UNIVERSITI SAINS MALAYSIA

CKC111 Discrete Structures

Advanced Counting Techniques

Summary

1: Applications of Recurrence Relations

2: Solving Linear Recurrence Relations

- Homogeneous Recurrence Relations
- Nonhomogeneous Recurrence Relations

3: Generating Functions

4: Inclusion-Exclusion

Applications of Recurrence Relations

Section Summary

Applications of Recurrence Relations

- Fibonacci Numbers
- Counting Problems

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots$ and suppose that $a_0 = 2$.

What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = 5 + 3 = 8$

$a_3 = 8 + 3 = 11$

Rabbits and the Fibonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

Find a **recurrence relation** for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

Rabbits and the Fibonacci Numbers

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers 3

Solution: Let f_n be the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.

The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

Counting Bit Strings

Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

Now assume that $n \geq 3$.

The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length $n - 1$ with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.

The bit strings of length n ending with 0 without two consecutive 0s are the bit
strings of length $n - 2$ with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.

We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Number of bit strings of length n with no consecutive 0s

End with a 1: Any bit string of length n-1 with no two consecutive 0s | **1** | a_{n-1}

End with a 0: Any bit string of length n-2 with no two consecutive 0s | **1 0** | a_{n-2}

Total= $\boldsymbol{a_n = a_{n-1} + a_{n-2}}$

Bit Strings 3

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

a1	a2	a3	a4	
0	00	000	0000	1101
1	01	001	0001	1110
	10	010	0011	1111
	11	011	0010	=8
		100	0100	
=2	=3	101	0101	
		110	0110	
		111	0111	
		=5	1000	
			1001	
			1010	
			1011	
			1100	

Exercise 1

How many bit string of length seven contain no two consecutive 0s?

Solving Linear Recurrence Relations

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

- it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

Examples of Linear Homogeneous Recurrence Relations

$P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one

$f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two

$a_n = a_{n-1} + a^2_{n-2}$ not linear

$H_n = 2H_{n-1} + 1$ not homogeneous

$B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

The basic approach is to look for solutions of the form

$a_n = r^n$, where r is a constant.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r_{n-k}.$$

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.

The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation.

The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $(r^2 - c_1)(\,r - c_2\,) = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation

if and only if $a_n = c_1 a_{n-1} + \ c_2 a_{n-2}$

$$\boxed{a_n^{(h)} = c_1 \cdot r_1^n + c_2 \cdot r_2^n}$$

for $n = 0,1,2,\ldots$, where c_1 and c_2 are constants.

Example 1

What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:

1. Characteristic equation for this recurrent relation

$$a_n = r^n \longrightarrow r^n = r^{n-1} + 2r^{n-2}$$

is $r^2 - r - 2 = 0$

2. Factor the quadratic

$$(r-2)(r+1) = 0$$

3. The possible value (called roots) $r_1 = 2$ and $r_2 = -1$

4. The general solution to the homogeneous part of the recurrence relation is then given by $a_n = c_1 2^n + c_2(-1)^n$,

$$a_n^{(h)} = c_1 \cdot r_1^n + c_2 \cdot r_2^n$$

5. where c_1 and c_2 are constants that can be determined using the initial conditions $a_0 = 2$ and $a_1 = 7$?

6. For n=0

$$a_0 = c_1 2^0 + c_2(-1)^0 \qquad a_1 = c_1 2^1 + c_2(-1)^1$$

$$= c_1 + c_2 \qquad 2c_1 - c_2$$

$$= 2 \qquad = 7$$

7. Solving these equations, we find that $c_1 = 3$ and $c_2 = -1$.

8. Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3\cdot 2^n - 1.\ (-1)^n$

Exercise 2

Solve this recurrence relation with initial condition given. $n \geq 2$

$$a_n = 5a_{n-1} - 6a_{n-2}$$

For $n \geq 2,\ a_0 = 1, a_1 = 0$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 1

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k} + F(n),$$

$$\boldsymbol{a_n = a_n^{(h)} + a_n^{(p)}}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Example 3

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution:

The associated linear homogeneous equation is $a_n - 3a_{n-1} = 0$

1. Character equation r-3 = 0

   root= 3

2. Its Homogeneous solutions are $a_n^{(h)} = \alpha \cdot 3^n$, where α is a constant.

3. Because $F(n) = 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n,

$p_n = cn + d$, is a solution where c and d are constants.

4. Then $a_n = 3a_{n-1} + 2n$

a_n - $3a_{n-1}$ -$2n$ = 0

5. becomes $cn + d - 3(c(n-1) + d) - 2n= 0$

6. Simplifying yields cn + d – 3cn + 3c – 3d - 2n = 0

-2cn – 2n – 2d + 3c = 0

-2n(c+1) + (-2d+ 3c) = 0

C+1= 0 and $-2d + 3c = 0$.

7. Therefore, $cn + d$ is a solution with c = − 1 and d = − 3/2.

8. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

9. solutions are of the form $a_n = + a_n^{(h)} + a_n^{(p)} = \alpha.3^n - n - 3/2$, where α is a constant.

