Part I

Unit 1: Algebra, statistics, and functions

Topic 1

Arithmetic sequences and series

1.1 Arithmetic sequences

A **sequence** is simply a list of numbers, where the **order** of the numbers is important, such as: 1, 2, 3, 4. The numbers in the list are called **terms (*t*)**. For example, the second term of the sequence would be t_2. This gives us the general rule that the n^{th} term is t_n. These sequences can also be either finite or infinite.

This course looks at **recursive sequences** only, which means these lists of numbers are defined by a **rule.** This rule tells you how to get from one term to the next in the sequence.

The rule for **arithmetic sequences** only involves adding or subtracting a fixed amount from a term. This means that every arithmetic sequence will have a **common difference (*d*)** between successive terms.

For example, for the following arithmetic sequence:

$$1, 3, 5, 7$$

... the rule is to simply add 2 to the first term (1) to get to the second term (3), where the common difference is 2. However, to find the fourth term of the sequence given only the first term, it would be tedious to individually add the common difference to each term, finding every term prior to the fourth term. As such, the formula below is used, which is provided in the formula sheet.

To find the n^{th} term in an arithmetic sequence:

$$t_n = t_1 + (n-1)d$$

- t_n is the n^{th} term in the sequence
- t_1 is the first term in the sequence
- n is the number representing what term we're looking for
- d is the common difference

To make sure we're clear, the n^{th} term just means whatever term in the sequence you are after; this could be the first, second, third, or the hundredth. Also, be sure to recognise the difference between t_n and n. For the sequence 1, 3, 5, 7, the fourth term (t_4) is 7, but the value of n is 4.

You may also encounter subscripts such as t_{n-1}. This just refers to the term immediately before t_n (e.g. if $n = 3$, then $t_{n-1} = t_2$), so don't be intimidated!

Furthermore, you may be asked to find the rule of the arithmetic sequence – all you have to do in these questions is find an equation for the n^{th} term by substituting t$_1$ and d into $t_n = t_1 + (n-1)\,d$.

Example 1.1

Given that t_n is the n^{th} term in an arithmetic sequence, find n if $t_1 = 7$, $t_2 = 11$, and $t_n = 227$.

Since 7 and 11 are consecutive terms: $d = 4$.

Substituting known values gives us:

$$\begin{aligned} t_n &= t_1 + (n-1)\,d \\ 227 &= 7 + (n-1)4 \\ 227 &= 7 + 4n - 4 \\ 224 &= 4n \\ n &= 56 \end{aligned}$$

Now that you are getting familiar with using this formula, let's try something involving more algebraic skill!

Example 1.2

For an arithmetic sequence, find t_6 if $t_{10} = 31$ and $t_{20} = 61$.

While this may seem daunting due to the lack of information, when you see 'arithmetic sequence' immediately write down the relevant formula and substitute values that you know:

$$\begin{aligned} t_n &= t_1 + (n-1)\,d \\ t_{10} &= 31 \\ &= t_1 + 9d \\ t_{20} &= 61 \\ &= t_1 + 19d \end{aligned}$$

Now we have two equations:

$$\begin{aligned} 31 &= t_1 + 9d \quad (1) \\ 61 &= t_1 + 19d \quad (2) \end{aligned}$$

There are two unknowns and two equations, so we can use simultaneous equations to eliminate one variable and solve for the other. The variable t_1 is the easiest to eliminate since it has a coefficient of 1 in both equations, so subtract equation (1) from (2):

$$\begin{aligned} &(1) - (2): \\ 61 - 31 &= t_1 + 19d - (t_1 + 9d) \\ 30 &= 19d - 9d \\ 30 &= 10d \\ d &= 3 \end{aligned}$$

Now we have one variable, so we can use either equation (1) or (2) to find t_1 (or, to double check, you can use both separately – both should give the same answer):

$$\begin{aligned} 31 &= t_1 + 9 \times 3 \\ t_1 &= 4 \end{aligned}$$

Don't forget to remember what the original question asked! To find t_6:

$$\begin{aligned} t_6 &= 4 + (6-1) \times 3 \\ t_6 &= 19 \end{aligned}$$

Key Point :
Solving simultaneously is an extremely useful tool that allows you to find two unknowns with two different equations. If you are not already confident with this technique, consider putting extra time into practising this. You will most likely be using simultaneous equations often throughout this course. Algebraic skills like this will always be relevant, no matter what the content is.

Remember that the goal is to eliminate one variable so you can solve for the other variable. Furthermore, be sure to label your equations and clearly convey to the marker what is happening (e.g. equation 1 minus 2) – they will not waste time trying to figure something out if it is messy!

