PHYS1001B College Physics IB

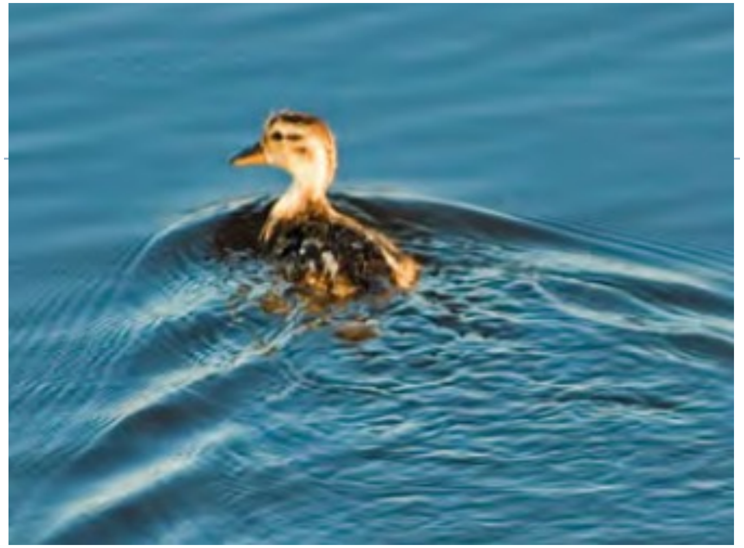
Waves I — Mechanical Waves (Ch. 15)

Introduction

- This chapter focuses on *waves travelling along a stretched string*, such as on a guitar

- The next chapter focuses on *sound waves*, such as those produced by a guitar string being played

- Before we do all this, though, our first job is to classify the countless waves of the everyday world into basic types



Outline

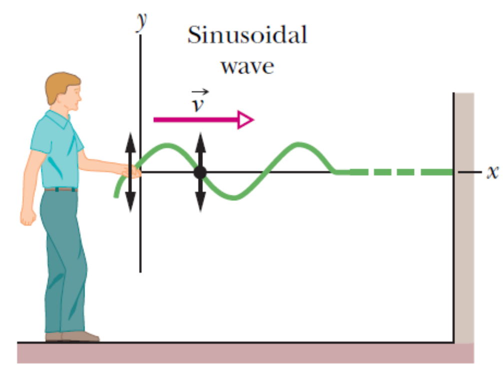
- 15-1 Types of Transverse Waves
- 15-2 Periodic Waves
- 15-3 Mathematical Description of a Wave
- 15-4 Speed of a Transverse Waves
- 15-5 Energy in Wave Motion
- 15-6 Wave Interference, Boundary Conditions and Superposition
- 15-7 Standing Waves on a String
- 15-8 Normal Modes of a String

15-1 Transverse Waves

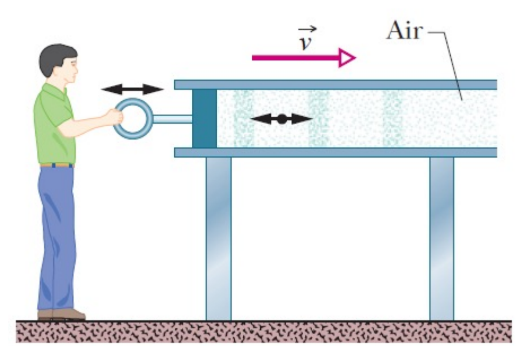
- Mechanical waves (機械波)
  - are governed by Newton's laws, and exist only within a material *medium*
  - Examples include *water waves*, *sound waves*, and seismic waves
- Electromagnetic waves (電磁波)
  - require *no material medium* to exist
  - Examples include visible and ultraviolet *light*, radio and television waves, microwaves, x rays, and radar waves
- Matter waves (物質波)
  - are associated with electrons, protons, and other fundamental particles, and even atoms and molecules

15-1 Transverse Waves

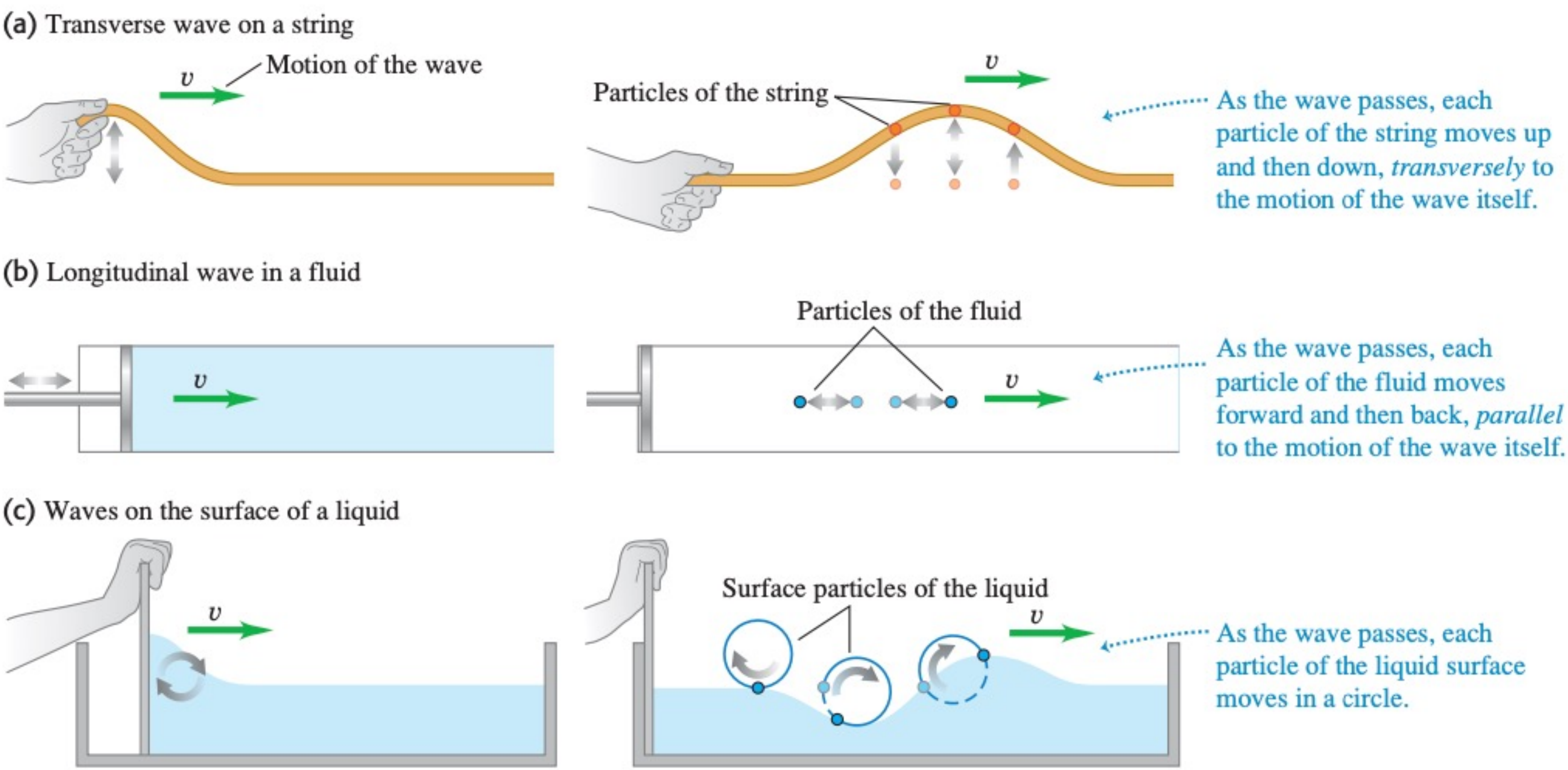
- Transverse wave (橫波)
  - the displacement of every such oscillating element along the wave is *perpendicular* to the direction of travel of the wave
    - e.g. string waves, water waves



- Longitudinal wave (縱波)
  - the motion of the oscillating particles is *parallel* to the direction of the wave's travel
    - e.g. sound waves



15-1 Transverse Waves



15.1 Three ways to make a wave that moves to the right. (a) The hand moves the string up and then returns, producing a transverse wave. (b) The piston moves to the right, compressing the gas or liquid, and then returns, producing a longitudinal wave. (c) The board moves to the right and then returns, producing a combination of longitudinal and transverse waves.

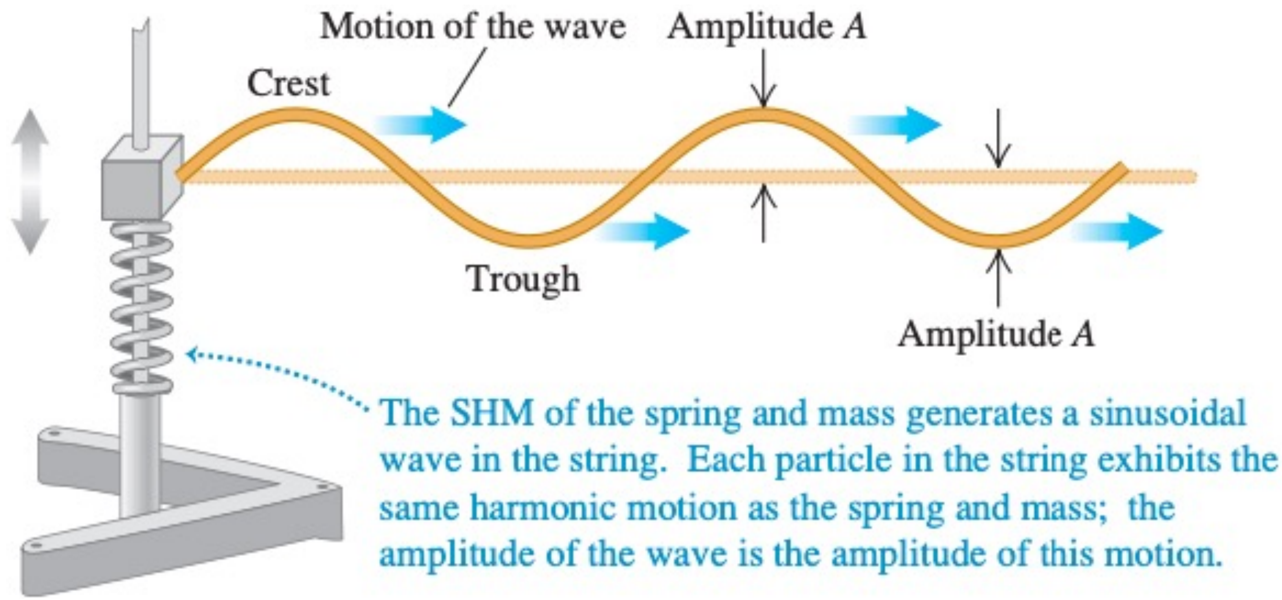
15-1 Transverse Waves

15.2 "Doing the wave" at a sports stadium is an example of a mechanical wave: The disturbance propagates through the crowd, but there is no transport of matter (none of the spectators moves from one seat to another).

