UNIVERSITI SAINS MALAYSIA

Chapter 4

Graphs

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Chapter Summary

Section 4.1: Graphs and Graph Terminology

Section 4.2: Representing Graphs

Section 4.3: Connectivity

Section 4.4: Shortest-Path Problems

Graphs and Graph Terminology

Section 4.1

Section Summary

Graph

Graph Terminology

Some Special Types of Graphs

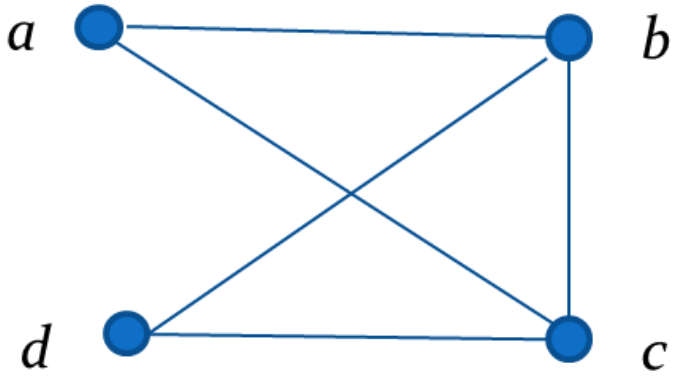
Bipartite Graphs

New Graphs from Old

Graphs

Definition: A *graph G =* (*V, E*) consists of a nonempty set *V* of *vertices* (or *nodes*) and a set *E* of *edges.* Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

Example:
This is a graph with four vertices and five edges.



Directed Graphs 1

Definition: A *directed graph* (or *digraph*) $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *directed edges* (or *arcs*). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u,v) is said to *start at* u and *end at* v.

Remark:

- Graphs where the end points of an edge are not ordered are said to be *undirected graphs*.

Some Terminology 1

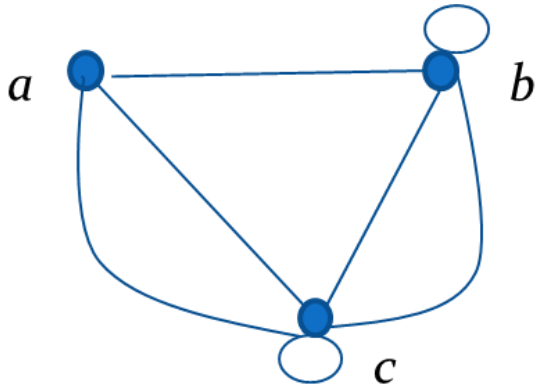
In a *simple graph* each edge connects two different vertices and no two edges connect the same pair of vertices.

Multigraphs may have multiple edges connecting the same two vertices. When *m* different edges connect the vertices *u* and *v*, we say that {*u*,*v*} is an edge of *multiplicity m*.

An edge that connects a vertex to itself is called a *loop*.

A *pseudograph* may include loops, as well as multiple edges connecting the same pair of vertices.

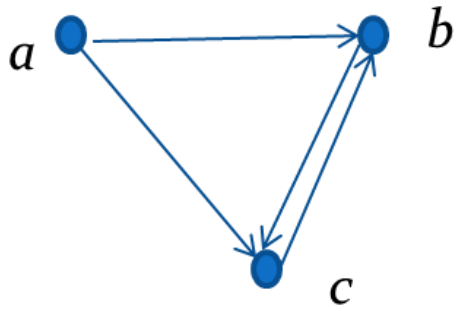
Example:
This pseudograph has both multiple edges and a loop.



Some Terminology 2

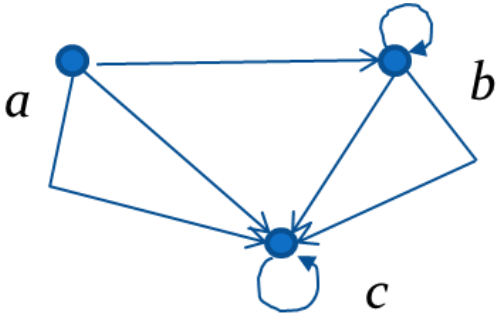
A *simple directed graph* has no loops and no multiple edges.

