Discrete Structures

(CKC 111)

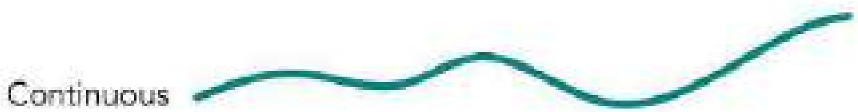
Introduction

Introduction

Mathematics

Continuous Mathematics

Continuous mathematics is the branch of mathematics dealing with objects that can vary smoothly.



eg: calculus, real numbers, etc.

Discrete Mathematics

Discrete mathematics is the branch of mathematics dealing with objects that can assume only distinct, separated values.

eg: graphs, integers, logic-based statements, etc.

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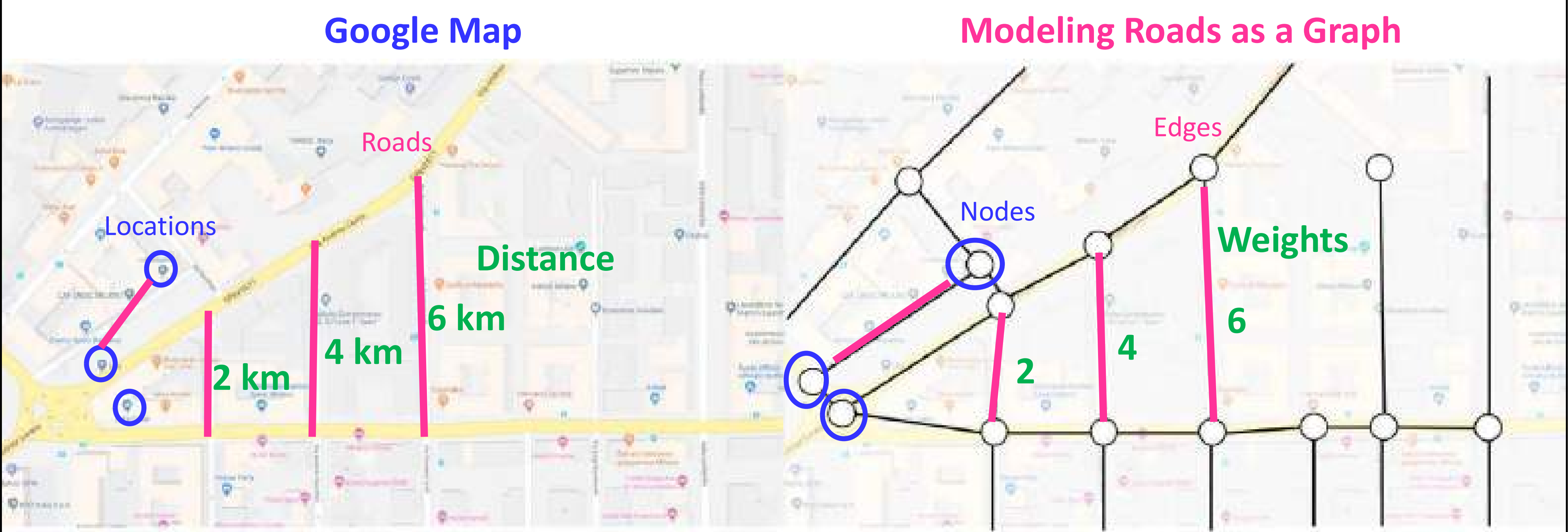
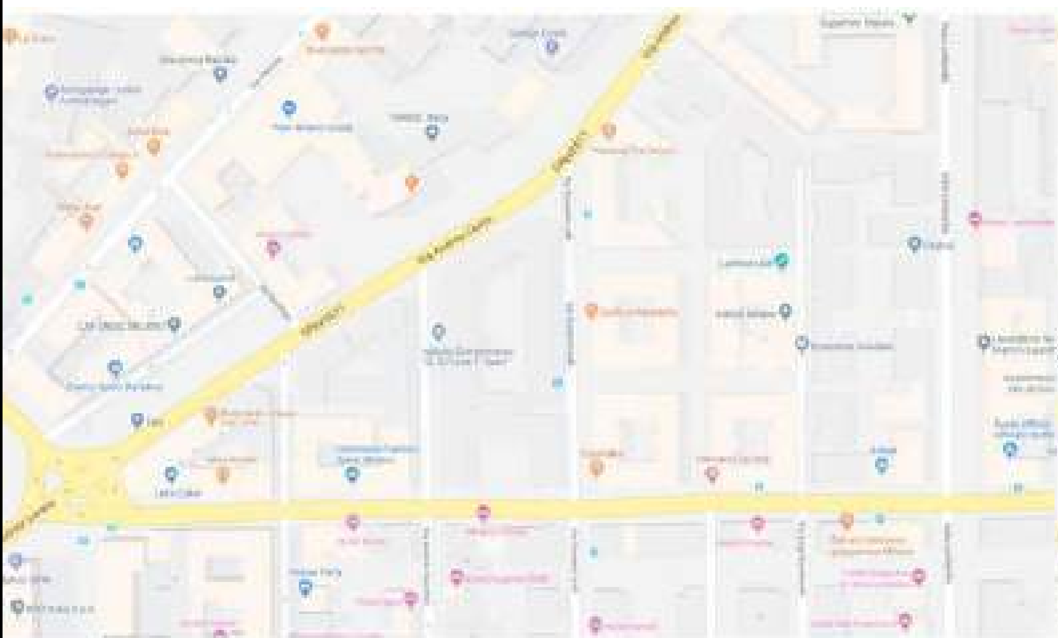
Applications of Discrete Mathematics



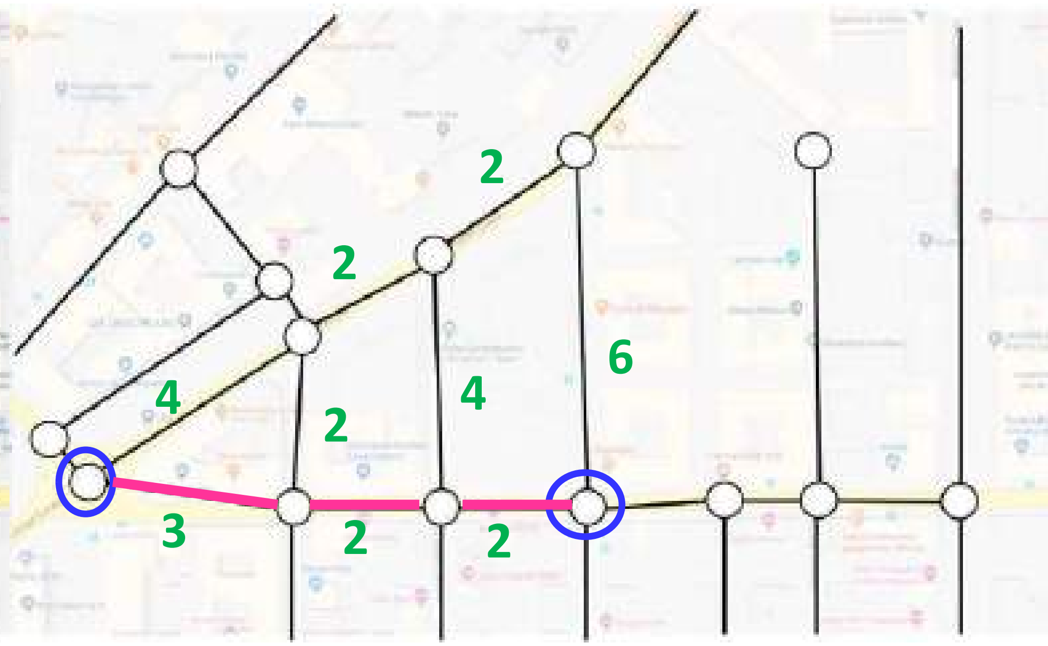
•**Nodes (Vertices)**: These represent intersections or points of interest (POIs) on a map.
•**Edges**: These represent the roads connecting those intersections or locations. Edges can be directed (one-way streets) or undirected (two-way streets).
•**Weights on Edges**: The edges often have weights assigned to them, representing the distance, time, or cost (e.g., fuel or tolls) to travel between two nodes.

Applications of Discrete Mathematics

Google Map



Shortest path



We have to use some logic to find shortest path.
Then we have to proof the path what we have found is the shortest path.

Basic Structures

Section Summary

Sets

- The Language of Sets
- Set Operations
- Set Identities

Functions

- Types of Functions
- Operations on Functions
- Computability

Sequences and Summations

- Types of Sequences
- Summation Formulae

Set Cardinality

- Countable Sets

Matrices

- Matrix Arithmetic

Sets

Section Summary[1]

- ✓ Definition of sets
- ✓ Describing Sets
  - Roster Method
  - Set-Builder Notation
- ✓ Some Important Sets in Mathematics
- ✓ Empty Set and Universal Set
- ✓ Subsets and Set Equality
- ✓ Cardinality of Sets
- ✓ Tuples
- ✓ Cartesian Product

Sets

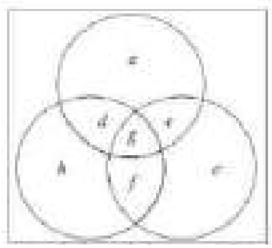
- A *set* is an **unordered collection of objects**.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the ***elements***, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an **element** of the set A.
- If a is not a member of A, write $a \notin A$

Sets

At least there are 3 ways to write a set;

1. List all the elements using "{" and "}", separated by commas. E.g., The set V of vowels is written as: $\boldsymbol{V} = \{\boldsymbol{a}, \boldsymbol{e}, \boldsymbol{i}, \boldsymbol{o}, \boldsymbol{u}\}$

2. Using set formula. E.g., The set E of all even positive integers is written as: $\{x \mid (x = 2n) \text{ and } (n \in Z^{+})\}$

3. Using a Venn diagram: E.g., 

Describing a set: Roster Method

$$S = \{a, b, c, d\}$$

Order not important: $S = \{a, b, c, d\} = \{b, c, a, d\}$

Each **distinct object** is either a member or not; listing more than once does not change the set: $S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$

Elapses (…) may be used to describe a set without listing all of the members when the pattern is clear. $S = \{a, b, c, d, \dots \dots, z\}$

Describing a set: Roster Method (example)

- Set of all vowels in the English alphabet:

  $V = \{a, e, i, o, u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3, \dots \dots \dots, 99\}$

- Set of all integers less than 0:

  $S = \{\dots., -3, -2, -1\}$

Some Important Sets

N

natural numbers = {0,1,2,3....}

Z

integers = {...,-3,-2,-1,0,1,2,3,...}

Z^+

positive integers = {1,2,3,.....}

R

set of *real numbers*

R^+

set of *positive real numbers*

C

set of *complex numbers.*

Q

set of rational numbers

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid P(x)\}$

Example: $S = \{x \mid \text{Prime}(x)\}$

Positive rational numbers:

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Describing Sets - Set-Builder Notation
Q&A

- A set O of all odd positive integers less than 10

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  OR

  specifying the universe as the set of positive integers, as

  $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$

- A set of an integer between 5 and 8

  $A = \{ x \in Z \mid 5 < x < 8 \}$

- The set of all the even numbers

  $A = \{ x \mid x = 2n, n \text{ is an integer} \}$

- set Q^+ of all positive rational numbers

  $Q^+ = \{x \in R \mid x = p/q \text{ , for some positive integers } p \text{ and } q\}$

Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- *Sometimes implicit*
- *Sometimes explicitly stated.*
- *Contents depend on the context.*

Venn Diagram

John Venn (1834-1923)
Cambridge, UK

The **empty set** is the set with no elements. Symbolized ∅, but {} also used.

