The Foundations: Logic and Proofs

Chapter Summary

1.1 Propositional Logic	• The Language of Propositions. • Applications. • Logical Equivalences.
1.2 Predicate Logic	• The Language of Quantifiers.
1.3 Proofs	• Rules of Inference. • Proof Methods. • Proof Strategy.

Section 1.1

Proposition Logic

Propositions

A *proposition* is a declarative sentence that is either true or false.

Examples of propositions:

a) The Moon is made of green cheese.

b) Toronto is the capital of Canada.

d) 1 + 0 = 1

e) Snow is white.

f) The piano is a multi stringed instrument.

g) Snow is green.

h) Oranges are orange.

i) The highest speed reached by any polar bear on 11 January 2004 was 31.35 kilometers per hour.

j) I am hungry.

Examples that are not propositions.

a) Sit down!
b) What time is it?
c) $x + 1 = 2$
d) $x + y = z$

Examples of NOT Propositions

When are you joining the institute?	**QUESTION**
Close the door	**COMMAND/ INSTRUCTION**
Stop it!	**EXCLAMATION !**
Let us go for dinner tonight	**PROPOSAL**

Propositional Logic

Constructing Propositions

Propositional Variables: $p, q, r, \ldots$

The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.

Compound Propositions; constructed from logical connectives and other propositions.

- Negation $\neg$
- Conjunction $\wedge$ (and)
- Disjunction $\vee$ (or)
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Compound Propositions: Negation

The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Example: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

The *disjunction* of propositions p and q is denoted by $p \lor q$ and has this truth table:

p	q	$P \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \lor q$ denotes "I am at home or it is raining."

The Connective Or in English

In English "or" has two distinct meanings.

- "Inclusive Or" - In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
- "Exclusive Or" - When reading the sentence "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."

In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication [1]

In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.

These implications are perfectly fine, but would not be used in ordinary English.

- "If the moon is made of green cheese, then I have more money than Bill Gates. "
- "If the moon is made of green cheese then I'm on welfare."
- "If $1 + 1 = 3$, then your grandma wears combat boots."

Understanding Implication [2]

One way to view the logical conditional is to think of an obligation or contract.

- "If I am elected, then I will lower taxes."
- "If you get 100% on the final, then you will get an A."

If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \rightarrow q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

Converse, Contrapositive, and Inverse

From $p \rightarrow q$ we can form new conditional statements.

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$.
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$.
- $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$.

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q ." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

Some alternative ways “*p* if and only if *q*” is expressed in English:

- *p* **is necessary and sufficient for** *q.*
- **if** *p* **then** *q*, **and conversely**.
- *p* **if** *q.*

Truth Tables For Compound Propositions

Construction of a truth table:

Rows.

Need a row for every possible combination of values for the atomic propositions.

Columns.

Need a column for the compound proposition (usually at far right).

Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

- This includes the atomic propositions.

Example Truth Table

Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

Two propositions are *equivalent* if they always have the same truth value.

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

How many rows are there in a truth table with *n* propositional variables?

Solution: 2^n

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Section 1.2

Applications of Propositional Logic

Translating English Sentences

Steps to convert an English sentence to a statement in propositional logic.

- Identify atomic propositions and represent using propositional variables.
- Determine appropriate logical connectives.

"If I go to Harry's or to the country, I will not go shopping."

- p: I go to Harry's.
- q: I go to the country.
- r: I will go shopping.

If p or q then not r.

$$\left(p \vee q\right) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let a, c, and f represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \rightarrow (c \vee \neg f)$$

System Specifications

System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."

$$q \rightarrow \neg p$$

Propositional Equivalences

Section 1.3

Tautologies, Contradictions, and Contingencies

A *tautology* is a proposition which is always true.

- Example: $p \lor \neg p$.

A *contradiction* is a proposition which is always false.

- Example: $p \land \neg p$.

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences 1

Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$

Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$

Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$

Double Negation Law: $\neg(\neg p) \equiv p$

Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

Key Logical Equivalences 2

Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$

Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$

$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing New Logical Equivalences

We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs 1

Example: Show that $\neg\left(p \vee \left(\neg p \wedge q\right)\right)$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg\left(p \vee \left(\neg p \wedge q\right)\right) \equiv \neg p \wedge \neg\left(\neg p \wedge q\right)$	by the second De Morgan law
$\equiv \neg p \wedge \left[\neg\left(\neg p\right) \vee \neg q\right]$	by the first De Morgan law
$\equiv \neg p \wedge \left(p \vee \neg q\right)$	by the double negation law
$\equiv \left(\neg p \wedge p\right) \vee \left(\neg p \wedge \neg q\right)$	by the second distributive law
$\equiv F \vee \left(\neg p \wedge \neg q\right)$	because $\neg p \wedge p \equiv F$
$\equiv \left(\neg p \wedge \neg q\right) \vee F$	by the commutative law for disjunction
$\equiv \left(\neg p \wedge \neg q\right)$	By the identity law for **F**

Equivalence Proofs 2

Example: Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Solution:

$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ by truth table for $\to$

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the first De Morgan law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by associative and commutative laws laws for disjunction

$\equiv T \vee T$ by truth tables

$\equiv T$ by the domination law