Poisson’s and Laplace’s Equations

Poisson's Equation

From

$$\nabla \cdot \vec{D} = \rho_v$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{E} = -\nabla V$$

It obtains

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = -\varepsilon \nabla \cdot \nabla V = \rho_v$$

Poisson's Equation

$$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

Laplace's Equation (1)

From Poisson's equation, setting medium is dielectric $(\rho_v = 0 \ \mathrm{C/m^3})$

$$\nabla^2 V = 0$$

∇^2 : Laplacian Operator

Laplace's Equation (2)

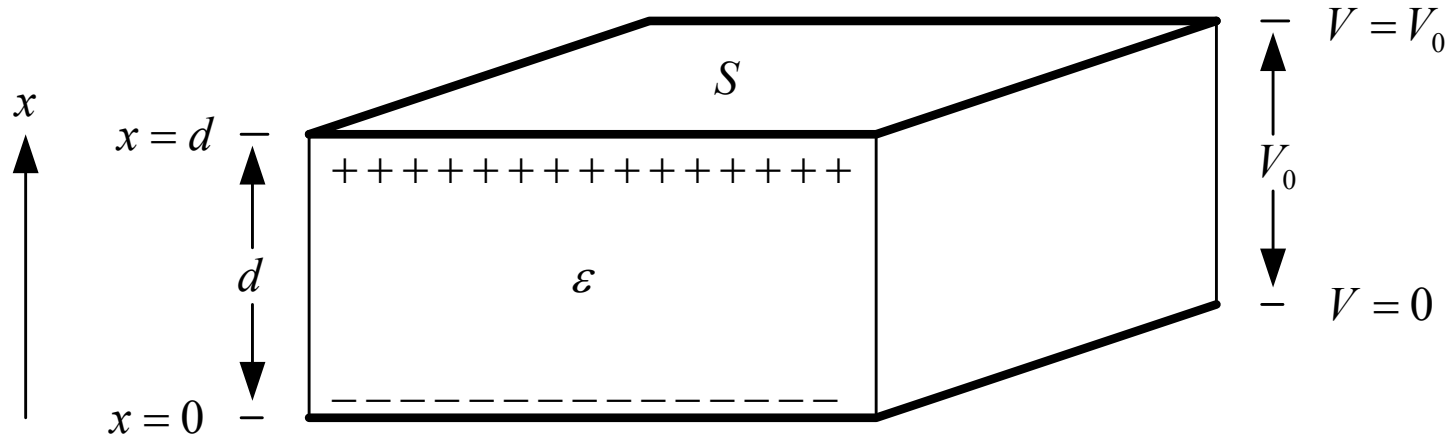
- Definition of Laplacian

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V$$

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} V \right) + \frac{1}{\rho^2} \left(\frac{\partial^2}{\partial \phi^2} V \right) + \frac{\partial^2}{\partial z^2} V$$

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} V \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} V \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} V$$

Parallel-Plate Capacitor (1)



จากสมการ Laplace

$$\frac{d^2}{dx^2}V = 0$$

$$\frac{d}{dx}V = A$$

$$V = Ax + B$$

Parallel-Plate Capacitor (2)

กำหนดให้ $x = 0$ m มี $V = 0$ V และ $x = d$ m มี $V = V_0$ V จะได้

$$A = \frac{V_0}{d} \qquad B = 0$$

นั่นคือ

$$V = \frac{V_0}{d} x \quad \text{V}$$

$$\vec{E} = -\nabla V = -\frac{d}{dx} V \vec{a}_x = -\frac{V_0}{d} \vec{a}_x \quad \text{V/m}$$

