A user guide to the single-phase stokes equation

The single-phase stokes system of equations in 2D: with a regular staggered grid ($\Delta x = \Delta z$)

Mass conservation: For an incompressible medium

$$\nabla \vec{v} = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(eq. 1)}$$

Momentum conservation: For variable background viscosity

$$\nabla(\eta \nabla \vec{v}) - \nabla P + \rho g = 0$$

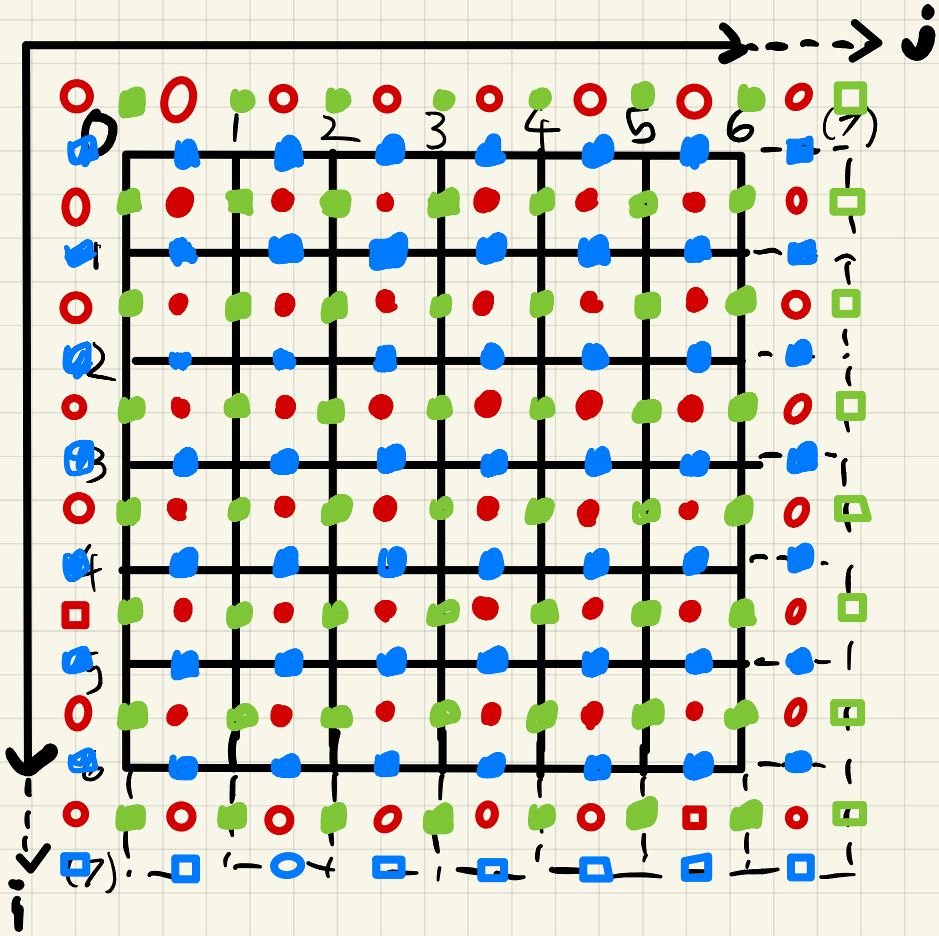
v_x)

$$2\frac{\partial}{\partial x}\left(\eta \frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial z}\left(\eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right) - \frac{\partial P}{\partial x} = 0 \; (+\rho g_x) \qquad \text{(eq. 2a)}$$

v_z)

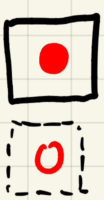
$$2\frac{\partial}{\partial z}\left(\eta \frac{\partial v_z}{\partial z}\right) + \frac{\partial}{\partial x}\left(\eta\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right) - \frac{\partial P}{\partial z} = \rho g_z \qquad \text{(eq. 2b)}$$

The staggered grid system



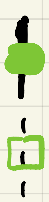
For a g'rid nx by nz,
With staggered Vx & Vz nodes:
Vx = nx+1 by nz+2
Vz = nx+2 by nz+1

We add ghost nodes to make the 3 grids equally sized to nx+2 by nz+2



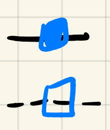
Middle/P grid

$P, \eta_p, T, \alpha, C_p, \rho, H_r, V_{x_{mid}}, V_{z_{mid}}$



Staggered Vx grid

V_x, ρ_{vx}



Staggered Vz grid

V_z, ρ_{vz}

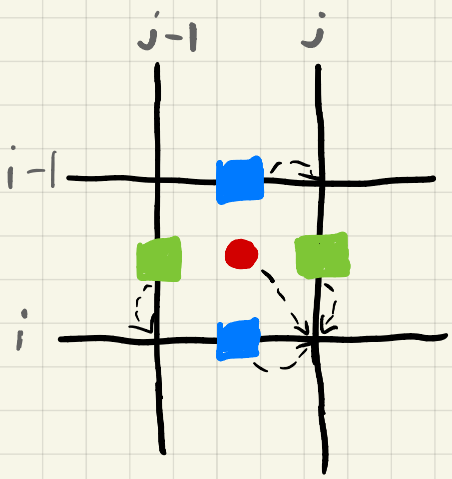
Staggered edge grid

η_e, ρ_e

Discretising mass conservation

Let's start simply by discretising the mass conservation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(eq.1)}$$