Example:
This is a directed graph with three vertices and four edges.



A *directed multigraph* may have multiple directed edges. When there are *m* directed edges from the vertex *u* to the vertex *v*, we say that (*u*,*v*) is an edge of *multiplicity m*.

Example:
In this directed multigraph the multiplicity of (*a*,*b*) is 1 and the multiplicity of (*b*,*c*) is 2.



Graph Terminology: Summary

To understand the structure of a graph and to build a graph model, we ask these questions:

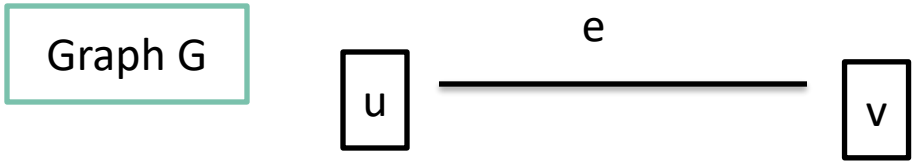
- Are the edges of the graph undirected or directed (or both)?
- If the edges are undirected, are multiple edges present that connect the same pair of vertices? If the edges are directed, are multiple directed edges present?
- Are loops present?

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Basic Terminology

Definition 1. Two vertices *u*, *v* in an undirected graph *G* are called *adjacent* (or *neighbors*) in *G* if there is an edge *e* between *u* and *v*. Such an edge *e* is called *incident with* the vertices *u* and *v* and *e* is said to *connect u* and *v*.



e is called *incident with* the vertices *u* and *v*

vertices *u*, *v* called *adjacent* (or *neighbors*) in *G*

N (u) = {v}

N (v) = {u}

Definition 2. The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

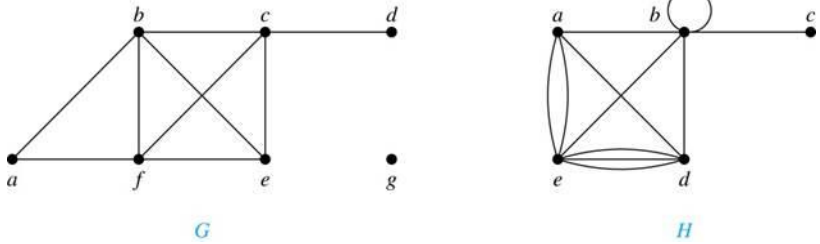
$$N(A) = \bigcup_{v \in A} N(v).$$

Definition 3. The *degree of a vertex in a undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex *v* is denoted by deg(*v*).

deg(v) = 3

Degrees and Neighborhoods of Vertices

Example: What are the degrees and neighborhoods of the vertices in the graphs G and H?



Solution:

G: $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$,

$\deg(e) = 3$, $\deg(g) = 0$.

$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$,

$N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, $N(g) = \varnothing$.

H: $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, $\deg(d) = 5$.

$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$,

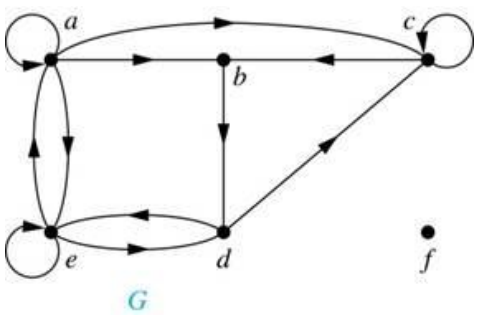
$N(d) = \{a, b, e\}$, $N(e) = \{a, b, d\}$.

