Relations

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Chapter Summary

Section 1: Relations and Their Properties

Section 2: Representing Relations

Section 3: Equivalence Relations

Relations and Their Properties

Section 1

Section Summary

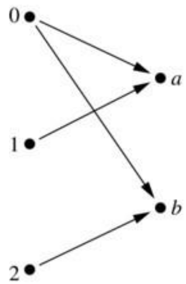
- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Jump to long description

Binary Relations on a Set[1]

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Reflexive

- **Definition:** R is **reflexive** iff $(a, a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if:

$$\forall x\left[\mathrm{x} \in U \rightarrow (x, x) \in R\right]$$

- **Example:**

  The relation $R = \{(a, a), (b, b)\}$ on set $X = \{a, b\}$ is **reflexive**.

Exercise 1

Example:

Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

a) $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$
b) $R = \{(1, 2), (2, 3), (3, 1)\}$
c) $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
d) $R = \{(1, 1), (2, 2), (3, 3)\}$

Symmetric

- **Definition:** R is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $(a, b) \in A$. Written symbolically, R is symmetric if and only if:

$$\forall x \forall y \left[\left(x, y \right) \in R \to \left(y, x \right) \in R \right]$$

- **Example:**

  The relation $R = \{(1,2), (2,1), (3,2), (2,3)\}$ on set $A = \{1,2,3\}$ is **symmetric**.

Exercise 2

Example:

Are the following relations on $\{1, 2, 3, 4\}$ symmetric?

a) $R = \{(1, 1), (1, 2), (2, 3), (3, 4), (4, 4)\}$

b) $R = \{(1, 2), (2, 2), (3, 1)\}$

c) $R = \{(1, 2), (2, 1), (2, 3), (3, 2), (4, 4)\}$

Antisymmetric

- **Definition:** A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$ is called **antisymmetric**. Written symbolically, R is antisymmetric if and only if:

$$\forall x \forall y \left[\left(x, y \right) \in R \wedge \left(y, x \right) \in R \rightarrow x = y \right]$$

- **Example:**

The relation $R = \{(x, y) \rightarrow N | x \leq y\}$ is **antisymmetric** since $x \leq y$ and $y \leq x$ implies $x = y$.

Exercise 3

Example:

Are the following relations on $\{1, 2, 3\}$ antisymmetric?

a) $R = \{(1, 1), (1, 2), (2, 3)\}$

b) $R = \{(1, 3), (1, 2), (3, 1)\}$

c) $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}$

Transitive

- **Definition:** A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for all $a, b, c \in A$. Written symbolically, R is transitive if and only if:

$$\forall x \forall y \forall z \left[\left(x, y\right) \in R \wedge \left(y, z\right) \in R \rightarrow \left(x, z\right) \in R \right]$$

- **Example:**

The relation $R = \{\ (1,2)\ ,(2,3)\ ,(1,3)\}$ on set $A = \{1,2,3\}$ is **transitive**.

Exercise 4

Example:

Are the following relations on $\{1, 2, 3, 4\}$ transitive?

a) $R = \{(1, 1)\ ,(1, 2)\ ,(2, 3),(3,4), (4,4)\}$

b) $R = \{(1, 2), (2, 3), (3, 1)\}$

c) $R = \{(1, 1)\ , (1, 2), (1, 3)\ , (2,1), (2, 2), (2, 3), (4,4)\}$

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2)(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

