Discrete Structures

(CKC111)

Week 11 & Week 12

Induction and Recursion

Section Topics

Induction and Recursion

.1 Mathematical & Strong Induction

.2 Recursive Definitions and Structural Induction

.3 Recursive Algorithms

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Mathematical & Strong Induction

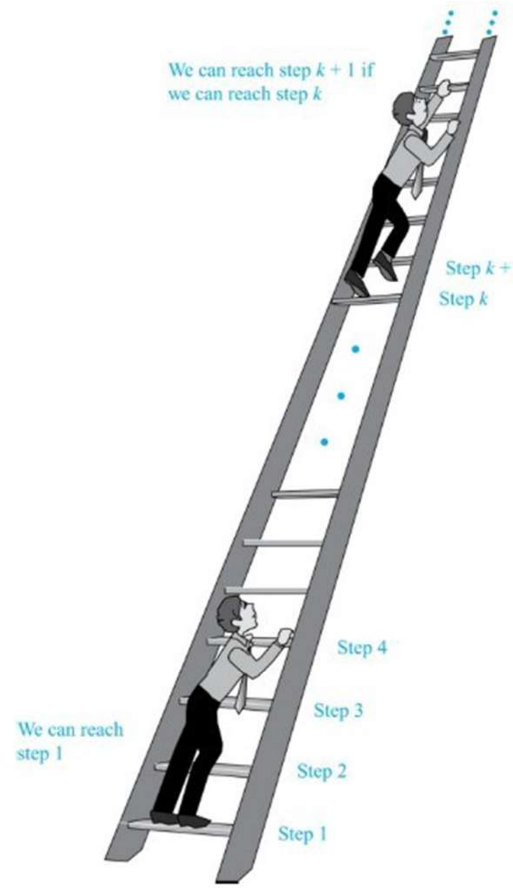
Section Summary

- ✓ Mathematical Induction
- ✓ Examples of Proof by Mathematical Induction
- ✓ Guidelines for Proofs by Mathematical Induction
- ✓ Strong Induction
- ✓ Example Proofs using Strong Induction

Mathematical Induction

Climbing an Infinite Ladder

- Suppose we have an infinite ladder:
- We can reach the first rung of the ladder.
- If we can reach a particular rung of the ladder, then we can reach the next rung.
- From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.
- This example motivates proof by mathematical induction.



Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. P(k) is called inductive hypothesis.

BASIS STEP: $P(1)$ is true.

INDUCTIVE STEP: For all positive integers k, if P(k) is true, then P(k + 1) is true.

$$\forall k(P(k) \rightarrow P(k + 1))$$

After completing the basis and inductive steps of a proof that P(n) is true for all positive integers n. Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \land \forall k(P(k) \rightarrow P(k + 1))) \rightarrow \forall\, nP(n)$$

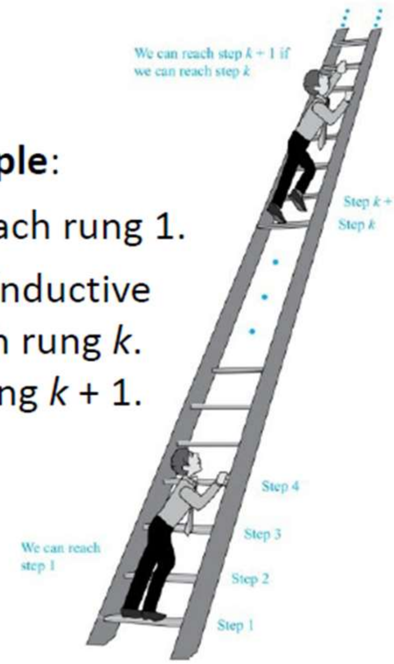
when the domain is the set of positive integers

Principle of Mathematical Induction

Example:

Climbing an Infinite Ladder Example:

- ***Basis Step***: By (1), we can reach rung 1.
- ***Inductive Step***: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k + 1$.



Hence, ***P*(*k*) → *P*(*k* + 1)** is true for all positive integers k. We can reach every rung on the ladder.

Remembering How Mathematical Induction Works

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Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.

