Fourier Series and Transform

Fourier Series

$2L$-periodic real form (for 2π-periodic: put $L = \pi$)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{N} \left(a_k \cos\left(kx\frac{\pi}{L}\right) + b_k \sin\left(kx\frac{\pi}{L}\right)\right) dx$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(kx\frac{\pi}{L}\right) dx$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(kx\frac{\pi}{L}\right) dx$$

```
a(k, expression, x, L = pi)
b(k, expression, x, L = pi)
fourierseries(n_terms, expression, x, L = pi)
```

Value at discontinuities: average of left- and right-side limits.
Fourier sine series: if $f(x)$ is odd, $a_n = 0$, and we only have the sine part.
Fourier cosine series: if $f(x)$ is even, $b_n = 0$, and we only have the cosine part.

$2L$-periodic complex form (for 2π-periodic: put $L = \pi$)

> Reason for negative exponent in γ_k: complex inner product requires conjugating the second function

$$f(x) = \sum_{k=-N}^{N} \gamma_k e^{ikx\frac{\pi}{L}}$$

$$\gamma_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ikx\frac{\pi}{L}}\, dx$$

```
γ(k, expression, x, L = pi)
complexfourierseries(n_terms, expression, x, L = pi)
```

Conversion between real and complex forms

$$\gamma_0 = \frac{1}{2}a_0$$

$$\gamma_k = \frac{1}{2}(a_k - ib_k)$$

$$\gamma_{-k} = \frac{1}{2}(a_k + ib_k)$$

Parseval's theorem

If $f(x)$ is a real-valued function defined over $[-\pi, \pi]$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2}{a_0}^2 + \sum_{k=1}^{N}\left[{a_k}^2 + {b_k}^2\right]$$

Fourier Transform

Non-periodic real form (unofficial, for completeness only)

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(f_c(\omega)\cos(\omega x)+f_s(\omega)\sin(\omega x)\right)d\omega \\
\hat{f}_c(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx \\
\hat{f}_s(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx
\end{aligned}
$$

Fourier sine transform: if $f(x)$ is odd, $\hat{f}_c(\omega)=0$, and we only have the sine part.
Fourier cosine transform: if $f(x)$ is even, $\hat{f}_s(\omega)=0$, and we only have the cosine part.

Non-periodic complex form (official definition)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega \\
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx
\end{aligned}
$$

Fourier sine transform: if $f(x)$ is odd, we can write $\hat{f}(\omega)=-2i\hat{f}_s(\omega)$, i.e.
$f(x)=\frac{2}{\pi}\int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,d\omega$ and $\hat{f}_s(\omega)=\int_0^\infty f(x)\sin(\omega x)\,dx$.
Fourier cosine transform: if $f(x)$ is even, we can write $\hat{f}(\omega)=2\hat{f}_c(\omega)$, i.e.
$f(x)=\frac{2}{\pi}\int_0^\infty \hat{f}_c(\omega)\cos(\omega x)\,d\omega$ and $\hat{f}_c(\omega)=\int_0^\infty f(x)\cos(\omega x)\,dx$.

Conversion between real and complex forms

$$
\hat{f}(\omega)=2\hat{f}_c(\omega)-2i\hat{f}_s(\omega)
$$

Properties

> The following theorems are written in η-reduced form ($\mapsto$ denotes lambda function)

Notations $\mathcal{F}[f]=\hat{f}$ and $\mathcal{F}^{-1}[\hat{f}]=f$ are inverse mappings.

Linearity: $\mathcal{F}[af+bg]=a\mathcal{F}[f]+b\mathcal{F}[g]$
Linearity: $\mathcal{F}^{-1}[a\hat{f}+b\hat{g}]=a\mathcal{F}^{-1}[\hat{f}]+b\mathcal{F}^{-1}[\hat{g}]$
Time-scaling: $\mathcal{F}[x\mapsto f(ax)]=\omega\mapsto\frac{1}{a}\mathcal{F}[f]\left(\frac{\omega}{a}\right)$ // Only when $a>0$
Time-reversal: $\mathcal{F}[x\mapsto f(-x)]=\omega\mapsto\mathcal{F}[f](-\omega)$ // Note that RHS is positive!
Time-shift: $\mathcal{F}[x\mapsto f(x-x_0)]=\omega\mapsto e^{-i\omega x_0}\mathcal{F}[f](\omega)$
Freq-shift: $\mathcal{F}^{-1}[\omega\mapsto\hat{f}(\omega-\omega_0)]=x\mapsto e^{i\omega_0 x}\mathcal{F}^{-1}[\hat{f}](x)$
Symmetry: $\mathcal{F}[\mathcal{F}[f]]=x\mapsto 2\pi f(-x)$
Conjugation: $\mathcal{F}[x\mapsto\overline{f(x)}]=\omega\mapsto\overline{\mathcal{F}[f](-\omega)}$