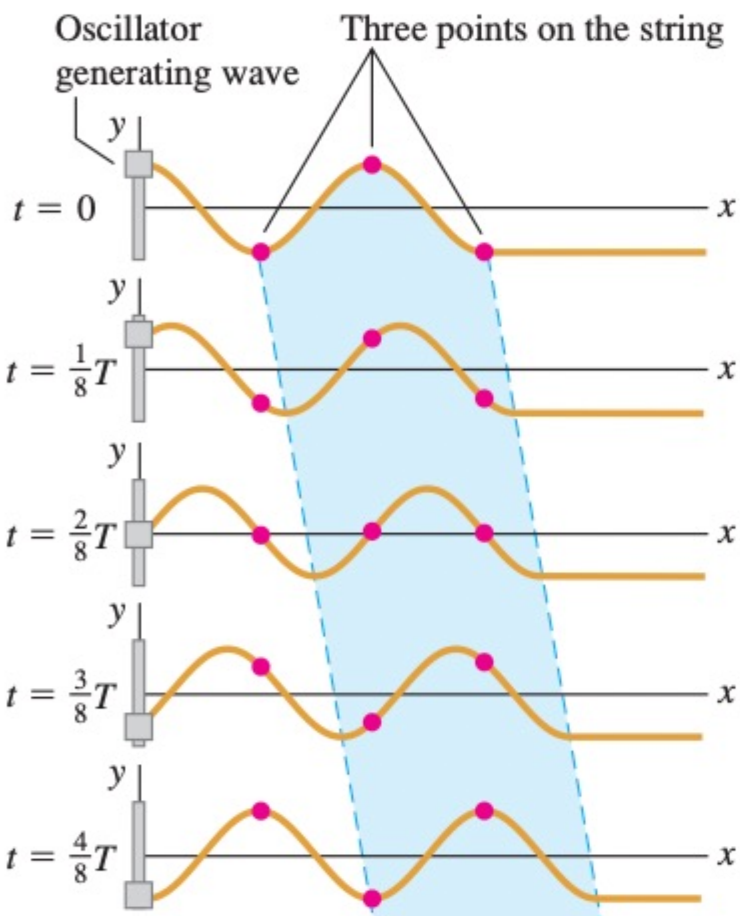
Waves transport energy, but not matter, from one region to another

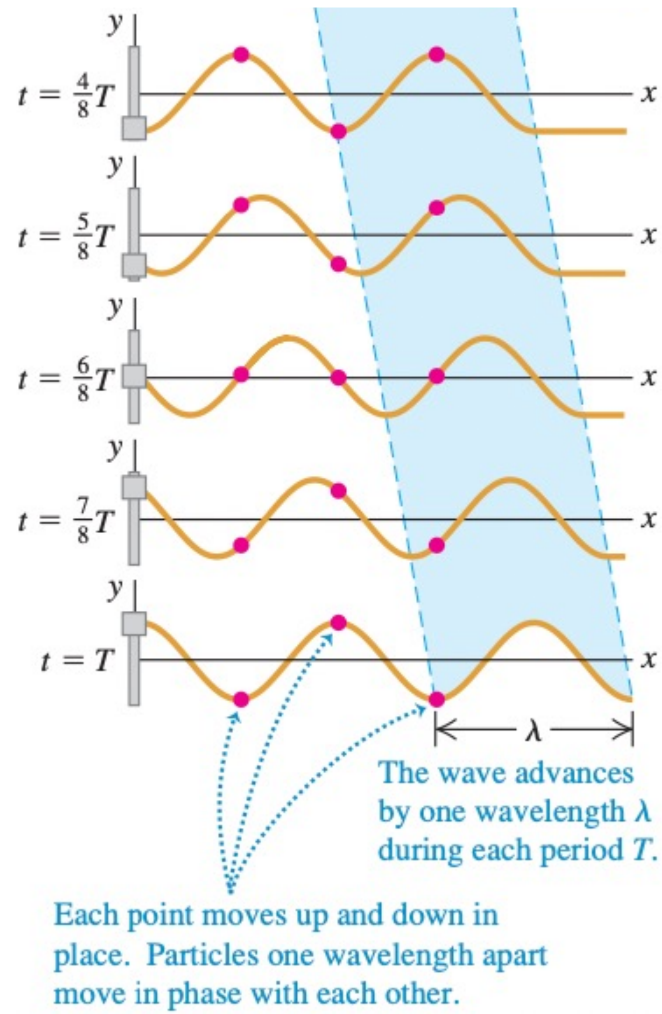
15-2 Periodic Waves

A more interesting situation develops when we give the free end of the string a repetitive, or *periodic,* motion. Then each particle in the string also undergoes periodic motion as the wave propagates, and we have a **periodic wave.**

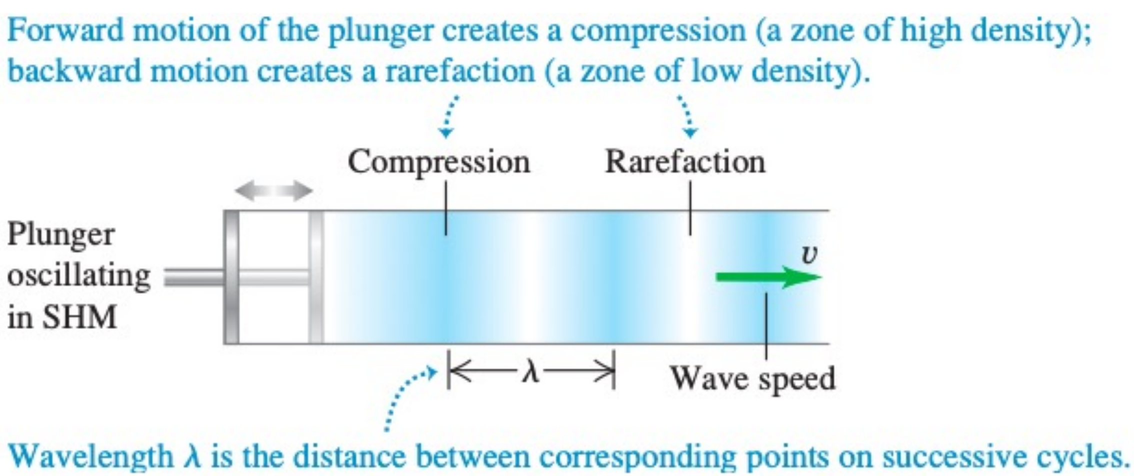


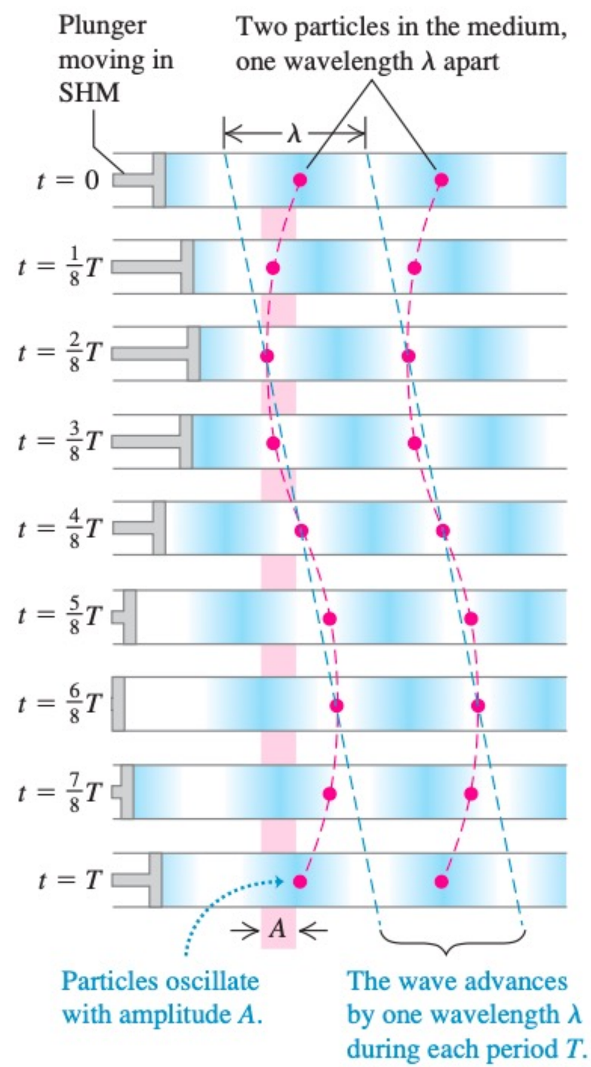
Periodic Transverse Waves





Periodic Longitudinal Waves





15-2 Periodic Waves

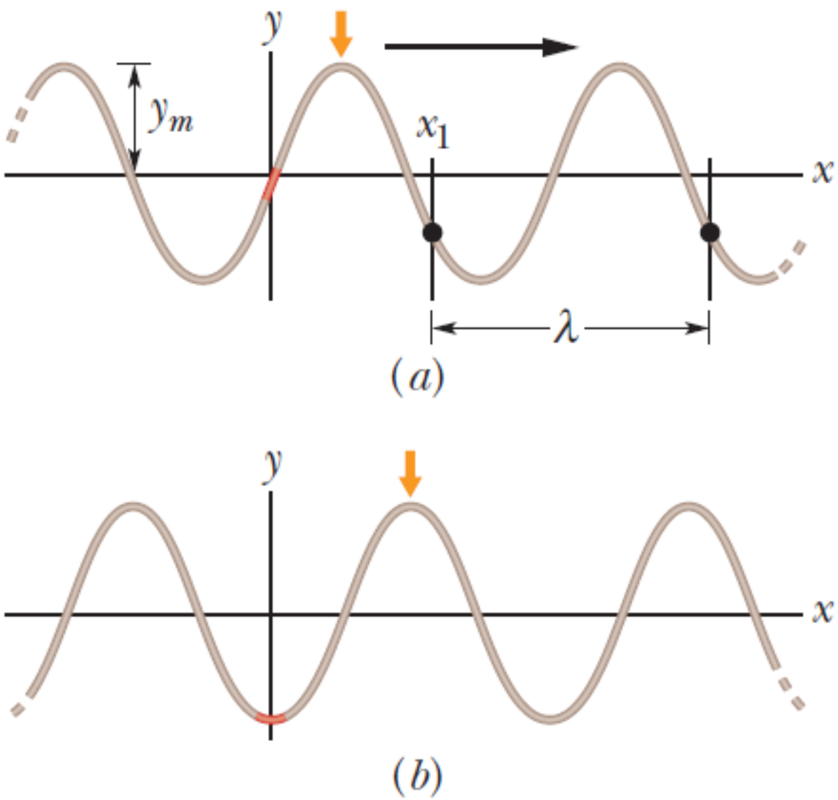
$$v = \lambda f \qquad \text{(periodic wave)}$$

λ wavelength, v wave speed, f frequency

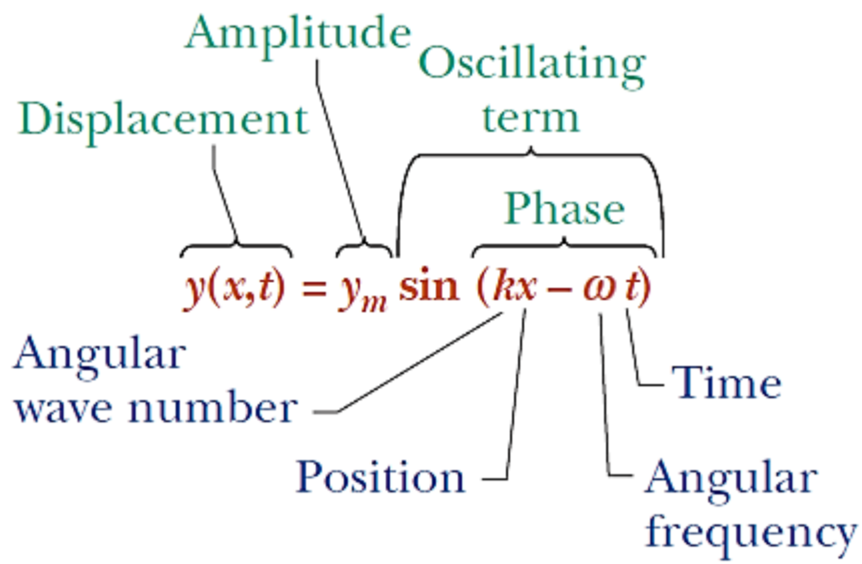
CAUTION **Wave motion vs. particle motion** Be very careful to distinguish between the motion of the *transverse wave* along the string and the motion of a *particle* of the string. The wave moves with constant speed v *along* the length of the string, while the motion of the particle is simple harmonic and *transverse* (perpendicular) to the length of the string.