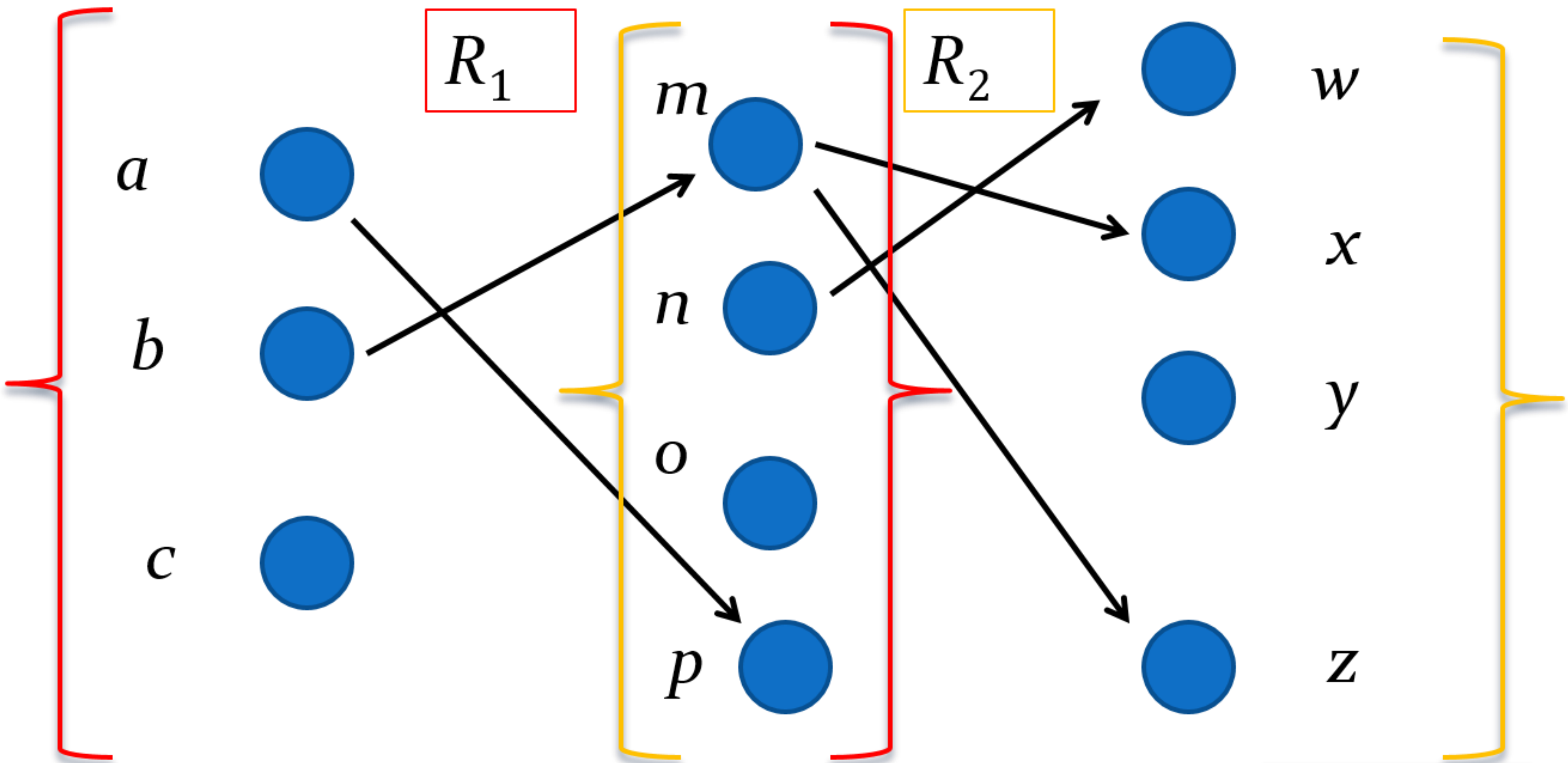
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of Relations



$$\boxed{R_2} \circ \boxed{R_1} = \{(b,x),(b,z)\}$$

Exercise 5

Given two relations R and S. R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with $S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$.

What is the composite of the relations R and S?

Representing Relations

Section 2

Section Summary

Representing Relations using Matrices

Representing Relations using Digraphs

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices [1]

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations Using Matrices 2

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Exercise 6

Represent each of these relations on {1,2,3} with a matrix.

a) {(1,1), (1,2), (1,3)}

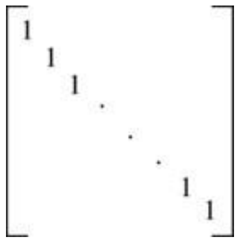
b) {(1,2), (2,1), (2,2), (2,3)}

c) {(1,1), (1,2),(1,3),(2,2),(2,3),(3,3)}

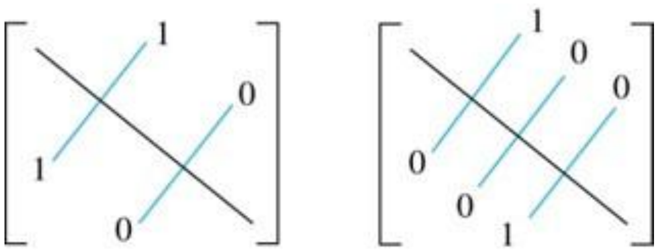
d) {(1,3),(3,1)}

Matrices of Relations on Sets

If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

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Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

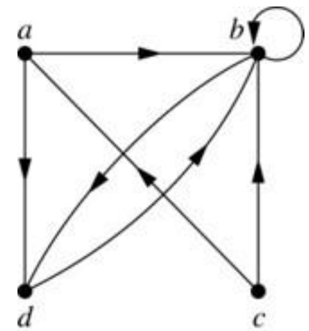
Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set *V* of *vertices* (or *nodes*) together with a set *E* of ordered pairs of elements of *V* called *edges* (or *arcs*). The vertex *a* is called the *initial vertex* of the edge (*a*,*b*), and the vertex *b* is called the *terminal vertex* of this edge.