Time-derivative: $\mathcal{F}[\mathcal{D}[f]]=\omega\mapsto i\omega\mathcal{F}[f](\omega)$
Freq-derivative: $\mathcal{D}[\mathcal{F}[f]]=\mathcal{F}[x\mapsto -ixf(x)]$

Derivatives for sine and cosine transforms:
$\mathcal{F}_c[\mathcal{D}[f]](\omega)=-f(0)+\omega\mathcal{F}_s[f](\omega)$
$\mathcal{F}_s[\mathcal{D}[f]](\omega)=-\omega\mathcal{F}_c[f](\omega)$
$\mathcal{F}_c[\mathcal{D}^2[f]](\omega)=-\mathcal{D}[f](0)-\omega^2\mathcal{F}_c[f](\omega)$
$\mathcal{F}_s[\mathcal{D}^2[f]](\omega)=\omega f(0)-\omega^2\mathcal{F}_s[f](\omega)$

Convolution theorem

$$\begin{aligned}
\text{Convolution:} \quad & f * g = x \mapsto \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du \\
\text{Theorem:} \quad & \mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g] \\
& \mathcal{F}^{-1}[\hat{f} \cdot \hat{g}] = \mathcal{F}^{-1}[\hat{f}] * \mathcal{F}^{-1}[\hat{g}] \\
& \mathcal{F}[f \cdot g] = \frac{1}{2\pi}(\mathcal{F}[f] * \mathcal{F}[g]) \\
& \mathcal{F}^{-1}[\hat{f} * \hat{g}] = 2\pi\, \mathcal{F}^{-1}[\hat{f}] \cdot \mathcal{F}^{-1}[\hat{g}]
\end{aligned}$$

Energy theorem

If $f(x)$ is a real-valued function defined over $\mathbb{R}$:

$$\int_{-\infty}^{\infty} [f(x)]^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega$$

Dirac delta "function" δ

Shifting property axiom: for any continuous g, we have $\int_{-\infty}^{\infty} g(x)\delta(a-x)dx = g(a)$.

$$\begin{aligned}
\text{Fourier axiom 1:} \quad & \hat{\delta}(\omega) \overset{?}{=} 1 \\
\text{Fourier axiom 2:} \quad & \delta(x) \overset{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} d\omega \\
& \overset{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} d\omega \\
\text{Example:} \quad \mathcal{F}[\cos\omega_0 x](\omega) & = \int_{-\infty}^{\infty} \left(\frac{e^{i\omega_0 x} + e^{-i\omega_0 x}}{2}\right) e^{i\omega x} dx \\
& = \frac{1}{2} \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)x} dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)x} dx \\
& = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)
\end{aligned}$$

Ordinary Differential Equations

Analytic Solutions

> In this section, variables like x, y can be seen as **coordinates in a Euclidean space** that satisfy certain **constraints**.
> "Solving an ODE" $\Leftrightarrow$ "simplifying (rewriting) an equation until there are no derivative terms".

Order: the order of the highest derivative.
Degree: the power of the highest derivative, with fractional powers removed.

> In stating the following theorems, we use gray font for universally quantified functions, in order to assist in "human-powered pattern matching" and "human-powered E-unification". Remember to $+C$ after evaluating integrals.

First-order ODEs

Normalisation step: solve for $\frac{dy}{dx}$ and move all terms to the LHS. Explicit normal form: $\frac{dy}{dx} + F(y, x) = 0$.

- **Separable:** if $\frac{dy}{dx} + P(x)Q(y) = 0$, then $\frac{1}{Q(y)}\frac{dy}{dx} + P(x) = 0$, and we can solve $\int \frac{1}{Q(y)}\frac{dy}{dx}dx + \int P(x)dx = 0$ *(usually direct).*
- **Homogeneous:** if $\frac{dy}{dx} + P(\frac{y}{x}) = 0$, let $u := \frac{y}{x}$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d(xu)}{du}\frac{du}{dx} = u + x\frac{du}{dx}$,

and we can solve $u + x\frac{du}{dx} + P(u) = 0$ *(usually recursive)*.