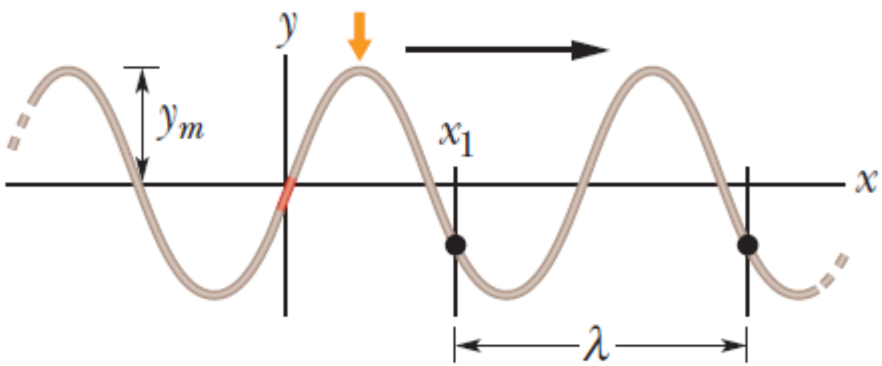
15-3 Mathematical Description of a Wave

- A string wave travelling in the positive direction of an *x* axis is shown below



$$y(x,t) = y_m \sin(kx - \omega t)$$





$$y(x,t) = y_m \sin(kx - \omega t)$$

- Amplitude (振幅), y_m
  - is the magnitude of the maximum displacement of the elements from their equilibrium positions
- Phase (相位/相角) of the wave, $(kx - \omega t)$
  - is the argument of the sine function. As the wave sweeps through a string element at a particular position *x*, the phase changes linearly with time *t*
- Wavelength (波長), λ
  - is the distance between repetitions of the shape of the wave

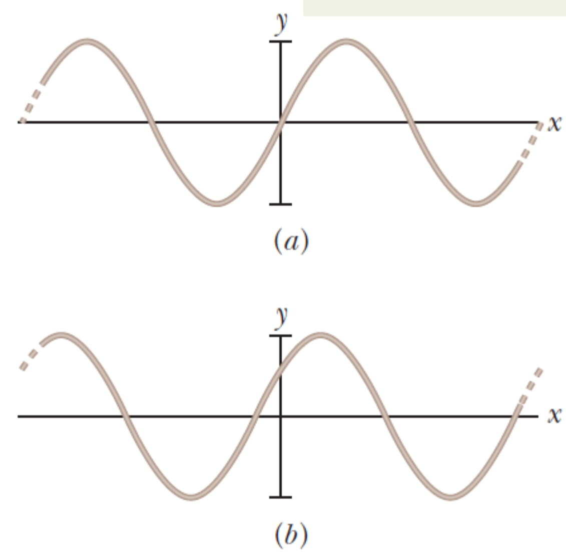
15-3 Mathematical Description of a Wave

- (Angular) wave number (波數), k
  - is the number of wavelengths per 2π units of distance, $k = \dfrac{2\pi}{\lambda}$
  - SI unit: rad/m

- Period (週期) (T), angular frequency (角頻率) (ω), and frequency (頻率) (f)
  - $\omega = \dfrac{2\pi}{T} = 2\pi f$

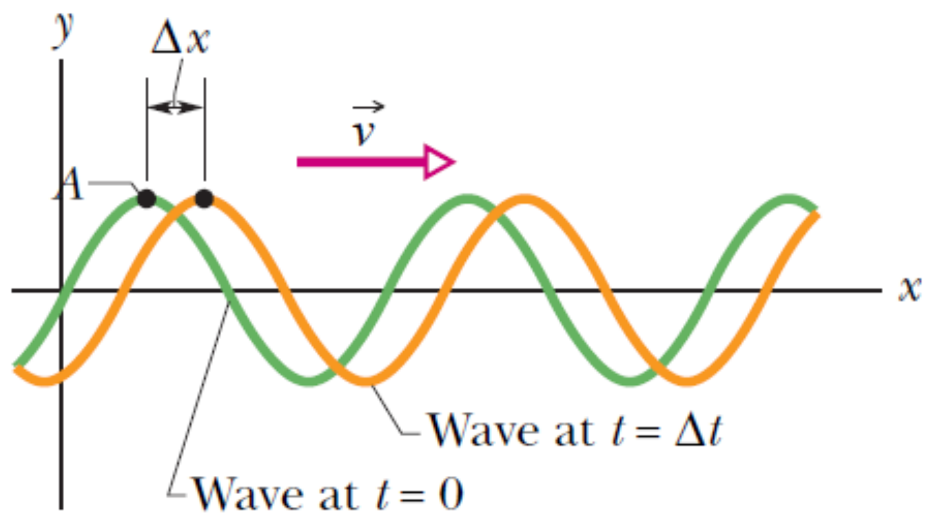
- Phase constant (相位常數), ϕ
  - $y(x,t) = y_m \sin(kx - \omega t + \phi)$
    - E.g. (a) $\phi = 0$; (b) $\phi = \frac{\pi}{5}$

The effect of the phase constant ϕ is to shift the wave.



15-3 Mathematical Description of a Wave

- As a wave moves, each point of the moving wave form, such as point *A* on a peak, retains its displacement *y*
  - $y = y_m \sin(kx - \omega t) = constant$
  - $kx - \omega t = constant$
  - $k\dfrac{dx}{dt} - \omega = 0$
  - $\dfrac{dx}{dt} = v = \dfrac{\omega}{k}$
  - $v = \dfrac{\omega}{k} = \dfrac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \dfrac{\lambda}{T} = \lambda f$



Sample Problem

- A wave traveling along a string is described by
$$y(x,t) = 0.00327 \sin(72.1x - 2.72t),$$
- in which the numerical constants are in SI units (0.00327 m, 72.1 rad/m, and 2.72 rad/s).
- (a) What is the amplitude of this wave?

$$y_m = 0.00327 \text{ m} = 3.27 \text{ mm}$$

- (b) What are the wavelength, period, and frequency of this wave?

$$\lambda = \frac{2\pi}{k} = \frac{2\pi \text{ rad}}{72.1 \text{ rad/m}} = 0.0871 \text{ m} = 8.71 \text{ cm.}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{2.72 \text{ rad/s}} = 2.31 \text{ s}$$

$$f = \frac{1}{T} = \frac{1}{2.31 \text{ s}} = 0.433 \text{ Hz}$$

Sample Problem

- (c) What is the velocity of this wave?

$$v = \frac{\omega}{k} = \frac{2.72 \text{ rad/s}}{72.1 \text{ rad/m}} = 0.0377 \text{ m/s}$$
$$= 3.77 \text{ cm/s.}$$

- (d) What is the displacement y of the string at x = 22.5 cm and t = 18.9 s?

$$y = 0.00327 \sin(72.1 \times 0.225 - 2.72 \times 18.9)$$
$$= (0.00327 \text{ m}) \sin(-35.1855 \text{ rad})$$
$$= (0.00327 \text{ m})(0.588)$$
$$= 0.00192 \text{ m} = 1.92 \text{ mm.} \qquad ($$

Sample Problem

- (e) What is u, the *transverse velocity* of the element of the string at x = 22.5 cm and t = 18.9 s?

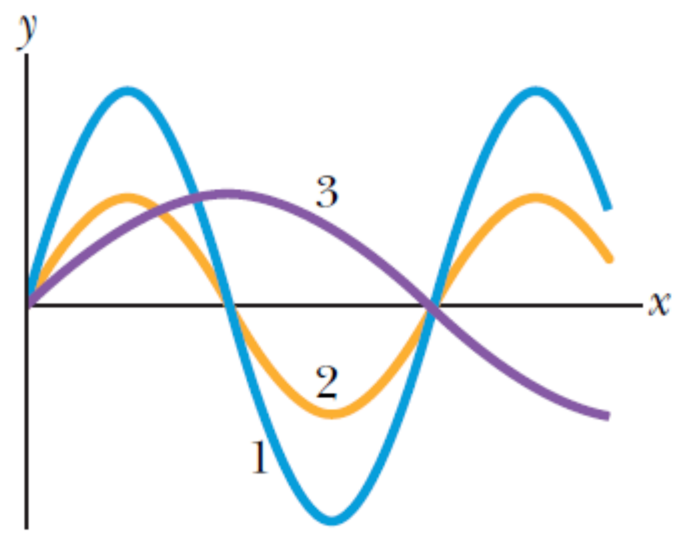
$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$$

$$\begin{aligned} u &= (-2.72 \text{ rad/s})(3.27 \text{ mm}) \cos(-35.1855 \text{ rad}) \\ &= 7.20 \text{ mm/s.} \end{aligned}$$

- Remarks
  - For an element at a certain location x, we find the rate of change of y by taking the derivative with respect to t while *treating x as a constant*
  - A derivative taken while one (or more) of the variables is treated as a constant is called a *partial derivative* and is represented by the symbol $\frac{\partial}{\partial t}$ rather than $\frac{d}{dt}$

Questions

- Figure shows three waves that are separately sent along a string that is stretched under a certain tension along an *x* axis.



- Rank the waves according to their (a) wavelengths, (b) speeds, and (c) angular frequencies, greatest first.

