Discrete Structures

(CKC111)

Discrete Probability

1 Basic Probability

2 Probability Theory

Basic Probability

Section Summary

✓ Finite Probability

✓ Probabilities of Complements and Unions of Events

Finite Probability

Probability of an Event

• Pierre-Simon Laplace (1749-1827)

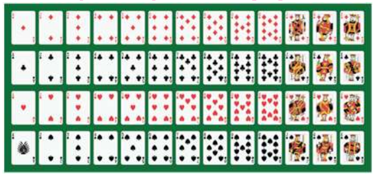
- We first study Pierre-Simon Laplace's classical theory of probability, which he introduced in the 18th century, when he analyzed games of chance.
- We first define these key terms:
  - An ***experiment*** is a procedure that yields one of a given set of possible outcomes.
  - The ***sample space*** of the experiment is the set of possible outcomes.
  - An ***event*** is a subset of the sample space.

Example:

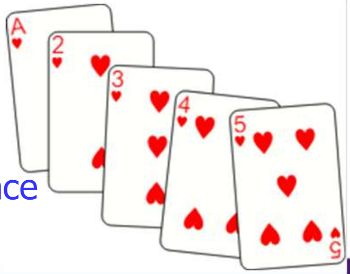
- Experiment
  A single die is rolled.
- Sample space, (S)
  { 1, 2, 3, 4, 5, 6 }
- Event (E) – occurring 2
  { 2 }

- Experiment
  Tossing a Coin
- Sample space, (S)
  {H, T}
- Event (E) – occurring Head
  { H }

- Experiment
  Drawing cards
- Sample space, (S)



- Event (E) – selecting an ace



Probability of an Event

- Pierre-Simon Laplace (1749-1827)

- Here is how Laplace defined the probability of an event:
- **Definition**: If S is a finite sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is

$$p(E) = \frac{|E|}{|S|}$$

Example:

- Experiment
  A single die is rolled.
- Sample space, (S)
  { 1, 2, 3, 4, 5, 6 }
- Event (E) – occurring 2
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  Tossing a Coin
- Sample space, (S)
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- Event (E) – occurring Head
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  Drawing cards
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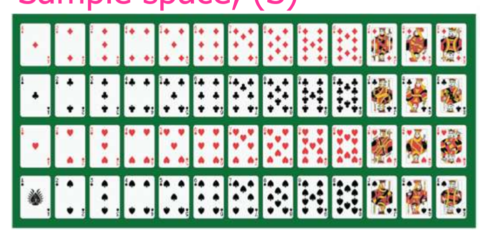

- Event (E) – selecting an ace

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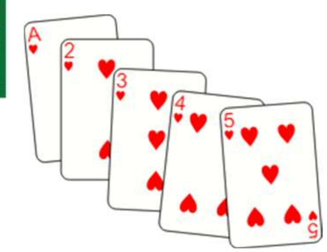

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$$p(E) = \frac{|E|}{|S|}$$

- Pierre-Simon Laplace (1749-1827)

Example:

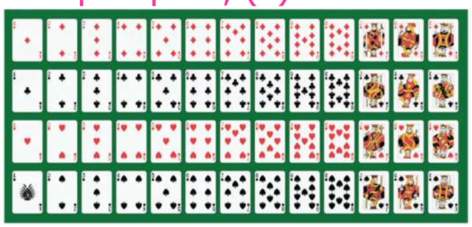
- Experiment
  A single die is rolled.
- Sample space, (S)
  { 1, 2, 3, 4, 5, 6 }
- Event (E) – occurring 2
  { 2 }

$$p(E) = \frac{|E|}{|S|} = \frac{1}{6}$$

- Experiment
  Tossing a Coin
- Sample space, (S)
  {H, T}
- Event (E) – occurring Head
  { H }

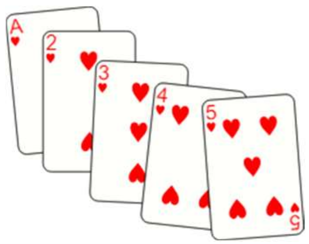
$$p(E) = \frac{|E|}{|S|} = \frac{1}{2}$$

- Experiment
  Drawing cards
- Sample space, (S)



- Event (E) – selecting an ace

$$p(E) = \frac{|E|}{|S|} = \frac{4}{52} = \frac{1}{13}$$



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What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

- Experiment
- Sample space, (S)

|S| =

- Event (E) – the sum of the numbers on the two dice is 7

|E| =

What is the probability?