- **Linear:** if $\frac{dy}{dx} + P(x)y + Q(x) = 0$, let $I := e^{\int P(x)dx}$, then $I\frac{dy}{dx} + IP(x)y + IQ(x) = 0$ gives $\frac{d}{dx}(Iy) + IQ(x) = 0$, and we can solve $Iy + \int IQ(x)dx = 0$ *(usually direct)*.
- **Bernoulli:** if $\frac{dy}{dx} + P(x)y + Q(x)y^n = 0$, let $u := y^{1-n}$, then $\frac{dy}{dx} = \left(\frac{du}{dy}\right)^{-1}\frac{du}{dx} = ((1-n)y^{-n})^{-1}\frac{du}{dx} = \frac{y^n}{1-n}\frac{du}{dx}$, so the original equation rewrites to $\frac{y^n}{1-n}\frac{du}{dx} + P(x)y + Q(x)y^n = 0$, and we can solve $\frac{du}{dx} + (1-n)P(x)u + (1-n)Q(x) = 0$ *(usually recursive: first-order linear)*.
- **Exact:** if $\frac{dy}{dx} + \frac{P(x,y)}{Q(x,y)} = 0$, see the last section.
- **Linear with constant coefficients:** if $\frac{dy}{dx} + \alpha_0 y + f(x) = 0$,
  - Solve for y_{CF}: solve $\frac{dy}{dx} + \alpha_0 y = 0$ — this is separable.
  - Find an y_{PI} using Ansatz:
    If $f(x)$ is polynomial, we could try a polynomial one like $Ax^2 + Bx + C$ *(direct)*.
    If $f(x) = e^{bx}$, we could try Ae^{bx} or $A(x)e^{bx}$ *(usually recursive: "method of variation of parameters")*.
    If $f(x)$ is polynomial times e^{bx}, we could try $A(x)e^{bx}$ *(usually recursive, same as above)*.
    We may also decompose $f(x)$ into a sum of simpler ones, and find an Ansatz for each.

Second-order ODEs

Normalisation step: solve for $\frac{d^2y}{dx^2}$ and move all terms to the LHS. Explicit normal form: $\frac{d^2y}{dx^2} + F(\frac{dy}{dx}, y, x) = 0$.

- **Reducible - without y:** if $\frac{d^2y}{dx^2} + F(\frac{dy}{dx}, x) = 0$, let $u := \frac{dy}{dx}$, then we can solve $\frac{du}{dx} + F(u, x) = 0$ *(usually recursive: first-order)*.
- **Reducible - without x:** if $\frac{d^2y}{dx^2} + F(\frac{dy}{dx}, y) = 0$, let $u := \frac{dy}{dx}$, then $\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u = \frac{d}{dy}\left(\frac{1}{2}u^2\right)$, then we can solve either $\frac{du}{dy}u + F(u, y) = 0$ or $\frac{d}{dy}\left(\frac{1}{2}u^2\right) + F(u, y) = 0$ *(usually recursive: first-order)*.
- **Linear with constant coefficients:** if $\frac{d^2y}{dx^2} + \alpha_1\frac{dy}{dx} + \alpha_0 y + f(x) = 0$,
  - Solve for y_{CF}: solve $\frac{d^2y}{dx^2} + \alpha_1\frac{dy}{dx} + \alpha_0 y = 0$. Use Ansatz $e^{\lambda x}$, solve the *characteristic equation*.
    If $\lambda_1 \neq \lambda_2$, we are done. In case $\lambda_1 = \lambda_2$, use another Ansatz $A(x)e^{\lambda_1 x}$ for y_2.
  - Find an y_{PI}:
    If $f(x)$ is polynomial, we could try a polynomial one like $Ax^2 + Bx + C$ *(direct)*.
    If $f(x) = e^{bx}$, we could try Ae^{bx}, Axe^{bx} or $A(x)e^{bx}$ *(usually recursive: "reduction of order")*.
    If $f(x)$ is polynomial times e^{bx}, we could try $A(x)e^{bx}$ *(usually recursive, same as above)*.
    We may also decompose $f(x)$ into a sum of simpler ones, and find an Ansatz for each.

> Problems involving initial conditions: equivalently speaking, **you get some equations for free**. Be wise to apply them **as soon as possible** to reduce the amount of future work. (except LCC methods)

General linear ODEs

The coefficient $\alpha_n(x)$ of the highest derivative can be normalised to 1.

$$\mathcal{L}[y](x) + f(x) = \sum_{k=0}^{n} \alpha_k(x)\frac{d^ky}{dx^k} + f(x) = 0$$

If $f(x) = 0$ this is called a *homogeneous* linear ODE.
If α_k's do not depend on x this is called a linear ODE *with constant coefficients*.

- General solution $= y_{CF} + y_{PI}$
- nth order homogeneous linear ODE has n-dimensional solution space *(no proof)*. Use Wronskian to confirm linear independence of solutions.
- **First- and second-order linear ODEs with constant coefficients:** see above.