15-3 Mathematical Description of a Wave

$$y(x, t) = A\cos(kx - \omega t)$$

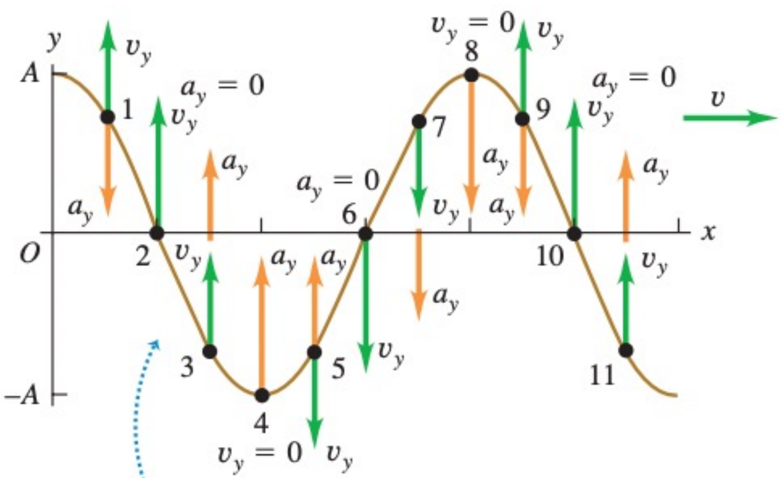
Wave function

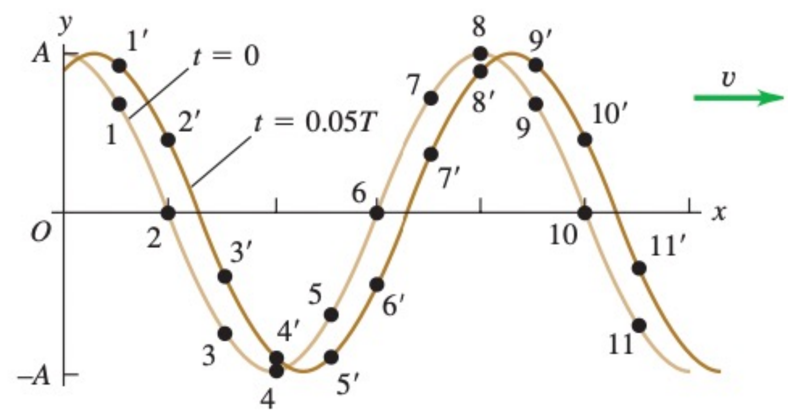
$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t)$$

Velocity

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x, t)$$

Acceleration





15-3 Mathematical Description of a Wave

$$\begin{cases} a_y(x,t) = \dfrac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t) \\ \dfrac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x,t) \end{cases}$$

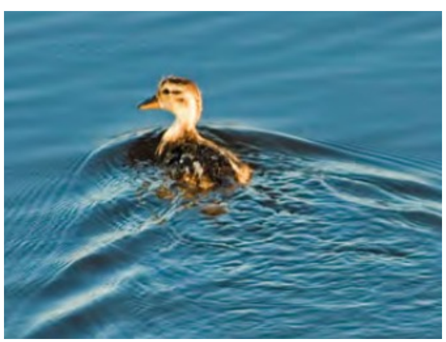
$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad \text{(wave equation)}$$

15-3 Mathematical Description of a Wave

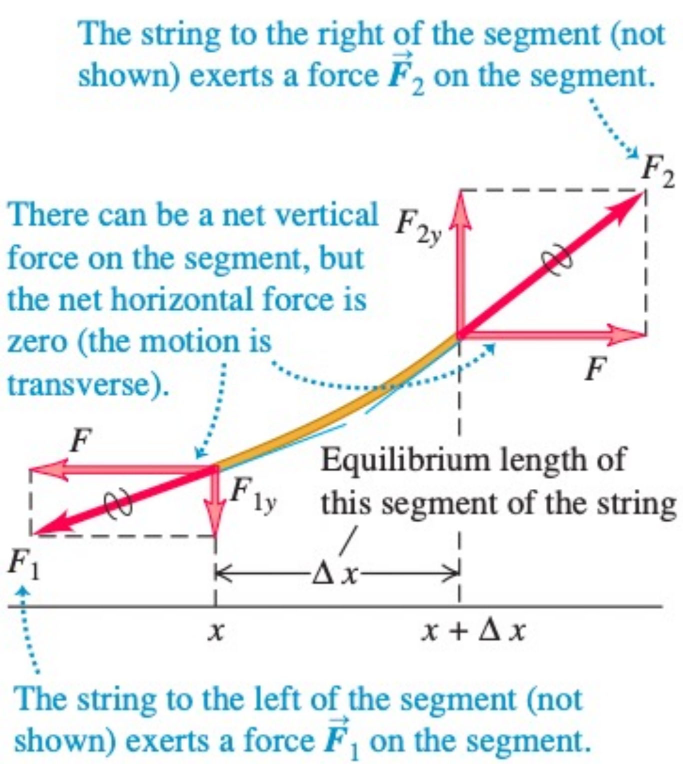
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- The wave equation is an important *second-order linear partial differential equation (PDE)* for the description of waves – as they occur in physics – such as sound waves, light waves and water waves



15-4 Speed of a Transverse Waves

15.13 Free-body diagram for a segment of string. The force at each end of the string is tangent to the string at the point of application.



$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \qquad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$

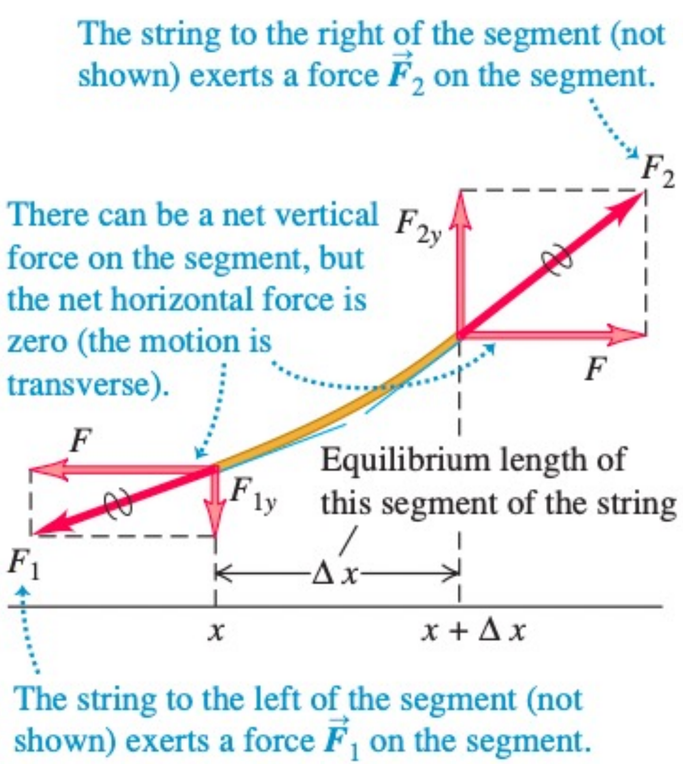
$$F_y = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu\,\Delta x\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

15-4 Speed of a Transverse Waves

15.13 Free-body diagram for a segment of string. The force at each end of the string is tangent to the string at the point of application.



$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

$$v = \sqrt{\frac{F}{\mu}}$$

15-4 Speed of a Transverse Waves

General form

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

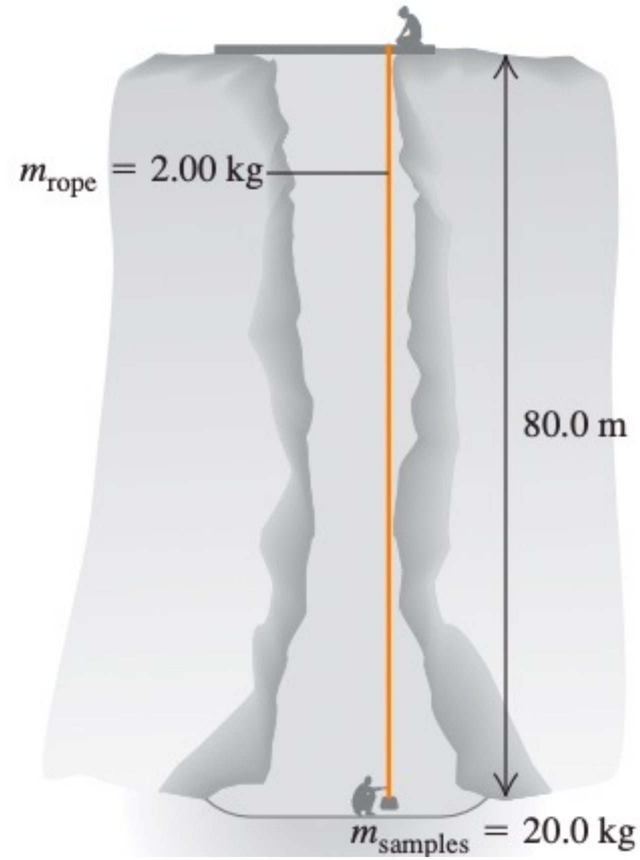
The tension F in the string plays the role of the restoring force; it tends to bring the string back to its undisturbed, equilibrium configuration.

The linear mass density m provides the inertia that prevents the string from returning instantaneously to equilibrium.

Sample Problem

Example 15.3 Calculating wave speed

One end of a 2.00-kg rope is tied to a support at the top of a mine shaft 80.0 m deep (Fig. 15.14). The rope is stretched taut by a 20.0-kg box of rocks attached at the bottom. (a) The geologist at the bottom of the shaft signals to a colleague at the top by jerking the rope sideways. What is the speed of a transverse wave on the rope? (b) If a point on the rope is in transverse SHM with $f = 2.00$ Hz, how many cycles of the wave are there in the rope's length?



