1. Introduction to Matrices

- **D1** column vectors

  $\mathbb{R}^n =$ set of column vectors of height n with **entries** $a_i \in \mathbb{R}$

- **D2** zero vector

  $\mathbf{0}_n =$ column vector of height n with all entries $a_i = 0$

- **D3** standard basis vectors

  $\mathbf{e}_k =$ column vector of height n with $a_k = 1$ and all other entries $a_i = 0, i \neq k$

- **R1**

  $\forall \mathbf{v}, \mathbf{e}_k \in \mathbb{R}^n, \lambda \in \mathbb{R},$

  - $\mathbf{v} + \mathbf{0}_n = \mathbf{0}_n + \mathbf{v} = \mathbf{v}$
  - $0\mathbf{v} = \lambda\mathbf{0}_n = \mathbf{0}_n$
  - $\mathbf{v} \cdot \mathbf{0}_n = 0$
  - $\mathbf{v} \cdot \mathbf{e}_k = v_k$ (kth entry of $\mathbf{v}$)
- **D5** linear combination

  $\forall \mathbf{v}_1 \ldots \mathbf{v}_k \in \mathbb{R}^n, c_1 \ldots c_k \in \mathbb{R},$

  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ **(linear combination)**

- **D6** span

  $\forall \mathbf{v}_1 \ldots \mathbf{v}_k \in \mathbb{R}^n,$

  - $span\{\mathbf{v}_1 \ldots \mathbf{v}_k\} =$ the set of all linear combinations of $\mathbf{v}_1 \ldots \mathbf{v}_k$
  - $\forall \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \in span\{\mathbf{e}_1 \ldots \mathbf{e}_n\}$
- **D7** length/norm

  $\forall \mathbf{v} \in \mathbb{R}^n, ||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

  - $||\mathbf{0}_n|| = 0$
  - $\mathbf{v}_n \neq \mathbf{0}_n \implies ||\mathbf{v}|| > 0$
- **D8** unit vector

  $\mathbf{v} \in \mathbb{R}^n, ||\mathbf{v}|| = 1$ **(unit vector)**

- **E3**
  - $\forall \mathbf{0}_n \neq \mathbf{v} \in \mathbb{R}^n, \mathbf{u} := \frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector **(normalizing)**
  - $\mathbf{e}_k$ are unit vectors
- **D9** distance

  $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, dist(\mathbf{u}, \mathbf{v}) := ||\mathbf{u} - \mathbf{v}||$

- **D10** matrix
  - an $n \times m$ matrix has n rows and m columns
  - column vectors of height n are $n \times 1$ matrices
  - row vectors of length n Are $1 \times n$ matrices
- **D11** matrix entries
  - The (i, j) entry of a matrix is the entry in row i and column j

- $M = (a_{ij})$ is a matrix whose (i, j) entry is a_{ij}

- **D12** zero matrix

  $\mathbf{0}_{n\times m} =$ matrix with all entries $a_{ij} = 0$

- **D14** transpose
  - $\forall A = (a_{ij}), A^T = (a_{ji})$
  - Transpose = reflexion in the **leading diagonal** (the $(1, 1), (2, 2), \ldots$ entries)
  - $(A^T)^T = A$
- **D16** identity matrix
  - $I_n := (a_{ij}) \in \mathbb{R}^{n\times n} : \forall 1 \le i, j \le n, a_{ij} = \begin{cases} 0, i \ne j \\ 1, i = j \end{cases}$
  - Identity matrix = square matrix with entries on the leading diagonal 1 and the rest 0
- **D17** matrix-vector multiplication

  $\forall A = (a_{ij}) \in \mathbb{R}^{n\times m}, \mathbf{v} \in \mathbb{R}^m,$

  $A\mathbf{v} \in \mathbb{R}^n$ with kth entry $= \sum_{j=1}^{m} a_{kj}v_j = \mathbf{v} \cdot (k$th row of $A)^T$

- **L1** kth column of a matrix

  $\forall A \in \mathbb{R}^{n\times m}, \mathbf{e}_k \in \mathbb{R}^m,$

  $A\mathbf{e}_k \in \mathbb{R}^n =$ the kth column of A

- **E8**
  - ith row of $I_n = (\mathbf{e}_i)^T$
  - $\forall \mathbf{v} \in \mathbb{R}^n, I_n\mathbf{v} = \mathbf{v}$
- **E9**