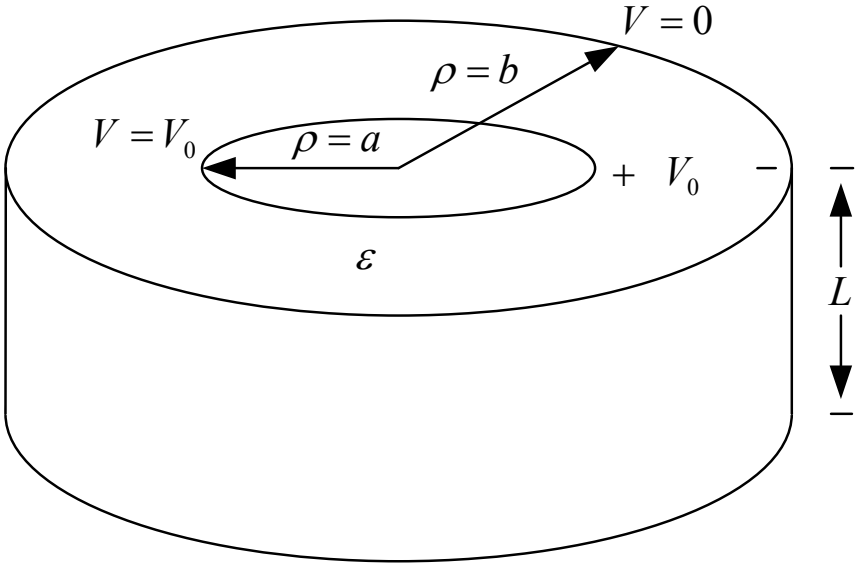
Parallel-Plate Capacitor (3)

$$\vec{D} = \varepsilon \vec{E} = -\frac{\varepsilon V_0}{d}\vec{a}_x \qquad \mathrm{C/m^2}$$

$$Q = \oint_S \vec{D} \cdot d\vec{S} = -\frac{\varepsilon V_0}{d}\vec{a}_x \cdot \left(-S\vec{a}_x\right) = \frac{\varepsilon V_0 S}{d} \qquad \mathrm{C}$$

$$C = \frac{Q}{V} = \frac{\varepsilon S}{d} \qquad \mathrm{F}$$

Coaxial Capacitor (1)



จากสมการ Laplace

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}V\right) = 0$$

$$\rho\frac{d}{d\rho}V = A$$

$$V = A\ln(\rho) + B$$

Coaxial Capacitor (2)

กำหนดให้ $\rho = b$ m มี $V = 0$ V และ $\rho = a$ m มี $V = V_0$ V จะได้

$$0 = A\ln(b) + B$$

$$V_0 = A\ln(a) + B$$

แก้สมการได้

$$A = \frac{V_0}{\ln(a/b)} \qquad B = -\frac{V_0}{\ln(a/b)}\ln(b)$$

นั่นคือ

$$V = \frac{V_0}{\ln(a/b)}\ln(\rho) - \frac{V_0}{\ln(a/b)}\ln(b) = \frac{V_0}{\ln(a/b)}\ln\left(\frac{\rho}{b}\right) \quad \text{V}$$

Coaxial Capacitor (3)

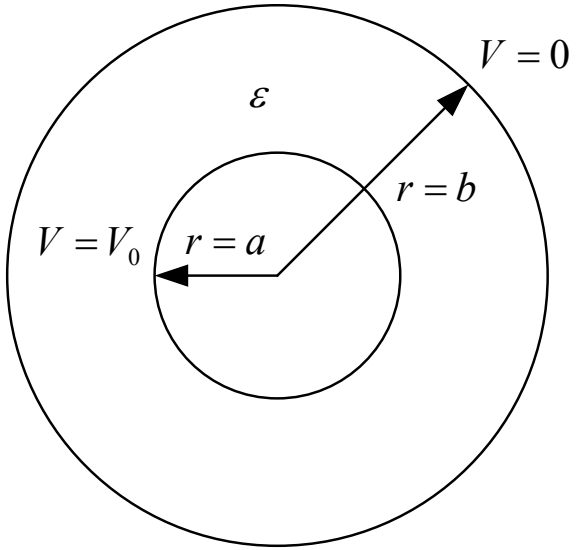
$$\vec{E} = -\nabla V = -\frac{d}{d\rho} V \vec{a}_\rho = \frac{V_0}{\ln(b/a)\rho} \vec{a}_\rho \quad \text{V/m}$$

