively Defined Set</u>

BASIS STEP:

$a_0 = 1$

RECURSIVE STEP:

$a_{n+1} = 2a_n$ for $n = 0, 1, 2, \ldots$.

The terms of the sequence are found from previous terms.

- To prove results about recursively defined sets we use a method called *structural induction*.

- Structural induction: a technique for proving results about recursively defined sets

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Recursively Defined Functions

Recursively Defined Functions

Definition: A *recursive* or *inductive definition* of a function consists of two steps.

Basis Step: Specify the value of the function at **zero**.

Recursive Step: Give a rule for finding its value at an integer from its values at smaller integers.

A function $f(n)$ is the same as a sequence $a_0, a_1, \ldots$, where a_i is a real number for every nonnegative integer i , where $f(i) = a_i$. *This was done using recurrence relations in Section 2.4 (main reference book).*

Recursive definition of a function: a definition of a function that specifies an initial set of values and a rule for obtaining values of this function at integers from its values at smaller integers.

Recursively Defined Functions

Student Work

Example 1:

Suppose that f is defined recursively by

BASIS STEP: $f(0)=3$

RECURSIVE STEP: $f(n+1)=2f(n)+3$

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Solution: From the recursive definition it follows that

$$f(1)=2f(0)+3=2\cdot 3+3=9,$$

$$f(2)=2f(1)+3=2\cdot 9+3=21,$$

$$f(3)=2f(2)+3=2\cdot 21+3=45,$$

$$f(4)=2f(3)+3=2\cdot 45+3=93.$$

Recursively Defined Functions

Student Work

Example 2:

Suppose that f is defined recursively by

BASIS STEP: $f_0 = 0, f_1 = 1$

RECURSIVE STEP: $f_n = f_{n-1} + f_{n-2}$

Find the Fibonacci numbers, $f_{2,}\ f_3, f_4$ and f_5

Solution: From the recursive definition it follows that

$$f_2 = f_1 + f_0 = 0 + 1 = 1$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

Recursively Defined Functions

Example 3: Give a recursive definition of a^n, where a is a nonzero real number and n is a nonnegative integer

Solution:

$f(n) = a^n$

BASIS STEP: $f(0) = a^0 = 1$

RECURSIVE STEP:

Find the rule for $f(n+1)$ from $f(n)$,

$$\begin{aligned}
f(1) &= a^1 = a \\
f(2) &= a^2 = a \,.\, a & &= a \,.\, a^1 \\
f(3) &= a^3 = a \,.\, a \,.\, a & &= a \,.\, a^2 \\
f(4) &= a^4 = a \,.\, a \,.\, a \,.\, a & &= a \,.\, a^3 \\
&\vdots & &\vdots \\
f(n) &= a^n = a \,.\, a \,.\, \ldots\ldots \, a & &= a \,.\, a^{n-1}
\end{aligned}$$

$$f(n+1) = a^{n+1} = a \cdot a^n$$

$$f(n+1) = a \,.\, f(n) \quad \text{for } n = 0, 1, 2, 3, \ldots$$

Recursively Defined Functions

Student Work

Example 4:

Give a recursive definition of the factorial function $n!$

Solution:

$f(n)= n!$

BASIS STEP: $f(0)= 0! = 1$

RECURSIVE STEP:

Find the rule for $f(n+1)$ from $f(n)$,

$f(1)=1,$

$f(2)=(2)\cdot 1=2,$

$f(3)=(3)\cdot 2=6,$

$f(4)=(4)\cdot 6=24.$

.

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$f(n+1) = (n+1) \ . \ f(n),$

for $n = 0, 1, 2, 3,\ldots$

$f(0)= 0! = 1$

$f(1)=1! = 1$

$f(2)=2! = 2 \text{ x } 1 \quad = 2 \text{ x } 1! \quad = 2 \, . \, f(1)$

$f(3)=3! = 3 \text{ x } 2 \text{ x } 1 \quad = 3 \text{ x } 2! \quad = 3 \, . \, f(2)$

$f(4)=4! = 4 \text{ x } 3 \text{ x } 2 \text{ x } 1 \quad = 4 \text{ x } 3! \quad = 4 \, . \, f(3)$

.