For each node $P_{(i,j)}$ we can solve $v_{x(i,j)}$, $v_{x(i,j-1)}$, $v_{z(i,j)}$, $v_{z(i-1,j)}$

eq.1 becomes:

$$P_{scale}\left(\frac{v_{x(i,j-1)} - v_{x(i,j)}}{\Delta x} + \frac{v_{z(i-1,j)} - v_{z(i,j)}}{\Delta z}\right)\left(+\, P_{(i,j)}\, c\, \frac{\Delta x \Delta z}{\eta_{(i,j)}}\right) = 0 \qquad \text{(eq.3)}$$

pressure scaling coefficient, more later

this term should equal zero but in order to make the A matrix complete, we add pressure scaling coefficients

$\Delta x \to 0$, $\Delta z \to 0$, $\frac{\Delta x \Delta z}{\eta} \to 0$

So eq.3 becomes:

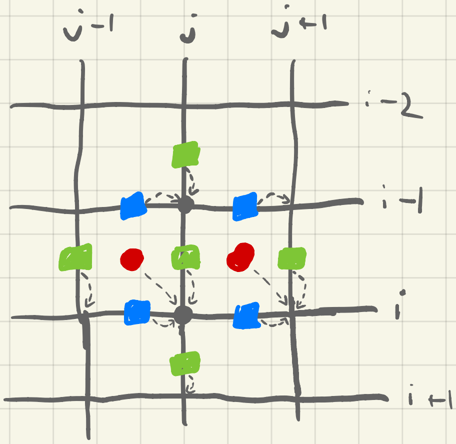
$$\frac{P_{scale}}{\Delta x} v_{x(i,j-1)} - \frac{P_{scale}}{\Delta x} v_{x(i,j)} + \frac{P_{scale}}{\Delta z} v_{z(i-1,j)} - \frac{P_{scale}}{\Delta z} v_{z(i,j)} \left(+\, \frac{\Delta x \Delta z}{\eta_{(i,j)}} P_{(i,j)}\right) = 0 \qquad \text{(eq.4)}$$

Where the black terms fill the A matrix and the coloured terms are part of the solution vector

Momentum conservation discretisation:

X-momentum term:

$$2\frac{\partial}{\partial x}\left(\eta\frac{\partial v_x}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\right)-\frac{\partial P}{\partial x}=0\ (+\rho g_x) \quad \text{(eq. 2a)}$$



For each $v_{x(i,j)}$ node, we can solve $v_{x(i,j)}, v_{x(i-1,j)}, v_{x(i+1,j)}, v_{x(i,j-1)}, v_{x(i,j+1)}$, $v_{z(i,j)}, v_{z(i,j+1)}, v_{z(i-1,j+1)}, v_{z(i-1,j)}$. $\grave{P}_{(i,j)}$, $\grave{P}_{(i,j+1)}$ implicitly

equation 2a becomes:

$$2\eta_{p(i,j+1)}\frac{v_{x(i,j+1)}-v_{x(i,j)}}{\Delta x^2}-2\eta_{p(i,j)}\frac{v_{x(i,j)}-v_{x(i,j-1)}}{\Delta x^2}$$

$$+\eta_{e(i,j)}\left(\frac{v_{x(i+1,j)}-v_{x(i,j)}}{\Delta z^2}+\frac{v_{z(i,j+1)}-v_{z(i,j)}}{\Delta x\Delta z}\right)+\eta_{e(i-1,j)}\left(\frac{v_{x(i,j)}-v_{x(i-1,j)}}{\Delta z^2}+\frac{v_{z(i-1,j+1)}-v_{z(i-1,j)}}{\Delta x\Delta z}\right)$$

$$+P_{scale}\frac{P_{(i,j+1)}-P_{(i,j)}}{\Delta x}=0\left(+\frac{\rho_{(i-1,j)}+\rho_{(i,j)}}{2}g_x\right)$$

(eq. 5)

note that η_p is on • & η_e is on +

which is discretised to:

$$\frac{\eta_{e(i-1,j)}}{\Delta z^2}v_{x(i,j+1)}+\frac{\eta_{e(i,j)}}{\Delta z^2}v_{x(i,j-1)}\ \frac{-2\eta_{p(i,j+1)}}{\Delta x^2}v_{x(i,j+1)}-\frac{2\eta_{p(i,j)}}{\Delta x^2}v_{x(i,j-1)}$$