Some things to remember

Sets can **be elements** of **sets**.

$$\{\{1,2,3\}, a, \{b, c\}\}$$

$$\{\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}\}$$

The empty set is different from a set containing the empty set.

$$\emptyset \neq \{\emptyset\}$$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore, if A and B are sets, then A and B are equal if and only if:

$$\forall x\,(x \in A \leftrightarrow x \in B)$$

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3,5,1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set *A* is a ***subset*** of *B*, if and only if every element of *A* is also an element of *B*.

- The notation $A \subseteq B$ is used to indicate that *A* is a **subset** of the set *B*.
- $A \subseteq B$ holds if and only if $\forall x (x \in A \rightarrow x \in B)$ is TRUE.

1. Because $a \in \emptyset$ ⍰ is always false, $\emptyset \subseteq S$,for every set S.
2. Because $a \in S$ $a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if ***x*** belongs to ***A***, then ***x*** also belongs to ***B***.

Showing that A is not a Subset of B: To show that *A* is not a subset of *B*, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a **subset** of all students at your University.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Subsets

THEOREM 1

For every set S,

(i) $\emptyset \subseteq$ **S**

(ii) $S \subseteq S$

$\forall x(x \in A \rightarrow x \in B)$ is true.

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Let S be a set.
- To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true.
- Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false.
- It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true.
- Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true.
- This completes the proof of (i).
- Note that this is an example of a vacuous proof.

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff;

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalence, we have that $A = B$ iff;

$$\forall x\left[\left(x \in A \rightarrow x \in B\right) \wedge \left(x \in B \rightarrow x \in A\right)\right]$$

This is equivalent to $A \subseteq B$ and $B \subseteq A$

Review Questions

Given { 1, {2, 3}, 4 }

1. How many elements in the set?

3

2. Write the elements.

1 {2, 3} 4

3. State True or False

a. $1 \in \{ 1, \{2, 3\}, 4 \}$ True

b. $\{2, 3\} \in \{ 1, \{2, 3\}, 4 \}$ True

c. $2 \notin \{ 1, \{2, 3\}, 4 \}$ True

d. $\{4\} \in \{ 1, \{2, 3\}, 4 \}$ False

4. State True or False

a. $\{1\} \subseteq \{ 1, \{2, 3\}, 4 \}$ True

b. $\{2, 3\} \not\subseteq \{ 1, \{2, 3\}, 4 \}$ True

c. $x \notin \emptyset$ True

d. $\emptyset \subseteq S$ True

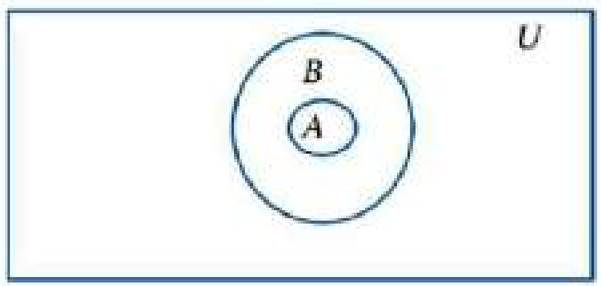
e. $\emptyset \in \{ 1, \{2, 3\}, 4 \}$ False

f. $\emptyset \subseteq \{ 1, \{2, 3\}, 4 \}$ True

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

If $A \subset B$, then; $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$ is true.



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Subsets

What are the Subset of A = {1, 2, 3}?

```
{} (the empty set)
{1}
{2}
{3}
{1, 2}
{1, 3}
{2, 3}
{1,2,3}
```

Proper Subsets

What are the Proper Subset of A = {1, 2, 3}?

```
{} (the empty set)
{1}
{2}
{3}
{1, 2}
{1, 3}
{2, 3}
```

kami memimpin

Power Sets

Definition: The set of all subsets of a set A, denoted $\boldsymbol{P(A)}$, is called the *power set* of $\boldsymbol{A}$.

Example: If $A = \{a, b\}$ then;

$$\boldsymbol{P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}}$$

If a set has *n* elements, then the cardinality of the power set is 2^n.

Power Set – Q & A

Write the power set for A = {a, b, c}

Number of elements in set A (n)= 3

Number of elements in power set A = 2^n = 2^3 = 8

Power set

{{},{a},{b},{c},{a, b},{a, c},{b, c},{a, b, c}}

Review Questions

1. Write the notation and match the following

	Notation	
Subset	$\subseteq$	$\forall x(x \in A \leftrightarrow x \in B)$ is true
Proper subset	$\subset$	$A \times B = \{(a, b) \mid a \in A \land b \in B\}$
Set equality	$=$	$\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$ is true
Cartesian Products	X	$\forall x(x \in A \rightarrow x \in B)$ is true

Set Cardinality

Definition: If there are exactly n distinct elements in ***S*** where ***n*** is a **nonnegative integer**, we say that <u>***S* is *finite***</u>. Otherwise, it is <u>*infinite*</u>.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

- $|\emptyset| = 0$
- Let S be the letters of the English alphabet. Then $|S| = 26$
- $|\{1,2,3\}| = 3$
- $|\{\emptyset\}| = 1$
- The set of integers is infinite.

Terminology and Notation

- A set that has no elements is called the **empty set** or **null set** and is denoted $\varnothing$

- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}

- Note the subtlety in $\varnothing \neq \{\varnothing\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

Tuples

- Tuples: *finite sequence of objects; a structure containing multiple parts*
- The *ordered* ***n-tuple*** $(a_1,a_2,\ldots\ldots,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called ***ordered pairs***.
- The ordered pairs (a,b) and (c,d) are **equal** if and only if $a = c$ and $b = d$.

Cartesian Product[1]

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

<u>Example:</u>

$A = \{a, b\} \quad B = \{1,2,3\}$

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Definition: A subset *R* of the Cartesian product $A \times B$ is called a ***relation*** from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product₂

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = \mathbf{1}, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $\boldsymbol{A \times B \times C} =$

$$\{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$$

Cartesian Product – Q&A

1. What is the Cartesian product of A = {1, 2} and B = {a, b, c}?

The Cartesian product $A \times B$ is

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

2. Show that the Cartesian product B × A is not equal to the Cartesian product A × B, where A and B are as in Question 1.

The Cartesian product $B \times A$ is

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

This is not equal to $A \times B$, which was found in **Question 1.**

Section Summary

Sets
- The Language of Sets
- Set Operations
- Set Identities

Functions
- Types of Functions
- Operations on Functions
- Computability

Sequences and Summations
- Types of Sequences
- Summation Formulae

Set Cardinality
- Countable Sets

Matrices
- Matrix Arithmetic

Set Operations: Summary

✓ Set Operations

- Union
- Intersection
- Complementation
- Difference

✓ More on Set Cardinality

✓ Set Identities

Union

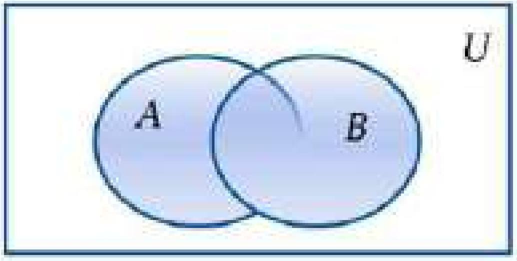
Definition: Let A and B be sets. The ***union*** of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3,4,5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is;

$$\{x \mid x \in A \vee x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

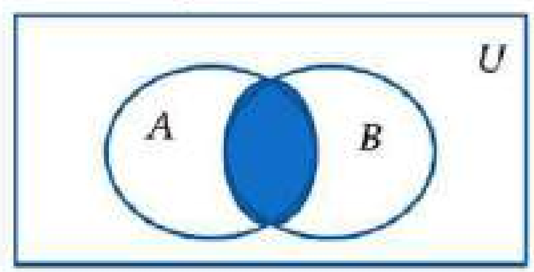
Example: What is $\{1,2,3\} \cap \{3,4,5\}$?