Jump to long description

Directed Graphs

Definition: The *in-degree of a vertex v*, denoted $deg^-(v)$, is the number of edges which terminate at *v*. The *out-degree of v*, denoted $deg^+(v)$, is the number of edges with *v* as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

Example: In the graph *G* we have



$$\deg^-(a)=2, \deg^-(b)=2, \deg^-(c)=3, \deg^-(d)=2,$$
$$\deg^-(e)=3, \deg^-(f)=0.$$

$$\deg^+(a)=4, \deg^+(b)=1, \deg^+(c)=2, \deg^+(d)=2,$$
$$\deg^+(e)=3, \deg^+(f)=0.$$

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Directed Graphs

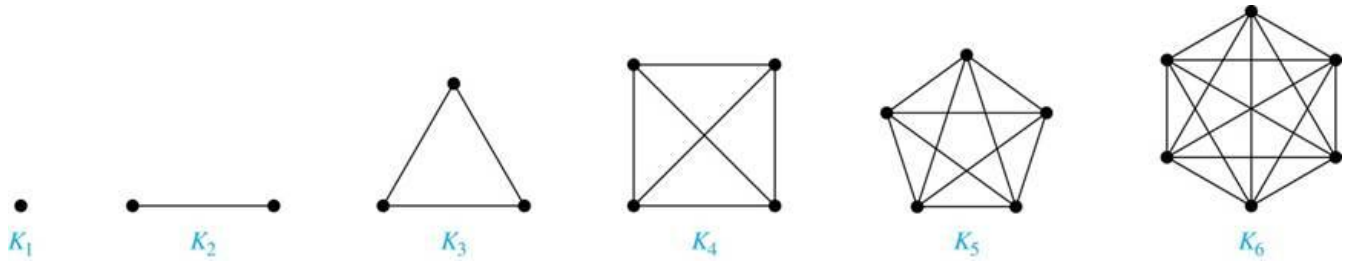
Theorem 3: Let $G = (V, E)$ be a graph with directed edges. Then:

$$|E| = \sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v).$$

Proof: The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal the number of edges in the graph.

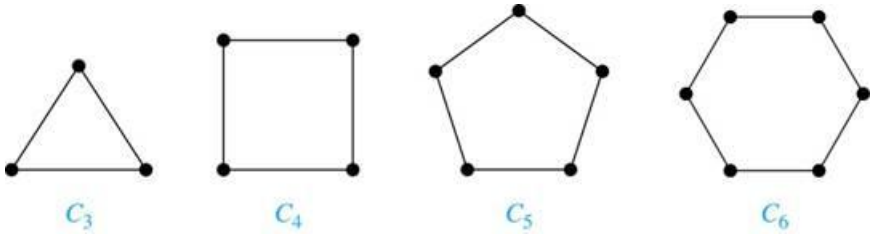
Special Types of Simple Graphs: Complete Graphs

A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

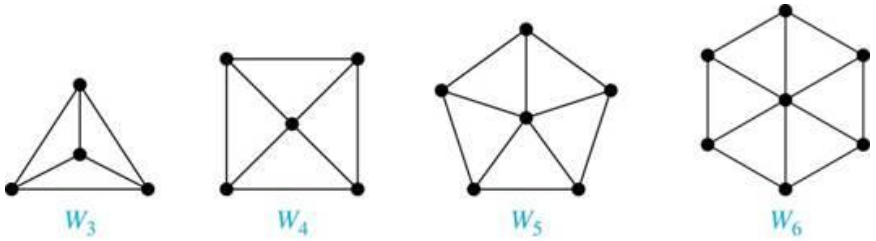


Special Types of Simple Graphs: Cycles and Wheels

A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



A *wheel* W_n is obtained by adding an additional vertex to a cycle C_n for $n \geq 3$ and connecting this new vertex to each of the n vertices in C_n by new edges.