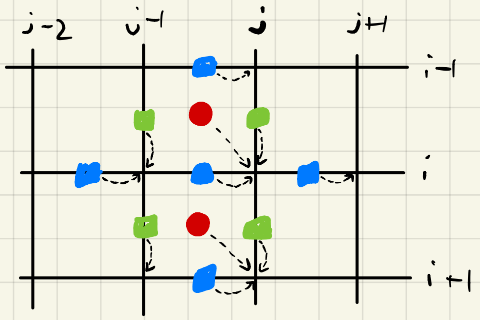
$$+\left(-\frac{2(\eta_{p(i,j+1)}+\eta_{p(i,j)})}{\Delta x^2}-\frac{(\eta_{e(i-1,j)}+\eta_{e(i,j)})}{\Delta z^2}\right)v_{x(i,j)}$$

$$+\frac{\eta_{e(i-1,j)}}{\Delta x\Delta z}v_{x(i-1,j)}+\frac{\eta_{e(i,j)}}{\Delta x\Delta z}v_{x(i,j+1)}-\frac{\eta_{e(i-1,j)}}{\Delta x\Delta z}v_{x(i,j)}-\frac{\eta_{e(i,j)}}{\Delta x\Delta z}v_{x(i-1,j+1)}$$

$$+\frac{P_{scale}}{\Delta x}\grave{P}_{(i,j+1)}-\frac{P_{scale}}{\Delta x}\grave{P}_{(i,j)}=0\left(+\frac{\rho_{(i-1,j)}+\rho_{(i,j)}}{2}g_x\right) \quad \text{(eq. 6)}$$

And for the Vz momentum:

$$2\frac{\partial}{\partial z}\left(\eta \frac{\partial v_z}{\partial z}\right) + \frac{\partial}{\partial x}\left(\eta\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\right) - \frac{\partial P}{\partial z} = \rho g_z \qquad \text{(eq.2b)}$$



Foreach $v_{z(i,j)}$ node, we can solve:
$v_{z(i-1,j)}, v_{z(i+1,j)}, v_{z(i,j-1)}, v_{z(i,j+1)}, v_{z(i,j)},$
$v_{x(i,j)}, v_{x(i+1,j)}, v_{x(i+1,j-1)}, v_{x(i,j-1)}$
$P_{(i,j)}, P_{(i+1,j)}$
implicitly

eq.2 can be rewritten as:

$$2\eta_{p(i+1,j)}\frac{v_{z(i+1,j)}-v_{z(i,j)}}{\Delta z^2} - 2\eta_{(i,j)}\frac{v_{z(i,j)}-v_{z(i-1,j)}}{\Delta z^2}$$

$$+\eta_{e(i,j)}\left(\frac{v_{z(i,j+1)}-v_{z(i,j)}}{\Delta x^2} + \frac{v_{x(i+1,j)}-v_{x(i,j)}}{\Delta x\Delta z}\right) - \eta_{e(i,j-1)}\left(\frac{v_{z(i,j)}-v_{z(i,j-1)}}{\Delta x^2}+\frac{v_{x(i+1,j)}-v_{x(i,j-1)}}{\Delta x \Delta z}\right)$$

$$-\, pscale\,\frac{P'_{(i,j+1)}-P'_{(i,j)}}{\Delta z} = \frac{\rho_{(i,j-1)}-\rho_{(i,j)}}{2} \qquad \text{(eq.7)}$$

and is discretised to:

$$\frac{\eta_{e(i,j)}}{\Delta x^2} v_{z(i,j+1)} + \frac{\eta_{e(i,j-1)}}{\Delta x^2} v_{z(i,j-1)} - \frac{2\eta_{p(i+1,j)}}{\Delta z^2} v_{z(i+1,j)} - \frac{2\eta_{p(i,j)}}{\Delta z^2} v_{z(i-1,j)}$$

$$+\left(-\frac{2(\eta_{p(i+1,j)}+\eta_{p(i,j)})}{\Delta z^2} - \frac{(\eta_{e(i,j)}+\eta_{e(i,j-1)})}{\Delta x\Delta z}\right) v_{z(i,j)}$$

$$+\frac{\eta_{e(i,j)}}{\Delta x\Delta z} v_{x(i+1,j)} + \frac{\eta_{e(i,j-1)}}{\Delta x\Delta z} v_{x(i,j-1)} - \frac{\eta_{e(i,j)}}{\Delta x \Delta z} v_{x(i,j)} - \frac{\eta_{e(i,j-1)}}{\Delta x\Delta z} v_{x(i+1,j)}$$

$$+\frac{pscale}{\Delta z} P'_{(i,j)} - \frac{pscale}{\Delta z} P_{(i,j+1)} = \frac{\rho_{(i,j-1)}+\rho_{(i,j)}}{2} g_z \qquad \text{(eq.8)}$$

Creating the A-matrix

$A \cdot C = RHS$ → $C = A \backslash RHS$

Solution vector

backslash operator