.

$f(n)=n! = n \text{ x } \ldots.. 4 \text{ x } 3 \text{ x } 2 \text{ x } 1 \quad = n \text{ x } (n-1)!$

$f(n+1)=(n+1)! = (n+1) \text{ x } n\ldots\ldots\ldots 4 \text{ x } 3 \text{ x } 2 \text{ x } 1 = (n+1) \text{ x } n! \quad = (n+1) \, . \, f(n)$

Recursively Defined Functions

Student Work

Example 3:

Give a recursive definition of: $\sum_{k=0}^{n} a_k .$

Solution:

The first part of the definition is $\sum_{k=0}^{0} a_k = a_0 .$

The second part is $\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k \right) + a_{n+1}$

Recursively Defined Sets and Structures

Recursively Defined Sets and Structures

Recursive definitions of sets have two parts:

- The ***basis step*** specifies an initial collection of elements.
- The ***recursive step*** gives the rules for forming new elements in the set from those already known to be in the set.

Sometimes the recursive definition has an ***exclusion*** *rule*, which specifies that the set contains nothing other than those elements specified in the basis step and generated by applications of the rules in the recursive step.

We will always assume that the exclusion rule holds (is true), even if it is not explicitly/clearly mentioned.

Recursive definition of a set: a definition of a set that specifies an initial set of elements in the set and a rule for obtaining other elements from those in the set.

Recursively Defined Sets and Structures

https://www.youtube.com/watch?v=WstKQxUYgnY

Example 1:

Recursive definition of a set S containing the positive multiples of 3.

- Basis Step: $3 \in S$
- Recursive Step: If $n \in S$ then $n + 3 \in S$

Compute the elements in the set S.

$S = \{ 3, 6, 9, 12, \ldots\ldots\ldots\}$

Example 2:

What elements are in the set T defined by:

- Basis Step: $1 \in T$
- Recursive Step: If $n \in T$ then $n+1 \in T$ and $n-1 \in T$

Student Work

$T = \{ 1, 2, 0, 3, -1, 4, -2, \ldots\ldots\ldots\}$

$T = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

Recursively Defined Sets and Structures

Student Work

Example 1: Subset of Integers S:

Basis step: $3 \in S$.

Recursive step: If $x \in S$ and $y \in S$, then $x + y$ is in S.

Initially 3 is in S, then $3 + 3 = 6$, then $3 + 6 = 9$, etc.

Example 2: The natural numbers $\boldsymbol{N}$.

Basis step: $0 \in \boldsymbol{N}$.

Recursive step: If n is in $\boldsymbol{N}$, then $n + 1$ is in $\boldsymbol{N}$.

Initially 0 is in S, then $0 + 1 = 1$, then $1 + 1 = 2$, etc.

Recursively Defined Sets and Structures

https://www.youtube.com/watch?v=WstKQxUYgnY

Write out a recursive definition of a set containing the powers of 3 (starting at 1)
{1,3,9,27, 81....}

Student Work

Solution:

BASIS STEP: $1 \in S$

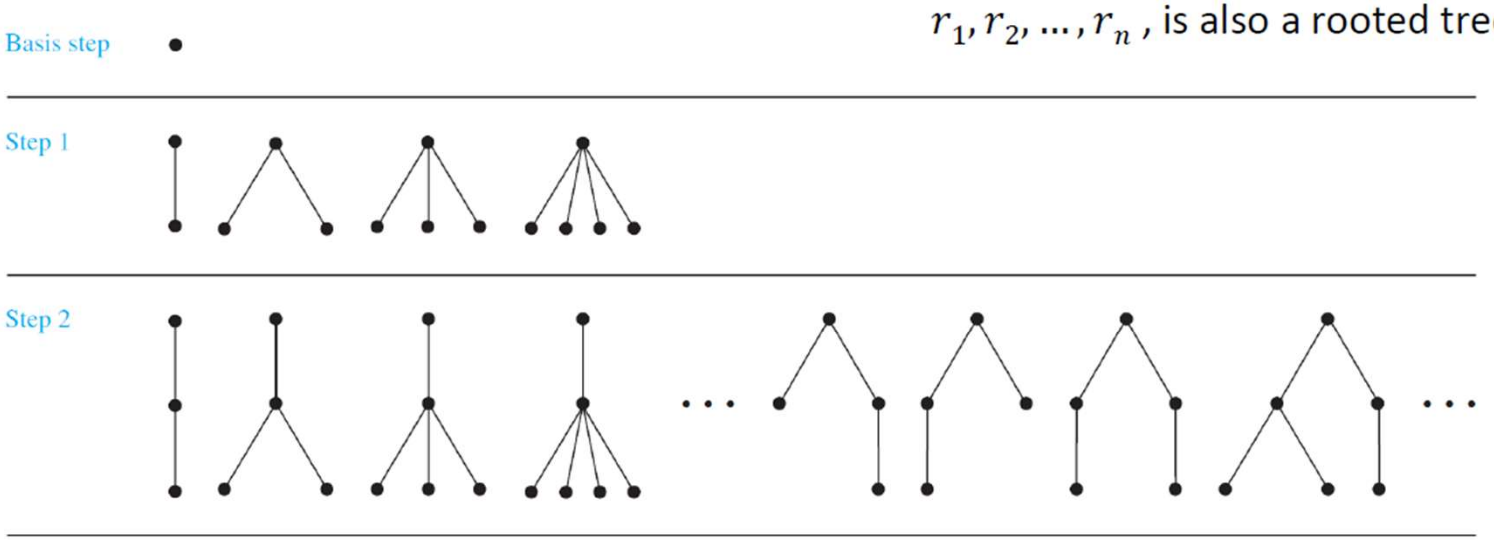
RECURSIVE STEP: If $n \in S$, then $3n \in S$

Rooted Trees

Definition: The set of *rooted trees,* where a rooted tree consists of a set of vertices containing a distinguished vertex called the ***root***, and **edges** connecting these vertices, can be defined recursively by these steps:

Basis step: A single vertex r is a rooted tree.

Recursive step: Suppose that $T_1, T_2, \dots, Tn$ are disjoint rooted trees with roots $r_1, r_2, \dots, r_n$, respectively. Then the graph formed by starting with a root r, which is not in any of the rooted trees $T_1, T_2, \dots, Tn$, and adding an edge from r to each of the vertices $r_1, r_2, \dots, r_n$, is also a rooted tree.



Some of rooted trees formed starting with the basis step and applying the recursive step one time and two times. Note that infinitely many rooted trees are formed at each application of the recursive definition.

Building Up Extended Binary Trees

Binary trees

- special type of rooted trees
- two binary trees are combined to form a new tree with one of these trees designated the left subtree and the other the right subtree.

In extended binary trees, the left subtree or the right subtree can be empty, but in full binary trees this is not possible. Binary trees are one of the most important types of structures in computer science.

Basis step: The empty set is an extended binary tree.

Recursive step: If T_1 and T_2 are disjoint extended binary trees, there is an extended binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2 when these trees are nonempty.