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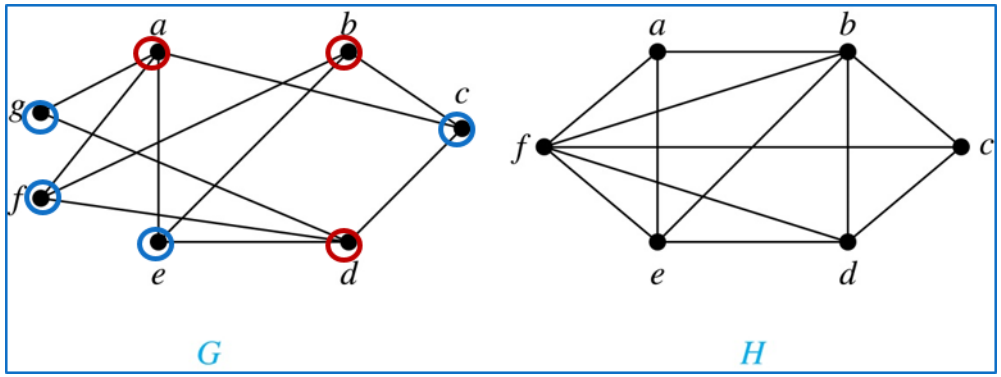
Bipartite Graphs[1]

Definition: A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

It is not hard to show that an equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

G is bipartite

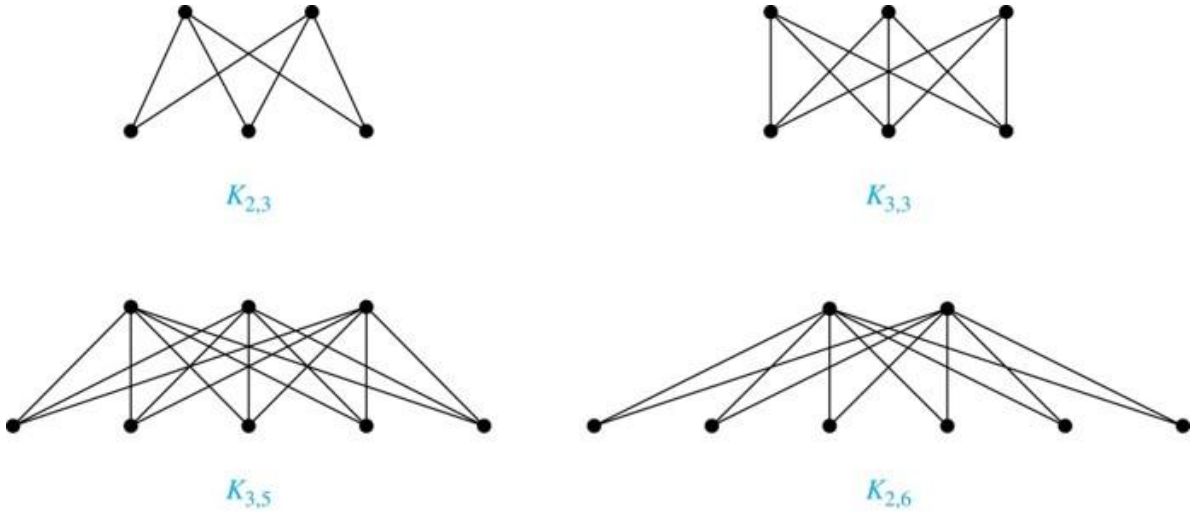


H is not bipartite since if we color a red, then the adjacent vertices f and b must both be blue.

Complete Bipartite Graphs

Definition: A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Example: We display four complete bipartite graphs here.