Solution: $\{3\}$

Example: What is $\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

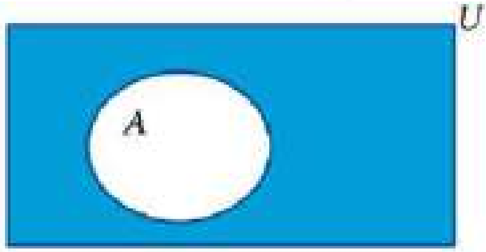
Definition: If A is a set, then the ***complement*** of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$;

$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

<u>Example:</u> If U is the positive integers less than 100, what is the complement of $\{\boldsymbol{x} \mid \boldsymbol{x} > \mathbf{70}\}$

Venn Diagram for Complement

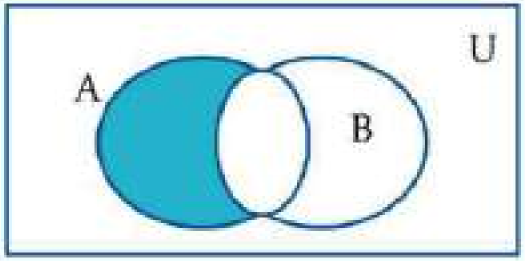


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution:

2. $A \cap B$

   Solution:

3. $\bar{A}$

   Solution:

4. $\bar{B}$

   Solution:

5. $A - B$

   Solution:

6. $B - A$

   Solution:

Review Questions (Answer)

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

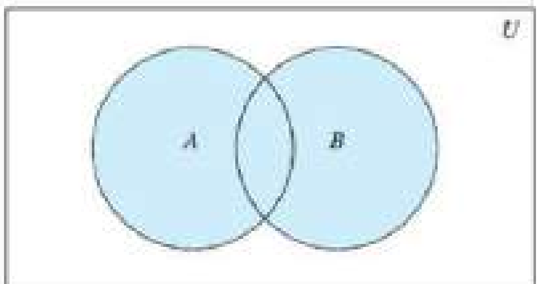
6. $B - A$

   Solution: $\{6,7,8\}$

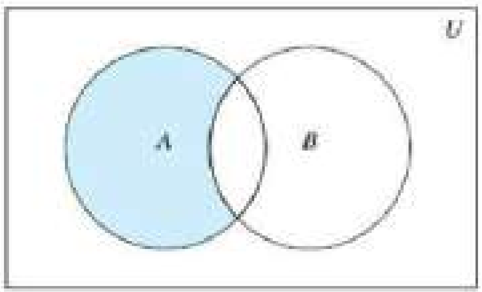
Set Operations – Q & A

1. Write the set operation and match the following

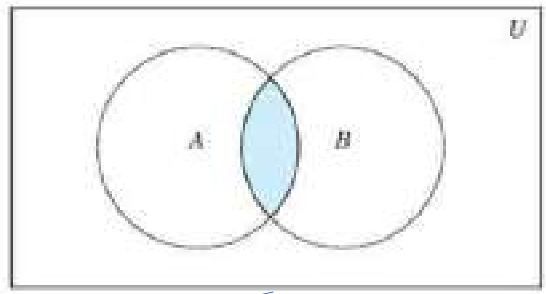
a. union



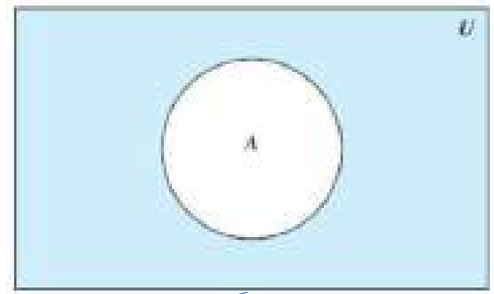
b. difference



c. intersection



d. complement



$A \cap B = \{x \mid x \in A \land x \in B\}$

$A = \{x \in U \,/\, x \notin A\}$

$A \cup B = \{x \mid x \in A \lor x \in B\}$

$A - B = \{x \mid x \in A \land x \notin B\}$

The Cardinality of the Union of Two Sets

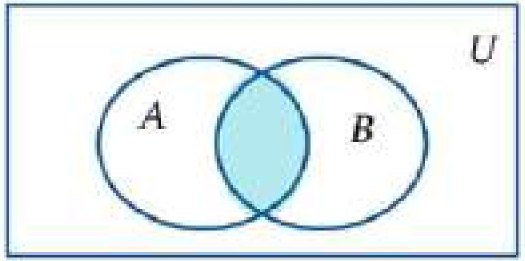
Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Venn Diagram for A, B, A ∩ B, A ∪ B



The Cardinality of the Union of Two Sets
Q & A

1. **Given two sets B = {a, b, c} and C = {d, e, f, g}, What is the cardinality of the union B∪C?**

   Solution:

   $|B| = 3$
   $|C| = 4$
   $B \cap C = \{\ \}$
   $|B \cap C| = 0$
   $|B \cup C| = |B| + |C| - |B \cap C|$
   $= 3 + 4 - 0$
   $= 7$

   <u>To check</u>
   $B \cup C = \{a, b, c, d, e, f, g\}$
   The cardinality of $B \cup C$ is i.e., $|B \cup C| = 7$

The Cardinality of the Union of Two Sets
Q & A

2. **Let D = {2, 4, 6} and E = {1, 2, 3, 4, 5, 6}. Find the cardinality of D∩E and D∪E .**
Solution:

The cardinality of D∩E:

D∩E = {2, 4, 6}
The cardinality of D∩E is:
|D∩E| = 3

The cardinality of D∪E:

|D| = 3
|E| = 6
|D∩E| = 3
|D∪E| = |D| + |E| - |D∩E|
= 3 + 6 – 3
= 6

To check

D∪E = {1, 2, 3, 4, 5, 6}
The cardinality of D∪E is i.e., |D∪E| = 6

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Set Identities[1]

Identity laws: $A \cup \varnothing = A$ $\quad A \cap U = A$

Domination laws: $A \cup U = U$ $\quad A \cap \varnothing = \varnothing$

Idempotent laws: $A \cup A = A$ $\quad A \cap A = A$

Complementation law: $\left(\overline{\overline{A}}\right) = A$

Commutative laws: $A \cup B = B \cup A$ $\quad A \cap B = B \cap A$

Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Set Identities₂

Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\quad \overline{A \cap B} = \overline{A} \cup \overline{B}$

Absorption laws: $A \cup (A \cap B) = A$ $\quad A \cap (A \cup B) = A$

Complement laws: $A \cup \overline{A} = U$ $\quad A \cap \overline{A} = \varnothing$

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TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities

Methods of Proving Set Identities.

1. Subset method

   Show that each side of the identity is a subset of the other side.

2. Membership table

   For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side

3. Apply existing identities

   Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

1. Subset method

Two sets: $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$

each set is a subset of the other.

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

if x is in $\overline{A \cap B}$, then x must also be in $\overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$.	
$x \notin A \cap B$.	By the definition of complement
$\neg((x \in A) \wedge (x \in B))$	Using the definition of intersection
$\neg(x \in A)$ or $\neg(x \in B)$	By applying De Morgan's law for propositions
$x \notin A$ or $x \notin B$.	Using the definition of negation of propositions
$x \in \overline{A}$ or $x \in \overline{B}$.	Using the definition of the complement of a set
$x \in \overline{A} \cup \overline{B}$.	by the definition of union

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Proved

$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

if x is in $\overline{A} \cup \overline{B}$, then x must also be in $\overline{A \cap B}$.

$x \in \overline{A} \cup \overline{B}$.	
$x \in \overline{A}$ or $x \in \overline{B}$.	By the definition of union
$x \notin A$ or $x \notin B$.	Using the definition of complement
$\neg(x \in A) \vee \neg(x \in B)$	Using the definition of union
$\neg((x \in A) \wedge (x \in B))$	By applying De Morgan's law for propositions
$\neg(x \in A \cap B)$.	By the definition of intersection
$x \in \overline{A \cap B}$.	use the definition of complement

$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Proved

$\overline{A \cap B} = \overline{A} \cup \overline{B}$. Hence proved

Prove the second distributive law which states that
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
for all sets A, B, and C.

each set is a subset of the other.

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

if x is in $A \cap (B \cup C)$ then x must also be in $(A \cap B) \cup (A \cap C)$

$x \in A$ and $x \in B \cup C$

$x \in A$, and $x \in B$ or $x \in C$ — By the definition of union

$(x \in A) \wedge ((x \in B) \vee (x \in C))$

$((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$ — By the distributive law for conjunction over disjunction

$x \in A$ and $x \in B$, or $x \in A$ and $x \in C$

$x \in A \cap B$ or $x \in A \cap C$ — By the definition of intersection

$x \in (A \cap B) \cup (A \cap C)$ — Using the definition of union

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

if x is in $(A \cap B) \cup (A \cap C)$ then x must also be in $A \cap (B \cup C)$

$x \in (A \cap B) \cup (A \cap C)$.

$x \in A \cap B$ or $x \in A \cap C$ — By the definition of union

$x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$ — By the definition of intersection

$x \in A$, and $x \in B$ or $x \in C$

$x \in A$ and $x \in B \cup C$ — By the definition of union

$x \in A \cap (B \cup C)$. — By the definition of intersection

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Hence proved

1. Subset method

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Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A Membership Table for the Distributive Property							
A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

2. Membership table

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Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

$\overline{A \cup (B \cap C)} = \overline{A} \cap (\overline{B \cap C})$ by the first De Morgan law

$= \overline{A} \cap (\overline{B} \cup \overline{C})$ by the second De Morgan law

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$ by the commutative law for intersections

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$ by the commutative law for unions

3. Apply existing identities

TABLE 1 Set Identities.