1.2 Arithmetic series

An arithmetic **series** is the **sum of the terms** in an arithmetic **sequence.** We use the formula below (from the formula sheet) to calculate the sum of the terms in a sequence up to and including the n^{th} term. Note that this formula will always start from the beginning of the sequence.

To find the sum of the terms in an arithmetic sequence, we use:

$$S_n = \frac{n}{2}(2t_1 + (n-1)\,d) = \frac{n}{2}(t_1 + t_n)$$

Hence, there are two ways of finding S_n; you may choose to use either one depending on the given information.

Example 1.3

For the sequence 4, 8, 12..., find the value of n if $S_n = 180$.

Write down what you know and any relevant formulas:

$$d = 4 \quad t_1 = 4 \quad S_n = 180$$

Since no information is given about t_n, we'll use the formula $S_n = \frac{n}{2}(2t_1 + (n-1)\,d)$. Plugging in the values above and solving for n:

$$180 = \frac{n}{2}(2 \times 4 + (n-1) \times 4)$$
$$180 = \frac{n}{2}(4 + 4n)$$

Factorise the 4 out of the bracket and then divide both sides by 2 before expanding the bracket – this will eliminate unnecessary calculations.

$$180 = 2n(1 + n)$$
$$90 = n(1 + n)$$
$$90 = n + n^2$$

Hopefully, you recognise that this is a quadratic, which you can solve by moving all the terms to one side and factorising:

$$0 = n^2 + n + 90$$
$$0 = (n-9)(n+10)$$
$$n = 9 \text{ or } n = -10 \text{ (disregard negative as } n \text{ must be positive)}$$
$$\therefore n = 9$$

As this is an arithmetic series, the value of n starts at 1 and is only ever positive.

Therefore, the answer must be $n = 9$. Make sure to provide reasoning in this situation to explain why we discard the negative case.

Key Point :
Solving quadratics is another essential algebraic skill that you know how to do. While you can solve this without a calculator, if the test allows a calculator, be sure to utilise it to save time. You can do this by either graphing it and finding the roots (x-axis intercepts) or using the **equation solver on your graphics calculator.** If you need to refresh your knowledge of quadratics, see page 13.

It is important to note that in most of the previous examples, the question tells you that a certain sequence of numbers is arithmetic, so you already know what formula to use. However, you may not have this luxury in the more worded, 'problem solving' questions; always remember to identify what type of sequence you need. If you see a constant difference between terms, immediately make the connection to arithmetic sequences. This becomes more relevant because we will also cover geometric sequences later on, so you need to able to differentiate the two.

With that in mind, let's cover a problem solving question.

Example 1.4

A brand new sports club is founded with 40 members in its first year. Each following year, the number of new members exceeds the number of retirements by 15. Each member pays $120 p.a. in membership fees. Calculate the amount received from fees in the first 12 years of the club's existence.

The question is asking us to find out the amount received from fees paid by each member in 12 years, so we need to obtain some information about the number of members. It is helpful to write down all the information we know: the club starts with 40 members and gains 15 members each year, and each member pays $120 every year.

Your first thought might be to find the total number of members in 12 years and multiply that by 120 to find the total fees; however not all members in the club will have paid the same amount of money in total to the club, since they have been there for a different number of years. In other words, some members will have paid the $120 fee multiple times.

Instead, we are looking for the total number of times the $120 fee has been paid in 12 years. To do this, we can create an arithmetic sequence, with the first term as 40 and the constant difference as 15, giving us a sequence of 40, 65, 80, etc. The terms show that 40 people paid the fee in the first year, 65 people in the second year, and so on. By finding the sum of the terms to the 12^{th} term, we find the total number of times the $120 fee has been paid in 12 years.

$$\begin{aligned} S_n &= \frac{n}{2}(2t_1 + (n-1)\,d) \\ S_{12} &= \frac{12}{2}(2 \times 40 + (12-1) \times 15) \\ S_{12} &= 1,470 \end{aligned}$$

Now multiply this number by 120 to find the amount of money received by the club:

$$\begin{aligned} \text{total fees} &= 1,470 \times 120 \\ &= \$176,400 \end{aligned}$$

For questions like this one, provide your answer as a short, written response:

Therefore, in the first 12 years, the club received $176,400 in fees.

The process for solving this question may have been more apparent just because we are currently covering arithmetic series, so obviously the question will relate to this topic. Had this question appeared in a mixed revision sheet or an exam, you would have had to figure out for yourself that it involved arithmetic series. This thought process is very important – you must understand what the question is asking and then devise a plan to solve it – so you should strive to do mixed revision once you have learned all the content!