The *probability* of *E* is $p(E) = \frac{|E|}{|S|} =$

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What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

- Experiment

  Rolling two dice

- Sample space, (S)

{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), $\boxed{(1, 6)}$

(2, 1), (2, 2), (2, 3), (2, 4), $\boxed{(2, 5)}$, (2, 6)

(3, 1), (3, 2), (3, 3), $\boxed{(3, 4)}$, (3, 5), (3, 6)

(4, 1), (4, 2), $\boxed{(4, 3)}$, (4, 4), (4, 5), (4, 6)

(5, 1), $\boxed{(5, 2)}$, (5, 3), (5, 4), (5, 5), (5, 6)

$\boxed{(6, 1)}$, (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

}

$|S| = 36$

- Event (E) – the sum of the numbers on the two dice is 7

{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$|E| = 6$

<u>What is the probability?</u>

The *probability* of *E* is $p(E) = \dfrac{|E|}{|S|} = \dfrac{6}{36} = \dfrac{1}{6}$

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Probability of an Event

- According to Laplace's definition, the probability of an event is between 0 and 1.

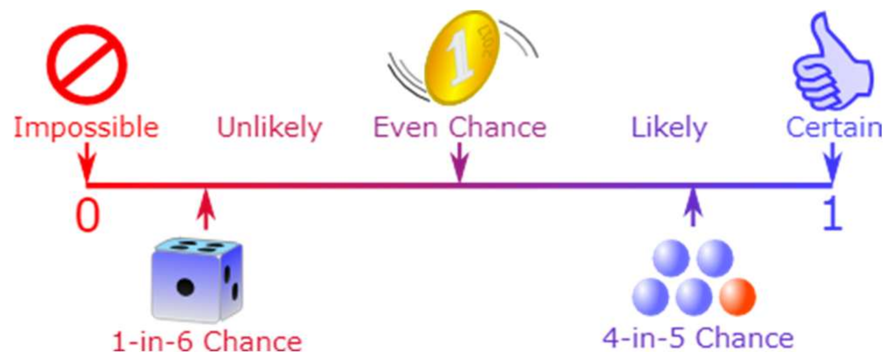
  To see this, note that if E is an event from a finite sample space S, then

  $0 \le |E| \le |S|$ (because $E \subseteq S$)

  $\frac{0}{|S|} \le \frac{|E|}{|S|} \le \frac{|S|}{|S|}$ ($\div$ $|S|$)

  $0 \le p(E) = |E|/|S| \le 1$.

- Pierre-Simon Laplace (1749-1827)



Applying Laplace's Definition

- **Example**: An urn contains **four** blue balls and **five** red balls. What is the probability that a ball chosen from the urn is blue?

- **Solution**: The probability that the ball is chosen is **4/9** since there are nine possible outcomes, and four of these produce a blue ball.

Applying Laplace’s Definition

- **Example**: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

- **Solution**: By the **product rule** there are $6^2 = 36$ possible outcomes. Six of these sum to 7. Hence, the probability of obtaining a 7 is 6/36 = **1/6**.

Applying Laplace's Definition

- **Example**: What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin with 50 balls labeled with the numbers 1,2, …, 50 if
  a) The ball selected is <u>not returned</u> to the bin.
  b) The ball selected is <u>returned</u> to the bin before the next ball is selected.