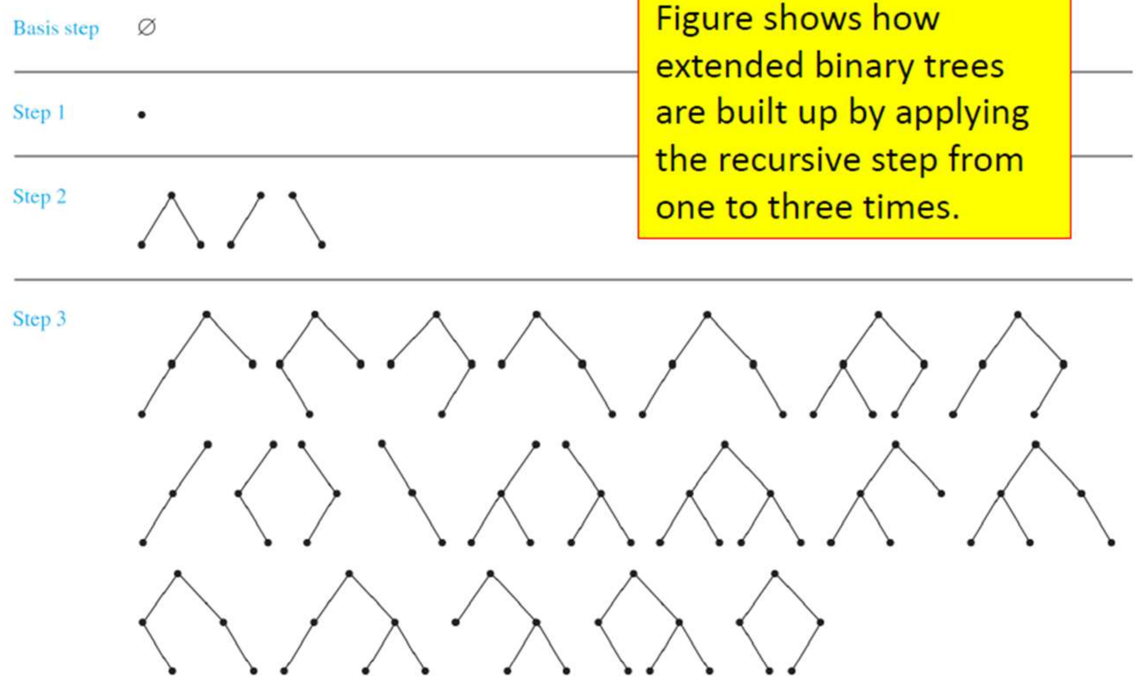


Figure shows how extended binary trees are built up by applying the recursive step from one to three times.

Building Up Full Binary Rooted Trees

Definition: Note that the difference between this recursive definition and that of extended binary trees lies entirely in the basis step.

Basis step: There is a full binary tree consisting only of a single vertex r.

Recursive step : If T_1 and T_2 are disjoint full binary trees, there is full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.

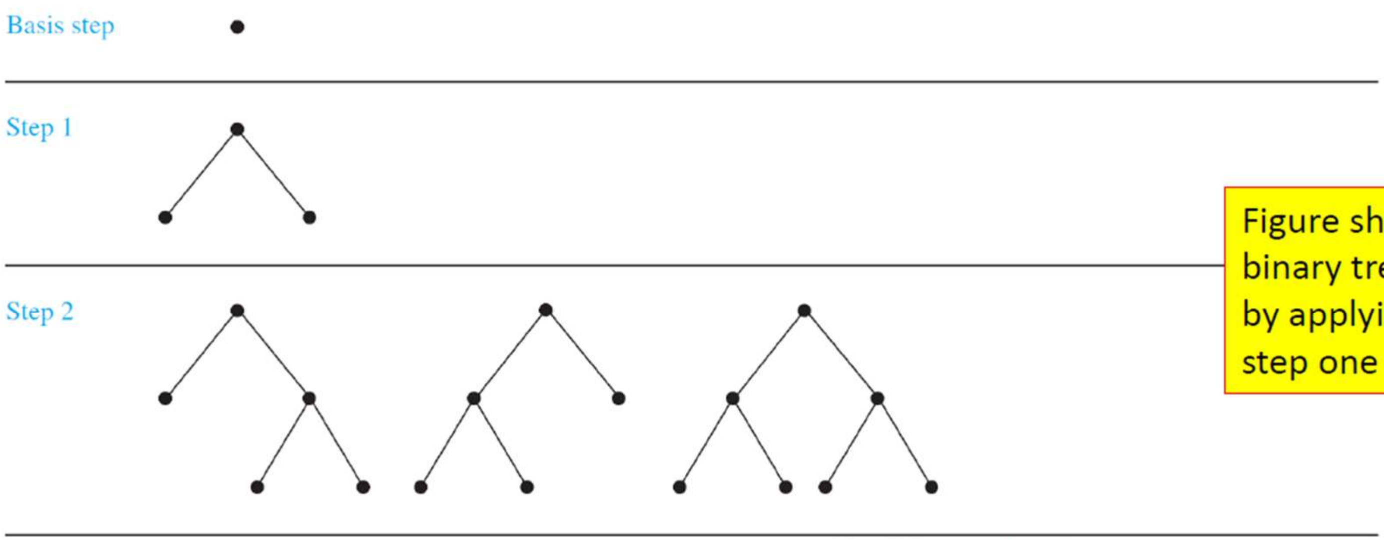


Figure shows how full binary trees are built up by applying the recursive step one to two times.