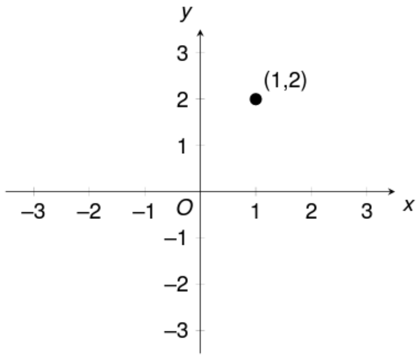
We will come back to the topic of sequences later when we cover geometric series sequences on page 41, but for now, we will move on to the topic of functions and graphs.

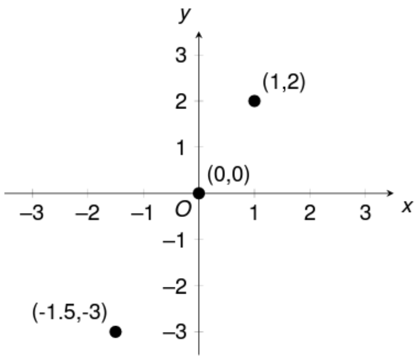
Topic 2

Functions and graphs

2.1 Reviewing relations, domain, and range

You have probably have already seen the Cartesian plane, which is shown on the right. With this, we can describe any point on the plane with an **ordered pair**, denoted as (x, y), which always appears in that order. The first number in the pair tells you where the point lies horizontally, as that number refers to the x-axis. The second number refers to y-axis and shows you where the point is vertically. For example, the point in the plane on the right is described as $(1, 2)$.

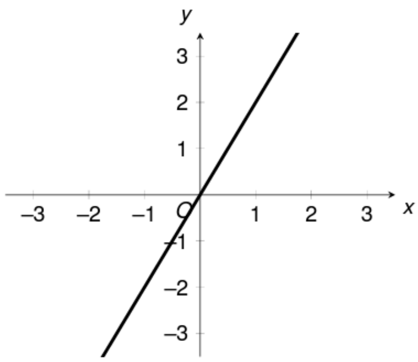




We can expand on this idea by introducing more points on our plane. We call a set of these points a **relation.** The relation (which we'll arbitrarily call R) shown on the left is described as: R = $(–1.5, –3)$, $(0, 0)$, $(1, 2)$. All the x-values of the ordered pairs are called the **domain,** and all the y-values are called the **range.** For example, the domain of R is $[–1.5, 1]$, and the range of R is $[–3, 2]$. This is just terminology that you have to remember.

Relations can also be defined by rules, such as $y = x$. This rule simply tells us that for any given x-value, the y-value is the same. This makes it unnecessary to fully list the ordered pairs, since if we have the rule, we already know what the y-value of a point would be if we're given the x-value. We must also include the domain of the relation (we could also give the range, but by convention we give the domain) to give us a list of x-values which we can put into the rule and find our corresponding y-values. Remember, if you *define* a relation by a rule, you must include the domain as well.

Relations do not have to be individual points either; for example, the graph on the right shows the rule $y = 2x$ as a continuous line. The difference is that the domain does not consist of individual numbers like above, but is a subset of **real numbers.** This domain can be described as $[0, \infty)$ or $x \geq 0$. Remember the filled-in dot on the graph signifies that point is included, while a hollow dot would mean that point would not be included. Also, the square bracket in $[0, \infty)$ means that 0 is included, and the curly bracket means that ∞ is not included (the domain extends to infinity, but since infinity can never be reached, the curly bracket is used).



KEY POINT :
If a rule for relation is written without a domain, you assume that the domain is as large as possible (for the written rule), which is called the **implied domain** or **maximal domain.** There will be many instances where the domain is not written, as you'll see throughout this section.

2.2 Functions

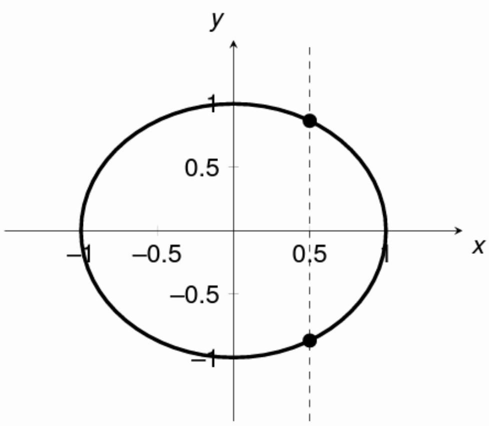
A function is a type of relation with a special rule: each x-value only corresponds to one y-value. For example, does the following set of ordered pairs define a function?