  $\forall \mathbf{u}_1 \ldots \mathbf{u}_m \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^m$, let $A := (\mathbf{u}_1 \ldots \mathbf{u}_k) \in \mathbb{R}^{n\times m}$,

  $A\mathbf{x} = x_1u_1 + \ldots + x_mu_m$ is the linear combination of $\mathbf{u}_1 \ldots \mathbf{u}_m$

2. Systems of Linear Equations

- **D20** solutions to linear system of equations

  A **consistent** system has solutions where **inconsistent** ones do not.

- **L2/D21** coefficient and augmented matrix
  - $\forall A = (a_{ij}) \in \mathbb{R}^{n\times m}$ **(coefficient matrix)** , $\mathbf{b} \in \mathbb{R}^n, \mathbf{v} \in \mathbb{R}^m$

    $(v_1 \ldots v_m)$ is a solution to the system $\iff A\mathbf{v} = \mathbf{b}$
  - $(A|\mathbf{b}) =$ adding $\mathbf{b}$ as an extra column to A **(augmented matrix)**

- **D22** row operation
  - $r_i(\lambda)$: multiply the ith row by $0 \ne \lambda \in \mathbb{R}$
  - r_{ij} : swap row i with row j
  - $r_{ij}(\lambda)$: add λ times row i to row j
- **P1** operations do not affect solutions

  Let $(A|\mathbf{b}) \xrightarrow{r} (A'|\mathbf{b}'), A\mathbf{v} = \mathbf{b} \iff A'\mathbf{v} = \mathbf{b}'$

- **D23** leading entry

  The left-most non-zero entry in a non-zero row

- **D24/25** echelon form/**row reduced echelon (RRE) form**
  - The leading entry in each non-zero row equals 1
  - The leading 1 in each non-zero row is to the right of the leading 1 in any rows above
  - All zero rows are below all non-zero rows
  - **The leading 1 in each non-zero row is the only non-zero entry in its column**
- **E14** cases of RRE form

  Consider $(A|\mathbf{b})$ representing a system of linear equations in matrix form,

  - $A = I_n : \mathbf{b}$ is a unique solution
  - $A = \begin{pmatrix} I_n \\ \mathbf{0}_{k\times n} \end{pmatrix}$
    - $\forall n < i \le n+k, b_i = 0 : \mathbf{b}$ is a unique solution
    - Otherwise, the system is inconsistent
  - The ith column does not contain a leading 1
    - The variable x_i can be set to any values **(free variable**, as opposed to **basic variable)**
    - The system has ∞ solutions **(underdetermined)**
- **D26** pivots
  - Pivot position = a leading entry in a matrix in RRE form
  - Pivot column = a column containing a pivot position
- **P2** put any matrix $A = (a_{ij})$ into RRE form
  - **Forward phase**: put into echelon form

    Starting from the first row, for each non-zero column k,

    1. Find a row j such that $a_{jk} \neq 0$, multiply it by a_{jk}^{-1} (to get a leading 1) and swap with the current row;
    2. Subtract $a_{j'k}$ times the current row from each succeeding row j' to create 0s;
    3. Move on to the next row.
  - **Backward phase**: put into RRE form

    Starting from the bottom row,

    1. If the leading 1 is at the kth row, subtract a_{jk} times the current row from each preceding row j to clear the column;
    2. Move on to the previous row.
  - The RRE form of a matrix is unique
- **P3** number of solutions

  A system of linear equations has either $0, 1, \infty$ solutions

3. Matrix Multiplication

- **D27** matrix multiplication

  $\forall A = (a_{ij}) \in \mathbb{R}^{n\times m}, B = (b_{ij}) \in \mathbb{R}^{m\times l}$,

$AB = (p_{ij}) \in \mathbb{R}^{n\times l} : p_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$

- **R3** remarks for matrix multiplication

  Let rows of A be $\mathbf{r}_1 \dots \mathbf{r}_n \in \mathbb{R}^m$ and columns of B be $\mathbf{c}_1 \dots \mathbf{c}_l \in \mathbb{R}^m$,