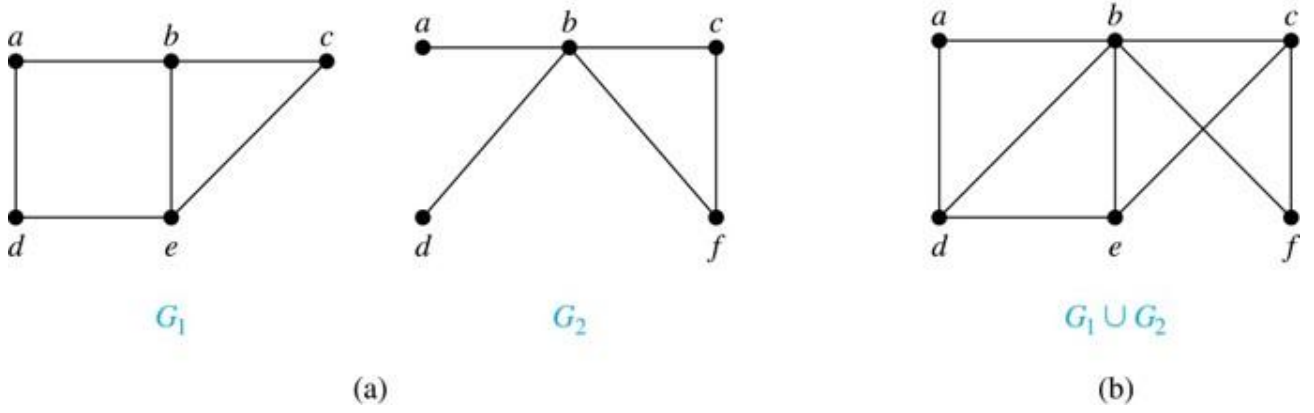


Jump to long description

New Graphs from Old

Definition: The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example:



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Representing Graphs

Section 4.2

Section Summary 3

Adjacency Lists

Adjacency Matrices

Incidence Matrices

Representing Graphs: Adjacency Lists

Definition: An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

Example:

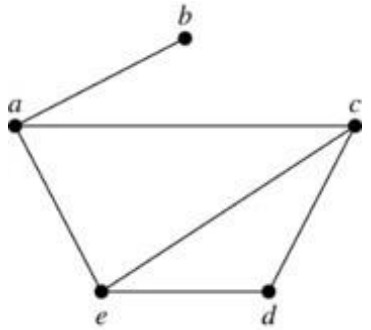


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	*b, c, e*
b	*a*
c	*a, d, e*
d	*c, e*
e	*a, c, d*

Example:

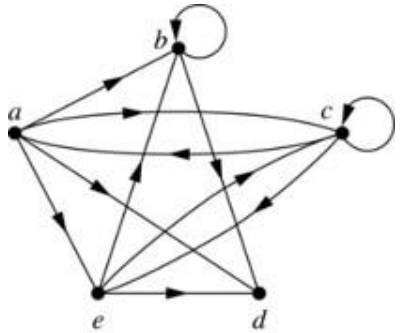


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	*b, c, d, e*
b	*b, d*
c	*a, c, e*
d	
e	*b, c, d*

Jump to long description

Representation of Graphs: Adjacency Matrices 1

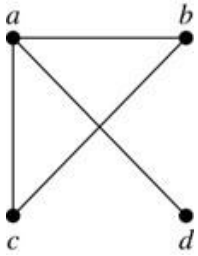
Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.

- In other words, if the graphs adjacency matrix is $\mathbf{A}_G = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

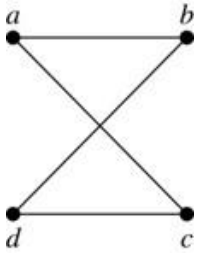
Representation of Graphs: Adjacency Matrices 2

Example:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.

When a graph is sparse, that is, it has few edges relatively to the total number of possible edges, it is much more efficient to represent the graph using an adjacency list than an adjacency matrix. But for a dense graph, which includes a high percentage of possible edges, an adjacency matrix is preferable.

Note: The adjacency matrix of a simple graph is symmetric, i.e., $a_{ij} = a_{ji}$

Also, since there are no loops, each diagonal entry a_{ii} for $i = 1, 2, 3, \ldots, n$, is 0.