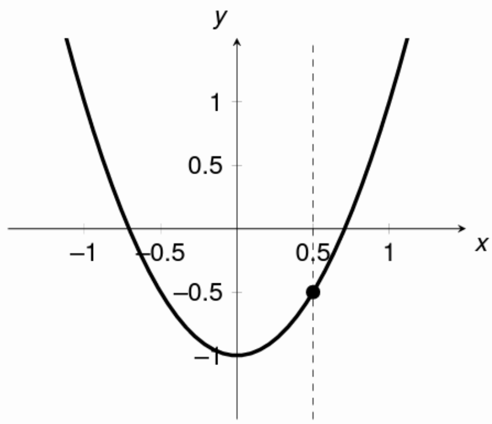
$$(4, 1), (4, 3), (-1, 2), (-2, -3)$$

Since the x-value of 4 has two y-values (1 and 3), this is not a function. Essentially, a function cannot include ordered pairs with the same x-value. However, since you'll probably be focusing on relations shown on graphs more, there is another way of determining if a relation is a function: using a graph!

2.2.1 Vertical line test

The graphs below show the rules $x^2 + y^2 = 1$ (left) and $y = x^2 - 1$ (right). To determine if it is a function, we conduct the vertical line test. This means if a vertical line is drawn anywhere and it only intersects the graph **once,** the relation is a function.





As seen above, the dotted line intersects the graph of $x^2+y^2 = 1$ twice, so this relation is *not* a function.

By contrast, the dotted line only intersects the graph of $y = x^2 - 1$ once, and this will be true no matter where on the domain we draw the line, so this relation *is* a function.

2.2.2 Function notation

You've probably seen something like '$y = x+1$' on a graph before. Typically, you'll be familiar with equations in the form of '$y = ...$' and each x-value can only have one corresponding y-value.

When we know that something is a function, we can rewrite it to specifically show it is a function in the form $f(x) = x + 1$. All we do is simply replace the y in the equation with $f(x)$. We do this because the y-value is obtained by taking an x-value and substituting it in the equation; hence, the y-value is a function of the x-value. Functions are usually written with lowercase letters such as f, g, h, and so on, with f being the most common.

Key Point :
You might find yourself using the $f(x) =$ and the $y =$ notations interchangeably in your working; while they both imply functions, is it better to stick with the notation that the question has given you. Using $f(x)$ becomes beneficial when dealing with multiple functions, as you can distinguish them, like $f(x)$ and $g(x)$.

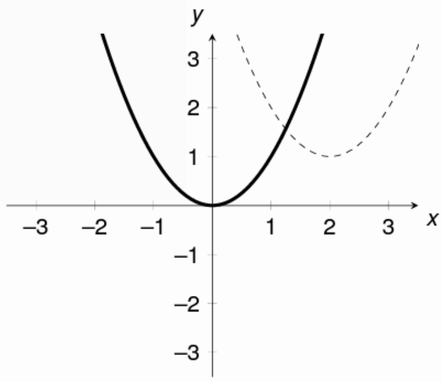
Function notation also helps to clarify what the independent and dependant variables are; if y is a function of x, then the y-value depends on the value of x. Thus, y is the dependant variable, and x is the independent variable.

2.2.3 Translations of functions

Translating a function means to move it up or down, left or right on a graph. More specifically, we refer the movement up or down as translating in the positive or negative direction of the y-axis. Similarly, translations left or right are written as translating in the negative or positive direction of the x-axis. While this is easy to do for a single point, applying translations to graphs involves a process that is shown in the example below.

Example 2.1

Below are two parabolas: one curve has the equation $y = x^2$, and the dotted curve is translation 2 units in the positive x-axis direction and 1 unit in the positive y-axis direction. Find the equation of the translated curve.



Let (x, y) be a point on the original curve, and let $\left(x', y'\right)$ be the translated point. To get from x to x', we need to add 2 units in the positive x-axis direction. To get from y to y', we need to add 1 unit in the positive y-axis direction. Then we rearrange both equations to make x and y the subject:

$$\begin{aligned} x + 2 &= x' \\ x &= x' - 2 \\ y + 1 &= y' \\ y &= y' - 1 \end{aligned}$$

Now we are able to substitute these equations into our original $y = x^2$, and as with all functions, we try to make y the subject:

$$\begin{aligned} y &= x^2 \\ y' - 1 &= \left(x' - 2\right)^2 \\ y' &= \left(x' - 2\right)^2 + 1 \end{aligned}$$

Therefore, the equation of the translated curve is $y = (x - 2)^2 + 1$. We give our answer without the dashes, as they were only there to allow us to differentiate between the original and translated points.

This process will work with any transformation of a graph, not just translations – which we will get to soon – so be sure to make this process stick in your mind.

