Discrete Structures

(CKC111)

Counting

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Section Topics

4. Counting

4.1 Basic Principles

4.2 The Pigeonhole Principle

4.3 Permutations and Combinations

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Basic Principles

Section Summary

✓ The Product Rule

✓ The Sum Rule

✓ The Subtraction Rule

✓ The Division Rule

The Product Rule

The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$$

different possible license plates.

$$\underbrace{\_\_\ \ \_\_\ \ \_\_}_{\substack{\text{26 choices}\\ \text{for each}\\ \text{letter}}}\quad \underbrace{\_\_\ \ \_\_\ \ \_\_}_{\substack{\text{10 choices}\\ \text{for each}\\ \text{digit}}}$$

Counting Functions

Counting Functions:

How many functions are there from a set with m elements to a set with n elements?

Solution:

Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting Functions

Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to one with n elements?

Solution:

Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n - 1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set:

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

(In Chapter 5.1 of main ref., mathematical induction was used to prove this same result.)

Solution:

When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$, are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , … , and an element in A_m.

By the product rule, it follows that: $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_1| \cdots |A_m|$.

Revision Questions

1. Your favourite shop offers breakfast sets. Based on the men, choose one type of bread to complement with one type of gravy.

Menu A

- *Roti canai*
- *Roti nan*
- *Roti jala*

Menu B

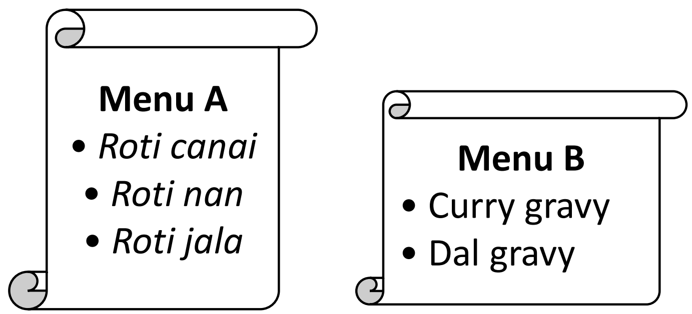
- Curry gravy
- Dal gravy

a. Specify the number of ways that can be done.

b. Determine the number of choices if the shop also includes four types of drinks into the menu.

Revision Questions

1. Your favourite shop offers breakfast sets. Based on the men, choose one type of bread to complement with one type of gravy.



a. Specify the number of ways that can be done.

3 types of *roti* × 2 types of gravy = 6 ways to choose a breakfast set

b. Determine the number of choices if the shop also includes four types of drinks into the menu.

3 types of *roti* × 2 types of gravy × 4 types of drinks = 24 ways to choose a breakfast set

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Revision Questions

2. Determine the number of ways to toss a dice and a piece of coin simultaneously.

3. Find the number of ways a person can guess a 4-digit code to access a cell phone if the digits can be repeated.

4. You need to create a 4-character code where each character can be any of the 5 letters (A, B, C, D, E). How many possible codes can be made if characters can repeat?

Revision Questions

2. Determine the number of ways to toss a dice and a piece of coin simultaneously.

The number of ways to toss a dice and a piece of coin simultaneously is
$6 \times 2 = 12.$

3. Find the number of ways a person can guess a 4-digit code to access a cell phone if the digits can be repeated.

The number of ways a person can guess the 4-digit code to access a cell phone is
$10 \times 10 \times 10 \times 10 = 10\,000.$

4. You need to create a 4-character code where each character can be any of the 5 letters (A, B, C, D, E). How many possible codes can be made if characters can repeat?

Each of the 4 positions has 5 possible choices, and characters can repeat.
$5 \times 5 \times 5 \times 5 = 625$

The Sum Rule

The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are $\mathbf{37}$ members of the mathematics faculty and $\mathbf{83}$ mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The Sum Rule in Terms of Sets

The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \varnothing \text{ for all } i, j.$$

The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Combining the Sum and Product Rule

Example:

Suppose statement labels in a programming language can be either a single letter **or** a letter followed by a digit. Find the number of possible labels.