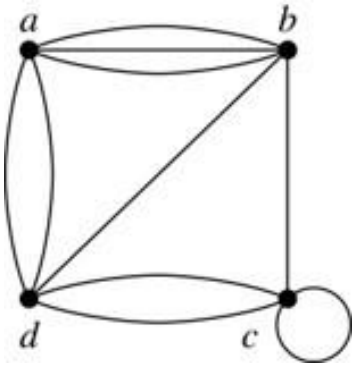
Representation of Graphs: Adjacency Matrices 3

Adjacency matrices can also be used to represent graphs with loops and multiple edges.

A loop at the vertex v_i is represented by a 1 at the (i, j)th position of the matrix.

When multiple edges connect the same pair of vertices v_i and v_j, (or if multiple loops are present at the same vertex), the (i, j)th entry equals the number of edges connecting the pair of vertices.

Example: We give the adjacency matrix of the pseudograph shown here using the ordering of vertices a, b, c, d.



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Jump to long description

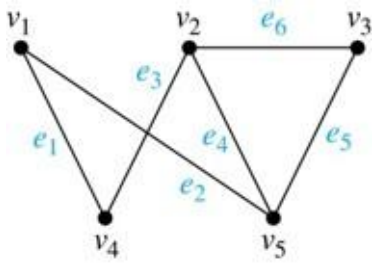
Representation of Graphs: Incidence Matrices 1

Definition: Let $G = (V, E)$ be an undirected graph with vertices where $v_1, v_2, \ldots v_n$ and edges $e_1, e_2, \ldots e_m$. The incidence matrix with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Representation of Graphs: Incidence Matrices 2

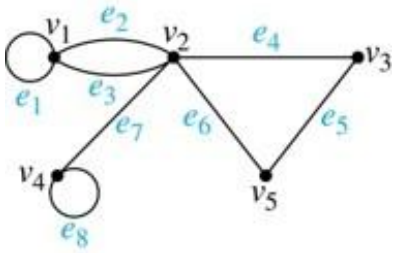
Example: Simple Graph and Incidence Matrix



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The rows going from top to bottom represent v_1 through v_5 and the columns going from left to right represent e_1 through e_6.

Example: Pseudograph and Incidence Matrix



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The rows going from top to bottom represent v_1 through v_5 and the columns going from left to right represent e_1 through e_8.

Jump to long description

Connectivity

Section 4.3

Section Summary 4

Paths

Connectedness in Undirected Graphs

Connectedness in Directed Graphs

Paths 1

Informal Definition: A *path* is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

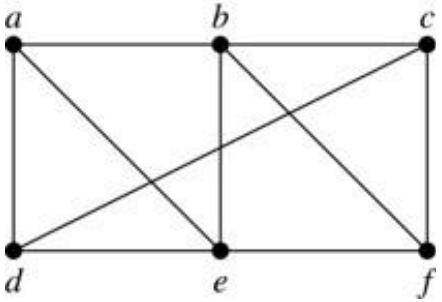
As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these.

Paths 2

Definition: Let n be a nonnegative integer and G an undirected graph. A *path* of *length n* from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i.

- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$(since listing the vertices uniquely determines the path).
- The path is a *circuit* if it begins and ends at the same vertex ($u = v$) and has length greater than zero.
- The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ and *traverse* the edges $e_1, \ldots, e_n$.
- A path or circuit is *simple* if it does not contain the same edge more than once.

Paths 3



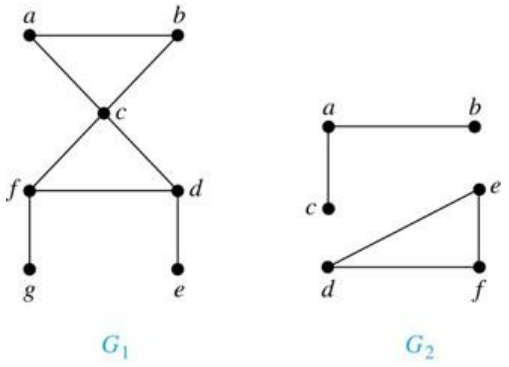
Example: In the simple graph here:

- *a*, *d*, *c*, *f*, *e* is a simple path of length 4.
- *d*, *e*, *c*, *a* is not a path because *e* is not connected to *c*.
- *b*, *c*, *f*, *e*, *b* is a circuit of length 4.
- *a*, *b*, *e*, *d*, *a*, *b* is a path of length 5, but it is not a simple path.