Key Point :
You might have realised that the end result is a parabola written in **turning point form:** $y = a(x - h)^2 + k$. If you knew and remembered to use this form, you would have the ability to go straight to the answer, simply by substituting the constants h and k with the information from the question. A question may or may not ask you to provide working out, but in any case this would allow you to check your answer. If you don't remember what turning point form is, see page 13.

2.2.4 Dilations of functions

Dilating a graph will stretch or compress it, which can be done from the x-axis or the y-axis. That might sound confusing, but all we have to do is apply the same technique from the example above. When a graph is dilated, it is dilated by a certain factor. If this factor is larger than 1, the graph will stretch, if it is less than 1, the graph will compress.

Example 2.2

Determine the equation of the image when the graph of $y = x^2$ is dilated by factor of 3 from the x-axis.

As before, let (x, y) be a point on the original graph, and $\left(x', y'\right)$ be a point on the dilated graph, also known as the image.

'Dilated by a factor of 3 from the x-axis' means that x' will be 3 times larger than x. Therefore, if we start with x, then we would have to multiply it by 3 to get to x', which is written as the following, which we rearrange to make x the subject:

$$3x = x'$$
$$x = \frac{x'}{3}$$

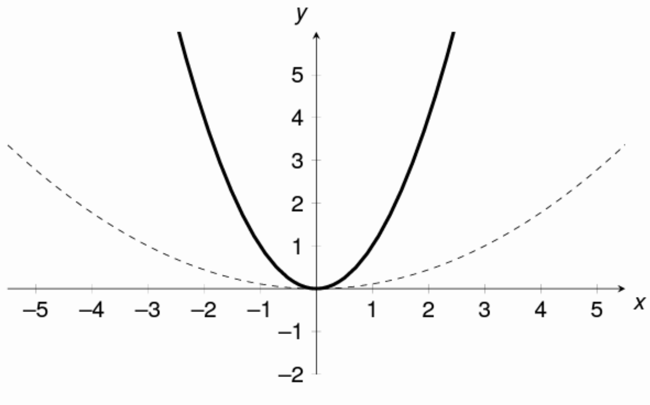
Keep in mind that no change is being made to the y-value, so: $y = y'$.

Substitute into the original rule of the graph and simplify:

$$y = \left(\frac{x'}{3}\right)^2$$
$$y = \frac{x^2}{9}$$

Therefore, the equation of the image is $y = \dfrac{x^2}{9}$.

To make more sense of this dilation, the original and dilated (dotted line) graphs are shown below.

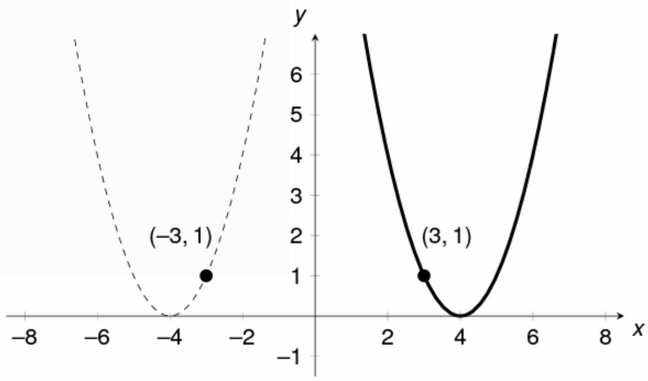


Since the dilation factor was more than 1, the dilated graph appears wider than the original. The same process applies when dilating from the y-axis.

2.2.5 Reflections of functions

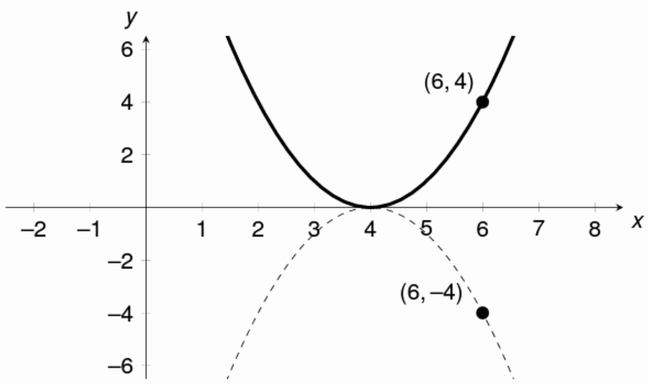
Reflecting a function will mirror the curve about an axis, either the x-axis or y-axis. In the graph below, a parabola has been reflected in the y-axis, represented by the dotted parabola. When comparing a single point on the original curve $(3, 1)$, its reflected point $(-3, 1)$ has the same y-value, but the x-value is now negative. So, when reflecting in the y-axis, remember this:

$$\text{Reflection in } y\text{-axis: } (x, y) \longrightarrow \left(-x', y'\right)$$



In the next graph, the same parabola has now been reflected in the x-axis, shown again by the dotted curve. The point $(6, 4)$ on the original curve has been reflected to the point $(6, -4)$; the x-value stays the same but the y-value become negative. Therefore:

$$\text{Reflection in } x\text{-axis: } (x, y) \longrightarrow \left(x', -y'\right)$$



Example 2.3

Find the equation of the image obtained when the graph of $y = \sqrt{x}$ is reflected in the x-axis.