- **Solution**: Use the product rule in each case.
  a) *<u>Sampling without replacement</u>*: The probability is 1/254,251,200 since there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254{,}251{,}200$ ways to choose the five balls.
  b) *<u>Sampling with replacement</u>*: The probability is $1/50^5 = 1/312{,}500{,}000$ since $50^5 = 312{,}500{,}000$.

The Probability of Complements and Unions of Events

The Probability of Complements and Unions of Events

- **Theorem 1**: Let *E* be an event in sample space *S*.
  The probability of the event $\overline{E}$ = *S* − *E*, the complementary event of *E*, is given by;

$$p\left(\overline{E}\right) = 1 - p(E).$$

- **Proof**: Using the fact that $|\overline{E}| = |S| - |E|$,

$$p\left(\overline{E}\right) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

The Probability of Complements and Unions of Events

- **Example**: A sequence of **10 bits** is chosen randomly. What is the probability that **at least one** of these bits is **0**?

- **Solution**: Let E be the event that at least one of the 10 bits is 0. Then is the event that all of the bits are 1s. The size of the sample space S is $\mathbf{2^{10}}$. Hence,

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

The Probability of Complements and Unions of Events

- **Theorem 2**: Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

- **Proof**: Given the inclusion-exclusion formula from Section 2.2, $|A \cup B| = |A| + |B| - |A \cap B|$, it follows that:

$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2). \end{aligned}$$

The Probability of Complements and Unions of Events

- **Example**: What is the probability that a **positive integer** selected at random from the set of positive integers **not exceeding 100** is **divisible by either 2 or 5**?

- **Solution**: Let E_1 be the event that the integer selected at random is divisible by 2.
  Let E_2 be the event that the integer selected at random is divisible by 5.
  $E_1 \cup E_2$ is the event that it is divisible by either 2 or 5.
  $E_1 \cap E_2$ is the event that it is divisible by both 2 and 5.

$$\boxed{|E_1| = \left\lfloor \frac{100}{2} \right\rfloor = 50} \quad \boxed{|E_2| = \left\lfloor \frac{100}{5} \right\rfloor = 20} \quad \boxed{|E_1 \cap E_2| = \left\lfloor \frac{100}{2.5} \right\rfloor = 10}$$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}$$

Exercises

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Exercise - 1

What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?

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Exercise - 1

What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?

$\boxed{|S| = 100}$ $\boxed{|E| = 50}$

$$p(E) = \frac{|E|}{|S|}$$

$$= \frac{50}{100}$$

$$= \frac{1}{2}$$

Exercise - 2

What is the probability that the sum of the numbers on two dice is even when they are rolled?

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Exercise - 2

What is the probability that the sum of the numbers on two dice is even when they are rolled?

$$|S| = 36$$

$$|E| = 18$$

$$p(E) = \frac{|E|}{|S|}$$

$$= \frac{18}{36}$$

$$= \frac{1}{2}$$

- Sample space, (S)

$\{\boxed{(1, 1)}, (1, 2), \boxed{(1, 3)}, (1, 4), \boxed{(1, 5)}, (1, 6)$
$(2, 1), \boxed{(2, 2)}, (2, 3), \boxed{(2, 4)}, (2, 5), \boxed{(2, 6)}$
$\boxed{(3, 1)}, (3, 2), \boxed{(3, 3)}, (3, 4), \boxed{(3, 5)}, (3, 6)$
$(4, 1), \boxed{(4, 2)}, (4, 3), \boxed{(4, 4)}, (4, 5), \boxed{(4, 6)}$
$\boxed{(5, 1)}, (5, 2), \boxed{(5, 3)}, (5, 4), \boxed{(5, 5)}, (5, 6)$
$(6, 1), \boxed{(6, 2)}, (6, 3), \boxed{(6, 4)}, (6, 5), \boxed{(6, 6)}$
$\}$

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Exercise - 3

What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

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Exercise - 3

What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

Let E be the event that the integer selected at random is divisible by 3.