Connectedness in Undirected Graphs

Definition: An undirected graph is called *connected* if there is a path between every pair of vertices. An undirected graph that is not *connected* is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example: G_1 is connected because there is a path between any pair of its vertices, as can be easily seen. However G_2 is not connected because there is no path between vertices *a* and *f*, for example.

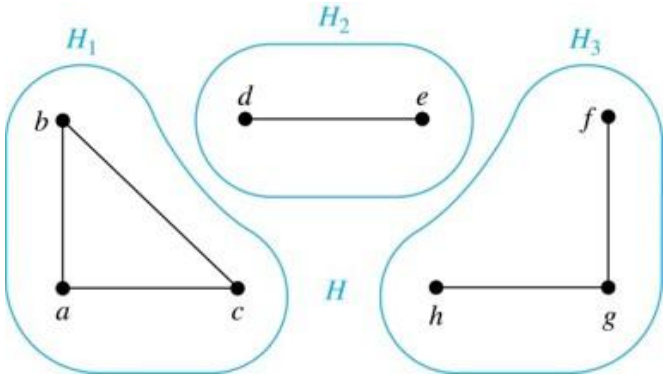


Jump to long description

Connected Components

Definition: A *connected component* of a graph *G* is a connected subgraph of *G* that is not a proper subgraph of another connected subgraph of *G*. A graph *G* that is not connected has two or more connected components that are disjoint and have *G* as their union.

Example: The graph *H* is the union of three disjoint subgraphs H_1, H_2, and H_3, none of which are proper subgraphs of a larger connected subgraph of *G*. These three subgraphs are the connected components of *H*.



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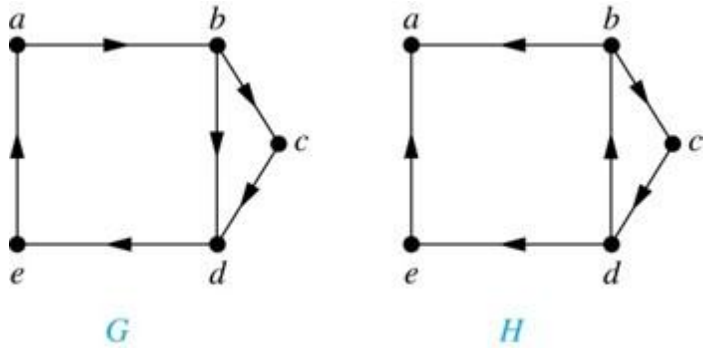
Connectedness in Directed Graphs 1

Definition: A directed graph is *strongly connected* if there is a path from *a* to *b* and a path from *b* to *a* whenever *a* and *b* are vertices in the graph.

Definition: A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph, which is the undirected graph obtained by ignoring the directions of the edges of the directed graph.

Connectedness in Directed Graphs 2

Example: *G* is strongly connected because there is a path between any two vertices in the directed graph. Hence, *G* is also weakly connected. The graph *H* is not strongly connected, since there is no directed path from *a* to *b*, but it is weakly connected.



Definition: The subgraphs of a directed graph *G* that are strongly connected but not contained in larger strongly connected subgraphs, that is, the maximal strongly connected subgraphs, are called the *strongly connected components* or *strong components* of *G*.

Example (*continued***)**: The graph *H* has three strongly connected components, consisting of the vertex *a*; the vertex *e;* and the subgraph consisting of the vertices *b*, *c*, *d* and edges (*b*,*c*), (*c*,*d*), and (*d*,*b*).