We know that the first domino is knocked down, i.e., $P(1)$ is **true** .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

***P*(*n*) is true for all positive integers *n*.**

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Examples of Proof by Mathematical Induction

PROVING SUMMATION FORMULAE

Example 1: Show that if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Solution: Let P(n) be the proposition that $1 + 2 + \cdots n = n(n+1)/2$.

BASIS STEP: P(1) is true, because $1 = \frac{1(1+1)}{2}$

$1 = 1$

INDUCTIVE STEP: Inductive hypothesis that P(k) is true for an arbitrary positive integer k.
The inductive hypothesis P(k) is

$1 + 2 + \cdots k = \frac{k(k+1)}{2}$ for all positive integers *k*.

Prove that if P(k) is true, then P(k + 1), which is the statement that

$$1 + 2 + 3 + \ldots\ldots. + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true.

Continue ….

PROVING SUMMATION FORMULAE

$$1 + 2 + \cdots k \overset{IH}{=} \frac{k(k+1)}{2}$$

$$1 + 2 + \cdots k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

Add k + 1 to both sides of the equation

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$1 + 2 + \cdots k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Hence proved.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ is true for all positive integers n.

PROVING SUMMATION FORMULAE

Example 2: Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: Let P(n) be the proposition that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for n positive odd integers

BASIS STEP: $P(1)$ is true, because $1 = 1^2$

$$1 = 1$$

INDUCTIVE STEP: Inductive hypothesis that $P(k)$ is true for an arbitrary positive integer k.

The inductive hypothesis $P(k)$ is

$1 + 3 + 5 + \cdots + (2k - 1) = k^2$ for all positive integers *k*.

Prove that if P(k) is true, then P(k + 1), which is the statement that

$$1 + 3 + 5 + \cdots + (2k - 1) + [2(k+1) - 1] = (k + 1)^2$$

$$1 + 3 + 5 + \cdots + (2k - 1) + [2k + 2 - 1] = (k + 1)^2$$

$1 + 3 + 5 + \cdots + (2k - 1) + (2k+1) = (k + 1)^2$ is also true.

Continue ….

PROVING SUMMATION FORMULAE

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k+1) = [1 + 3 + 5 + \cdots + (2k - 1)] + (2k+1)$$

$$\overset{IH}{=} k^2 + (2k+1) \qquad \textit{(by inductive hypothesis)}$$

$$= k^2 + 2k+1$$

$$= (k + 1)^2$$

Hence proved.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is true for all positive integers n.

End

PROVING INEQUALITIES

Example 1: Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

Solution: Let P(n) be the proposition that $n < 2^n$.

BASIS STEP: P(1) is true, because $1 < 2^1$

INDUCTIVE STEP: Inductive hypothesis that P(k) is true for an arbitrary positive integer k. ($k \geq 1$)
The inductive hypothesis P(k) is $k < 2^k$ for all positive integers k.

Prove that if P(k) is true, then P(k + 1), which is the statement that $k + 1 < 2^{k+1}$ is true.

$$k \overset{IH}{<} 2^k$$

$$k + 1 < 2^k + 1 \qquad \textit{(Add 1 to both sides)}$$

$$\leq 2^k + k \qquad (k = 1)$$

$$< 2^k + 2^k \qquad \textit{(by the inductive hypothesis } k < 2^k\textit{)}$$

$$< 2 \cdot 2^k \qquad \textit{(addition)}$$

$$k + 1 < 2^{k+1} \qquad \textit{(Multiplication)}$$

Hence proved.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n.

PROVING INEQUALITIES

Example 2: Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$.

Solution: Let P(n) be the proposition that $2^n < n!$.

BASIS STEP: P(4) is true, because $2^4 < 4!$

$16 < 24$

INDUCTIVE STEP: Inductive hypothesis that P(k) is true for an arbitrary integer k with $k \geq 4$.

The inductive hypothesis P(k) is $2^k < k!$ for the positive integer k with $k \geq 4$.

Prove that if P(k) is true, then P(k + 1), which is the statement that $2^{k+1} < (k+1)!$ is true.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\ &\overset{\text{IH}}{<} 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1)k! && \text{because } 2 < k+1 \\ 2^{k+1} &< (k+1)! && \text{by definition of factorial function} \end{aligned}$$

Hence proved.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $2^n < n!$ for every integer n with $n \geq 4$.

PROVING DIVISIBILITY RESULTS

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Solution: Let P(n) be the proposition that $n^3 - n$ is divisible by 3 .