Let (x, y) be a point on the original curve, and $\left(x', y'\right)$ be a point on the image.

Since we are reflecting in the x-axis, we must remember that (x, y) turns into $\left(x', -y'\right)$, meaning that the new y-value becomes negative. If we think of the original y-value, we must make it negative in order to reflect it in the x-axis:

$$\begin{aligned} -y &= y' \\ y &= -y' \end{aligned}$$

There is no change in the x value, so $x = x'$.

Substituting into the original curve:

$$-y' = \sqrt{x'}$$
$$y' = -\sqrt{x'}$$

Therefore, the equation of the image is $y = -\sqrt{x}$.

The three types of transformations that we have looked at above – translations, dilations, and reflections – can also be applied in a sequence; for example, a graph could be dilated by a factor of 2 from the x-axis, then reflected in the y-axis. To find the image of the graph, you would just consider one transformation at a time while following the process above.

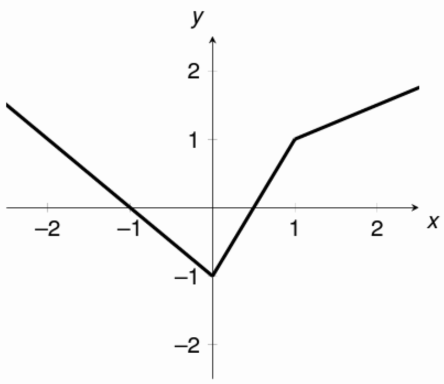
Key Point :
If you have to apply multiple transformations, remember the acronym **DRT – dilations, reflections, translations –** to describe the order in which transformations should be applied.

2.2.6 Piecewise functions

So far, we have only been considering one function at a time. Now it's time to introduce piecewise functions, which consist of **multiple functions.** Each function has its own specified domain, and functions will not overlap domains – if they could, it would not be a function anymore as our vertical line test would cut through multiple curves. For example, piecewise functions are written like:

$$f(x) = \left\{ \begin{array}{cc} -x-1 & \text{for } x < 0 \\ 2x-1 & \text{for } 0 \leq x \leq 1 \\ 0.5x+0.5 & \text{for } x > 1 \end{array} \right\}$$

This looks like:



The three functions each have specified domains that makes them part of the overall piecewise function $f(x)$. However, keep in mind that the functions do not have to join up; all that is required is that the **domains do not overlap.**

Piecewise functions are useful to describe scenarios where the relationship changes as the input value (x) exceeds certain boundaries, given by the domain.

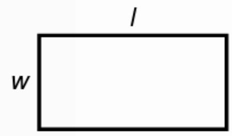
2.2.7 Applying functions

With our knowledge of function notation, let's see how we can use functions to model 'real' scenarios and solve problems.

Example 2.4

A farmer wishes to construct a rectangular enclosure for his animals that provides the largest maximum area with a 12 m length of wire. By sketching a graph, what is the maximum area that can be fenced and the corresponding length and width of the enclosure.

We know that the area is rectangular and that a 12 m wire is being used. That is, the perimeter of our rectangle is 12 m. At this stage, it is very helpful to draw a picture to understand the question better. Of course, we haven't figured out yet what length and width will give us our maximum area, so we will just draw a general rectangle.



We can see that our perimeter consists of $l + w + l + w$. Therefore, $12 = 2l + 2w$.

Since it is a rectangle, we know that *area* $= l \times w$. The question hints at using a graph to figure this question out; we are going to have to create some kind of function to do so. Remember that a function has an input and output value; you input an x-value, and what comes out is a function of that value. In the equation area $= l \times w$, the area is our output, but we get that by using two inputs: length and width. But our function can only have one input value, so how do we turn two inputs into one?

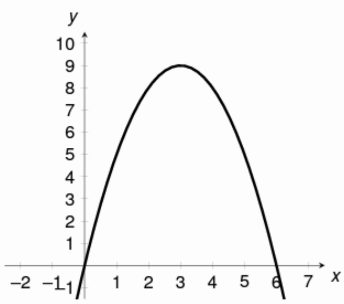
Fortunately, we also know that length and width are related by the perimeter too: $12 = 2l + 2w$, which we can simplify and rearrange:

$$6 = l + w$$
$$w = 6 - l$$

We now have an expression for width which we can substitute into area $= l \times w$, resulting in only one input value: length. We can then turn this into a proper function with area as our output value and a function of length:

$$\begin{aligned} \text{area} &= l \times (6 - l) \\ &= 6l - l^2 \\ f(l) &= 6l - l^2 \end{aligned}$$

Now we can sketch the function!