- **nth order linear ODEs with constant coefficients:** if $\sum_{k=0}^{n} \alpha_k \frac{d^k y}{dx^k} + f(x) = 0$,
  - Solve for y_{CF}: solve $\sum_{k=0}^{n} \alpha_k \frac{d^k y}{dx^k} = 0$. Use Ansatz $e^{\lambda x}$, solve the *characteristic equation*.
    If all λ_i's are distinct, we are done. (Wronskian $=$ $\det$ Vandermonde matrix, so non-zero.)
    In case there are d repeated roots and $k-d$ distinct ones, a basis is
    $\{e^{\lambda_1 x}, \ldots, e^{\lambda_r x}, xe^{\lambda_r x}, \ldots, x^{d-1}e^{\lambda_r x}, \ldots, e^{\lambda_{k-d+1} x}\}$.
    (No proof)
  - Find an y_{PI}:
    If $f(x)$ is polynomial, we could try a polynomial one like $Ax^2 + Bx + C$ *(direct)*.
    If $f(x) = e^{bx}$, we could use the following Ansatz:

    If for all i, $b \neq \lambda_i$, use Ae^{bx}.
    If for some $i \neq r$, $b = \lambda_i$, use Axe^{bx}.
    If $b = \lambda_r$, use $Ax^d e^{bx}$.

    If $f(x)$ is polynomial times e^{bx}, we could try $A(x)e^{bx}$ *(usually recursive)*.
    We may also decompose $f(x)$ into a sum of simpler ones, and find an Ansatz for each.
- **Euler-Cauchy ODE (note the different normalisation):** if $\sum_{k=0}^{n} \alpha_k x^k \frac{d^k y}{dx^k} + f(x) = 0$, let $u := \log x$, then $x = e^u$, and the original equation rewrites to

$$\begin{aligned}
&\sum_{k=0}^{n} \alpha_k x^k \frac{d}{du}\left(\frac{d^{k-1}y}{dx^{k-1}}\right) \cdot \frac{du}{dx} + f(x) \\
&= \sum_{k=0}^{n} \alpha_k x^k \frac{d}{du}\left(\frac{d}{du}\left(\frac{d^{k-2}y}{dx^{k-2}}\right) \cdot \frac{du}{dx}\right) \cdot \frac{du}{dx} + f(x) \\
&= \sum_{k=0}^{n} \alpha_k x^k \left(\frac{d^2}{du^2}\left(\frac{d^{k-2}y}{dx^{k-2}}\right) \cdot \frac{du}{dx} + \frac{d}{du}\left(\frac{d^{k-2}y}{dx^{k-2}}\right) \cdot \frac{d}{du}\frac{du}{dx}\right) \cdot \frac{du}{dx} + f(x) \\
&= \sum_{k=0}^{n} \alpha_k x^k \left(\frac{d^2}{du^2}\left(\frac{d^{k-2}y}{dx^{k-2}}\right) \cdot \frac{1}{x} + \frac{d}{du}\left(\frac{d^{k-2}y}{dx^{k-2}}\right) \cdot \left(-\frac{1}{x}\right)\right) \cdot \frac{1}{x} + f(x) \\
&= \ldots
\end{aligned}$$

  Eventually we get an equivalent equation that falls under the "nth order linear with constant coefficients" category *(recursive)*.
- **Use Fourier transform:** first take the Fourier transform of the LHS and RHS to get an equivalent equation. Let $\hat{y} := \mathcal{F}[y]$, we might be able to further rewrite this equation in terms of $\hat{y}$ (e.g. using facts such as $\mathcal{F}[\frac{dy}{dx}] = i\omega\, \hat{y}(\omega)$), and then solve for $\hat{y}$ *(usually recursive)*.

Systems of first-order linear ODEs

Any nth-order linear ODE can be turned into a system of first-order linear ODEs (simply set $\vec{y} := \left(y, y', y'', \ldots, y^{(n)}\right)^\top$ and add constraints in the form of $\frac{d}{dx}y^{(n-1)} - y^{(n)} = 0$).

$$\frac{d\vec{y}}{dx} + A(x)\vec{y} + \vec{f}(x) = \vec{0}$$

If $\vec{f}(x) = \vec{0}$ this is called a *homogeneous* system of first-order linear ODEs.
If A not depend on x this is called a system of first-order linear ODEs *with constant coefficients*.

