Algebra and Geometry

Yongting Huang 黄咏婷
College of Science, HITSZ
ythuang@hit.edu.cn

September 16th, 2025

- Linear algebra is built on the two operations:

  adding vectors and multiplying by scalars.

- Allowing all choices of c and d, the **linear combinations**

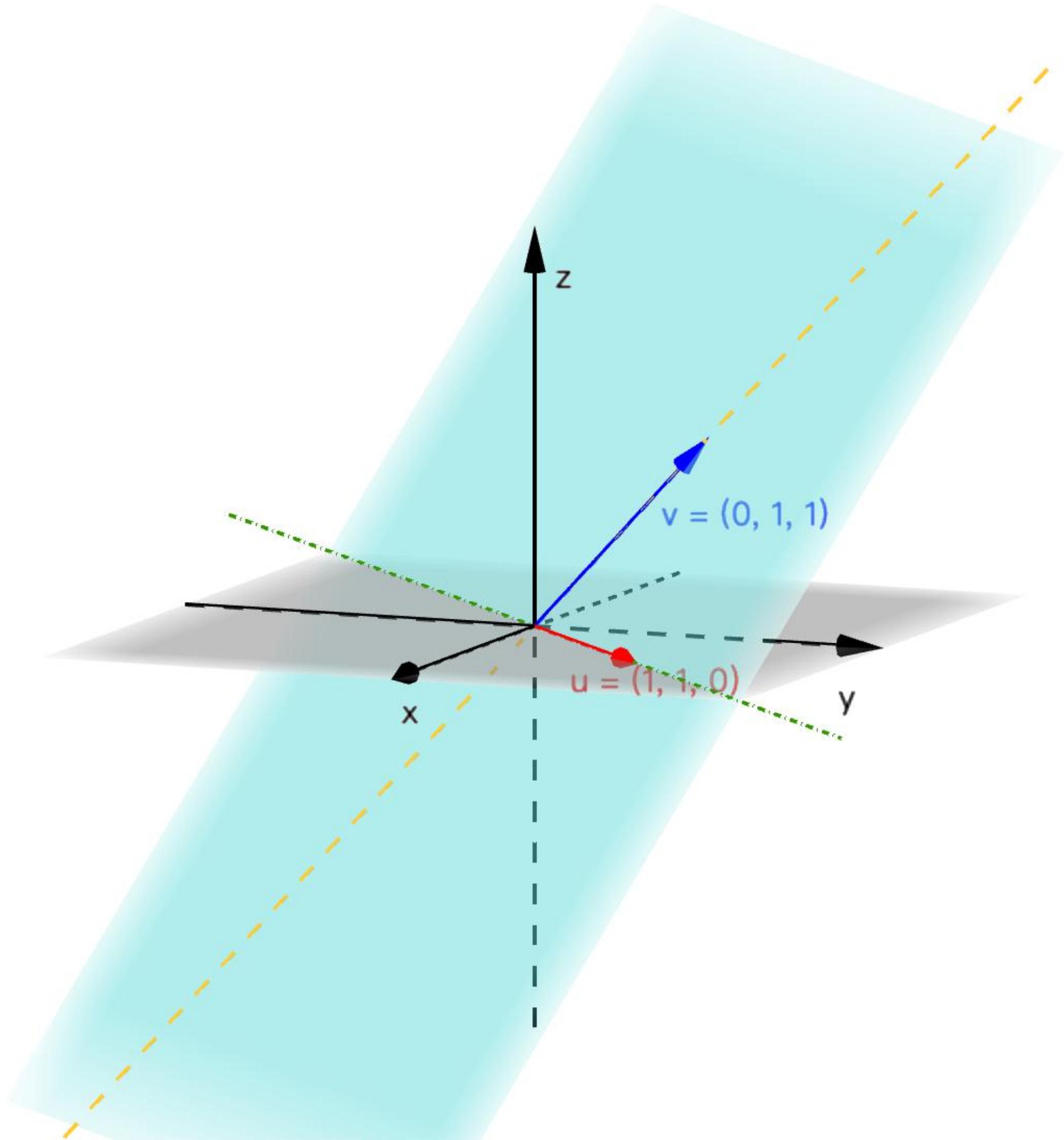
$$c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

  fill the xy plane unless $\boldsymbol{v}$ is in the line with $\boldsymbol{w}$.