$$\vec{D} = \varepsilon \vec{E} = \frac{\varepsilon V_0}{\ln(b/a)\rho} \vec{a}_\rho \quad \text{C/m}^2$$

$$Q = \oint_S \vec{D} \cdot d\vec{S} = \frac{\varepsilon V_0}{\ln(b/a)\rho} \vec{a}_\rho \cdot 2\pi\rho L \vec{a}_\rho = \frac{2\pi\varepsilon V_0 L}{\ln(b/a)} \quad \text{C}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln(b/a)} \quad \text{F}$$

Spherical Capacitor (1)



จากสมการ Laplace

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}V\right) = 0$$

$$r^2\frac{d}{dr}V = A$$

$$V = -\frac{A}{r} + B$$

Spherical Capacitor (2)

กำหนดให้ $r = b$ m มี $V = 0$ V และ $r = a$ m มี $V = V_0$ V จะได้

$$0 = -\frac{A}{b} + B$$

แก้สมการได้

$$V_0 = -\frac{A}{a} + B$$

$$A = -\frac{V_0}{\frac{1}{a} - \frac{1}{b}} \qquad B = -\frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)}$$

นั่นคือ

$$V = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r} - \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)b} = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} \quad \text{V}$$

Spherical Capacitor (3)

$$\vec{E} = -\nabla V = -\frac{d}{dr} V \vec{a}_r = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \vec{a}_r \quad \text{V/m}$$

$$\vec{D} = \varepsilon \vec{E} = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \vec{a}_r \quad \text{C/m}^2$$

$$Q = \oint_S \vec{D} \cdot d\vec{S} = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \vec{a}_r \cdot 4\pi r^2 \vec{a}_r = \frac{4\pi\varepsilon V_0}{\frac{1}{a} - \frac{1}{b}} \quad \text{C}$$

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} \quad \text{F}$$

Example

Parallel-Plate Capacitor มีพื้นที่หน้าตัด $S = 100\ \text{cm}^2$ Dielectric มี $\varepsilon_r = 10$ มีเงื่อนไขขอบเขตที่ $x = -5\ \text{cm}$ มี $V = -10\ \text{V}$ และที่ $x = 15\ \text{cm}$ มี $V = 20\ \text{V}$ จงหา V, V ที่ $x = 10\ \text{cm}$, $\vec{E}$, $\vec{D}$, Q และ C

Solution (1)

จากสมการ Laplace

$$\frac{d^2}{dx^2}V = 0$$

$$\frac{d}{dx}V = A$$

$$V = Ax + B$$

กำหนดให้ $x = -5$ cm มี $V = -10$ V และ $x = 15$ cm มี $V = 20$ V จะได้

Solution (2)

$$-10 = -5\times10^{-2}A + B$$

$$20 = 15\times10^{-2}A + B$$

แก้สมการได้

$$A = 150 \qquad B = -2.50$$

นั่นคือ

$$V = 150x - 2.50 \text{ V}$$

หา V ที่ $x = 10$ cm ได้

$$V = 150\times10\times10^{-2} - 2.50 = 12.50 \text{ V} \quad \#$$

Solution (3)

หา $\vec{E}$ ได้

$$\vec{E} = -\nabla V = -\frac{d}{dx} V \vec{a}_x = -150\vec{a}_x \ \text{V/m} \quad \#$$

หา $\vec{D}$ ได้

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = -\frac{1}{36\pi} \times 10^{-9} \times 10 \times 150\vec{a}_x = -13.26\vec{a}_x \ \text{nC/m}^2 \quad \#$$

หา Q ได้

$$Q = \oint_S \vec{D} \cdot d\vec{S} = -13.26\vec{a}_x \times 10^{-9} \cdot \left(-100 \times 10^{-4} \vec{a}_x\right) = 132.60 \ \text{pC} \quad \#$$