Solution:

Use the sum and product rule.

$$26 + 26 \cdot 10 = 286$$

Revision Questions

1. A restaurant offers 3 types of cakes and 2 types of pies. If a customer can order either a cake or a pie, how many dessert choices are available?

2. A pet store has 6 types of dogs and 4 types of cats. If you want to pick either a dog or a cat, how many choices do you have?

Revision Questions

1. A restaurant offers 3 types of cakes and 2 types of pies. If a customer can order either a cake or a pie, how many dessert choices are available?

Since a customer can order either a cake or a pie, but not both at the same time, we use the sum rule:

$$3 + 2 = 5$$

The customer has 5 dessert choices.

2. A pet store has 6 types of dogs and 4 types of cats. If you want to pick either a dog or a cat, how many choices do you have?

Since you can choose either a dog or a cat (not both), use the sum rule:

$$6 + 4 = 10$$

There are 10 different choices if you want either a dog or a cat.

The Subtraction Rule

The Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$, minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of* ***inclusion-exclusion***:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

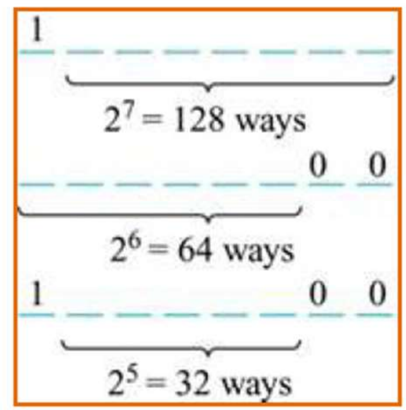
Counting Bit Strings

<u>Example</u>: How many bit strings of **length eight** either <u>start with a 1 bit</u> **or** <u>end with the two bits 00</u>?

<u>Solution</u>: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit

  and end with bits 00: $2^5 = 32$



Hence, the number is 128 + 64 − 32 = 160.

Revision Questions

1. In a class of 20 students, 5 are already occupied with another project. If you need to select any student from the class who is not occupied, how many students are available?

Revision Questions

1. In a class of 20 students, 5 are already occupied with another project. If you need to select any student from the class who is not occupied, how many students are available?

1. The total number of students = 20.
2. The number of occupied students = 5.
3. The number of available students. Using the subtraction rule:

$$20 - 5 = 15$$

So, 15 students are available for the task.

Revision Questions

2. You want to create a 3-digit code using the numbers 0–9, but codes starting with a 0 are not allowed. How many valid 3-digit codes can you form?

Revision Questions

2. You want to create a 3-digit code using the numbers 0–9, but codes starting with a 0 are not allowed. How many valid 3-digit codes can you form?

1. The total number of 3-digit codes. Each digit has 10 options, so:

$$10 \times 10 \times 10 = 1000$$

2. The number of codes that start with 0.

$$1 \times 10 \times 10 = 100$$

3. The number of codes that start without 0. Using the subtraction rule:

$$1000 - 100 = 900$$

So, there are 900 valid 3-digit codes that do not start with 0.

Revision Questions

3. In a school, there are 40 students in the drama club and 30 students in the debate club. If 15 students are in both clubs, how many unique students are in either the drama or the debate club?

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Revision Questions

3. In a school, there are 40 students in the drama club and 30 students in the debate club. If 15 students are in both clubs, how many unique students are in either the drama or the debate club?

Number of students in the drama club = $|A| = 40$
Number of students in the debate club = $|B| = 30$
Number of students in both clubs = $|A \cap B| = 15$

The unique students in either the drama or the debate club

$$\begin{aligned} |\boldsymbol{A} \cup \boldsymbol{B}| &= |A| + |B| - |A \cap B| \\ &= 40 + 30 - 15 \\ &= 55 \end{aligned}$$

The Division Rule

The Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.