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Use set builder notation and logical equivalences to establish the first De Morgan law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}$$

Topics

Basic Structures

Sets

Functions

Sequences and Summations

Set Cardinality

Matrices

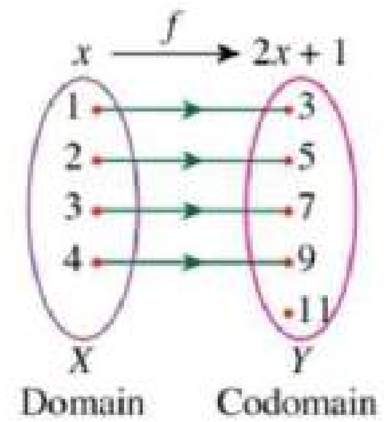
Functions

Section Summary

- ✓ Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- ✓ Injection, Surjection, Bijection
- ✓ Inverse Function
- ✓ Function Composition
- ✓ Graphing Functions
- ✓ Floor, Ceiling, Factorial

Introduce Functions

- $F: X \rightarrow Y$
- X maps to Y
- **Domain** (all element in X)
- **Codomain** (all element in Y)
- $x \in X \quad y \in Y$
- $f(x) = y$ (range, image)
- $y \in Y$ such that $f(x) = y$ is the range or the image
- $x \in X$ is pre-image



Domain = {1, 2, 3, 4}
Codomain= {3, 5, 7, 9, 11}
Range = {3, 5, 7, 9}

Introduce Functions

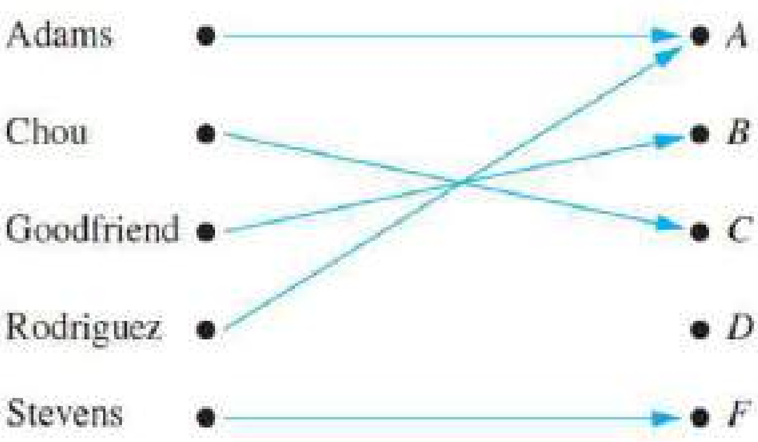
- Lots of terminology but remember the big picture.
- x_1 f maps $x \in X$ to $y \in Y$ = y_1 … and so on…
- **Range** is **part of codomain**
- Range is all the things that f can take elements in x to.
- Example $f(x) = x^2$

Introduce Functions

- Example $f(x) = x^2$
- $X \in Z$ (integer)
- Give example $x = 2$, so $y = 4$, $x = 0; y = 0$, $x = -4; y = 16$
- If we say that $X \in Z$ and $Y \in Z$, thus $f: Z \rightarrow Z$
- Always positive → W (whole number integer)
- **Codomain**: W (Range..?). Example if $y = 3$ (which is W), $x =?$ (answer is not an integer)
- Thus; Range: $\{x^2 \mid \sqrt{x}\ is\ an\ integer\}$

Function example

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {*A, B, C, D, F*}. And suppose that the grades are *A* for Adams, *C* for Chou, *B* for Goodfriend, *A* for Rodriguez, and *F* for Stevens. What are the domain, codomain, and range of the function that assigns grades to students?



The domain of *G* is the set

{Adams, Chou, Goodfriend, Rodriguez, Stevens}

The codomain is the set

{*A, B, C, D, F*}

The range of *G* is the set

{*A, B,C, F*}

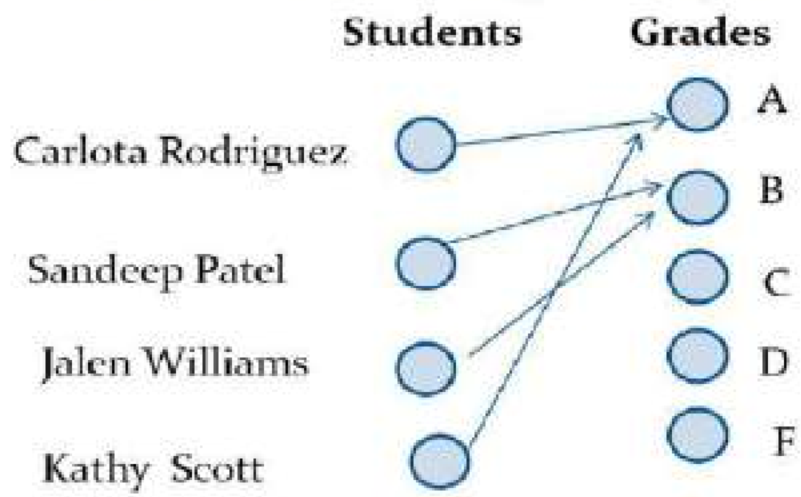
because each grade except *D* is assigned to some student.

Functions[1]

Definition: Let A and B be **nonempty** sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

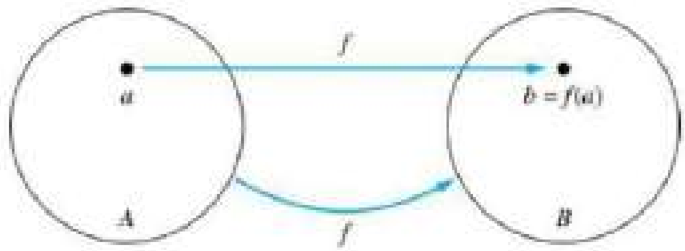
- Functions are sometimes called ***mappings*** or ***transformations***.



Functions[3]

Given a function $f: A \to B$:

- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in A under f. We denote it by $f(A)$.
- Two functions are ***equal*** when they have the **same domain**, the **same codomain** and **map** each element of the domain to the **same element** of the codomain.



Representing functions

Functions are specified in many different ways.

- explicitly state the assignments
- give a formula
- use a computer program to specify a function
- in terms of a relation from A to B

give a formula

1 . → . 2
2 . → . 4
3 . → . 6

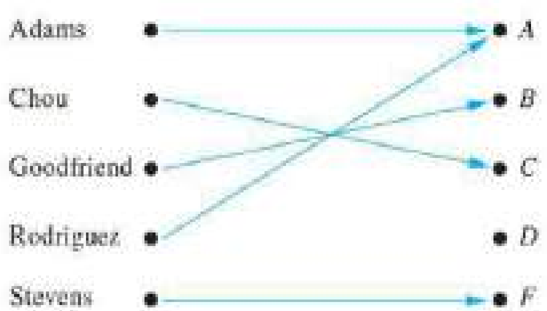
$f(a) = b$

$f(a) = 2a$

$\boxed{f(x) = 2x}$

Explicitly state the assignments

eg: Assignment of grades in a discrete mathematics class



in terms of a relation from A to B

A function $f : A \rightarrow B$ can also be defined in terms of a relation from A to B.

a relation from A to B is just a subset of $A \times B$.

A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B.

This function is defined by the assignment $f(a) = b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

Function – Example 1

Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22).

Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation

Solution

Let f be a function specified by R.

f (Abdul) = 22

f (Brenda) = 24

f (Carla) = 21

f (Desire) = 22

f (Eddie) = 24

f (Felicia) = 22

The domain

{Abdul, Brenda, Carla, Desire, Eddie, Felicia}

The codomain

The set of positive integers less than 100

(OR)

The set of all positive integers

(OR)

The set of positive integers between 10 and 90

The range of the function

{21, 22, 24}

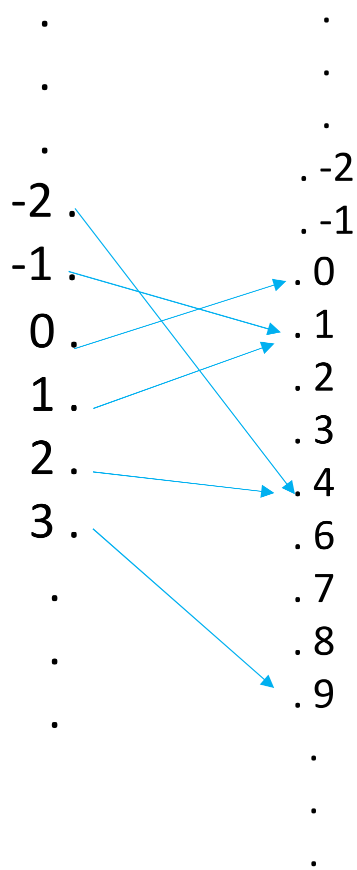
Function – Example 2

Let f : **Z** → **Z** assign the square of an integer to this integer.

<u>Solution</u>

$f\,(x) = x^2$

Domain   Codomain



<u>The domain of f</u>

The set of all integers

<u>The codomain of f</u>

The set of all integers

<u>The range of f</u>

The set of all integers that are perfect squares, namely, {0, 1, 4, 9,…}.

Questions

f(a) = ?

The image of d is ?

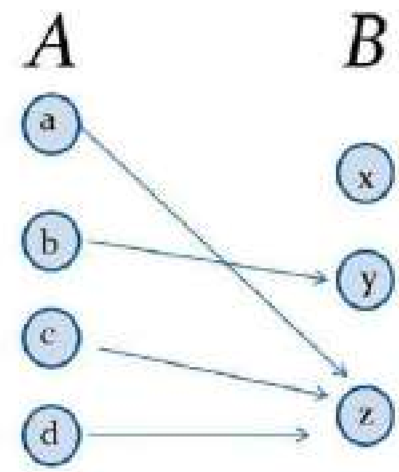
The domain of f is ?

The codomain of f is ?

The preimage of y is ?

$f(A)$ = ?

The preimage(s) of z is (are) ?



Answer

f(a) = ? z

The image of d is ? z

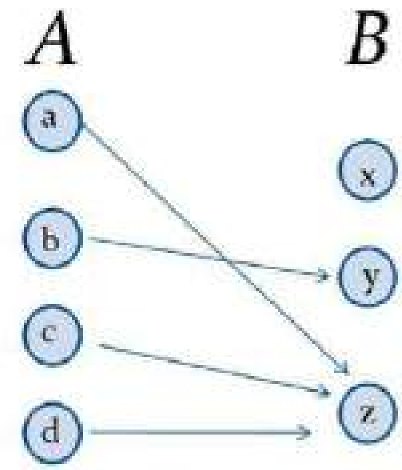
The domain of f is ? *A*

The codomain of f is ? *B*

The preimage of y is ? b

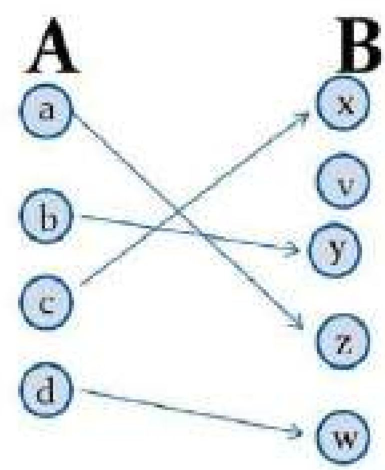
f(*A*) = ? {y,z}

The preimage(s) of z is (are) ? {a,c,d}



Injections

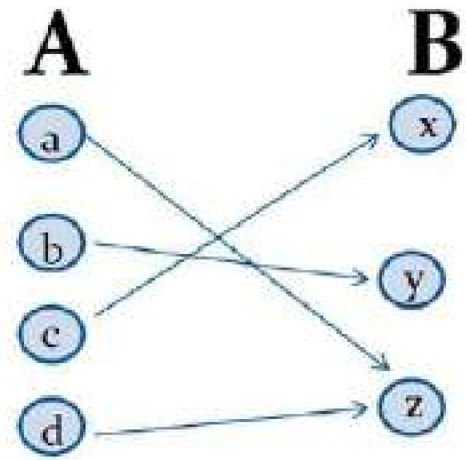
Definition: A function f is said to be ***one-to-one***, or ***injective***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an ***injection*** if it is ***one-to-one***.