Sample Problem

EXECUTE: (a) The tension in the rope due to the box is

$$F = m_{\text{box}}g = (20.0\ \text{kg})(9.80\ \text{m/s}^2) = 196\ \text{N}$$

and the rope's linear mass density is

$$\mu = \frac{m_{\text{rope}}}{L} = \frac{2.00\ \text{kg}}{80.0\ \text{m}} = 0.0250\ \text{kg/m}$$

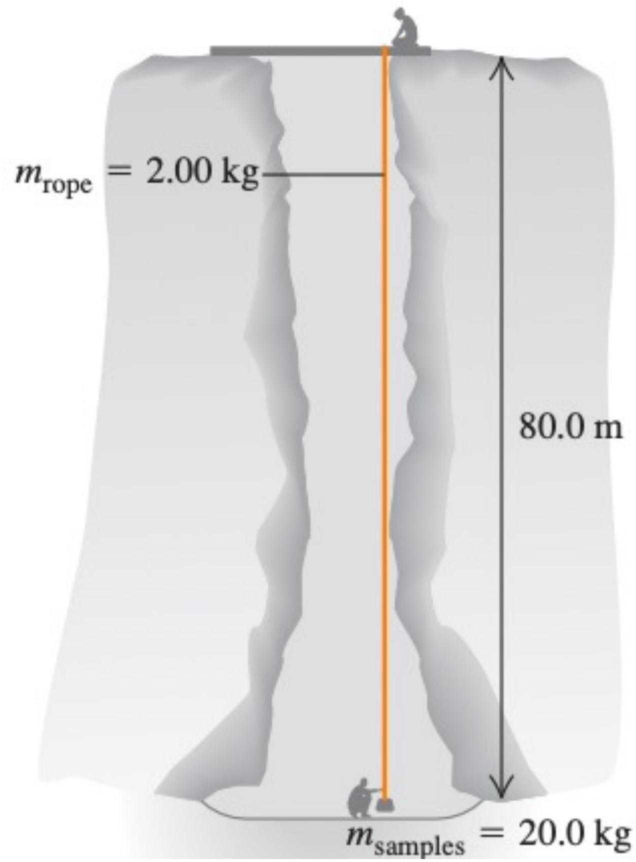
Hence, from Eq. (15.13), the wave speed is

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{196\ \text{N}}{0.0250\ \text{kg/m}}} = 88.5\ \text{m/s}$$

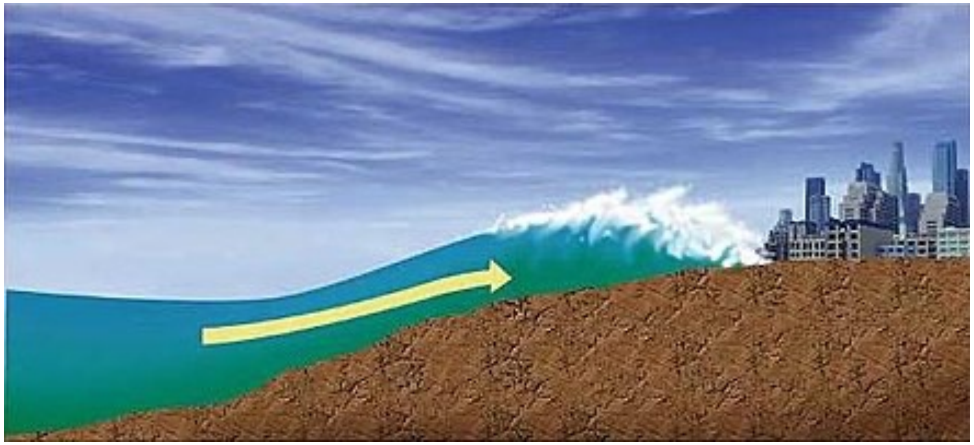
(b) From Eq. (15.1), the wavelength is

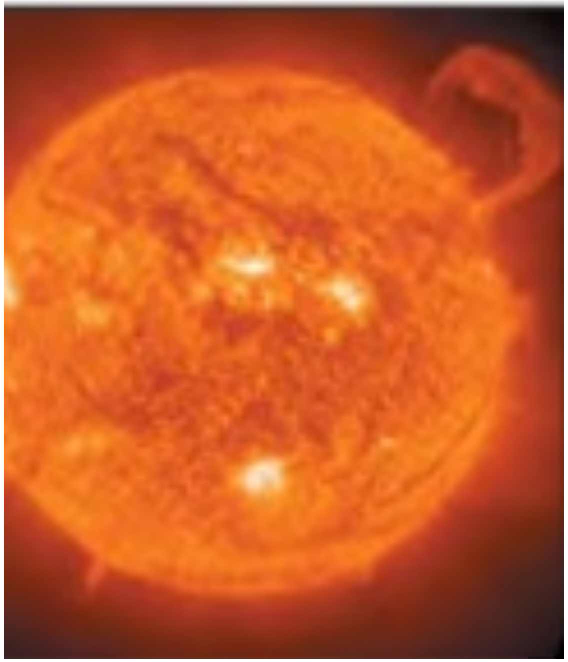
$$\lambda = \frac{v}{f} = \frac{88.5\ \text{m/s}}{2.00\ \text{s}^{-1}} = 44.3\ \text{m}$$

There are $(80.0\ \text{m})/(44.3\ \text{m}) = 1.81$ wavelengths (that is, cycles of the wave) in the rope.



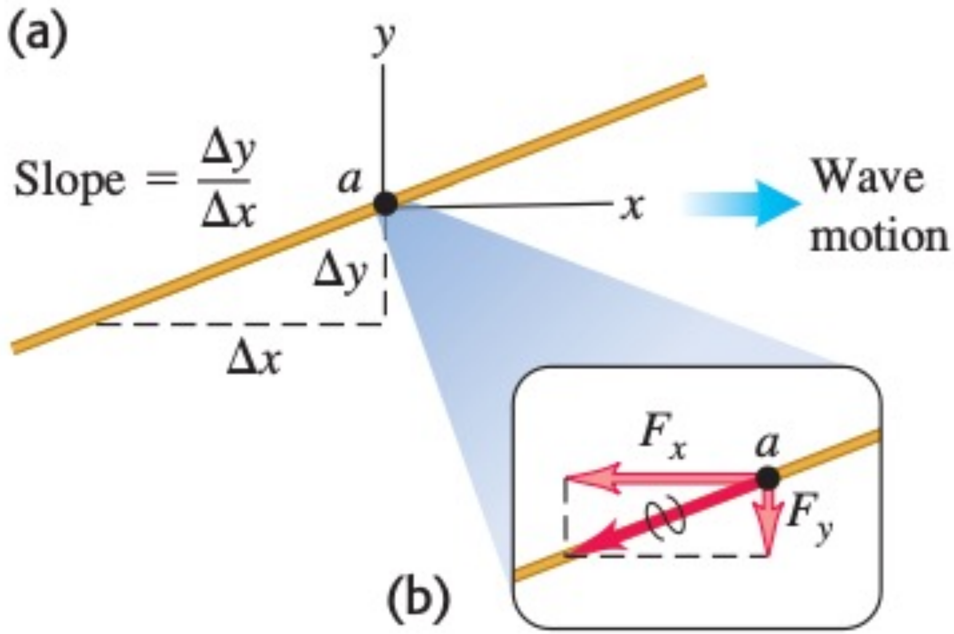
15-5 Energy in Wave Motion





Source: wikipeida, textbook

15-5 Energy in Wave Motion



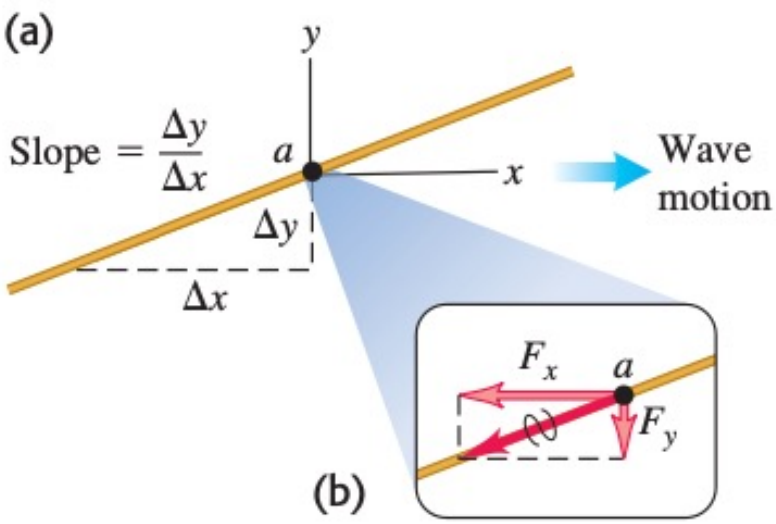
15.15 (a) Point a on a string carrying a wave from left to right. (b) The components of the force exerted on the part of the string to the right of point a by the part of the string to the left of point a.

$$F_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}$$

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

This power is the *instantaneous* rate at which energy is transferred along the string

15-5 Energy in Wave Motion



$$y(x, t) = A\cos(kx - \omega t)$$

$$\frac{\partial y(x, t)}{\partial x} = -kA\sin(kx - \omega t)$$

$$\frac{\partial y(x, t)}{\partial t} = \omega A\sin(kx - \omega t)$$

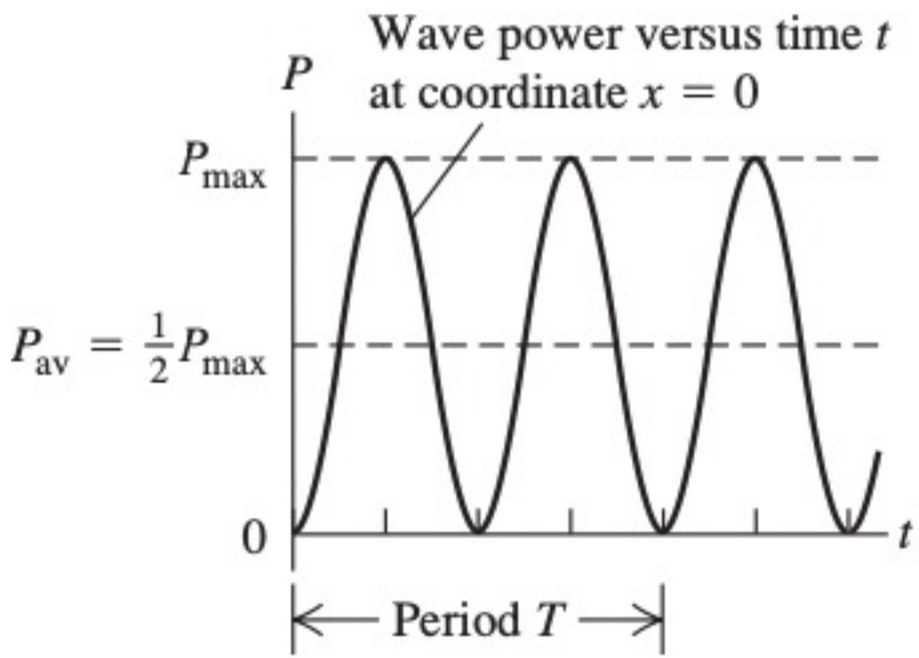
$$P(x, t) = Fk\omega A^2 \sin^2(kx - \omega t)$$

Use $\omega = vk$ $\quad v^2 = F/\mu$

$$P(x, t) = \sqrt{\mu F}\,\omega^2 A^2 \sin^2(kx - \omega t)$$

15-5 Energy in Wave Motion

$$P(x,t) = \sqrt{\mu F}\,\omega^2 A^2 \sin^2(kx - \omega t)$$



$$P_{max} = \sqrt{\mu F}\,\omega^2 A^2$$

$$P_{av} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$$

15-5 Energy in Wave Motion

Example 15.4 Power in a wave

(a) In Example 15.2 (Section 15.3), at what maximum rate does Throcky put energy into the clothesline? That is, what is his maximum instantaneous power? The linear mass density of the clothesline is $\mu = 0.250$ kg/m, and Throcky applies tension $F = 36.0$ N. (b) What is his average power? (c) As Throcky tires, the amplitude decreases. What is the average power when the amplitude is 7.50 mm?

Cousin Throckmorton holds one end of the clothesline taut and wiggles it up and down sinusoidally with frequency 2.00 Hz and amplitude 0.075 m. The wave speed on the clothesline is $v = 12.0$ m/s.