BASIS STEP: P(1) is true, because $1^3 - 1 = 0$ is divisible by 3

INDUCTIVE STEP: Inductive hypothesis that P(k) is true for an arbitrary integer k.
The inductive hypothesis P(k) is $k^3 - k$ is divisible by 3 for the positive integer k.

Prove that if P(k) is true, then P(k + 1), which is the statement that $(k+1)^3 - (k+1)$ is divisible by 3 is true.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 + 3k^2 + 3k - k \\
&= k^3 - k + 3k^2 + 3k \\
&= (k^3 - k) + 3(k^2 + k)
\end{aligned}
$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3.

The second term is divisible by 3 because it is 3 times an integer.

Hence proved.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

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Guidelines for Proofs by Mathematical Induction

Guidelines: Mathematical Induction Proofs

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "**Basis Step**." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "**Inductive Step**".
4. State, and clearly identify, the inductive hypothesis, in the form "**assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$.**"
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k + 1)$ says.
6. **Prove the statement $P(k + 1)$** making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the **conclusion** of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely, by mathematical induction, **$P(n)$ is true for all integers n with $n \geq b$.**

Review Questions

Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots n^2 = (\frac{n(n+1)(2n+1)}{6})$ for the positive integer n.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

(f) Explain why these steps show that this formula is true whenever n is a positive integer.

Review Questions

Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots n^2 = (\frac{n(n+1)(2n+1)}{6})$ for the positive integer n.

(a) What is the statement $P(1)$? a) $1^2 = 1 \cdot 2 \cdot 3/6$

(b) Show that $P(1)$ is true, completing the basis step of the proof. b) Both sides of P(1) shown in part (a) equal 1.

(c) What is the inductive hypothesis? c) $1^2+2^2+\cdots+k^2 = k(k+1)(2k+1)/6$

(d) What do you need to prove in the inductive step? d) For each $k \geq 1$ that P(k) implies P(k + 1) (OR) we can show $1^2+2^2+\cdots+k^2+(k+1)^2 = (k+1)(k+2)(2k+3)/6$

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

(f) Explain why these steps show that this formula is true whenever n is a positive integer.

f. We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

Review Questions

e. $(1^2+2^2+\cdots+k^2) + (k+1)^2 = \frac{[k(k+1)(2k+1)]}{6} + (k+1)^2$

$$= \frac{[(k+1)]}{6}[k(2k+1) + 6(k+1)]$$

$$= \frac{[(k+1)](2k^2 + k + 6k + 1)}{6}$$

$$= \frac{[(k+1)](2k^2 + 7k + 6)}{6}$$

$$= \frac{[(k+1)](k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

f. We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

Strong Induction

Strong Induction

- **Strong induction** is used when we cannot easily prove a result using mathematical induction.
- The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. However, the inductive steps in these two proof methods are different.

PRINCIPLE OF STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement

$$[P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k+1) \text{ is true for all positive integers } k.$$

The inductive hypothesis is P(j) is true for j = 1, 2,…, k.

Strong induction is sometimes called the **second principle of mathematical induction** or **complete induction**.

Example Proofs using Strong Induction

Completion of the proof of the Fundamental Theorem of Arithmetic

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as a product of primes.

- BASIS STEP: $P(2)$ is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is $P(j)$ is true for all integers j with $2 \leq j \leq k$. To show that $P(k + 1)$ must be true under this assumption, two cases need to be considered:
  - If $k + 1$ is prime, then $P(k + 1)$ is true.
  - Otherwise, $k + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k + 1$ can also be written as the product of those primes.

Hence, it has been shown that every integer greater than 1 can be written as the product of primes.

Proof using Strong Induction$_2$

- **Example**: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- **Solution**: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.
  - BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
    - $P(12)$ uses three 4-cent stamps.
    - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
    - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
    - $P(15)$ uses three 5-cent stamps.
  - INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
  - Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents. Hence, $P(n)$ holds for all $n \geq 12$.

Proof of Same Example using Mathematical Induction

- **Example**: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- **Solution**: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.
  - BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
  - INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$, we consider two cases:
    - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
    - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.
- Hence, $P(n)$ holds for all $n \geq 12$.

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Thank you