In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

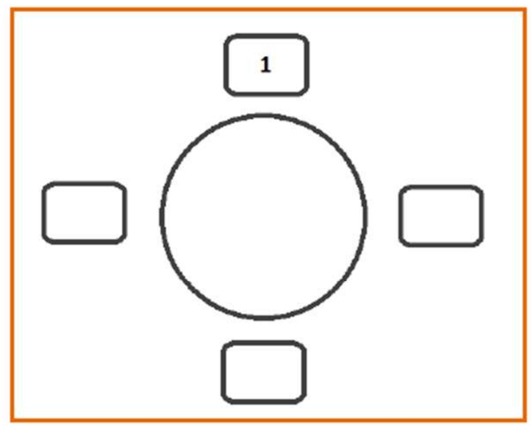
The Division Rule

Example: How many ways are there to seat four people around a circular table, where two seating are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are 4 ways to select the person for seat 1, 3 for seat 2, 2 for seat 3, and 1 way for seat 4. Thus, there are $4! = 24$ ways to order the four people.

But since two seating are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.



Revision Questions

1. Determine the number of ways to arrange six pupils to sit at a round table.

Revision Questions

1. Determine the number of ways to arrange six pupils to sit at a round table.

The number of pupils = 6

The number of ways to arrange the six pupils $= (6 \times 5 \times 4 \times 3 \times 2 \times 1) / 6$

$= 120$

Tree Diagrams

Tree Diagrams:

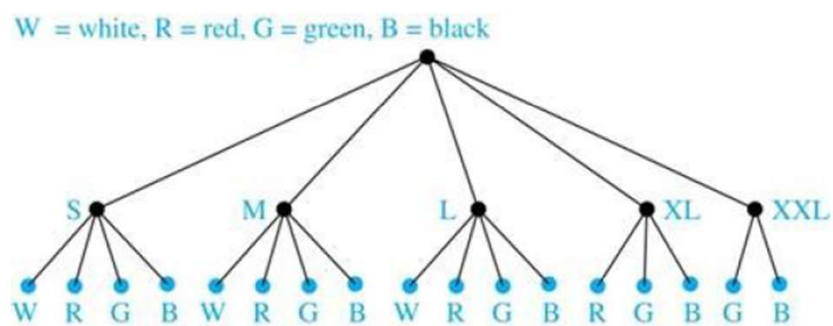
We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice, and the leaves represent possible outcomes.

Example:

Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus bookstore needs to stock to have one of each size and color available?

Solution:

Draw the tree diagram.
The store must stock 17 T-shirts.



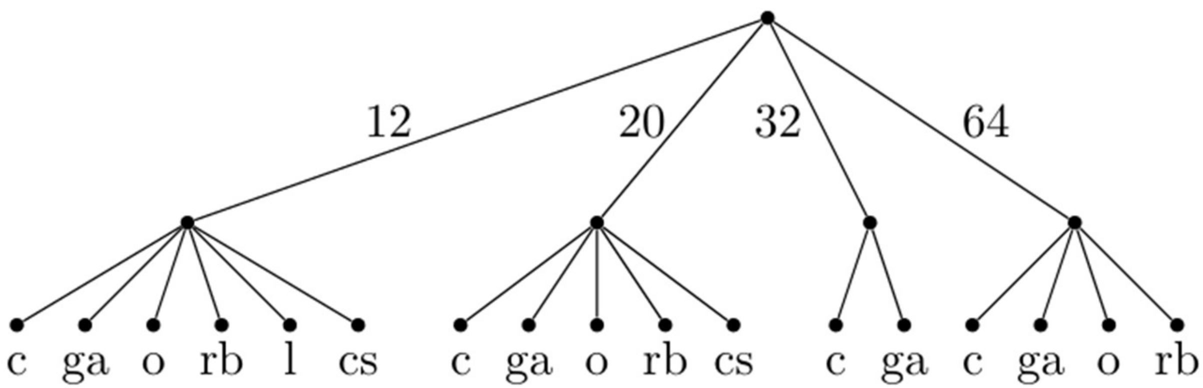
Exercises

a. Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles, the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64- ounce bottles?

b. Answer the question a. using counting rules

Exercises

a. Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles, the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64- ounce bottles?