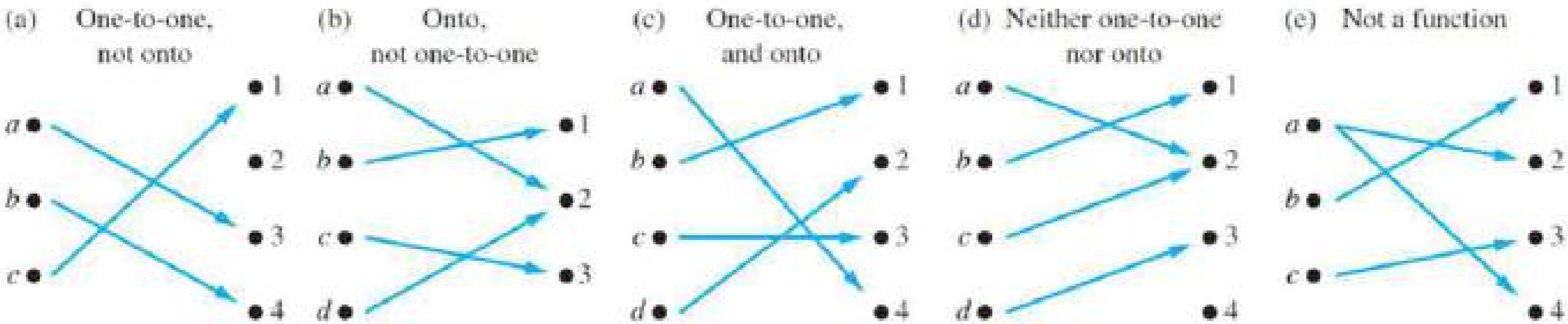
Surjections

Definition: A function *f* from *A* to *B* is called ***onto*** or ***surjective***, if and only if for **every** element $b \in B$, there is an element $a \in A$ with $f(a) = b$.

A function *f* is called a ***surjection*** if it is ***onto***.



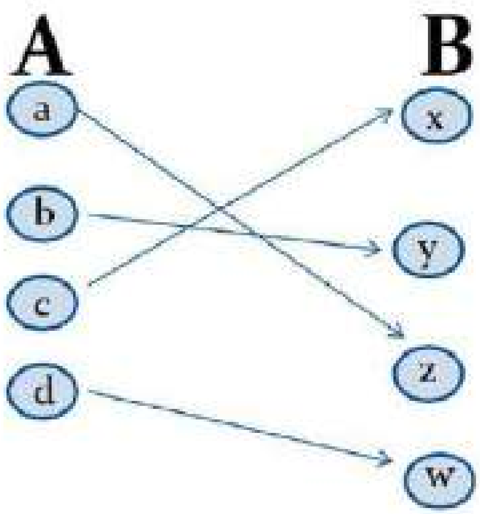
One-to-one & Onto



Examples of Different Types of Correspondences.

Bijections

Definition: A function f is a ***one-to-one correspondence***, or a ***bijection***, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Showing that f is one-to-one or onto

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions

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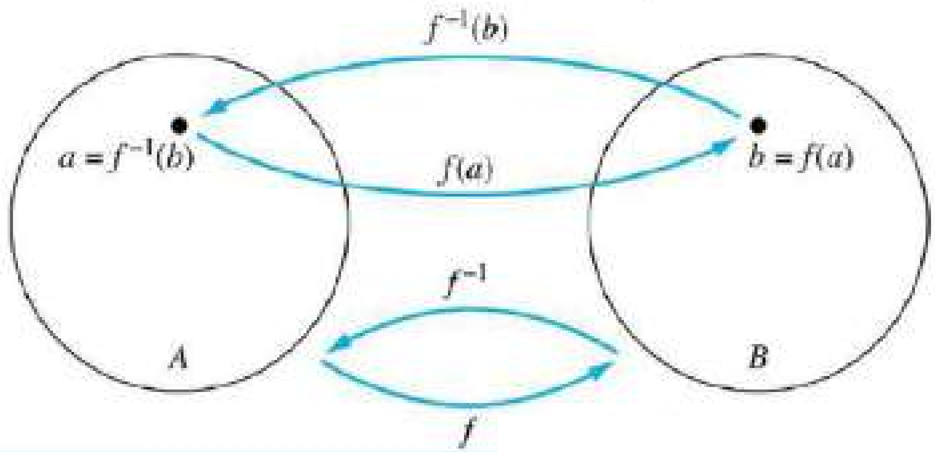
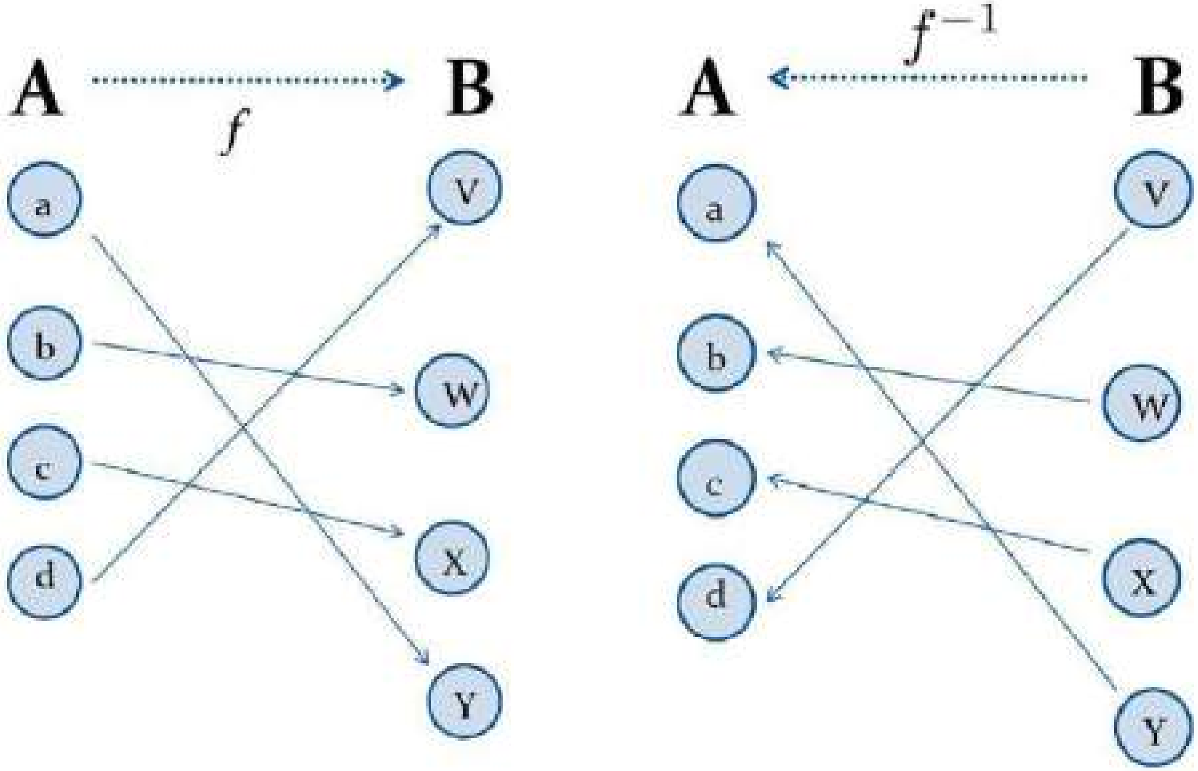
Definition: Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

No inverse exists unless f is a **bijection**. Why?



Inverse Functions



Inverse Functions

<u>No inverse exists unless f is a bijection. Why?</u>

Consider a one-to-one correspondence f from the set A to the set B. Because f is an onto function, every element of B is the image of some element in A. Furthermore, because f is also a one-to-one function, every element of B is the image of a unique element of A. Consequently, we can define a new function from B to A that reverses the correspondence given by f .

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Questions[1]

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions[2]

Example 2: Let f: $\mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions[3]

Example 3: Let f: **R** $\rightarrow$ **R** be such that $f(x) = x^2$
Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one.

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$Questions_4$

If $f: x \rightarrow 5x + 2$, find its inverse.