Jump to long description

Shortest-Path Problems

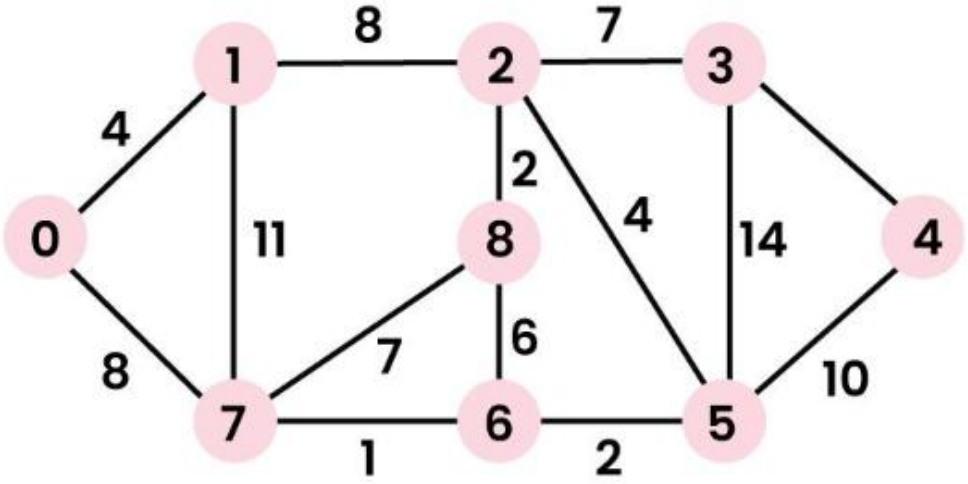
Section 4.4

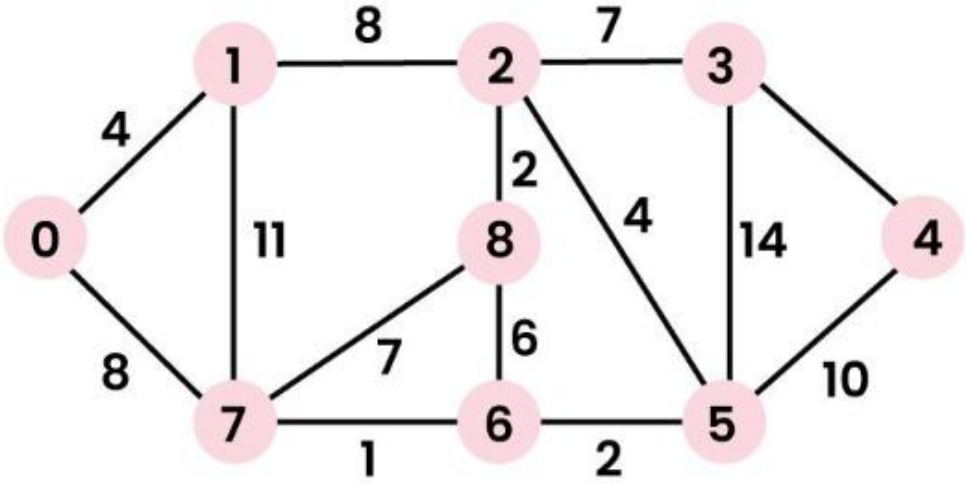
Shortest-Path Problems

Definition: The shortest path algorithms are the ones that focuses on calculating the minimum travelling cost from **source node** to **destination node** of a graph in optimal time and space complexities.

Dijkstra's Algorithm

Given a weighted graph and a source vertex in the graph, find the **shortest paths** from the source to all the other vertices in the given graph.





Explanation: *The distance from 0 to 1 = 4.*
The minimum distance from 0 to 2 = 12. 0->1->2
The minimum distance from 0 to 3 = 19. 0->1->2->3
The minimum distance from 0 to 4 = 21. 0->7->6->5->4
The minimum distance from 0 to 5 = 11. 0->7->6->5
The minimum distance from 0 to 6 = 9. 0->7->6
The minimum distance from 0 to 7 = 8. 0->7
The minimum distance from 0 to 8 = 14. 0->1->2->8

Output: *0 4 12 19 21 11 9 8 14*