- **System of first-order linear ODEs with constant coefficients:** if $\frac{d\vec{y}}{dx} + A\vec{y} + \vec{f}(x) = \vec{0}$,
  - Solve for $\vec{y}_{CF}$: solve $\frac{d\vec{y}}{dx} + A\vec{y} = \vec{0}$ or equivalently $\frac{d\vec{y}}{dx} = -A\vec{y}$.
    Diagonalise the matrix $(-A) = S\Lambda S^{-1}$, where:

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

```
S, Λ = Matrix([[...], ..., [...]]).diagonalize()
S, J = Matrix([[...], ..., [...]]).jordan_form()
S.inv()
```

- If $(-A)$ is diagonalisable, we are done, a basis is $B(x) = \{e^{\lambda_1 x}\vec{v}_1, \ldots, e^{\lambda_n x}\vec{v}_n\}$.

  (Proof: original equation implies $S^{-1}\frac{d\vec{y}}{dx} = S^{-1}(-A)(SS^{-1})\vec{y}$, which rewrites to $S^{-1}\frac{d\vec{y}}{dx} = \Lambda S^{-1}\vec{y}$, and if we let $\vec{u} := S^{-1}\vec{y}$ we have $\frac{d\vec{u}}{dx} = \Lambda\vec{u}$, which is easy to solve entry-by-entry).

- In case $(-A)$ is not diagonalisable, we have to find its Jordan normal form: $???$

  > Don't worry, just use SymPy.

  Then similarly, by letting $\vec{u} := S^{-1}\vec{y}$ we have $\frac{d\vec{u}}{dx} = J\vec{u}$, which can be solved entry-by-entry. A basis $B(x)$ can then be obtained from $\vec{y} = S\vec{u}$ (keeping the integration constants).

- Find an $\vec{y}_{PI}$:
  - If $(-A)$ is diagonalisable, solve $\frac{d\vec{u}}{dx} = \Lambda\vec{u} - S^{-1}\vec{f}(x)$. This can be done entry-by-entry *(recursive: first-order linear)*. In case $(-A)$ is not diagonalisable, try solve $\frac{d\vec{u}}{dx} = J\vec{u} - S^{-1}\vec{f}(x)$.
  - In either case we could try a vector Ansatz $B(x)\vec{v}(x)$ (where B is a basis as defined above and $\vec{v}(x)$ is undetermined). The original equation then rewrites to:

$$\begin{aligned} \frac{dB}{dx}\vec{v} + B\frac{d\vec{v}}{dx} + AB\vec{v} &= -\vec{f}(x) \\ (-A)B\vec{v} + B\frac{d\vec{v}}{dx} + AB\vec{v} &= -\vec{f}(x) \\ B\frac{d\vec{v}}{dx} &= -\vec{f}(x) \\ \frac{d\vec{v}}{dx} &= -B^{-1}\vec{f}(x) \\ \vec{v} &= -\int B^{-1}\vec{f}(x)\, dx \end{aligned}$$

    so $\vec{y}_{PI}$ can be $B(x)\vec{v}(x) = -B(x)\int B(x)^{-1}\vec{f}(x)\, dx$ *(direct)*.

Qualitative Behaviours (Stability)

Possible asymptotic behaviours near fixed points:

- Unstable ("diverge")
- Lyapunov stable ("around")
- Asymptotically stable ("converge")

Phase plane analysis

Consider a homogeneous system of first-order linear ODEs with constant coefficients:

$$\frac{d\vec{y}}{dt} = A\vec{y}$$

Then the **velocity** $\frac{d\vec{y}}{dt}$ is a vector-valued function defined at every point of the **phase space**, independent of t. At such, it forms a **vector field**. The velocities are tangent vectors to the so-called **trajectories**.

- Lines in the directions of the eigenvectors of A are *invariant*.
- If A is a 2×2 matrix, the phase space is a two-dimensional plane, and we could classify the behaviour of the ODE based on the trace and determinant of A: see Linear Phase Portraits.

Bifurcations

In linear systems, a **bifurcation** can be defined as a *change in stability of the system*.

In non-linear one-dimensional systems in the form of $\frac{dx}{dt} = f(x)$ where $f(x)$ is a polynomial:

- Saddle-node bifurcation ("⊂")
- Transcritical bifurcation ("exchange")
- Pitchfork bifurcation (subcritical: "3unstable1stable", supercritical: "3stable1unstable")
- Singular point ("undefined")

Linear stability analysis: if $\frac{df}{dx} > 0$ at some fixed point x, this fixed point is unstable; if $\frac{df}{dx} < 0$ at some fixed point x, this fixed point is asymptotically stable.