15-5 Energy in Wave Motion

EXECUTE: (a) From Eq. (15.24),

$$P_{max} = \sqrt{\mu F}\omega^2 A^2$$
$$= \sqrt{(0.250\ \mathrm{kg/m})(36.0\ \mathrm{N})}(4.00\pi\ \mathrm{rad/s})^2(0.075\ \mathrm{m})^2$$
$$= 2.66\ \mathrm{W}$$

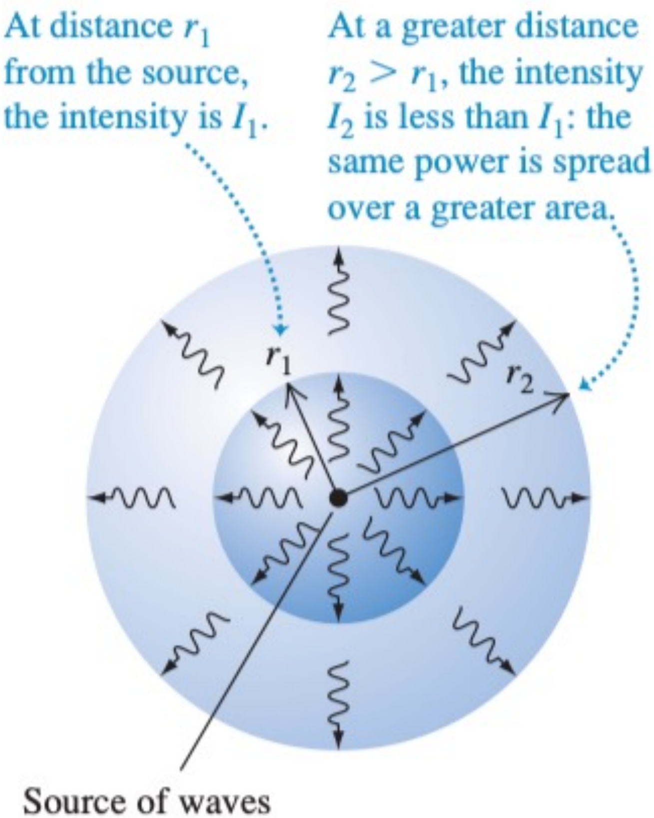
(b) From Eqs. (15.24) and (15.25), the average power is one-half of the maximum instantaneous power, so

$$P_{av} = \tfrac{1}{2}P_{max} = \tfrac{1}{2}(2.66\ \mathrm{W}) = 1.33\ \mathrm{W}$$

(c) The new amplitude is $\frac{1}{10}$ of the value we used in parts (a) and (b). From Eq. (15.25), the average power is proportional to A^2, so the new average power is

$$P_{av} = \left(\tfrac{1}{10}\right)^2(1.33\ \mathrm{W}) = 0.0133\ \mathrm{W} = 13.3\ \mathrm{mW}$$

15-5 Energy in Wave Motion



intensity (denoted by I): *the time average rate at which energy is transported by the wave, per unit area*

$$I_1 = \frac{P}{4\pi r_1^2}$$

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \qquad \text{(inverse-square law for intensity)}$$

15-5 Energy in Wave Motion

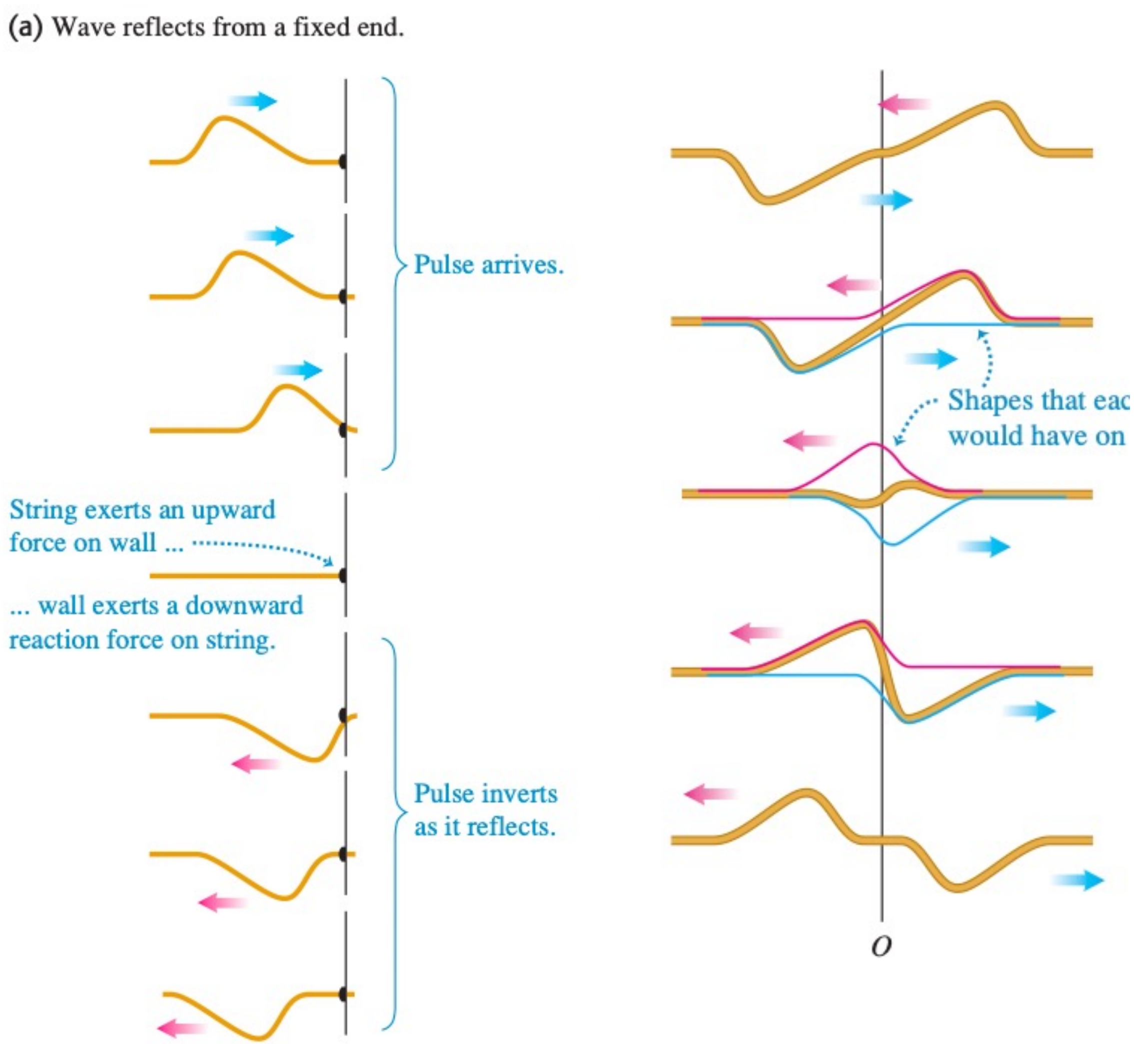
Example 15.5 **The inverse-square law**

A siren on a tall pole radiates sound waves uniformly in all directions. At a distance of 15.0 m from the siren, the sound intensity is $0.250\ \mathrm{W/m^2}$. At what distance is the intensity $0.010\ \mathrm{W/m^2}$?

$$r_2 = r_1\sqrt{\frac{I_1}{I_2}} = (15.0\ \mathrm{m})\sqrt{\frac{0.250\ \mathrm{W/m^2}}{0.010\ \mathrm{W/m^2}}} = 75.0\ \mathrm{m}$$