หา C ได้

$$C = \frac{Q}{V} = \frac{132.60 \times 10^{-12}}{20 - (-10)} = 4.42 \ \text{pF} \quad \#$$

Quiz 5

Coaxial Capacitor ยาว $L = 10$ cm Dielectric มี $\varepsilon_r = 12$ มีเงื่อนไขขอบเขตที่ $\rho = 2$ cm มี $V = 15$ V และที่ $\rho = 20$ cm มี $V = -5$ V จงหา V, V ที่ $\rho = 10$ cm, $\vec{E}$, $\vec{E}$, ที่ $\rho = 15$ cm, $\vec{D}$, $\vec{D}$ ที่ $\rho = 18$ cm, Q และ C

$$V = -8.69\ln(\rho) - 18.98 \text{ V}, \vec{E} = \frac{8.69}{\rho}\vec{a}_\rho \text{ V/m}, \vec{D} = \frac{922.04}{\rho}\vec{a}_\rho \text{ pC/m}^2$$

$$V = 1.03 \text{ V}, \ \vec{E} = 57.93\vec{a}_\rho \text{ V/m}, \ \vec{D} = 5.12\vec{a}_\rho \text{ nC/m}^2, \ Q = 579.33 \text{ pC}, \ C = 28.97 \text{ pF}$$

$L = 10$ cm
$\epsilon_r = 12$

$\rho = 2$ cm, $V = 15$ V
$\rho = 20$ cm, $V = -5$ V
$\rho = 10$ cm, $V = ?$ $Q = ?$
$\rho = 15$ cm, $\vec{E} = ?$ $C = ?$
$\rho = 18$ cm, $\vec{D} = ?$

$15 = \ln(2\times10^{-2})A + B$
$-5 = \ln(20\times10^{-2})A + B$

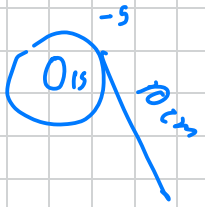
$A = -8.69$ $B = -18.98$
$V = -8.69\ln(\rho) - 18.98$ V
$V@10\text{cm} = -8.69\ln(10\times10^{-2}) - 18.98$
$V@10\text{cm} = 1.03$ V

$\vec{E} = -\frac{d}{d\rho} V \vec{a}_\rho$ V/m
$= -\frac{d}{d\rho}(-8.69\ln(\rho) - 18.98)\vec{a}_\rho$ V/m
$= -\frac{d}{d\rho}(-8.69)\frac{d}{d\rho}\ln(\rho) - \frac{d}{d\rho}(-18.98)$
$= -(-8.69)\frac{1}{\rho}$
$= \frac{8.69}{\rho}\vec{a}_\rho$ V/m

$D = \epsilon_0 \epsilon_r E$
$= \frac{1}{36\pi}\times10^{-9}\times12\pi\frac{8.69}{\rho}\vec{a}_\rho$ C/m²
$= \frac{922.09}{\rho}$ pC/m²

$E = \frac{8.69}{15\times10^{-2}}\vec{a}_\rho$ V/m
$E = 57.93\vec{a}_\rho$ V/m

$D@18\text{cm} = 5.12$ nC/m²



$Q = \oint_S \vec{D}\cdot d\vec{S}$
$= \frac{922.09\rho}{\rho}\vec{a}_\rho \times 2\pi\rho l$
$= 922.09\rho \times 2\pi 10\times10^{-2}$

$2\pi r = h$
$2\pi\rho l$

$Q = 579.33\rho$C

$C = \frac{Q}{V} = \frac{579.33\rho}{|-5-15|}$ F
$C = 28.97\rho$F

$V = -8.69\ln(\rho) - 18.98$ V, $\vec{E} = \frac{8.69}{\rho}\vec{a}_\rho$ V/m, $D = \frac{922.09}{\rho}\vec{a}_\rho$ pC/m²
$V = 1.03$V, $\vec{E} = 57.93\vec{a}_\rho$ V/m, $\vec{D} = 5.12\vec{a}_\rho$ nC/m², $Q = 579.33$pC, $C = 28.97$pF