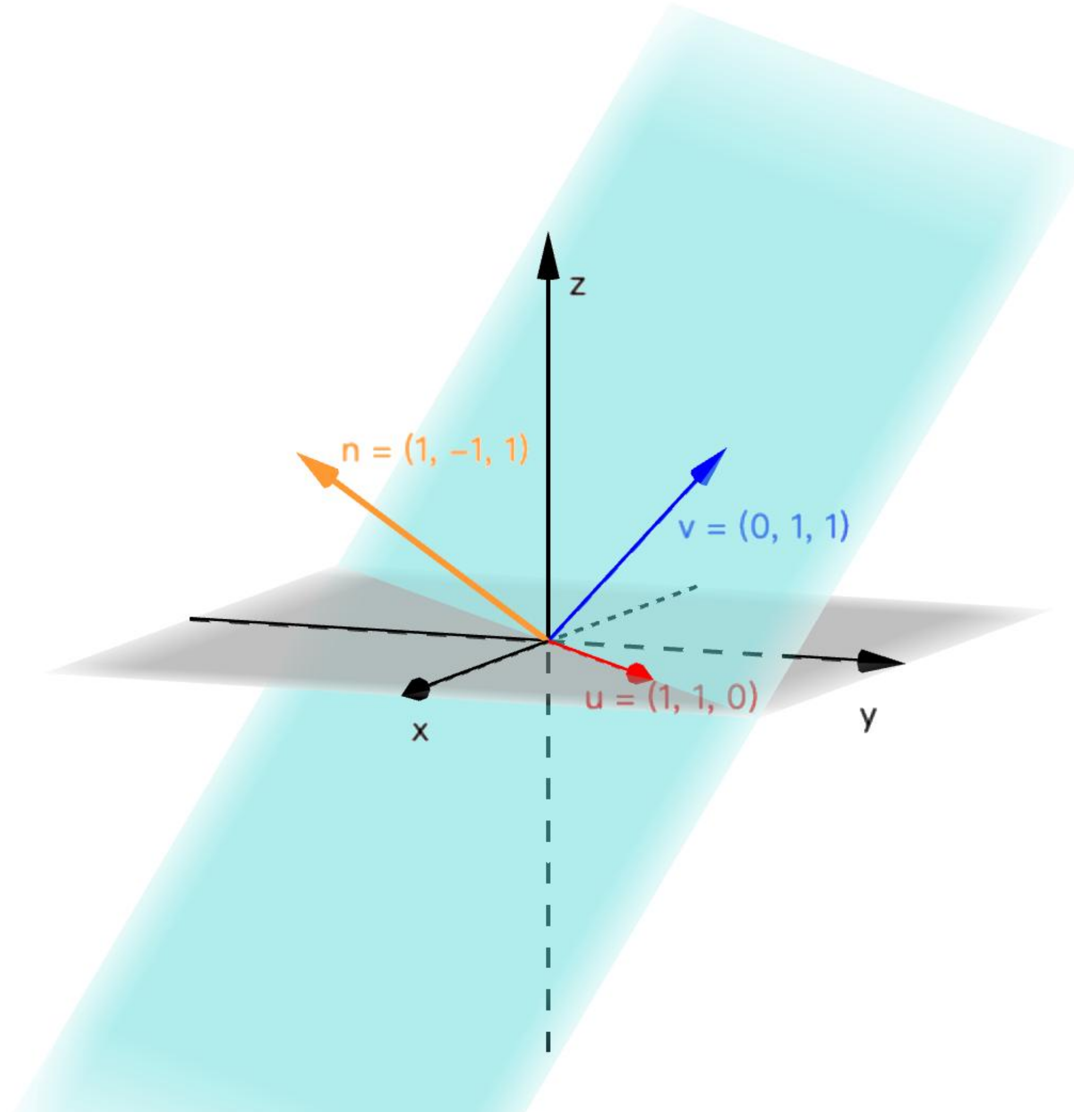
- Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Their linear combinations

$$a\boldsymbol{u} + b\boldsymbol{v} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ b \end{bmatrix}.$$