$$|E| = \left\lfloor \frac{100}{3} \right\rfloor = 33 \qquad |S| = 100$$

$$p(E) = \frac{|E|}{|S|}$$

$$= \frac{33}{100}$$

Probability Theory

Section Summary

✓ Assigning Probabilities

✓ Probabilities of Complements and Unions of Events

✓ Conditional Probability

✓ Independence

Assigning Probabilities

Recap…

- Here is how **Laplace** defined the probability of an event:
- **Definition**: If *S* is a finite sample space of equally likely outcomes, and *E* is an event, that is, a subset of *S*, then the *probability* of *E* is;

$$p(E) = \frac{|E|}{|S|}$$

The *probability* of *E* is $p(E) = \dfrac{\text{number of outcomes in E}}{\text{total number of outcomes in S}}$

Assigning Probabilities

- Laplace's definition from the previous section, assumes that **all outcomes are equally likely**. Now we introduce a more general definition of probabilities that avoids this restriction.
- Let S be a sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome s, so that:

$$i. \quad 0 \le p(s) \le 1 \text{ for each } s \in S$$

$$ii. \quad \sum_{s \in S} p(s) = 1$$

Condition (*i*) states that the probability of each outcome is a nonnegative real number no greater than 1.

Condition (*ii*) states that the sum of the probabilities of all possible outcomes should be 1

- The function p from the set of all outcomes of the sample space S is called a ***probability distribution***.

Assigning Probabilities

Example:

What probabilities should we assign to the outcomes H(heads) and T (tails) when a **fair** coin is flipped?

What probabilities should be assigned to these outcomes when the coin is biased so that **heads comes up twice** as often as tails?

Solution:

For a **fair** coin, we have $p(H) = p(T) = ½$.

For a **biased** coin, we have $p(H) = 2p(T)$.

Because $p(H) + p(T) = 1$, it follows that

$$2p(T) + p(T) = 3p(T) = 1.$$

Hence, $p(T) = 1/3$ and $p(H) = 2/3$.

Uniform Distribution

- **Definition**: Suppose that S is a set with n elements. The ***uniform distribution*** assigns the probability **$1/n$** to each element of S. (Note that we could have used Laplace's definition here.)

Example:

- Experiment
  A single die is rolled.
- Sample space, (S)
  { 1, 2, 3, 4, 5, 6 } $n = 6$
- Event (E) – n equally likely outcomes
  { 1 } { 2 } { 3 } { 4 } { 5 } { 6 }

$p(1) = \frac{1}{6}$ $p(3) = \frac{1}{6}$ $p(5) = \frac{1}{6}$

$p(2) = \frac{1}{6}$ $p(4) = \frac{1}{6}$ $p(6) = \frac{1}{6}$

The sum of probabilities = $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

- Experiment
  Tossing a Coin
- Sample space, (S)
  {H, T} $n = 2$
- Event (E) - n equally likely outcomes
  { H } { T }

$p(H) = \frac{1}{2}$ $p(T) = \frac{1}{2}$

The sum of probabilities = $\frac{1}{6} + \frac{1}{6} = 1$

Probability of an Event

- **Definition**: The probability of the event *E* is the sum of the probabilities of the outcomes in *E*.

$$p(E) = \sum_{s \in S} p(s)$$

- Note that now **no assumption is being made** about the distribution.

Example

Example: Suppose that a die is biased so that **3 appears twice** as often as each other number, but that the other five outcomes are equally likely. What is the probability that an **odd number** appears when we roll this die?

Solution: We want the probability of the event $E = \{1,3,5\}$.

We have $p(3) = 2/7$ and

$p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$.

Hence, $p(E) = p(1) + p(3) + p(5) = 1/7 + 2/7 + 1/7 = \mathbf{4/7}$.

Probabilities of Complements and Unions of Events

Probabilities of Complements and Unions of Events

- Complements: $p(\overline{E}) = 1 - p(E)$ still holds. Since each outcome is in either E or $\overline{E}$ but not both,

$$\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E}).$$

- Unions: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$ also, still holds under the new definition.