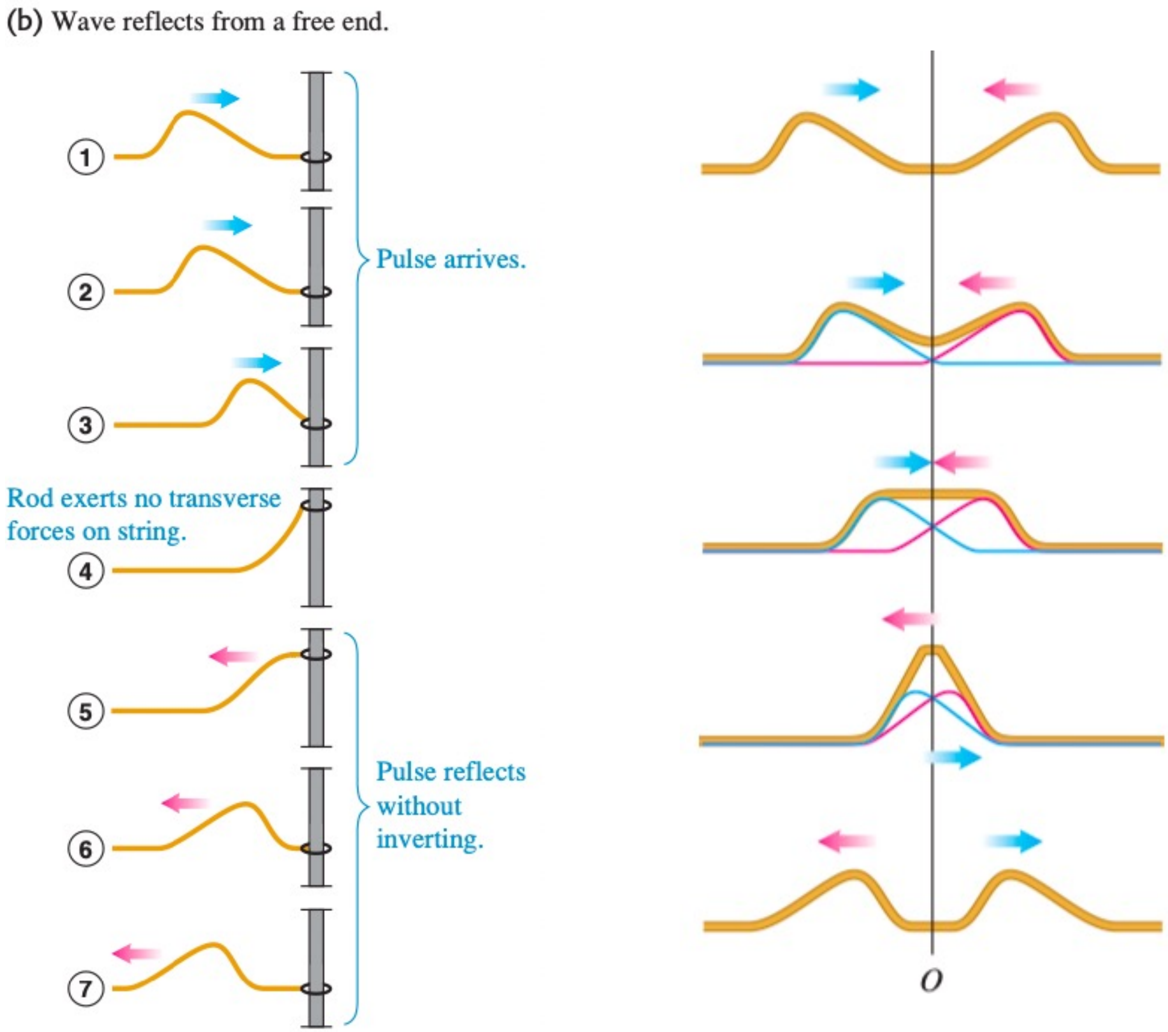
15-6 Boundary Conditions

Fixed end



15-6 Boundary Conditions

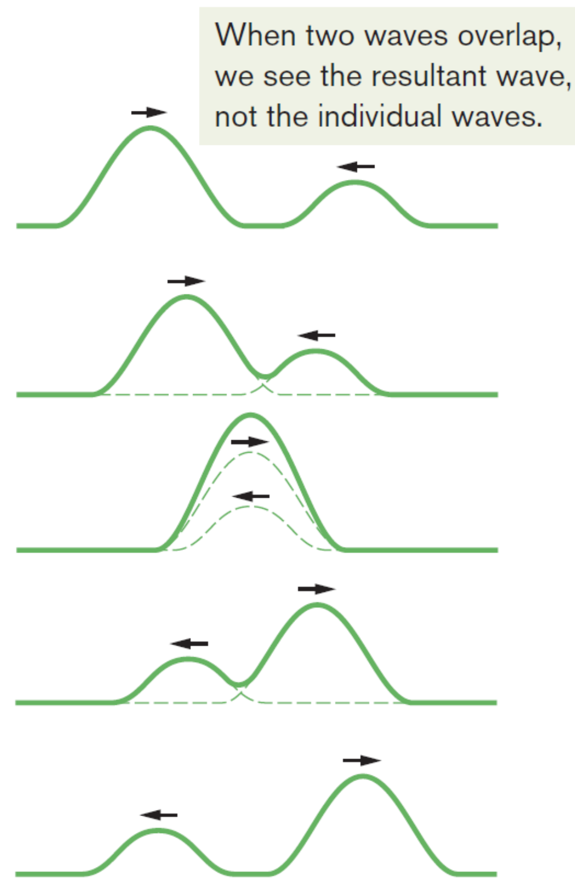
Free end

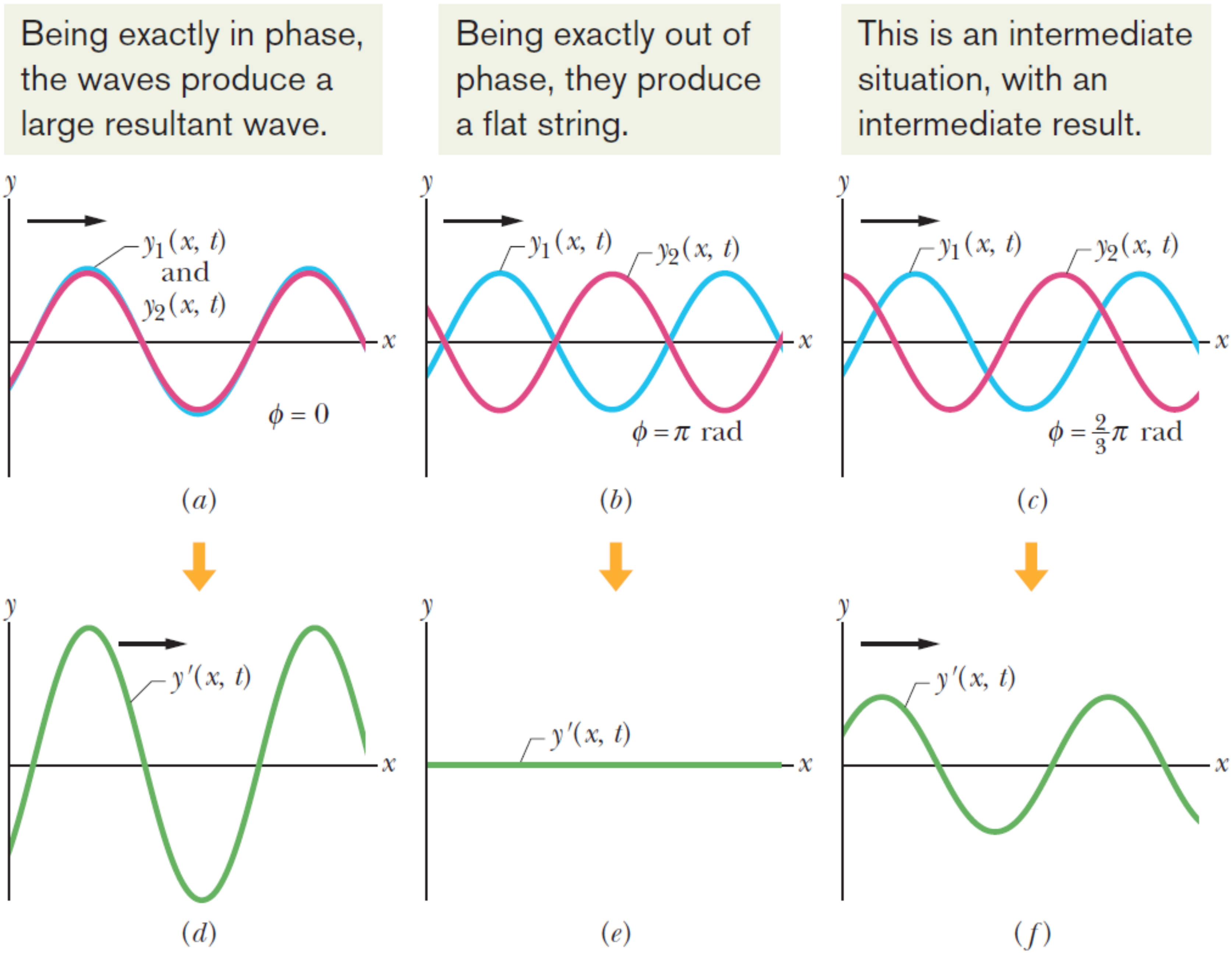


15-6 Interference (干涉) of Waves

- Let $y_1(x,t)$ and $y_2(x,t)$ be the displacements that the string would experience if each wave travelled alone

- The Principle of Superposition
  - $y'(x,t) = y_1(x,t) + y_2(x,t)$




Being exactly in phase, the waves produce a large resultant wave.
Being exactly out of phase, they produce a flat string.
This is an intermediate situation, with an intermediate result.
y
x
$y_1(x, t)$ and $y_2(x, t)$
$\phi = 0$
(a)
$y_1(x, t)$
$y_2(x, t)$
$\phi = \pi$ rad
(b)
$y_1(x, t)$
$y_2(x, t)$
$\phi = \frac{2}{3}\pi$ rad
(c)
$y'(x, t)$
(d)
$y'(x, t)$
(e)
$y'(x, t)$
(f)

15-6 Interference (干涉) of Waves

- Types of interference

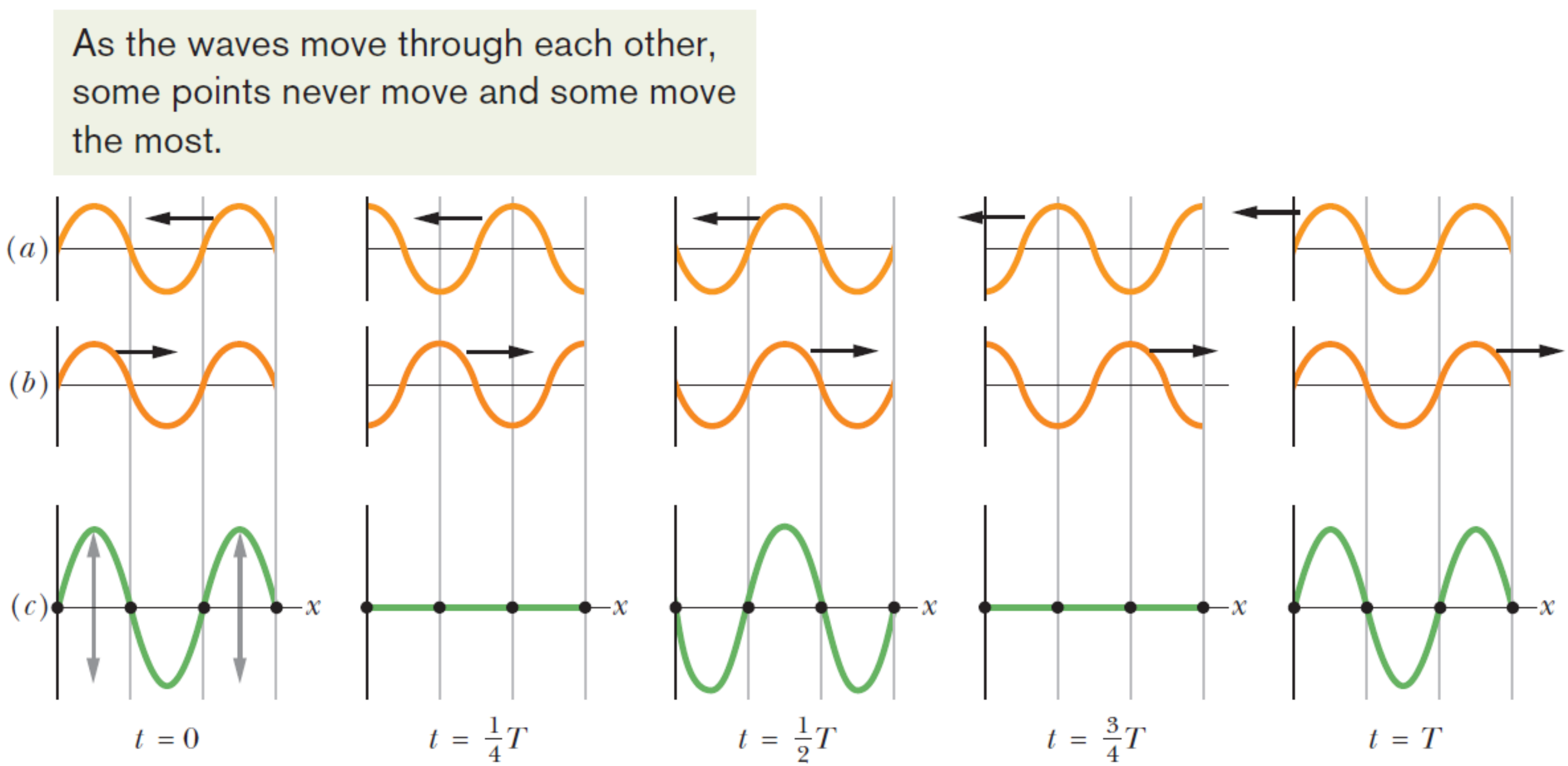
Phase Difference and Resulting Interference Types[a]

Phase Difference, in			Amplitude of Resultant Wave	Type of Interference
Degrees	Radians	Wavelengths		
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

[a]The phase difference is between two otherwise identical waves, with amplitude y_m, moving in the same direction.