Multivariate Calculus

Partial Differentiation

Multivariate limits

- **Multivariate limit:** we say the *limit* of the function $f(\vec{x})$ at point $\vec{x}_0$ is L when:
  $\vec{x}_0$ *is a limit point of* S and
  $$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall \vec{x} \in S,\ 0 < \left|\vec{x} - \vec{x}_0\right| < \delta \to \left|f(\vec{x}) - L\right| < \epsilon$$
  We write $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$ for the L satisfying the above relation.
- **Multivariate continuity:** the function $f(\vec{x})$ is said to be *continuous* at point $\vec{x}_0$ when $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

Properties (no proof)

- "Add, sub, smul, mul, div, pow, composition" laws also hold for multivariate limits.
- All elementary functions are continuous within their domain.
- **Extreme value theorem:** a *continuous* function defined on a *compact set* S must be bounded, and have maximum and minimum values.
- **Intermediate value theorem:** a *continuous* function defined on a *compact set* S will attain all values in the range from the minimum to the maximum.
- **Uniform continuity:** a *continuous* function defined on a *compact set* S must be uniformly continuous on S.

Multivariate derivatives

> "Total differential" is just **a notation for the Jacobian!**
> The "Jacobian" is just **a matrix of partial derivatives!**

- **Partial derivative:** for a function $f(x_1, x_2 \ldots, x_n)$ of multiple variables, the function $\frac{\partial f}{\partial x_1}$ (aka. $\mathcal{D}_1[f]$) is defined as

$$\frac{\partial f}{\partial x_1}\bigg|_{(x_1,x_2,\ldots,x_n)} = \mathcal{D}_1[f](x_1,x_2,\ldots,x_n) := \lim_{h\to 0}\frac{f(x_1+h,x_2,\ldots,x_n)-f(x_1,x_2,\ldots,x_n)}{h}$$

The same goes for the function $\frac{\partial f}{\partial x_2}$ (aka. $\mathcal{D}_2[f]$) and so on.

- **Directional derivative:** for a function $f(\vec{x})$ of multiple variables, and any unit vector $\vec{u}$, the function $\mathcal{D}_{\vec{u}}[f]$ is defined as $\mathcal{D}_{\vec{u}}[f](\vec{x}) := \lim_{h\to 0}\frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$.

  (Partial derivatives are a special case of directional derivatives.)

- **(Total) differential:** a function $f(\vec{x})$ is said to be *differentiable at the point* $\vec{x}_0$ when there exists a linear map $\mathbf{J}_{\vec{x}_0}$ such that $\lim_{\vec{h}\to\vec{0}}\frac{\left|f(\vec{x}_0+\vec{h})-f(\vec{x}_0)-\mathbf{J}_{\vec{x}_0}(\vec{h})\right|}{\left|\vec{h}\right|}=0$. The *(total) differential at the point* $\vec{x}_0$ is then defined as $df := \mathbf{J}_{\vec{x}_0}(d\vec{x})$ *(just a notation around the linear map.)*

  ("All partial derivatives exist" does not necessarily mean differentiable!)

- **Jacobian matrix:** if a function $f(\vec{x})$ is *differentiable at* $\vec{x}_0$ (i.e. the linear map $\mathbf{J}_{\vec{x}_0}$ indeed exists), then all partial derivatives exist at $\vec{x}_0$, and the linear map $\mathbf{J}_{\vec{x}_0}$ is equal to the following *Jacobian matrix*:

$$\mathbf{J}_{\vec{x}_0} = \begin{bmatrix} | & | & & | \\ \mathcal{D}_1[f](\vec{x}_0) & \mathcal{D}_2[f](\vec{x}_0) & \cdots & \mathcal{D}_n[f](\vec{x}_0) \\ | & | & & | \end{bmatrix}$$

  (If $\vec{f}(\vec{x})$ is a vector-valued function, then the entry on the i-th row, j-th column of $\mathbf{J}_{\vec{x}_0}$ is the derivative of the i-th component of $\vec{f}$ w.r.t. the j-th component of $\vec{x}$, evaluated at $\vec{x}_0$.)

  We often use the notation $\frac{df}{d\vec{x}}\Big|_{\vec{x}_0}$ for $\mathbf{J}_{\vec{x}_0}$ *(note that somehow we are "differentiating w.r.t. a vector" here)*.

Properties (no proof)

- If *partial derivatives* $\mathcal{D}_1[f], \mathcal{D}_2[f], \ldots, \mathcal{D}_n[f]$ *all exist* near a point $\vec{x}_0$, and *are all continuous at* $\vec{x}_0$, then f is differentiable at $\vec{x}_0$.
- **Clairaut's theorem:** if $\mathcal{D}_i[\mathcal{D}_j[f]]$ and $\mathcal{D}_j[\mathcal{D}_i[f]]$ are *both continuous* within a region S, then they are equal in S.
- **Chain rule:** for a (possibly vector-valued) function $f(\vec{u})$ of n variables and an n-vector-valued function $\vec{u}(\vec{x})$ of m variables, we have

$$\frac{df}{d\vec{x}} = \frac{df}{d\vec{u}}\frac{d\vec{u}}{d\vec{x}}$$

  i.e. the Jacobian of f w.r.t $\vec{x}$ (at some $\vec{x}_0$) is equal to the matrix product of the Jacobian of f w.r.t. $\vec{u}$ (at $\vec{u}(\vec{x}_0)$) and the Jacobian of $\vec{u}$ w.r.t. $\vec{x}$ (at $\vec{x}_0$).