  - The (i, j) entry of $AB = \mathbf{r}_i^T \cdot \mathbf{c}_j$
  - The jth column of $AB = A\mathbf{c}_j$
  - AB is defined $\iff$ number of columns of $A =$ number of rows of B
- **E17/L3** matrix multiplication as functions
  - $\forall A \in \mathbb{R}^{n\times m}, \exists T_A : \mathbb{R}^m \to \mathbb{R}^n, \mathbf{v} \mapsto A\mathbf{v}$
  - $T_B : \mathbb{R}^l \to \mathbb{R}^m, T_A \circ T_B = T_{AB} : \mathbb{R}^l \to \mathbb{R}^n$
  - $\forall \mathbf{v} \in \mathbb{R}^l, A(B\mathbf{v}) = (AB)\mathbf{v}$
- **P4** properties of matrix multiplication

  $\forall A, A' \in \mathbb{R}^{m\times n}, B, B' \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times q}, \lambda \in \mathbb{R}$

  - $A(BC) = (AB)C$ **(associativity)**
  - $A(B + B') = AB + AB'$ **(left distributivity)**

    $(A + A')B = AB + A'B$ **(right distributivity)**
  - $(\lambda A)B = \lambda(AB) = A(\lambda B)$
  - Not commutative
  - $AB = 0 \not\Longrightarrow A = \mathbf{0}_{m\times n} \lor B = \mathbf{0}_{n\times p}$
- **L4** behaviour of zero and identity matrices

  $\forall A \in \mathbb{R}^{n\times m}$,

  - $\forall k, \mathbf{0}_{k\times n}A = \mathbf{0}_{k\times m}, A\mathbf{0}_{m\times k} = \mathbf{0}_{n\times k}$
  - $I_nA = AI_m = A$
- **D28** diagonal matrix

  $D = (d_{ij}) \in \mathbb{R}^{n\times n} : \forall i \neq j, d_{ij} = 0$ (all entries 0 other than leading diagonal)

  - $\forall n, I_n$ and $\mathbf{0}_{n\times n}$ are diagonal
  - Let $D := diag(d_1, \dots, d_n), D' := diag(d'_1, \dots, d'_n)$,

    $DD' = diag(d_1d'1, ldots, d_nd'_n)$
- **D29** triangular matrix

  $\forall A = (a_{ij})$,

  - $\forall i > j, a_{ij} = 0$ **(upper triangular)**
  - $\forall i \geq j, a_{ij} = 0$ **(strictly upper triangular)**
  - $\forall i < j, a_{ij} = 0$ **(lower triangular)**
  - $\forall i \leq j, a_{ij} = 0$ **(strictly lower triangular)**
- **E22** special triangular matrices
  - A is both upper and lower triangular $\iff$ A is diagonal
  - A is both strictly upper and strictly lower triangular $\iff$ $A = \mathbf{0}_{n\times n}$
- **D30** inverse matrix

  $\forall A \in \mathbb{R}^{n\times n}$,

  - $\exists A^{-1} \in \mathbb{R}^{n\times n} : AA^{-1} = A^{-1}A = I_n$ (A **invertible**)

- $\nexists A^{-1}$ (A **singular**)
- **L5** uniqueness of inverse matrix
  - A **invertible** $\implies \exists! A^{-1}$
  - A **invertible** $\wedge (\exists B \in \mathbb{R}^{n \times n} : AB = I_n \vee BA = I_n) \implies B = A^{-1}$
- **L6** inverse of matrix product

  $(AB)^{-1} = B^{-1}A^{-1}$
- **L7/C1/C2** cases of non-invertibility

  $\forall A = (a_{ij}) \in \mathbb{R}^{n \times n}$,
  - $\exists \mathbf{0}_n \neq \mathbf{v} \in \mathbb{R}^{n \times n} : A\mathbf{v} = \mathbf{0}_n \implies A$ **non-invertible**
  - $\exists \mathbf{0}_{n \times n} \neq B \in \mathbb{R}^{n \times n} : AB = \mathbf{0}_{n \times n} \vee BA = \mathbf{0}_{n \times n} \implies A$ **non-invertible**
  - $\exists k : (\forall i, a_{ik} = 0) \vee (\forall j, a_{kj} = 0) \implies A$ **non-invertible**
- **E23** determinant