The number of leaves in the tree is 17, which is the answer.

b. Answer the question a. using counting rules

The 12-ounce bottle has 6,
the 20-ounce bottle has 5,
the 32-once bottle has 2, and
the 64-ounce bottle has 4.
Therefore 6+5+2+4 = 17 different types of bottles need to be stocked

The Pigeonhole Principle

Section Summary

✓ The Pigeonhole Principle

✓ The Generalized Pigeonhole Principle

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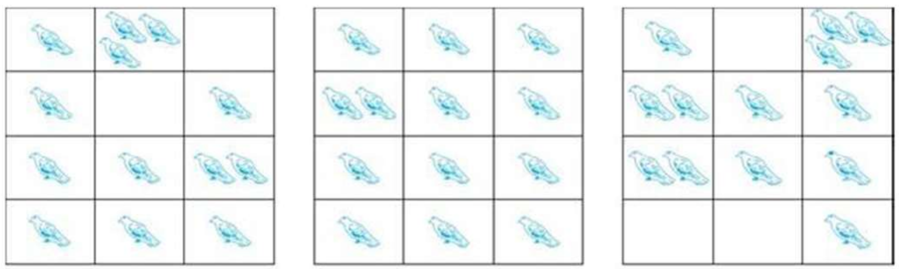
The Pigeonhole Principle

The Pigeonhole Principle

If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon. Figure illustrates the situation where we have more pigeons than pigeonholes.



(a) (b) (c)

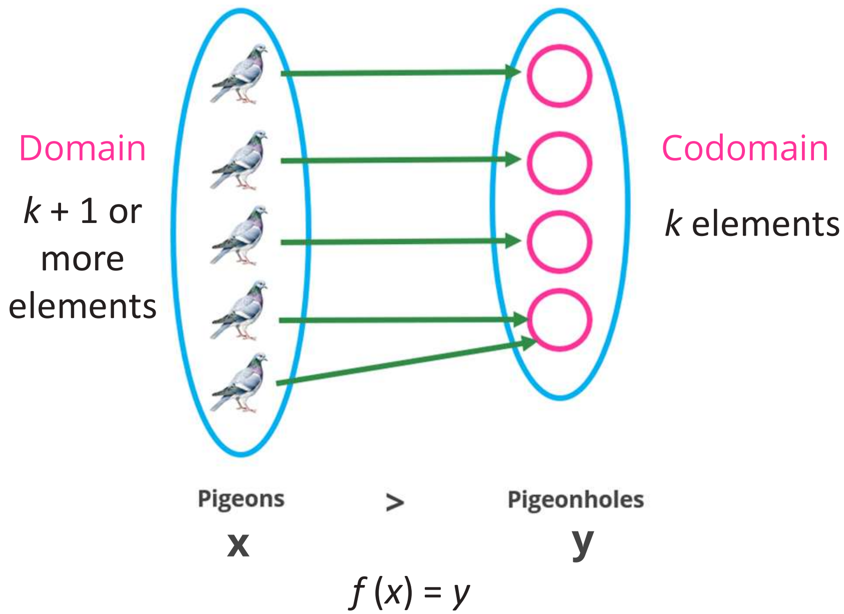
Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

Corollary

A function *f* from a set with $k + 1$ or more elements to a set with k elements is **not one-to-one**.

Example:



Proof:

Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that

$$f(x) = y.$$

Because
the domain contains $k + 1$ or more elements and
the codomain contains only k elements,
the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain.
This means that f cannot be one-to-one.

The pigeonhole principle

Example 1:

How do we know that in a crowd of 367 people, there will be at least two people with the same birthday?

Solution:

Example 2:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution:

The pigeonhole principle

Example 1:

How do we know that in a crowd of 367 people, there will be at least two people with the same birthday?