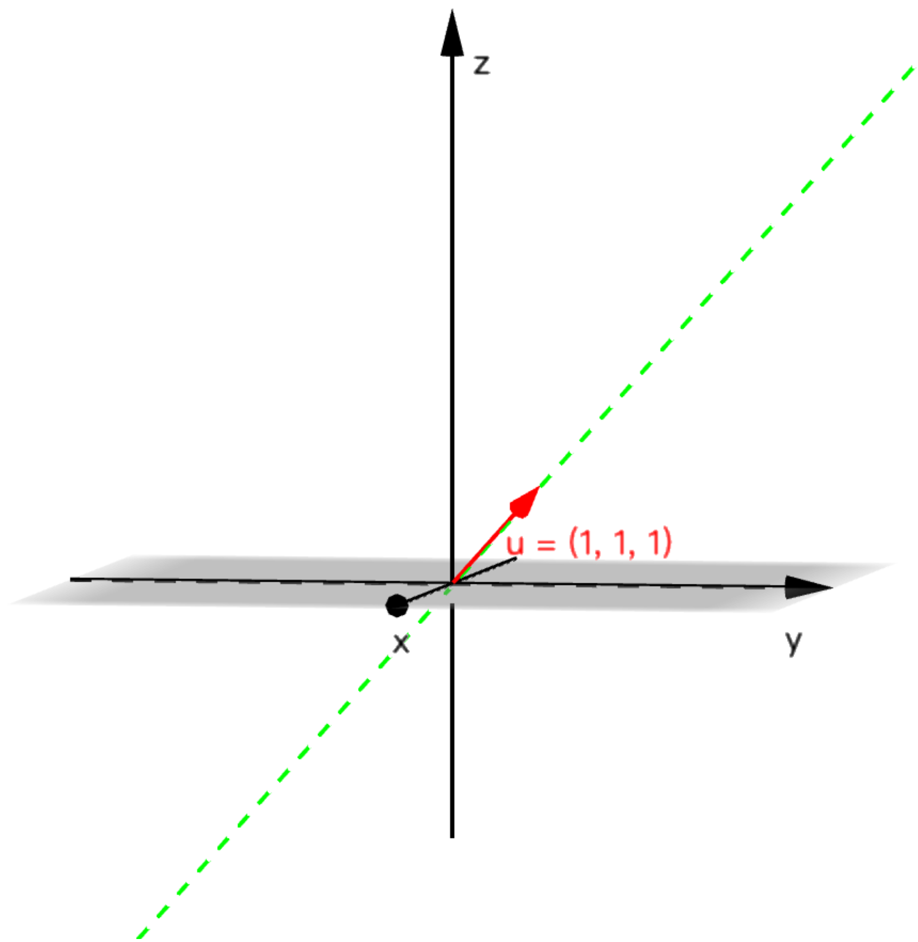
  fill a **plane** in the xyz space.



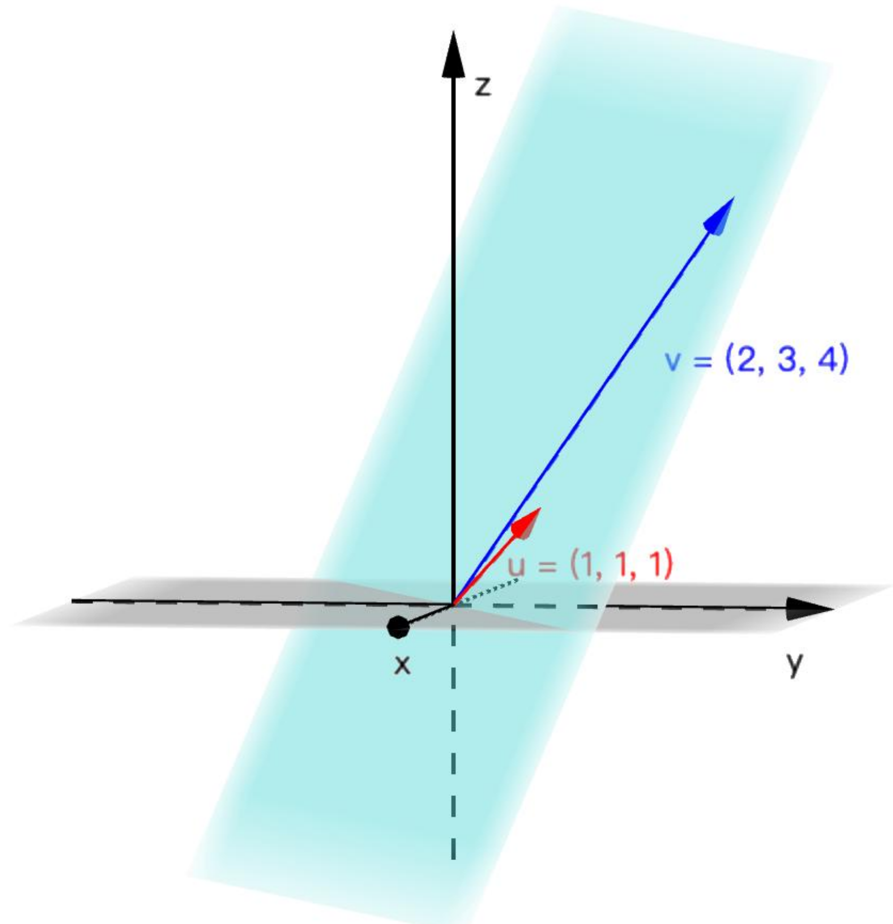
The linear combinations $a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ b \end{bmatrix}$ *fill a* **plane** *in the xyz space.*