- **Implicit function differentiation:** if the equational constraint $F(\vec{x})=0$ locally gives x_i as a function of other x_j's (for all $j \neq i$), then we could obtain partial derivatives of that function as

$$\frac{\partial x_i}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_i}} = -\frac{\mathcal{D}_j[F](\vec{x})}{\mathcal{D}_i[F](\vec{x})}$$

  (For possible confusion: [Proof of Multivariable Implicit Differentiation Formula - Mathematics Stack Exchange](). Will have to make things clear in the future...)

Functions of three variables

- The **gradient** of a scalar-valued function $f(\vec{x})$ is defined as $[\nabla f](\vec{x}) := \mathbf{J}_{\vec{x}}$ *(synonym for a Jacobian with only one row)*.
- The **divergence** of a 3-vector-valued function $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))^\top$ is defined as $[\nabla \cdot f](\vec{x}) := \mathcal{D}_1[f_1](\vec{x}) + \mathcal{D}_2[f_2](\vec{x}) + \mathcal{D}_3[f_3](\vec{x})$.
- The **curl** of a 3-vector-valued function $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))^\top$ is defined as $[\nabla \times f](\vec{x}) := \left([\mathcal{D}_2[f_3] - \mathcal{D}_3[f_2]](\vec{x}), [\mathcal{D}_3[f_1] - \mathcal{D}_1[f_3]](\vec{x}), [\mathcal{D}_1[f_2] - \mathcal{D}_2[f_1]](\vec{x})\right)^\top$.
- The **Laplacian** of a scalar-valued function $f(\vec{x})$ is the *divergence* of its *gradient*.

Taylor expansion (up to second-order term)

$$f(\vec{x} + \Delta\vec{x}) = f(x) + \left(\frac{df}{d\vec{x}}\right)\Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^\top \left(\frac{d^2 f}{d\vec{x}^2}\right)\Delta\vec{x} + \dots$$

where $\frac{df}{d\vec{x}}$ is the Jacobian matrix (gradient), and $\frac{d^2 f}{d\vec{x}^2}$ is the Hessian matrix defined as:

$$\left(\frac{d^2 f}{d\vec{x}^2}\right)_{ij} := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

(The Hessian is always symmetric, due to Clairaut's.)

Multivariate Integration

> Moved some AP Physics C (Electromagnetism) stuff here...

Common Jacobian matrices

- **From polar to Cartesian coordinates:** given that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$:

$$\frac{d\vec{x}}{d\vec{r}} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

- **From Cartesian to polar coordinates:** given that $\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \arctan\left(\frac{y}{x}\right) \end{bmatrix}$:

$$\frac{d\vec{r}}{d\vec{x}} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

  Note that $\frac{d\vec{x}}{d\vec{r}}$ and $\frac{d\vec{r}}{d\vec{x}}$ are inverses of each other!

Multiple integrals

$$\iint_D f(x,y)\; dA := \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\; \Delta A_i$$

$$\iiint_\Omega f(x,y,z)\; dV := \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\; \Delta V_i$$

$$\dots$$

These can be evaluated using iterated integration ("rewriting" $dA = dxdy$, $dV = dxdydz$, etc.)

- **Change of coordinates:** if $x = X(u,v)$ and $y = Y(u,v)$ for some X, Y, u, v, then we can rewrite the iterated integral in terms of u and v:

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx = \int_{u_0}^{u_1}\int_{v_0}^{v_1} f(X(u,v),Y(u,v))\left|\frac{d\vec{x}}{d\vec{u}}\right|\,dv\,du$$

where $\vec{x}=[x,y]^\top$, $\vec{u}=[u,v]^\top$ and $\left|\frac{d\vec{x}}{d\vec{u}}\right|$ is determinant of Jacobian.

This can be generalised to higher dimensions: if $\vec{x}=\vec{X}(\vec{u})$, we can rewrite like:

$$\iint\dots\int f(\vec{x})\,dx_1\,dx_2\dots dx_n = \iint\dots\int f(\vec{X}(\vec{u}))\left|\frac{d\vec{x}}{d\vec{u}}\right|\,du_1\,du_2\dots du_n$$

where $\vec{x}=[x_1,x_2,\dots,x_n]^\top$, $\vec{u}=[u_1,u_2,\dots,u_n]^\top$ and $\left|\frac{d\vec{x}}{d\vec{u}}\right|$ is determinant of Jacobian.