Remember that the y-axis represents the area, and the x-axis represents length. From the graph, the maximum area is 9, and occurs when the length is 3. Since we have an expression for width, we can work out what the corresponding width is for the length value:

$$\begin{aligned} w &= 6 - l \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

Therefore, the maximum area is 9 m^2, the length is 3 m, and the width is 3 m.

You are bound to see this type of question again, as these are common **optimisation questions** in Units 3&4, so be sure to fully understand the process behind solving this example.

2.3 Review of quadratic relationships

This next section is all about quadratic functions, also known as parabolas. First off, let's examine how we write a quadratic function.

Quadratics can be written in two forms: **polynomial form** and **turning point form.**

Polynomials are a type of function that are written in a certain way, which is: $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0$. Since that looks pretty confusing, I'll give you some examples.

$P(x) = x + 2$ is a polynomial function of degree 1, meaning the highest power of x is 1. The 2 is a constant, meaning it is a term which does not involve x. Functions can also have a constant of 0, but we wouldn't bother to write $f(x) = x^2 + 0$. Next, we consider $P(x) = x^2 + x + 2$, which is a polynomial of degree 2, because of the x with a power of 2. This is also known as a quadratic or parabola and is what we will be focusing on.

The general form is:

$$P(x) = ax^2 + bx + c$$

Note that the terms with variables are written with the highest power first, then the next highest, and so on. The constant is written last.

Quadratics can also be written in turning point form:

$$P(x) = a(x - h)^2 + k$$

Quadratics can be converted from polynomial form to turning point form by **completing the square** (covered on page 16), or a quadratic in turning point form can be converted to polynomial form by expanding the brackets.

The advantage of the turning point form is that you can easily how the quadratic has been transformed and identify key features – specifically the turning point! This also means you can graph the curve much more quickly than if it was in polynomial form.

- The value of a will widen or narrow the curve.
  - Also remember that a negative a value will flip the curve down – in other words, turn it from a 'smile' into a 'frown'.
- h will translate in the positive x-axis direction.
- k will translate in the positive y-axis direction.
- (h, k) is the turning point of the parabola.

You can also see quadratics as $y = a(x - b)(x - c)$, which is in factorised form. This just means that (on the right-hand side), a, $(x - b)$, and $(x - c)$ are all **multiplied** together, so they are factors. Compare this to the polynomial form, where terms are **added** together.

The polynomial form can also be referred as standard form, and turning point form as vertex form.

KEY POINT :
The main point to realise is that all quadratics have an x to the power of 2 (i.e. x^2). In polynomial form, it is already visible (e.g., $P(x) = x^2 - 2x + 1$). In turning point form: $P(x) = (x - 1)^2$, the x^2 is only visible when expanded out.

Now, let's look at some quadratic equations and solve them through different methods: factorisation, the quadratic formula, completing the square and with a calculator. We'll also need to know how to sketch the quadratics, so I'll show you after the examples.

A quick note before we start – when we talk about solving equations, we make the quadratic itself equal to 0 (e.g., $x^2 + 1 = 0$) and we try to solve for x. Usually, there are two answers for x, but in some cases there may only be one answer. We'll discuss why this is the case later!

2.3.1 Using factorisation

Example 2.5

Solve the equation $x^2 + 2x - 3 = 0$ *by factorisation.*

To factorise, think: "what two numbers multiply to give –3 (from the third term) and add to give 2 (from the coefficient of the second term)".

The factors of 3 are 1 and 3. Since one number must be negative to get to –3, let's say that –1 and 3 are multiplied together, giving –3. When they are added together: $-1 + 3 = 2$. Therefore, these two numbers satisfy our conditions.

The factorised form should look like $(x + b)(x + c)$. Now we just substitute our numbers in (the order doesn't matter):

$$(x-1)(x+3)$$

Now, let's go back to our original equation: $x^2 + 2x - 3 = 0$. We can rewrite this with our factorisation of the left-hand side:

$$(x-1)(x+3) = 0$$

The reason we factorised the left-hand side is because now we can use the **null factor theorem:** either $(x - 1) = 0$ or $(x + 3) = 0$. Since both factors multiply to give 0, one of these factors (or both) must equal 0.