- An edge of the form (*a*,*a*) is called a *loop*.

Example 1: A drawing of the directed graph with vertices *a*, *b*, *c*, and *d*, and edges (*a*, *b*), (*a*, *d*), (*b*, *b*), (*b*, *d*), (*c*, a), (*c, b*), and (*d*, *b*) is shown here.



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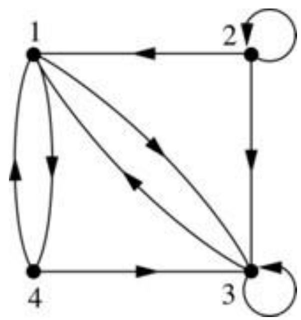
Examples of Digraphs Representing Relations

Example 2: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

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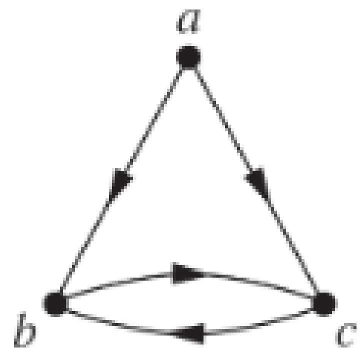
Exercise 7



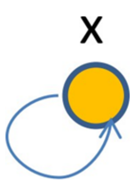
List the order paired in the relations represented by the directed graph.

Determining Properties a Relation has from its Digraph

- ***Reflexivity***: A **<u>loop</u>** must be present at all vertices in the graph.
- ***Symmetry***: If (x, y) is an edge, then so is (y, x).
- ***Antisymmetry***: If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- ***Transitivity***: If (x, y) and (y, z) are edges, then so is (x, z).

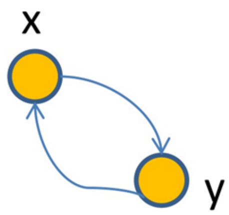
Properties a Relation Using Digraph

What do we could derive about the graphs representing a relation on a set (a relation from A to A)?

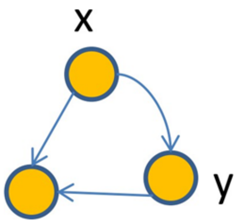


A relation is **reflexive** if for each point x, there is a loop at x:

A relation is **symmetric** if whenever there is an arrow from x to y there is also an arrow from y back to x:



A relation is **transitive** if whenever there are arrows from x to y and y to z, there is also an arrow from x to z:



Equivalence Relations

Section 3

Section Summary

Equivalence Relations

Equivalence Classes

Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Reflexive: A relation is said to be reflexive, if $(a, a) \in R$, for every $a \in A$.

Symmetric: A relation is said to be symmetric, if $(a, b) \in R$, then $(b, a) \in R$.

Transitive: A relation is said to be transitive if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation

Equivalence Classes

Let A be a nonempty set and let $\sim$ be an equivalence relation on the set A. Then,

We read $[a]$ as "the equivalence class of a

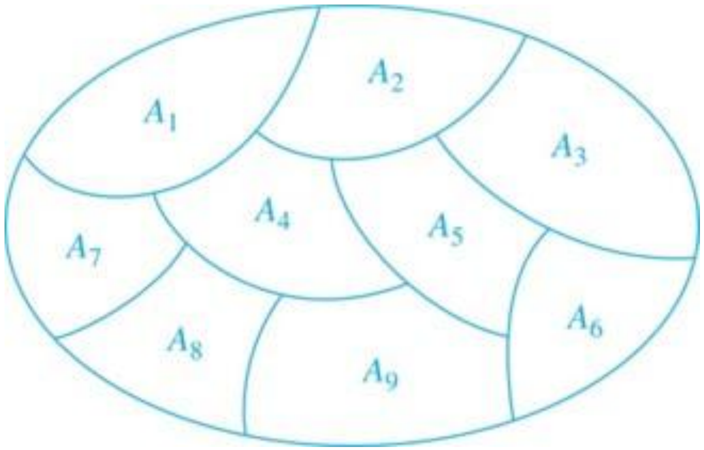
" or as "bracket a."

- ✓ For each $a \in A$, $a \in [a]$.
- ✓ For each $a, b \in A$, $a \sim b$ if and only if $[a] = [b]$,
- ✓ For each $a, b \in A$, $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

Exercise 8

Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition of S ?

a) {{m, o, c, k}, {r, u, b, s}}
b) {{c, o, m, b}, {u, s}, {r}}
c) {{b, r, o, c, k}, {m, u, s, t}}
d) {{u, m, b, r, o, c, k, s}}
e) {{b, o, o, k}, {r, u, m}, {c, s}}
f) {{u, m, b}, {r, o, c, k, s}, $\varnothing$}