- **Example:** we can rewrite double integrals "in polar coordinates" like $\iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$, where the r factor is the determinant of the corresponding Jacobian.

Line integral w.r.t. length

$$\int_L f(x,y,z)\,ds := \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\,\Delta s_i$$

where Δs_i is the length of the i-th arc, and λ is the maximum of all Δs_i.

- **Parametric integration:** when the curve L is parameterized as $\vec{r}(t)=(x(t),y(t),z(t))^\top$ for $t\in[a,b]$, we have:

$$\int_L f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

(Not a proof) one way to remember this: $ds=\sqrt{dx^2+dy^2+dz^2}=\sqrt{d\vec{x}\cdot d\vec{x}}$ so $\frac{ds}{dt}=\sqrt{\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}}$.

Line integral w.r.t. dot product

$$\int_L \vec{f}(x,y,z)\cdot d\vec{s} := \int_L \vec{f}(x,y,z)\cdot\hat{t}(x,y,z)\,ds$$

where the unit vector $\vec{t}$ is the tangent vector of the curve L.

- **Parametric integration:** when the curve L is parameterized as $\vec{r}(t)=(x(t),y(t),z(t))^\top$ for $t\in[a,b]$, we have:

$$\int_L \vec{f}(x,y,z)\cdot d\vec{s} = \int_a^b \vec{f}(x(t),y(t),z(t))\cdot\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)dt$$

(Not a proof) one way to remember this: $\frac{d\vec{s}}{dt}$ is the Jacobian $\left[\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right]^\top$.

Line integral w.r.t. cross product

$$\int_L \vec{f}(x,y,z)\times d\vec{s} := \int_L \vec{f}(x,y,z)\times\hat{t}(x,y,z)\,ds$$

where the unit vector $\vec{t}$ is the tangent vector of the curve L.

- **Parametric integration:** when the curve L is parameterized as $\vec{r}(t)=(x(t),y(t),z(t))^\top$ for $t\in[a,b]$, we have:

$$\int_L \vec{f}(x,y,z)\times d\vec{s} = \int_a^b \vec{f}(x(t),y(t),z(t))\times\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)dt$$

(Not a proof) one way to remember this: $\frac{d\vec{s}}{dt}$ is the Jacobian $\left[\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right]^\top$.

> When using parametric integration, indicate your vector-valued parametric equation (and its Jacobian) for that curve.

Exact ODEs

$$\frac{dy}{dx} + \frac{P(x,y)}{Q(x,y)} = 0$$

Let $u(x,y) = 0$ be a solution of the above ODE.
By implicit differentiation theorem, we have $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$, so the original equation rewrites to $\frac{P(x,y)}{Q(x,y)} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$.
Now assuming $P(x,y) = \frac{\partial u}{\partial x}$ and $Q(x,y) = \frac{\partial u}{\partial y}$, we get $\frac{\partial}{\partial y}P(x,y) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}Q(x,y)$.

- If this condition holds, our assumption is correct ($???$); such function $u(x,y)$ indeed exists and is called a **potential function**. We can try to find u by solving the **partial differential equations**

$$\begin{cases} \frac{\partial u}{\partial x} = P(x,y) \\ \frac{\partial u}{\partial y} = Q(x,y) \end{cases}$$

  by integrating the first equation w.r.t. x first, assuming on the RHS an integration constant $C(y)$ that potentially depends on y, and substitute that RHS into u in the second equation, and solve for $C(y)$.
- If the condition does not hold, we can try looking for a factor $\lambda(x)$ or $\lambda(y)$ such that $\lambda\frac{dy}{dx} + \lambda\frac{P(x,y)}{Q(x,y)} = 0$ is exact, and solve this new equation (it has the same solution).

Sketching functions of two variables

- Check continuity, find singularities
- Find asymptotic behaviour
- Obtain some level curves (e.g. $f(\vec{x}) = 0$)
- Find stationary points (i.e. where first partial derivatives are both zero), and use the eigenvalues of the Hessian to classify stationary points (since the Hessian is real symmetric, it can be orthogonally diagonalised and all its eigenvalues are real):
  If $\lambda_1, \lambda_2 > 0$, it is a minimum;
  If $\lambda_1, \lambda_2 < 0$, it is a maximum;
  If λ_1 and λ_2 have opposite signs, it is a saddle point;
  Otherwise, the Hessian test is inconclusive.