ชยาฤทธิ์ เกษมเทวินทร์

66011314

Assignment 5

Spherical Capacitor ประกอบด้วย Dielectric มี $\varepsilon_r = 9$ มีเงื่อนไขขอบเขตที่ $r = 10$ cm มี $V = 50$ V และที่ $r = 50$ cm มี $V = -40$ V จงหา V, V ที่ $r = 20$ cm, $\vec{E}$, $\vec{E}$ ที่ $r = 30$ cm, $\vec{D}$, $\vec{D}$ ที่ $r = 40$ cm, Q และ C

$$V = \frac{11.25}{r} - 62.50 \text{ V}, \vec{E} = \frac{11.25}{r^2}\vec{a}_r \text{ V/m}, \ \vec{D} = \frac{895.25}{r^2}\vec{a}_r \text{ pC/m}^2$$

$$V = -6.25 \text{ V}, \ \vec{E} = 125\vec{a}_r \text{ V/m}, \ \vec{D} = 5.60\vec{a}_r \text{ nC/m}^2, \ Q = 11.25 \text{ nC}, \ C = 125 \text{ pF}$$

$\varepsilon_r = 9$ $r = 10\,cm$ $V = 50\,V$

$r = 50\,cm$ $V = -40\,V$

$r = 20\,cm$ $V = ?$ $Q = ?$

$r = 30\,cm$ $\vec{E} = ?$ $C = ?$

$r = 48\,cm$ $\vec{D} = ?$

$$50 = -\frac{A}{10\times10^{-2}} + B$$

$$-40 = -\frac{A}{50\times10^{-2}} + B$$

$A = 11.25$ $B = -62.50$

$$V = \frac{11.25}{r} - 62.50\ V$$

$$V@20\,cm = \frac{11.25}{20\times10^{-2}} - 62.50\ V$$

$$V@20\,cm = -6.25\ V$$

$$\vec{E} = -\frac{d}{dr}\left(\frac{11.25}{r} - 62.50\right)$$

$$= -\frac{d}{dr}11.25r^{-1} - \cancel{\frac{d}{dr}62.50}^{0}$$

$$= -\frac{-11.25}{r^2}$$

$$= \frac{11.25}{r^2}\vec{a}_r\ V/m$$

$$E@30\,cm = \frac{11.25}{(30\times10^{-2})^2}\vec{a}_r\ V/m$$

$$E@30\,cm = 125\,\vec{a}_r\ V/m$$

$$\vec{D} = \varepsilon_0\varepsilon_r E$$

$$= \frac{1}{36\pi}\times10^{-9}\times9\,\frac{11.25}{r^2}\vec{a}_r\ C/m^2$$

$$= \frac{895.25}{r^2}\vec{a}_r\ pC/m^2$$

$$D@48\,cm = \frac{1}{36\pi}\times10^{-9}\times9\times\frac{11.25}{(48\times10^{-2})^2}$$

$$D@48\,cm = 5.60\,\vec{a}_r\ nC/m^2$$

$$Q = \oint_S D\cdot d\vec{s}$$

$$= \frac{895.25}{r^2}\times 4\pi\cancel{r^2}$$

$$Q = 11.25\ nC$$

$$C = \frac{Q}{V} = \frac{11.25n}{|-40-50|}\ F$$

$$C = 125\ pF$$

$$V = \frac{11.25}{r} - 62.50\,V,\ \vec{E} = \frac{11.25}{r^2}\vec{a}_r\ V/m,\ \vec{D} = \frac{895.25}{r^2}\vec{a}_r\ pC/m^2$$

$V = -6.25\,V$, $\vec{E} = 125\,\vec{a}_r\ V/m$, $\vec{D} = 5.60\,\vec{a}_r\ nC/m^2$, $Q = 11.25\,nC$, $C = 125\,pF$

ชวิตกฤษณ์ ทาสมเหมือนหวัง

66011314