$$f(x) = 5x + 2$$

$$f(x) = y$$

Let $y = 5x + 2$

$$5x + 2 = y$$

$$5x = y - 2$$

$$x = \frac{y - 2}{5} \quad \ldots\ldots\ldots (1)$$

$$x = f^{-1}(y) \quad \ldots\ldots\ldots (2)$$

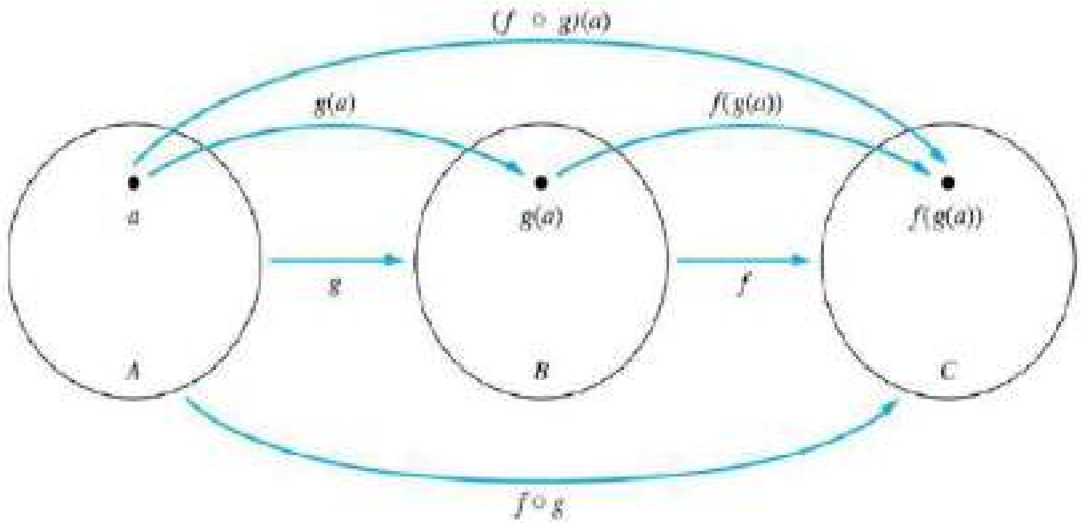
$$(1) = (2)$$

$$\boxed{f^{-1}(y) = \frac{y - 2}{5}}$$

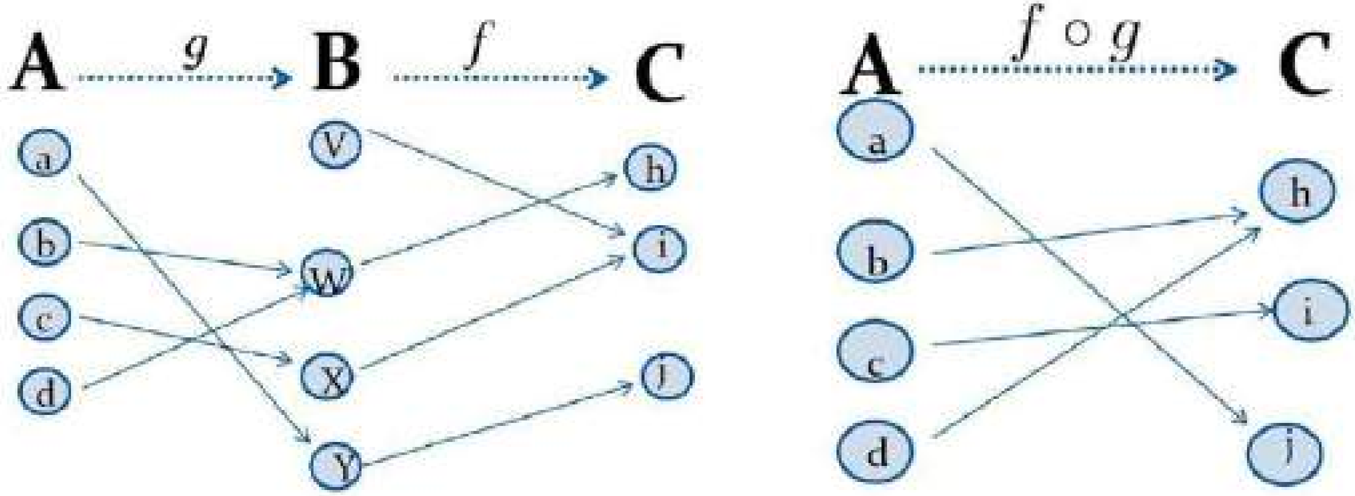
kami memimpin

Composition

Definition: Let $f: B \to C$, $g: A \to B$. The ***composition of f with g***, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f(g(x))$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$,

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and the composition of g and f ?

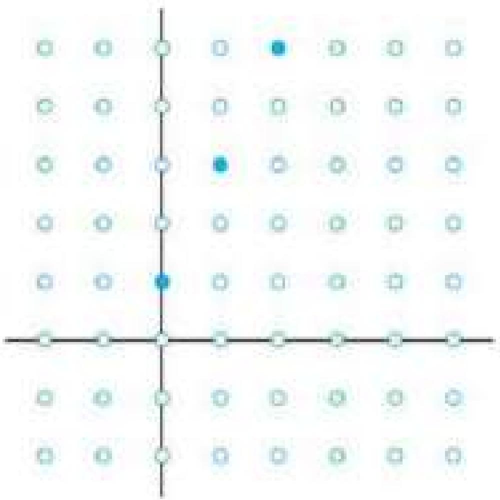
Solution: $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

and

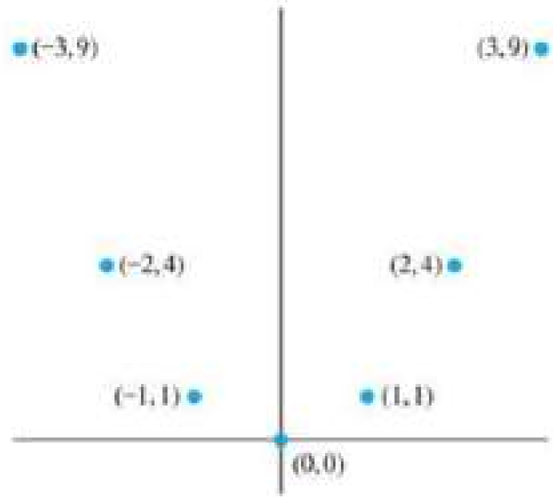
$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a)=b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

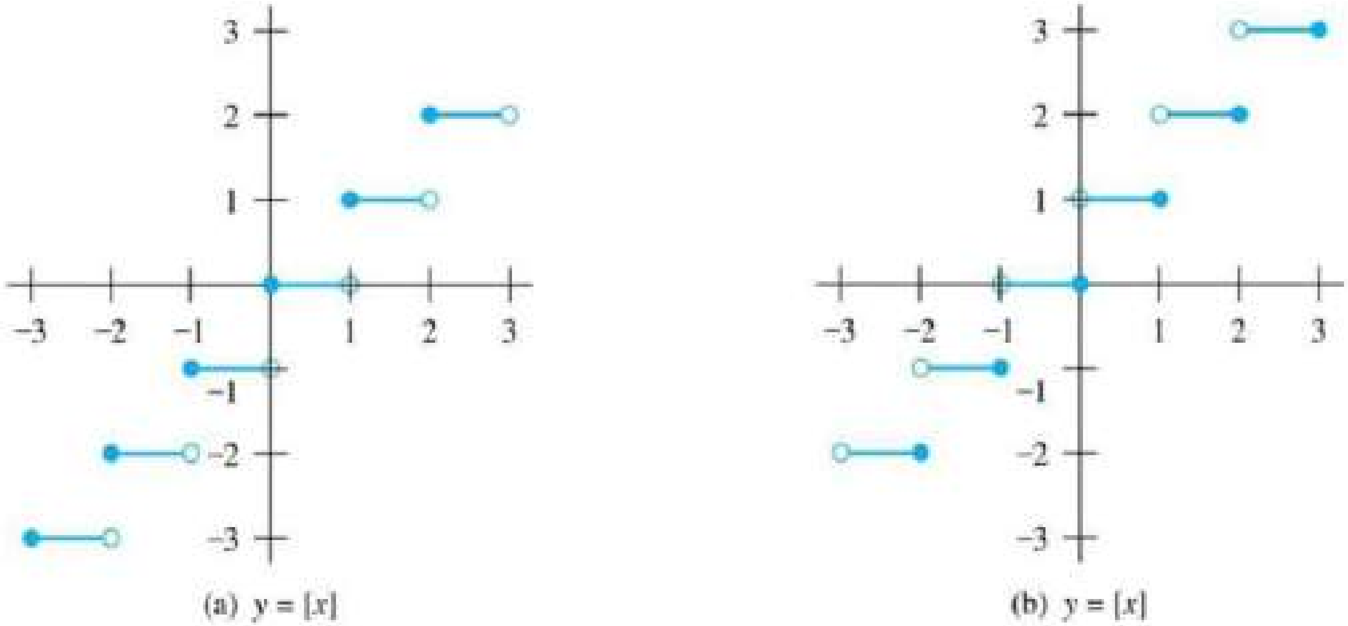
The ***floor*** function, denoted $f(x) = \lfloor x \rfloor$ is the **largest integer** less than or equal to x.

The ***ceiling*** function, denoted $f(x) = \lceil x \rceil$ is the **smallest integer** greater than or equal to x.

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



Graph of (a) **Floor** and (b) **Ceiling** functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Floor and Ceiling functions

Examples

a) $\lfloor 1/2 \rfloor = \lfloor 0.5 \rfloor = 0$

b) $\lceil 1/2 \rceil = \lceil 0.5 \rceil = 1$

c) $\lfloor -1/2 \rfloor = \lfloor -0.5 \rfloor = -\lceil 0.5 \rceil = -1$

d) $\lceil -1/2 \rceil = \lceil -0.5 \rceil = -\lfloor 0.5 \rfloor = 0$

e) $\lfloor 3.1 \rfloor = 3$

f) $\lceil 3.1 \rceil = 4$

g) $\lfloor 7 \rfloor = 7$

h) $\lceil 7 \rceil = 7$

i) $\lfloor 3.5 \rfloor = 3$

j) $\lceil 3.5 \rceil = 4$

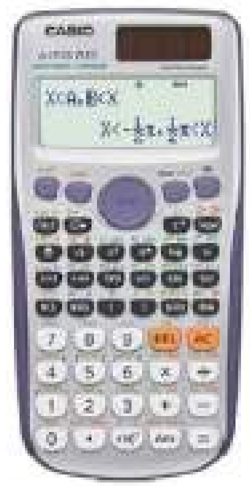
Factorial Function

Definition: f: **N** $\rightarrow$ **Z**$^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$
$$f(2) = 2! = 1 \cdot 2 = 2$$
$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$
$$f(20) = 2,432,902,008,176,640,000.$$



Basic Structures

Sets

Functions

Sequences and Summations

Set Cardinality

Matrices

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Sequences and Summations

Section Summary

✓ Sequences

- Examples: Geometric Progression, Arithmetic Progression

✓ Recurrence Relations

- Example: Fibonacci Sequence

✓ Summations

Introduction

Sequences are ordered lists of elements.

- 1, 2, 3, 5, 8 (sequence with five terms)
- 1, 3, 9, 27, 81, ...,3^n, ... (infinite)

Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.

We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.

The notation a_n is used to denote the **image** of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,…..} to S. We call a_n a *term* of the sequence.