Solution:

367 people (pigeons)

366 possible days in a year, including leap year (pigeonholes),

There must be at least two people with the same birthday because there are more people than possible birthdates.

Example 2:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution:

There are 101 possible scores on the final.

The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle:

If there are N pigeons to be mapped to k pigeonholes, then there is at least one pigeonhole with at least $\lceil N/k \rceil$ pigeons.

Example 1:

In a class of 52 students, there will be at least 10 students with the same grade (assume the grades are only A, B, C, D, or F).

Solution:

- 52 students/pigeons
- 5 grades/pigeonholes
- Based on this principle: $\lceil N/k \rceil = \lceil 52/5 \rceil = 11$

The Generalized Pigeonhole Principle

Example 2:

Some banks open on Saturdays, except the first Saturday of the month. Show that there is/are cases where those bank employees must work 4 Saturdays in one month.

Solution:

- 52 Saturdays (pigeons) in one year
- 12 months (pigeonholes) in one year
- Based on this principle: $\lceil N/k \rceil = \lceil 52/12 \rceil = 5$. Meaning – for 5 times (i.e., months), the employees must work on 4 Saturdays because they only take leave on 1 Saturday every month.

Permutations and Combinations

Section Summary

- ✓ Permutations
- ✓ Combinations
- ✓ Generalized Permutations and Combinations

Introduction

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.
- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Introduction

For example, let us say balls 1, 2 and 3 are chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

Permutations Combinations

Permutations

Permutations

Definition: A *permutation* of a set of distinct objects is an **ordered arrangement** of these objects. An ordered arrangement of r elements of a set is called an r-*permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n, r)$.

- The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

Exercises

1. List all the permutations of {*a, b, c*}.

2. P(6, 3)
   n = ?
   r = ?
   P(6, 3) = ?

3. P(8, 5)
   n = ?
   r = ?
   P(8, 5) = ?

4. In how many ways can we select three students from a group of five students to stand in line for a picture?

   n = ?
   r = ?

   P (5 , 3) =

Exercises

1. List all the permutations of {*a, b, c*}.

   abc, acb, bac, bca, cab, cba

2. P(6, 3)

   n = ? 6

   r = ? 3

   P(6, 3) = ? 6 . 5 . 4 = 120

3. P(8, 5)

   n = ? 8

   r = ? 5

   P(8, 5) = ? 8 . 7 . 6 . 5 . 4 = 6720

4. In how many ways can we select three students from a group of five students to stand in line for a picture?

   n = ? 5

   r = ? 3

   P (5 , 3) = 5 . 4 . 3 = 60 ways

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$ r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

Note that $P(n, 0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Solving Counting Problems by Counting Permutations

Example 1: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations

Example 2: Suppose that a saleswoman has to visit **eight** different cities. She must begin her trip in a specified city, but she can visit the other **seven cities in any order** she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations

Example 3: How many permutations of the letters $ABCDEFGH$ contain the string ABC?

Solution: We solve this problem by counting the permutations of six objects, ABC, D, E, F, G, and H.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements. The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, b, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

$C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

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Proof: The $P(n, r)$ r-permutations of the set can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways. Consequently, by the product rule,

$$P(n, r) = C(n, r) \cdot P(r, r)$$

$\div$

This implies that

$$\frac{P(n, r) = C(n, r)}{P(r, r)}$$

$$C(n, r) = \frac{P(n, r)}{P(r, r)}$$

$$= \frac{n!/(n-r)!}{r!/(r-r)!}$$

$$= \frac{n!}{(n-r)!} \div \frac{r!}{(r-r)!}$$

$$= \frac{n!}{(n-r)!} \text{ x } \frac{(r-r)!}{r!}$$

$$= \frac{n!}{(n-r)!} \text{ x } \frac{0!}{r!} \qquad (0! = 1)$$

$$= \frac{n!}{(n-r)!} \text{ x } \frac{1}{r!}$$

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Exercises

1. Let $S = \{1, 2, 3, 4, 5\}$.

   List all the 3-combinations of S.