Exercises

1. A coin is tossed. Verify that the sum of probabilities of all outcomes is 1.

Exercises

1. **A coin is tossed. Verify that the sum of probabilities of all outcomes is 1.**

- Sample space, (S)
  {H, T} $|S| = 2$
- Event (E)
  { H } { T }

$p(H) = \frac{1}{2}$ $p(T) = \frac{1}{2}$

The sum of probabilities of all outcomes

$= p(H) + p(T)$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

Hence proved

Condition (*i*) states that the probability of each outcome is a nonnegative real number no greater than 1.

Exercises

2. The probability of rain tomorrow is 0.3. What is the probability it won't rain?

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Exercises

2. **The probability of rain tomorrow is 0.3. What is the probability it won't rain?**

p(rain) = 0.3

p(no rain) =

$$p(\overline{E}) = 1 - p(E)$$

= 1 - 0.3

= 0.7

Condition (*ii*) states that the sum of the probabilities of all possible outcomes should be 1

Conditional Probability

Conditional Probability

- **Definition**: Let *E* and *F* be events with *p(F)* > 0. The conditional probability of *E* given *F*, denoted by *P*(*E*|*F*), is defined as:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

"the probability of E given F."

- **Example**: A bit string of length four is generated at random so that each of the 16-bit strings of length 4 is equally likely. What is the probability that it contains **at least two consecutive 0s**, given that its **first bit is a 0**?
- **Solution**: Let *E* be the event that the bit string contains at least two consecutive 0s, and *F* be the event that the first bit is a 0.
  - Since $E \cap F$ = {0000, 0001, 0010, 0011, 0100}, $p(E \cap F)$=5/16.
  - Because 8-bit strings of length 4 start with a 0, p(F) = 8/16= ½.
- Hence,

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{5/16}{1/2} = \frac{5}{8}.$$

Conditional Probability

- **Example:** What is the conditional probability that a family with **two children** has **two boys**, given that they have **at least one boy**. Assume that each of the possibilities *BB*, *BG*, *GB*, and *GG* is equally likely where *B* represents a boy and *G* represents a girl.
- **Solution**: Let *E* be the event that the family has two boys and let *F* be the event that the family has at least one boy. Then $E = \{BB\}$, $F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$.
  - It follows that p(F) = 3/4 and $p(E \cap F)=1/4$.
- Hence,

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Independence

Independence

- **Definition**: **The events *E* and *F* are <u>independent</u> if and only if**

$$p(E \cap F) = p(E)\, p(F).$$

- **Example 1**: Suppose *E* is the event that a randomly generated bit string of length four begins with a 1 and *F* is the event that this bit string contains an even number of 1s. Are *E* and *F* independent if the 16-bit strings of length four are equally likely?
- **Solution**: There are 8-bit strings of length four that begin with a 1-bit, and 8-bit strings of length four that contain an even number of 1s.
  - Since the number of bit strings of length 4 is 16,

$$p(E) = p(F) = 8/16 = 1/2.$$

  - Since $E \cap F$ = {1111, 1100, 1010, 1001}, $p(E \cap F)$ = 4/16=1/4.
    We conclude that E and F are **independent**, because

$$p(E \cap F) = 1/4 = (1/2)(1/2) = p(E)\, p(F)$$

Independence

- **Example 2**: Assume (as in the previous example) that each of the four ways a family can have two children (*BB*, *GG*, *BG*,*GB*) is **equally likely**. Are the events *E*, that a family with two children has two boys, and *F*, that a family with two children has at least one boy, **independent**?

- **Solution**: Because $E = \{BB\}$, $p(E) = 1/4$. We saw previously that that $p(F) = 3/4$ and $p(E \cap F) = 1/4$. The events *E* and *F* are **not independent** since;

$$p(E)\,p(F) = 3/16 \neq 1/4 = p(E \cap F).$$

Thank you