What is the solution with $a_1 = 3$?

$a_n = \alpha.3^n - n - 3/2$

$a_1 = \alpha.3^n - n - 3/2$

$3 = \alpha.3^n - n - 3/2$

$3 = \alpha.3^{(1)} - (1) - 3/2$

$3 = 3\alpha - 5/2$

$\alpha = 11/6$

$a_n = (11/6)3^n - n - 3/2$.

Exercise 3

Find the solution of this recurrence relation $a_n = 2a_{n-1} + n + 5$ with a0 = 4.

a) Determine values of the constants c and d such that $a_n = cn + d$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.

b) Find all solutions of this recurrence relation.

c) Find the solution of this a_0 =4

Generating Functions

Generating Functions for Finite Sequences

Example: What is the generating function for the sequence 1,1,1,1,1,1?

Solution:

The generating function of 1,1,1,1,1,1 is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

Example

Questions : How many ways to pick up n shirt from 4 shirts?

0 shirt = 1 way

1 shirt = 4 ways

2 shirts = 6 ways

3 shirts = 4 ways

4 shirts = 1 way

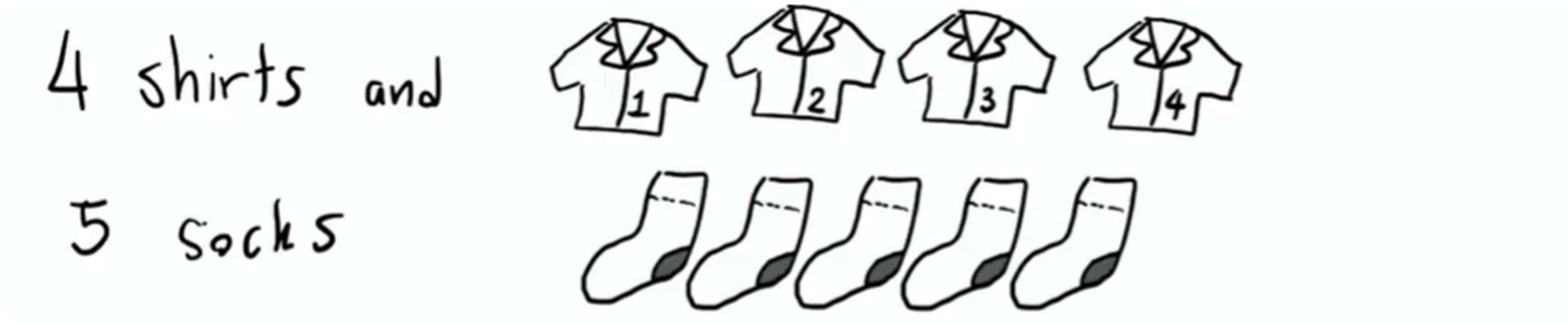
5 shirts = 0 way

So, the sequences is 1,4,6,4,1,0

n	0	1	2	3	4	5
	1	4	6	4	1	0

$$= 1 + 4x + 6x^2 + 4x^3 + 1x^4 + 0x^5$$

Exercise 4



What is the number of ways to pick either n shirts or n identical socks?

Generating Functions

Definition: The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Examples:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k$.
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function $\sum_{k=0}^{\infty} (k+1) x^k$.
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function $\sum_{k=0}^{\infty} 2^k x^k$.

Inclusion-Exclusion

Principle of Inclusion-Exclusion

In previous section we developed the following formula for the number of elements in the union of two finite sets:

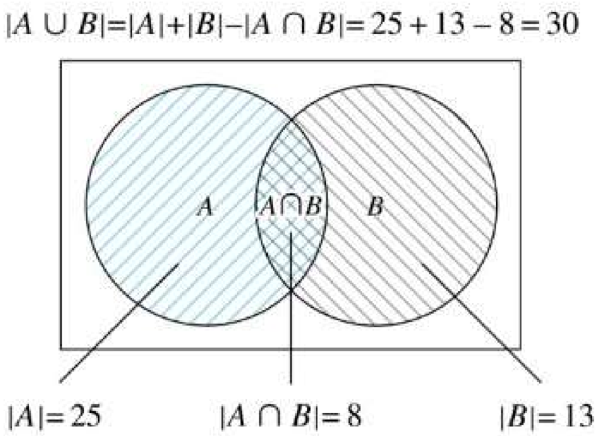
$$|A \cup B| = |A| + |B| - |A \cap B|$$

We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

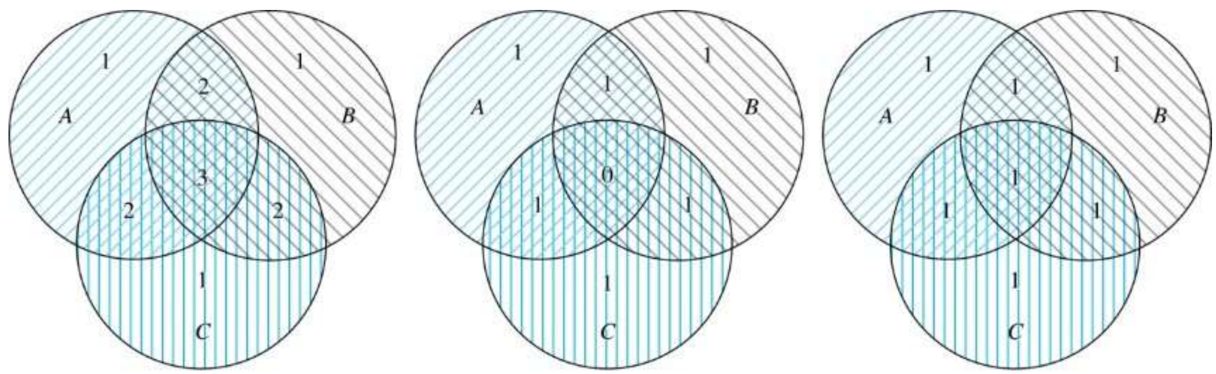
Solution:
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 25 + 13 - 8 = 30$$



Jump to long description

Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A| + |B| + |C|$

(b) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Jump to long description

Three Finite Sets

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 23$.

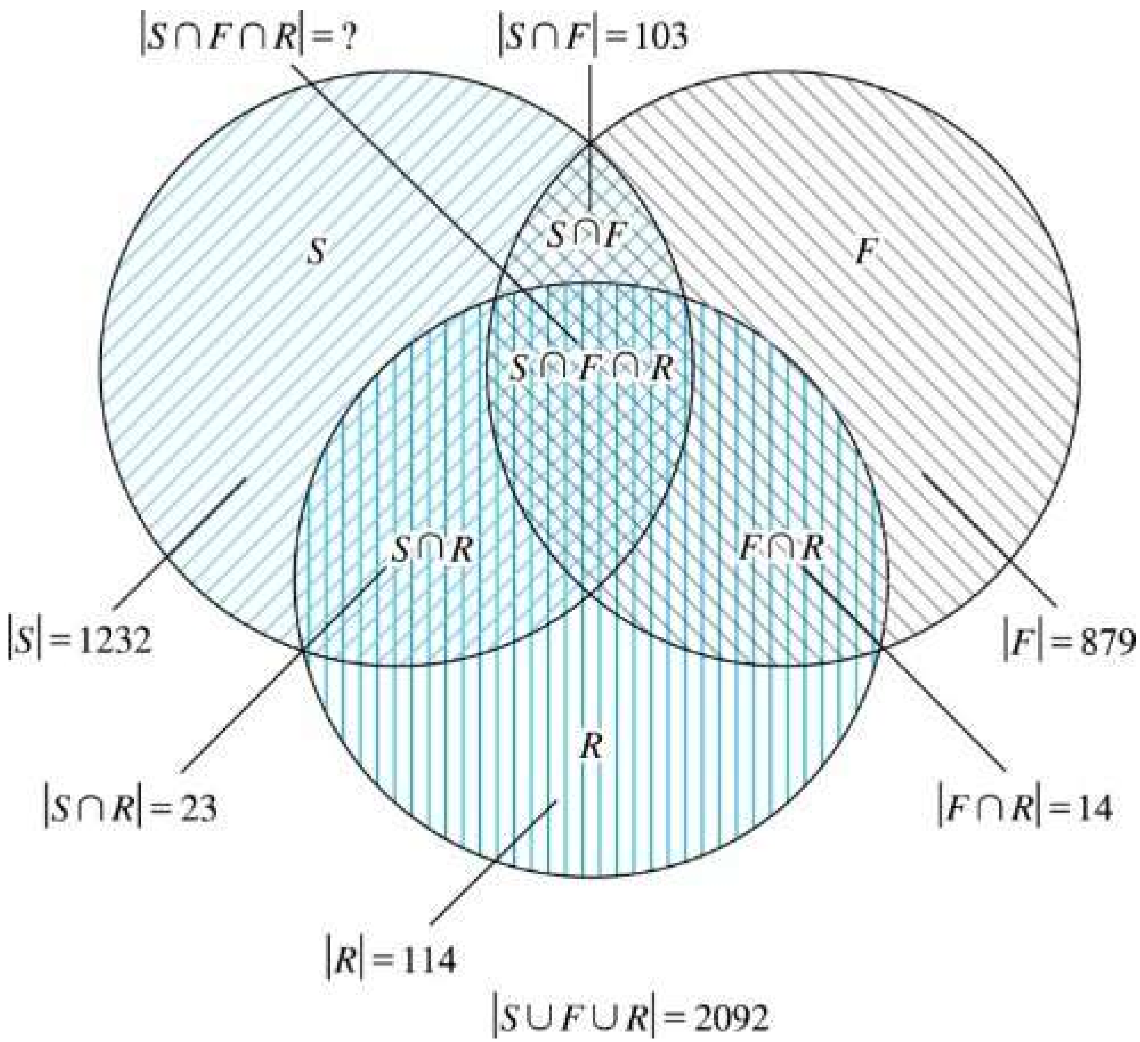
Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$,

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

Illustration of Three Finite Set Example



Exercise 5

There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses Java and C. If 189 of these student have taken course in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three subjects.

Thank you