15-7 Standing Waves (駐波) on a String

- If two sinusoidal waves of the *same amplitude and wavelength* travel in *opposite directions* along a stretched string, their interference with each other produces a standing wave



15-7 Standing Waves (駐波) on a String

- To analyse a standing wave, we represent the two combining waves with the equations
  - $y_1(x,t) = y_m \sin(kx - \omega t)$
  - $y_2(x,t) = y_m \sin(kx + \omega t)$

- The superposition principle (疊加原理) applies
  - $y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$

$$\left(\begin{array}{c} \because \sin\alpha + \sin\beta \\ = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta) \end{array} \right)$$

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \sin kx]}_{\substack{\text{Magnitude}\\ \text{gives}\\ \text{amplitude}\\ \text{at position } x}} \underbrace{\cos \omega t}_{\substack{\text{Oscillating}\\ \text{term}}}$$

15-7 Standing Waves (駐波) on a String

- Nodes (節點/波節)
  - where the string never moves (zero amplitude)

  - $y'(x,t) = [2y_m \sin kx] \cos \omega t$
  - $\sin kx = 0$
  - $kx = n\pi, \quad for\; n = 0,1,2,\ldots$
  - $x = n\frac{\pi}{k} = n\frac{\lambda}{2}, \quad for\; n = 0,1,2,\ldots \quad \left(\because k = \frac{2\pi}{\lambda}\right)$

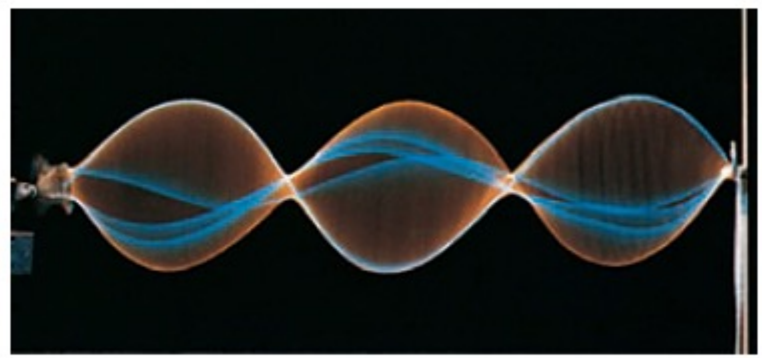
15-7 Standing Waves (駐波) on a String

- Antinodes (腹點/波腹)
  - where the amplitude of the net wave is a maximum

  - $y'(x,t) = [2y_m \sin kx] \cos \omega t$
  - $\sin kx = 1$
  - $kx = \left(n + \frac{1}{2}\right)\pi, \quad for\ n = 0,1,2, \ldots$
  - $x = \left(n + \frac{1}{2}\right)\frac{\pi}{k} = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad for\ n = 0,1,2, \ldots \quad \left(\because k = \frac{2\pi}{\lambda}\right)$

15-8 Normal Modes of a String

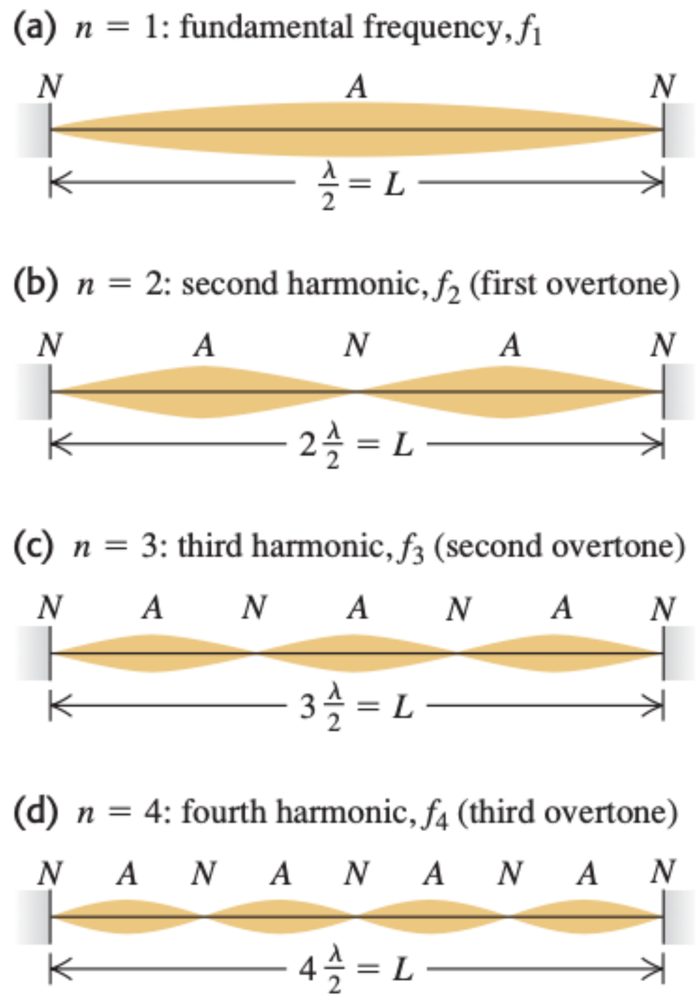
- Suppose we send a continuous sinusoidal wave of a certain frequency along the string, say, toward the right. When the wave reaches the right end, *it reflects and travels back* to the left. The interference may produce a standing wave pattern



- Such a standing wave is said to be produced at *resonance*, and the string is said to resonate at these certain frequencies, called *resonant frequencies*

15-8 Normal Modes of a String

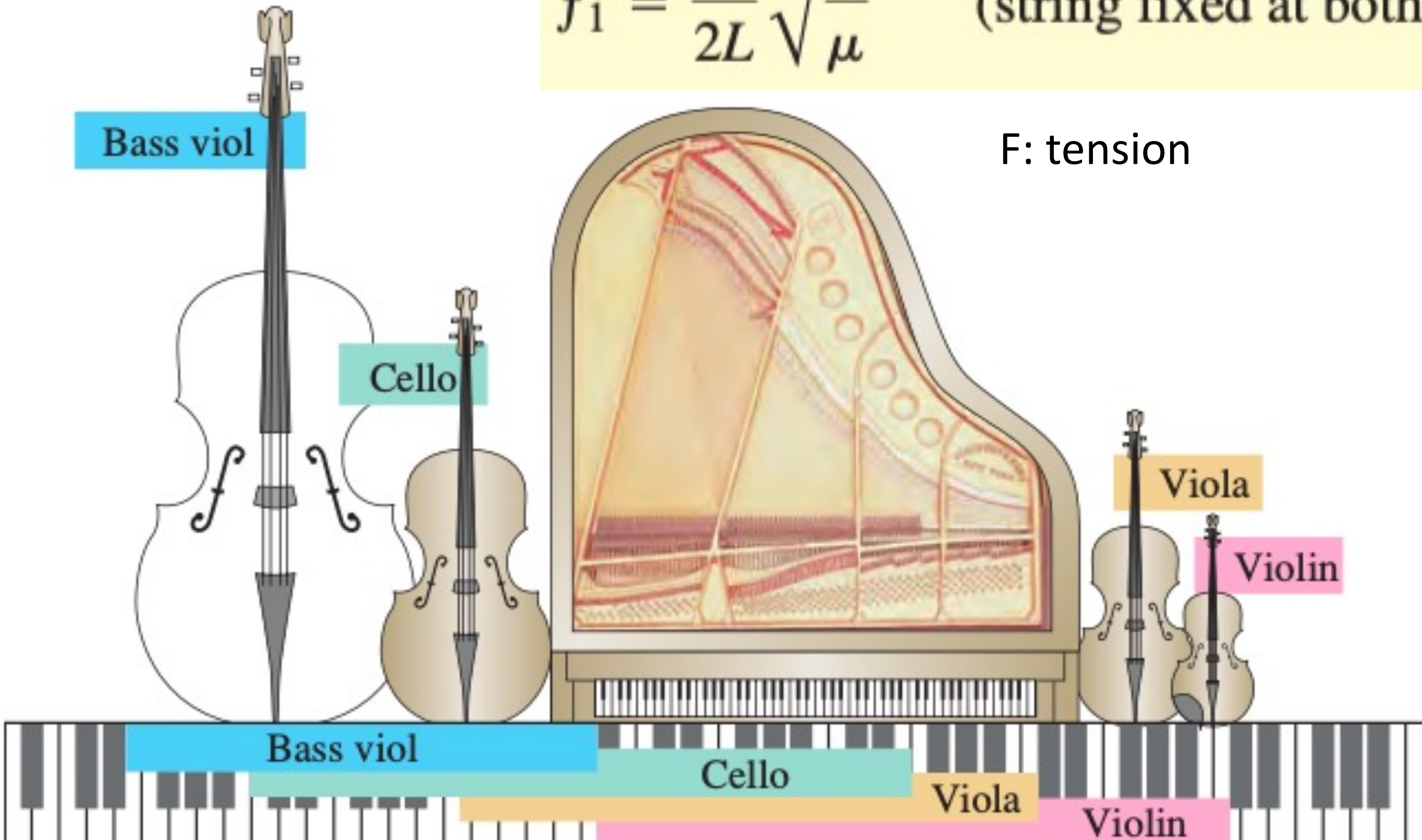
- Resonance can occur at wavelengths
  - $\lambda = \frac{2L}{n}, \quad for\, n = 1, 2, 3, \ldots$
- Resonant frequencies
  - $f = \frac{v}{\lambda} = n\frac{v}{2L}, \quad for\, n = 1, 2, 3, \ldots$
  - *n* is called the ***harmonic number*** of the *n*th harmonic



15-8 Normal Modes of a String

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \quad \text{(string fixed at both ends)}$$

F: tension



Sample Problem

Example 15.7 A giant bass viol

In an attempt to get your name in *Guinness World Records,* you build a bass viol with strings of length 5.00 m between fixed points. One string, with linear mass density 40.0 g/m, is tuned to a 20.0-Hz fundamental frequency (the lowest frequency that the human ear can hear). Calculate (a) the tension of this string, (b) the frequency and wavelength on the string of the second harmonic, and (c) the frequency and wavelength on the string of the second overtone.

EXECUTE: (a) We solve Eq. (15.35) for F:

$$F = 4\mu L^2 f_1{}^2 = 4(40.0 \times 10^{-3}\ \mathrm{kg/m})(5.00\ \mathrm{m})^2(20.0\ \mathrm{s}^{-1})^2$$
$$= 1600\ \mathrm{N} = 360\ \mathrm{lb}$$

Sample Problem

(b) From Eqs. (15.33) and (15.31), the frequency and wavelength of the second harmonic $(n = 2)$ are

$$f_2 = 2f_1 = 2(20.0\ \text{Hz}) = 40.0\ \text{Hz}$$

$$\lambda_2 = \frac{2L}{2} = \frac{2(5.00\ \text{m})}{2} = 5.00\ \text{m}$$

(c) The second overtone is the "second tone over" (above) the fundamental—that is, $n = 3$. Its frequency and wavelength are

$$f_3 = 3f_1 = 3(20.0\ \text{Hz}) = 60.0\ \text{Hz}$$

$$\lambda_3 = \frac{2L}{3} = \frac{2(5.00\ \text{m})}{3} = 3.33\ \text{m}$$