Structural Induction

Structural Induction

To prove results about recursively defined sets, we generally use some form of mathematical induction. This example illustrates the connection between recursively defined sets and mathematical induction.

Example 1: Show that the set S by specifying that $3 \in S$ and that if $x \in S$ and $y \in S$, then $x + y \in S$, is the set of all **positive integers** that are **multiples of 3**.

Solution: Let A be the set of all positive integers divisible by 3. To prove that $A = S$, show that A is a subset of S and S is a subset of A. To prove that A is a subset of S, we must show that every positive integer divisible by 3 in S.

Mathematical induction

$\boldsymbol{A \subset S}$**:**
Let $P(n)$ be the statement that $3n$ belongs to S.
Basis step: $3 \cdot 1 = 3 \in S$, by the first part of recursive definition.
Inductive step: Assume $P(k)$ is true. By the second part of the recursive definition, if $3k \in S$, then since $3 \in S$, $3k + 3 = 3(k + 1) \in S$. Hence, $P(k + 1)$ is true.

Recursively defined sets

$\boldsymbol{S \subset A}$**:**
Basis step: $3 \in S$ by the first part of recursive definition, and $3 = 3 \cdot 1$.
Recursive step: The second part of the recursive definition adds $x + y$ to S, if both x and y are in S. If x and y are both in A, then both x and y are divisible by 3.

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Structural Induction

We used mathematical induction over the set of positive integers and a recursive definition to prove a result about a recursively defined set. However, instead of using mathematical induction directly to prove results about recursively defined sets, we can use a more convenient form of induction known as **structural induction**. A proof by structural induction consists of two parts. These parts are

BASIS STEP: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.
RECURSIVE STEP: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

let $P(n)$ state that the claim is true for all elements of the set that are generated by n or fewer applications of the rules in the recursive step of a recursive definition.

Structural Induction

Let S be the subset of the set of integers defined recursively by:

Basis Step: $3 \in S$

Recursive Step: If $a \in S$ and $b \in S$, then $a + b \in S$

Use structural induction to show that $3|x$ for all $x \in S$.

$3|x$

$x = 3\ (m)$

Let $P(n)$ be that $3|x$ for all $x \in S$ after n applications of the recursive definition

Basis Step: $3 \in S,$

$3 \mid 3$

$3 = 3\ (1)$

Recursive Step:

$a \in S$, So $a = 3m$ $\qquad$ $b \in S$, So $b = 3n$

$a + b = 3m + 3n$

$a + b = 3(m + n)$

Therefore, $3|(a + b)$

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Structural Induction

Student Work

Let S be the subset of the set of ordered pairs of integers defines by:

Basis Step: $(0,0) \in S$

Recursive Step: If $(a, b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$, $(a+2, b+1) \in S$

Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$

Let $P(n)$ be that $a \leq 2b$ whenever $(a, b) \in S$ after n applications of the recursive definition

Basis Step: $(0,0) \in S$,

$$0 \leq 2(0)$$
$$0 \leq 0$$

Recursive Step:

	$(a, b+1) \in S$	$(a+1, b+1) \in S$	$(a+2, b+1) \in S$
$a \leq 2b$	$a \leq 2(b+1)$	$a+1 \leq 2(b+1)$	$a+2 \leq 2(b+1)$
	$0 \leq 2(0+1)$	$0+1 \leq 2(0+1)$	$0+2 \leq 2(0+1)$
	$0 \leq 2$	$1 \leq 2$	$2 \leq 2$

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Thank you