2. Find the value of each of these quantities.

   a) C(5, 1)

   b) C(8, 8)

   c) C(8, 0)

   d) C(12, 6)

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

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Exercises

1. Let $S = \{1, 2, 3, 4, 5\}$.

   List all the 3-combinations of S.

   123, 124, 125,
   134, 135,
   145,
   234, 235, 245,
   345

2. Find the value of each of these quantities.

   a) $C(5, 1) = 5$

   b) $C(8, 8) = 1$

   c) $C(8, 0) = 1$

   d) $C(12, 6) = 12.11.10.9.8.7/(6.5.4.3.2)$
   $= 924$

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960$$

COROLLARY 2 Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2 it follows that

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

$$C(n, n - r) = \frac{n!}{(n-r)!\;[n-(n-r)]!}$$

$$= \frac{n!}{(n-r)!\;(\underbrace{n-n}_{0}+r)!}$$

$$= \frac{n!}{(n-r)!\;r!}$$

$$C(n, n - r) = \frac{n!}{r!\;(n-r)!}$$

$$\boxed{\text{Hence, } C(n, r) = C(n, n-r)}$$

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Exercises

3. In how many ways can a set of two positive integers less than 100 be chosen?

kami memimpin

Exercises

3. In how many ways can a set of two positive integers less than 100 be chosen?

$C(99, 2) = 99 \,.\, 98 \,/\, 2 = 4851$

Generalized Permutations and Combinations

Generalized Permutations and Combinations

- In many counting problems, elements may be used repeatedly.
- E.g., license plate – a letter or digit can be used more than once.

- Sub-topics will be discussed:
  1. Permutations with repetition
  2. Combinations with repetition

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.		
Type	***Repetition Allowed?***	***Formula***
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition.

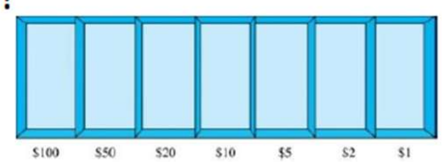
Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

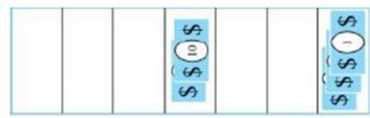
Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

Solution: Place the selected bills in the appropriate position of a cash box:



Some possible ways of placing the five bills:

The number of ways to **select five bills** corresponds to the number of ways to arrange **six bars** and **five stars** in a row.



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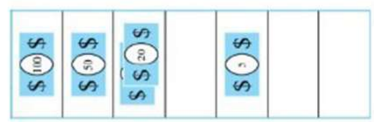

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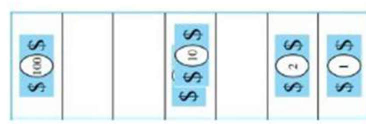

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This is the number of unordered selections of 5 objects from a set of 11.

Hence, there are $C(11,5) = \frac{11!}{5!6!} = 462$ ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is:

$$C(n+r-1, r) = C(n+r-1, n-1).$$

Combinations with Repetition

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three. By Theorem 2 it follows that there are;

$C(3 + 11 - 1,11) = C(13,11) = C(13,2) = 13 \cdot \frac{2}{1} \cdot 2 = 78$ solutions.

Theorem 2: $C(n + r - 1, r) = C(n + r - 1, n - 1)$.

Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? 

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.

By Theorem 2; $C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$ is the number of ways to choose six cookies from the four kinds.

Theorem 2: $C(n+r-1, r) = C(n+r-1, n-1)$.

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\,4!} \cdot \frac{4!}{2!\,2!} \cdot \frac{2!}{1!\,1!} \cdot \frac{1!}{3!\,2!\,1!\,1!} = 420$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

- **Proof**: By the product rule the total number of permutations is:
- $C(n, n_1)\ C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:
  - The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
  - Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
  - Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.
- The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\ \frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!}=\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Thank you