Sequence

- Use the notation $\{a_n\}$ to describe the sequence.
- a_n represents an individual term of the sequence $\{a_n\}$
- The list of the terms of this sequence, beginning with a_1, namely,

  $a_1, a_2, a_3, a_4, \ldots, \ldots\ldots$
- a_1 represents the first term of the sequence $\{a_n\}$
- Other letters or expressions may be used depending on the sequence under consideration.

  eg: $\{b_n\}$

Sequences

Example: Consider the sequence $\{a_n\}$ where;

$$a_n = \frac{1}{n} \qquad \{a_n\} = a_1, a_2, a_3 \ldots$$

The list of the terms of this sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the ***initial term a*** and the ***common ratio r*** are real numbers.

Examples :

1. Let $a = 1$ and $r = -1$. Then :

$$\{b_n\} = b_0, b_1, b_2, b_3, b_4, \ldots = 1, -1, 1, -1, 1, \ldots$$

2. Let $a = 2$ and $r = 5$. Then :

$$\{c_n\} = c_0, c_1, c_2, c_3, c_4, \ldots = 2, 10, 50, 250, 1250, \ldots$$

3. Let $a = 6$ and $r = 1/3$. Then :

$$\{d_n\} = d_0, d_1, d_2, d_3, d_4, \ldots = 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$$

Arithmetic Progression

Definition: A arithmetic progression is a sequence of the form:

$$a, a+d, a+2d, ..., a+nd, ...$$

where the ***initial term a*** and the ***common difference d*** are real numbers. **Examples :**

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = s_0, s_1, s_2, s_3, s_4, .. =$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = t_0, t_1, t_2, t_3, t_4, ... =$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = u_0, u_1, u_2, u_3, u_4, ... =$$

Arithmetic Progression

Definition: A arithmetic progression is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the ***initial term a*** and the ***common difference d*** are real numbers. **Examples :**

1. Let $a=-1$ and $d=4$:

$$\{s_n\} = s_0, s_1, s_2, s_3, s_4, .. = -1, 3, 7, 11, ..$$

2. Let $a=7$ and $d=-3$:

$$\{t_n\} = t_0, t_1, t_2, t_3, t_4, \ldots = 7, 4, 1, -2, -5, \ldots$$

3. Let $a=1$ and $d=2$:

$$\{u_n\} = u_0, u_1, u_2, u_3, u_4, \ldots = 1, 3, 5, 7, 9, ..$$

Strings

Definition: A ***string*** is a finite sequence of characters from a finite set (an alphabet).

Sequences of characters or bits are important in computer science.

The ***empty string*** is represented by λ (is the string that has no terms, empty string has length zero).

The string *abcde* has *length* 5.

Recurrence Relations

Definition: A ***recurrence relation*** for the sequence $\{a_n\}$ is an **equation** that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.

The ***initial conditions*** for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations

Example:

A sequence $\{a_n\}$

The recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$

(An equation that expresses a_n in terms of one or more of the previous terms of the sequence)

Initial term $a_0 = 2$

A solution of a recurrence relation

$a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = a_1 + 3 = 5 + 3 = 8$

$a_3 = a_2 + 3 = 8 + 3 = 11$

5 , 8, 11, ...

Recurrence Relations

Example 1:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$.

What are a_1, a_2 and a_3? [Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that;

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Recurrence Relations

Example 2:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that;

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$
$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

The **Fibonacci sequence,** also known as Fibonacci numbers, is defined as the **sequence of numbers** in which each number in the sequence is equal to the **sum of two numbers before it**.

Example: Find f_2, f_3, f_4, f_5 and f_6.

Fibonacci Sequence

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The **Fibonacci sequence,** also known as Fibonacci numbers, is defined as the **sequence of numbers** in which each number in the sequence is equal to the **sum of two numbers before it**.

Example: Find f_2, f_3, f_4, f_5 and f_6.

Answer :

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Solving Recurrence Relations

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

Such a formula is called a *closed formula*.

Solving Recurrence Relations

Method 1: Working upward, forward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, ...$ and suppose that $a_1 = 2$.

Solution: **Forward substitution -** we found successive terms beginning with the initial condition and ending with a_n.

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$
$$\vdots$$
$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Answer: $2 + 3(n - 1)$

Solving Recurrence Relations

Method 2: Working downward, backward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$.

Solution:

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \quad \boxed{\text{Final term}} \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\vdots \\
&= a_2 + 3(n-2) \\
&= (a_1 + 3) + 3(n-2) \\
&= 2 + 3(n-1).
\end{aligned}
$$

Initial term

Common difference

Number of term

$\boxed{\text{Answer: } 2 + 3(n-1)}$

Solving Recurrence Relations

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_n - 1 + 3$ for n = 1, 2, 3,…, and suppose that $a_1 = 2$. What are a_2 and a_3?

A solution of a recurrence relation

$$a_2 = a_1 + 3 = 2 + 3 = 5$$

$$a_3 = a_2 + 3 = 5 + 3 = 8$$

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_n - 1 + 3$ for n = 1, 2, 3,…, and suppose that $a_1 = 2$. What are a_2, a_{12}, and a_{50}?

Step 1: Find the closed formula

Step 2: Solve

Step 1: Find the closed formula

(Use **forward substitution** or **backward substitution** to find the closed formula. Refer previous example).

The closer formula: $a_n = 2 + 3(n - 1)$

Step 2: Solve

$$a_2 = 2 + 3(2 - 1) = 2 + 3(1) = 2 + 3 = 5$$

$$a_{12} = 2 + 3(12 - 1) = 2 + 3(11) = 2 + 33 = 35$$

$$a_{50} = 2 + 3(50 - 1) = 2 + 3(49) = 2 + 147 = 149$$

Solving Recurrence Relations

The sequence $\{s_n\}$

Closed formula: $s_n = -1 + 4n$

Initial term: -1

Common difference: 4

The list of terms $s_0, s_1, s_2, s_3, \ldots$

$-1, 3, 7, 11, \ldots,$

s_{30}

$s_{30} = -1 + 4\,(30)$

$= -1 + 120$

$= 119$

The sequences $\{t_n\}$

Closed formula: $t_n = 7 - 3n$

Initial term: 7

Common difference: -3

The list of terms $t_0, t_1, t_2, t_3, \ldots$

$7, 4, 1, -2, \ldots.$

t_{30}

$t_{30} = 7 - 3(30)$

$= 7 - 90$

$= -83$

Summations

Summations

Sum of the terms $a_m, a_{m+1}, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{m \leq j \leq n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the ***index of summation***. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations - Example

Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.

Solution:

The lower limit for the index of summation is 1,

The upper limit is 100.

The summation notation is

$$\sum_{j=1}^{100} \frac{1}{j}.$$

Summations

More generally for a set S:

$$\sum_{j \in s} a_j$$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\sum_{s \in S} s$ = 2 + 5 + 7 + 10

Summations - Example

What is the value of $\sum_{j=1}^{5} j^2$?

Solution:

The lower limit for the index of summation is 1,

The upper limit is 5.

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= 55$$

Summations - Example

Find $\sum_{k=50}^{100} k^2$

$$\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925$$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1 - x)^2}$

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THEOREM 1 If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\boxed{\sum_{k=0}^{n} ar^k \ (r \neq 0)}$	$\boxed{\dfrac{ar^{n+1} - a}{r-1}, r \neq 1}$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

THEOREM 1 If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j$$

If $r \neq 1$

$$S_n = a + ar + ar^2 + \cdots + ar^n \qquad \text{………… (1)}$$

To compute S, first multiply both sides of the equality by r

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1} \qquad \text{………… (2)}$$

(1) – (2)

$$S_n - rS_n = (a + ar + ar^2 + \cdots + ar^n) - (ar + ar^2 + ar^3 + \cdots + ar^{n+1})$$

$$S_n(1 - r) = a - ar^{n+1}$$

$$S_n = \frac{a(1 - r^{n+1})}{1 - r}$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$

If $r = 1$

$$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a\,1^j$$

$$S_n = a + a + a + \cdots + a \quad \text{(n+ 1 terms)}$$

$$S_n = (n + 1)a$$

Set Cardinality

Section Summary

✓ Cardinality

✓ Countable Sets

Introduction

Set

- **Finite set**
  - eg: P = { 0, 3, 6, 9}
  - **Cardinality of Finite Set**
    the number of elements in a set
    eg:
    |P| = 4
- **Infinite set**
  - eg: W= {0, 1, 2, 3, 4,...}
  - **Cardinality of Infinite Set**
    one-to-one correspondence with the set of positive integers.
  - **Countably infinite sets**
    eg: the set of odd positive integers is a countable set
  - **Uncountably infinite sets**
    eg: the set of real numbers between 0 and 1

1	2	3	4	5	6	7	8	9	10	11	12	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1	3	5	7	9	11	13	15	17	19	21	23	...

Introduction

The cardinality of a set A is **equal** to the cardinality of a set B.

If and only if there is a ***one-to-one correspondence*** *(i.e., a* *<u>bijection</u>) from A to B.*

$$|A| = |B|$$

The cardinality of A is **less than** or the **same** as the cardinality of B .

If there is a ***one-to-one function*** *(i.e., an* *<u>injection</u>) from A to B*

$$|A| \leq |B|$$

The cardinality of A is less than the cardinality of *B.*

$$|A| < |B|$$

Cardinality

Definition:

A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called ***countable***. A set that is not countable is ***uncountable***.

The set of real numbers **R** is an **uncountable** set.

When an **infinite** set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Countably infinite sets and Uncountably infinite sets - Examples

- The sets N (natural numbers), Z (integers), and Q (rational numbers) are countable.
- The set R (real numbers) is uncountable.
- The closed interval (eg[0,1]) is uncountable set.
- Any subset of a countable set is countable.
- Any superset of an uncountable set is uncountable.
- If A and B are countable then their cartesian product A X B is also countable.

Showing that a Set is Countable

An infinite set is countable if and only if it is **possible to list the elements** of the set in a sequence (indexed by the positive integers).

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}$$

Then *f* is a **bijection** from **N** to *E* since *f* is both one-to-one and onto.

Showing that a Set is Countable

Example 2: Show that the set of integers **Z** is countable.