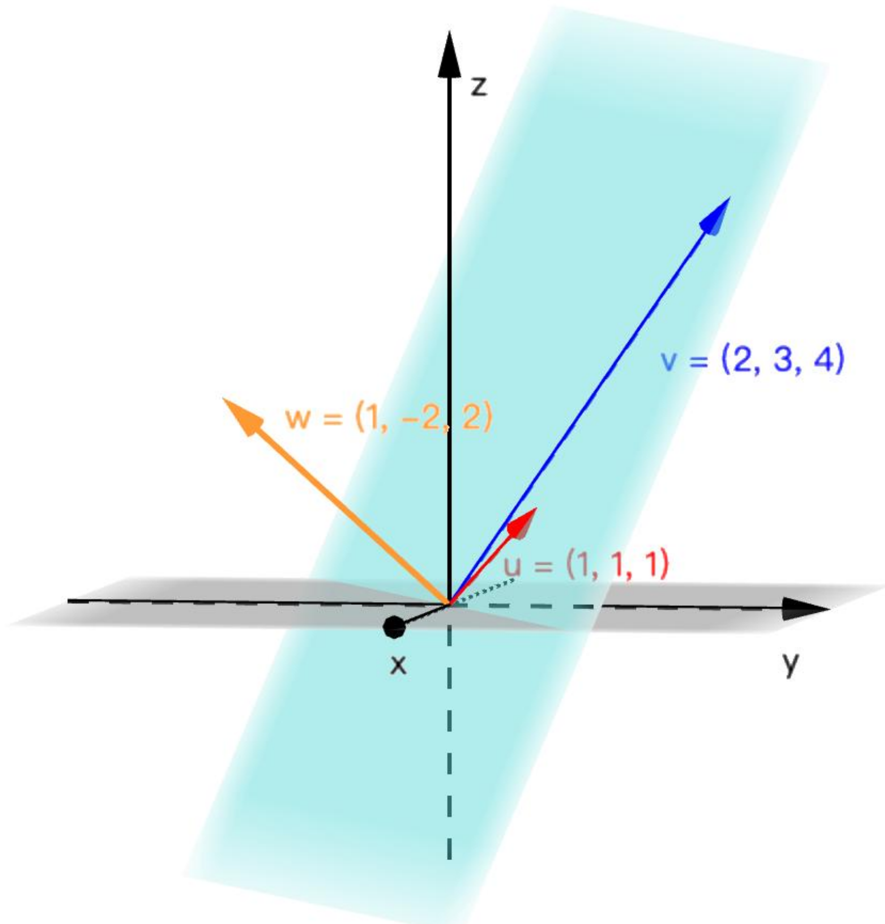
$$\boldsymbol{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ is perpendicular to the plane of } \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$



Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *The combinations* $a\boldsymbol{u}$ *fill a line through* $(0,0,0)$ *in the* xyz *space.*



The combinations of $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ *fill a plane through* $(0,0,0)$.



How can we prove the vector $\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ *is not on the plane of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*?*

Assignment for Section 1.1: Vectors and linear combinations

(1) If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, compute and draw the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

(2) From $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

(3) What combination $c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$?

Express the question as two equations for the coefficients c and d in the linear combination.

Note: in printing, a vector is denoted as a lowercase letter in boldface, e.g., $\boldsymbol{v}$.
In handwriting, we put an arrow over the letter to denote this vector, e.g., $\vec{v}$.