  $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}$
  - $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  - A **invertible** $\iff ad - bc \neq 0$ **(determinant)**
- **D31/R4** elementary matrix
  - $I_n \xrightarrow{r} R$ (differs by one row operation from identity matrix)
  - R **invertible**:
    - $R_i(\lambda)^{-1} = diag(1, \ldots, 1, \lambda^{-1}, 1, \ldots, 1) = R_i(\lambda^{-1})$
    - $R_{ij}^{-1} = R_{ij}$
    - $R_{ij}(\lambda)^{-1} = R_{ij}(-\lambda)$
- **L8/L9/P5** matrix invertability after row operations

  $\forall A \in \mathbb{R}^{n \times n}$,
  - $A \xrightarrow{r} A', A$ **invertible** $\iff A'$ **invertible**
  - $A' := A$ in RRE form, A' **invertible** $\iff A'$ has no zero rows
  - $A' := A$ in RRE form, A **invertible** $\iff A' = I_n$
  - A **invertible** $\iff \nexists \mathbf{0}_n \neq \mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}_n$
- **E24** compute inverse matrix

  If A **invertible**, $(A|I_n) \xrightarrow{RRE} (I_n|A^{-1})$

4. Vector Spaces

- **D4.1** vector space
  - a set $V, v \in V$ are referred as "vectors"
  - binary operation $+ : V \times V \to V$ **(addition)**
  - function $\mathbb{R} \times V \to V$ **(scalar multiplication)**

  With the following axioms hold $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \lambda_1, \lambda_2 \in \mathbb{R}$:
  - **(A1)** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

- **(A2)** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
    - **(A3)** $\exists \mathbf{0}_V \in V : \mathbf{u} + \mathbf{0}_V = \mathbf{u}$
    - **(A4)** $\exists - \mathbf{u} \in V : \mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$
    - **(M1)** $\lambda_1(\lambda_2 \mathbf{u}) = (\lambda_1 \lambda_2)\mathbf{u}$
    - **(M2)** $(\lambda_1 + \lambda_2)\mathbf{u} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{u}$
    - **(M3)** $\lambda_1(\mathbf{u} + \mathbf{v}) = \lambda_1 \mathbf{u} + \lambda_1 \mathbf{v}$
    - **(M4)** $1 \cdot \mathbf{u} = \mathbf{u}$
- **E4.1** some examples of vector spaces

  $\mathbb{R}, \mathbb{R}^n, \mathbb{R}[X] \subset \mathbb{R}^{\mathbb{R}} := \{f : \mathbb{R} \to \mathbb{R}\}, \mathbb{R}^{n \times m}, \mathbb{R}^0 = \{\mathbf{0}_V\}$

- **L4.1** more properties of vector spaces

  $\forall \mathbf{x} \in V$ a vector space,

    - $\forall n \in \mathbb{N}, n\mathbf{x} = \mathbf{x} + \mathbf{x} + \ldots + \mathbf{x}$ (n terms)
    - $0\mathbf{x} = \mathbf{0}_V$
    - $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}_V$ (**additive inverse**)
- **D4.2** subspace

  A subset $U \subset V$:

    - $\mathbf{0}_V \in U$ (not empty)
    - $\forall \mathbf{x}, \mathbf{y} \in U, \mathbf{x} + \mathbf{y} \in U$
    - $\forall \mathbf{x} \in U, \forall \lambda \in \mathbb{R}, \boldsymbol{\lambda} \mathbf{x} \in U$
- **E4.2** proper subspace

  $\forall V$ a vector space, V and $\mathbf{0}_V \subset V$ are subspaces, all other subspaces are **proper subspaces**

- **L4.2** operations of subspaces

  $\forall V$ a vector space, $\forall U, W \in V$ are subspaces,

    - $U \cap W$ **subspace**
    - $U \cup W$ **subspace** $\iff U \subseteq W \vee W \subseteq U$