Solution:

Can list in a sequence:

0, 1, − 1, 2, − 2, 3, − 3 ,……….

$$\begin{array}{ccccccccc} \{0 & 1 & -1 & 2 & -2 & 3 & -3 & \dots & \} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ \{1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots & \} \end{array}$$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

Cardinality of Sets

- The cardinality of a singleton set is 1.
- The cardinality of the empty set is 0.
- The cardinality of a finite set is n
- The cardinality of a power set is 2^n
- The cardinality of cartesian product is m x n
- The cardinality of a countably infinite set is $\aleph_0$
- The cardinality of a uncountably infinite set is $> \aleph_0$

Cardinality of Sets
Q & A

Determine whether each of these sets is **finite**, **countably infinite**, or **uncountable**. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) The negative integers Countably infinite, −1, −2, −3, −4, …

b) The even integers Countably infinite, 0, 2, −2, 4, −4, …

c) The integers less than 100 Countably infinite, 99, 98, 97, …

d) The real numbers between 0 and ½ Uncountable

e) The positive integers less than 1,000,000,000 Finite

f) The integers that are multiples of 7 Countably infinite, 0, 7, −7, 14, −14,…

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Cardinality of Sets
Q & A

1. Find the cardinality of set $A = \{ x^2 \mid x < 6 \text{ and } x \in Z^+\}$

    $A = \{ x^2 \mid x < 6 \text{ and } x \in Z^+\}$

    $A = \{ 1, 4, 9, 16, 25\}$

    $n = |A| = 5$

2. What is the cardinality of the power set of the set C={0,1,2}?

    $C = \{0,1,2\}$

    $n = |C| = 3$

    The cardinality of a power set is 2^n

    $|P(C)| = 2^n$

    $= 2^3$

    $= 8$

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Cardinality of Sets
Q & A

3. Match the following sets with their cardinality

B = { ……… -1, 0, 1, 2, ……….}	$> \aleph_0$
[0, 1]	4
A = {1, 2, 5, 9}	$\aleph_0$

4. Given two sets C = {1,2,6} and D = {8,3}. Find the cartesian product $C \times D$ and $|C \times D|$.

$C \times D = \{(1,8),(1,3),(2,8),(2,3),(6,8),(6,3)\}$

$m = |C| = 3$

$n = |D| = 2$

$|C \times D| = m \times n = 3 \times 2 = 6$

Matrices

Section Summary

- ✓ Definition of a Matrix
- ✓ Matrix Arithmetic
- ✓ Transposes and Powers of Arithmetic
- ✓ Zero-One matrices

Matrix

Definition: A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Notation

Let m and n be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The ith row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The jth column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

The (i,j)th *element* or *entry* of **A** is the element a_{ij}. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Matrix

Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$A = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix}$$

A = B

Because both the matrices have the same size (order) and each corresponding element is equal.

$$C = \begin{bmatrix} 3 & 9 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

C ≠ D

because both the matrices do not have the same size (order). The order of C is 1 × 2 while the order of D is 2 × 1

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Matrix Q & A

1. It is given that three matrices,

$P = [3 \;\; -7 \;\; 9]$

$Q = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 5 & -2 & 0 \\ 3 & 7 & 2 & 8 \\ -4 & 11 & 6 & 1 \\ 9 & 3 & -1 & 5 \end{bmatrix}$

Determine

(a) the size of each matrix,

The size of matrix P is 1×3

The size of matrix Q is 2×1

The size of matrix R is 4×4

(b) the element

(i) at the first row and 3rd column of matrix P, p_{13}, 9

(ii) at the 2nd row and first column of matrix Q, q_{21}, 5

(iii) at the 3rd row and 4th column of matrix R, r_{34}. 1

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Matrix Q & A

2. Match the following

a. $$\begin{bmatrix} 1 & 5 & -2 & 0 \\ 3 & 7 & 2 & 8 \\ -4 & 11 & 6 & 1 \\ 9 & 3 & -1 & 5 \end{bmatrix}$$

b. $$\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & 5 & -2 & 0 \\ 3 & 7 & 2 & 8 \end{bmatrix}$$

d. $[3 \ -7 \ 9]$

Rectangular matrix

Square matrix

Row matrix

Column matrix

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Matrix Q & A

3. $P = \begin{bmatrix} x & 7 \\ 0 & 5-3z \end{bmatrix}$ $\quad Q = \begin{bmatrix} 5 & y+1 \\ 0 & 2z \end{bmatrix}$

Determine the values of *x*, *y* and *z* if *P* = *Q*.

P = Q, hence all the corresponding elements are equal.

$x = 5$

$7 = y + 1$

$y + 1 = 7$

$-$

$y = 7 - 1$

$y = 6$

$5 - 3z = 2z$

$2z = 5 - 3z$

$+$

$5z = 5$

x

$z = 1$

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Matrix Arithmetic: Addition

Definition: Let $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ be $m \times n$ matrices. The sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} + \boldsymbol{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, $\boldsymbol{A} + \boldsymbol{B} = [a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes can not be added!

Matrix Arithmetic: Addition

$$C = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix} \qquad D = \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$$

Calculate $C + D$

$$C + D = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix} + \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 + 14 & -8 + (-2) & 4 + 1 \\ 6 + (-3) & -11 + 5 & 7 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -10 & 5 \\ 3 & -6 & 16 \end{bmatrix}$$

Matrix Arithmetic: Multiplication

Definition: Let **A** be an $m \times k$ matrix and **B** be a $k \times n$ matrix. The *product* of **A** and **B**, denoted by **AB**, is the $m \times n$ matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from the *i*th row of **A** and the *j*th column of **B**. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \text{x} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is **undefined** when the **number of columns in the first matrix is not the same as the number of rows in the second**.

Matrix Arithmetic: Multiplication

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (2)(6)+(3)(-2) & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix} = \begin{bmatrix} 6 & \\ & \end{bmatrix}$$

The element at row 1 and column 1

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & (2)(-7)+(3)(1) \\ \boxed{} & \boxed{} \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ & \end{bmatrix}$$

The element at row 1 and column 2

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ (1)(6)+(5)(-2) & \boxed{} \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & \end{bmatrix}$$

The element at row 2 and column 1

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & (1)(-7)+(5)(1) \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & -2 \end{bmatrix}$$

The element at row 2 and column 2

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Matrix Arithmetic: Multiplication

Q & A

It is given that matrix $E = \begin{bmatrix} 5 & -1 & 0 \\ -4 & 8 & 7 \end{bmatrix}$, matrix $F = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ and matrix $G = [4 \quad 3]$. Calculate

(a) GE (b) FG (c) GF

(a)
$$GE = [4 \quad 3]_{1\times 2}\begin{bmatrix} 5 & -1 & 0 \\ -4 & 8 & 7 \end{bmatrix}_{2\times 3}$$
$$= [4(5) + 3(-4) \quad 4(-1) + 3(8) \quad 4(0) + 3(7)]$$
$$= [8 \quad 20 \quad 21]_{1\times 3}$$

(b)
$$FG = \begin{bmatrix} -2 \\ 7 \end{bmatrix}_{2\times 1}[4 \quad 3]_{1\times 2}$$
$$= \begin{bmatrix} (-2)(4) & (-2)(3) \\ 7(4) & 7(3) \end{bmatrix}$$
$$= \begin{bmatrix} -8 & -6 \\ 28 & 21 \end{bmatrix}_{2\times 2}$$

(c)
$$GF = [4 \quad 3]_{1\times 2}\begin{bmatrix} -2 \\ 7 \end{bmatrix}_{2\times 1}$$
$$= [4(-2) + 3(7)]$$
$$= [13]_{1\times 1}$$

Matrix Multiplication is not Commutative

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Does **AB = BA**?

Solution:

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ $BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$

AB ≠ BA

Identity Matrix and Powers of Matrices

Definition: The *identity matrix of order n* is the $m \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathrm{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

when **A** is an $m \times n$ matrix

$$a \times 1 = a$$
$$1 \times a = a$$

Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathrm{A}^0 = \mathrm{I}_n \qquad \mathrm{A}^r = \underbrace{\mathrm{AAA} \cdots \mathrm{A}}_{\text{r times}}$$

Identity Matrix
Q & A

Write the identity matrix based on the order given below.

(a) 1×1 (b) 2×2

(c) 4×4 (d) 5×5

Solution:

(a) $[1]$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Transposes of Matrices[1]

Definition: Let $\boldsymbol{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^t$,is the $n \times m$ matrix obtained by interchanging the rows and columns of $\boldsymbol{A}$.

If $\boldsymbol{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$.

The **transpose** of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Transposes of Matrices[2]

Definition: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \le i \le n$ and $1 \le j \le n$.

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is **square.**

Zero-One Matrices[1]

Definition: A matrix all of whose entries are either **0 or 1** is called a ***zero-one matrix***.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Algorithms operating on discrete structures represented by zero-one matrices are based on **Boolean arithmetic** defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Zero-One Matrices[2]

Definition: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be an $m \times n$ zero-one matrices.

- The ***meet*** of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by $\mathbf{A} \wedge \mathbf{B}$.
- The ***join*** of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by $\mathbf{A} \vee \mathbf{B}$.

x	y	$x \vee y$	$x \wedge y$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Meets and Joins of Zero-One Matrices

Example: Find the meet and join of the zero-one matrices.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

x	y	$x \vee y$	$x \wedge y$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Solution: The meet of **A** and **B** is:

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The join of **A** and **B** is:

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean Product of Zero-One Matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of **A** and **B**, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ zero-one matrix with (i,j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

Example: Find the Boolean product of **A** and **B**, where;

$$\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathrm{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

x	y	$x \vee y$	$x \wedge y$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Solution: The Boolean product **A** $\odot$ **B** is given by;

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Powers of Zero-One Matrices

Definition: Let **A** be a square zero-one matrix and let r be a positive integer.

The rth Boolean power of **A** is the <u>**Boolean product**</u> of ***r* factors** of **A**, denoted by $\mathbf{A}^{[r]}$. Hence, $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

$$A^{[r]} = \underbrace{A \odot A \odot \ldots \odot A}_{\text{r times}}.$$

We define $\mathbf{A}^{[r]}$ to be $\mathbf{I}_n$.

(The Boolean product is well defined because the Boolean product of matrices is associative.)

Boolean Powers of Zero-One Matrices

Example: Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^n$ for all positive integers *n*.

Solution:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$A^{[n]} = A^5$ for all positive integers n with $n \geq 5$

Thank you