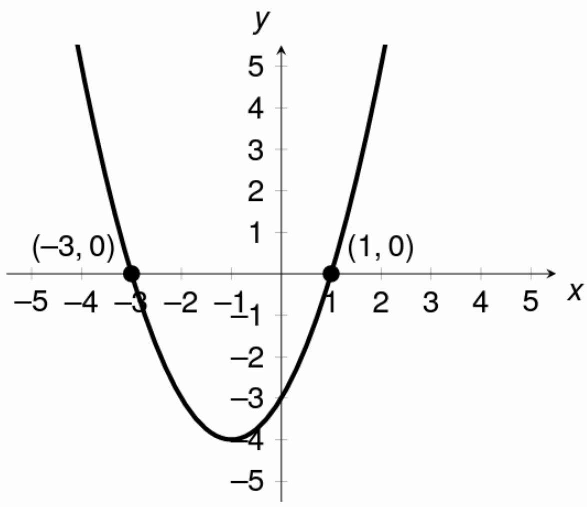
We can now easily solve both equations:

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \\ x + 3 &= 0 \\ x &= -3 \end{aligned}$$

Therefore, the two answers for x are 1 and –3.

Key Point :
This approach only works if the coefficient of x^2 in the equation is 1. If the coefficient is not 1, you have to use the **cross method** or **harbor bridge** method to factorise.

Now, let's make more sense of this equation by displaying its graph:



As you can see, the answers for x we found are the x-intercepts of the graph. That is why the solutions to quadratics are often called 'roots' or 'zeros'. The equation graphing the quadratic is $y = x^2 + 2x - 3$. As I said before, when we solve it, we make the y equal to zero: $0 = x^2 + 2x - 3$. This is why our answers for x relate to the x-intercepts, as x-intercepts have a y-value of zero.

But what if you were asked to graph this equation on your own? We already know the x-intercepts, but we need one more key bit of information: the turning point. This is the point where the function begins to curve and turn back in the other direction, creating the parabolic shape. It is also important to note that if we draw a vertical line that runs through the turning point, the parabola is symmetrical about this line.

With that information, it is easy to find the x-value of the turning point as it is halfway between the two x-intercepts, so our x-value is –1. With the x-value, you can find the corresponding y-value by plugging it into the original equation: $y = (-1)^2 + 2(1) - 3$. The turning point is $(-1, -4)$.

Now you are able to graph the parabola yourself; simply plot out important points like intercepts and turning points, and draw out the shape of a parabola through those points.

> **KEY POINT :**
> When you **solve** a quadratic equation, you are **finding the *x*-intercepts** of the curve on a graph.

As a quick aside, there is another quadratic you might see: $y = a(x - b)(x - c)$. In this form, this quadratic is incredibly easy to solve. Why? Remember that one of our options for solving quadratics was to factorise the quadratic – here it is already factorised for you. All you have to do now is use the null factor theorem to solve it.

2.3.2 Using the quadratic formula

The second method of solving quadratics is to use the quadratic formula, which is shown below:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $ax^2 + bx + c = 0$

Interestingly, this formula is not on the QCAA formula sheet and you may not need it for your exams (especially with the other methods of solving quadratics); however, I'd recommend memorising this formula anyway since you will be using it often.

Let's look at an example to demonstrate how to use this formula.

Example 2.6

Solve the equation $3x^2 - 4x - 2 = 0$ *using the quadratic formula.*

Referring back to the general form of quadratics in polynomial form: $ax^2 + bx + c$, we can establish the values of $a = 3$, $b = -4$, and $c = 2$. Now, substitute the values into the quadratic formula and simplify:

$$\begin{aligned} x &= \frac{4 \pm \sqrt{4^2 - 4 \times 3 \times -2}}{2 \times 3} \\ &= \frac{4 \pm \sqrt{16 + 24}}{6} \\ &= \frac{4 \pm 2\sqrt{10}}{6} \\ &= \frac{2\left(2 \pm \sqrt{10}\right)}{6} \\ &= \frac{2 \pm \sqrt{10}}{3} \end{aligned}$$

Therefore, $x = \frac{2+\sqrt{10}}{3}$ and $x = \frac{2-\sqrt{10}}{3}$. You should generally leave your answers in exact form (rather than rounding off a decimal number) where possible, unless the question states otherwise.

KEY POINT :
If you cannot factorise a quadratic using easier methods or 'by inspection' (i.e. by just looking at the equation), you can use the quadratic formula. You will encounter quadratics that you can't factorise using the methods shown previously, whereas the quadratic formula will *always* work. However, it may take longer to plug in the numbers, especially without a calculator, so try to find a quicker method where possible!

2.3.3 Completing the square

Now, let's look at completing the square. As I said before, you can use this technique to convert a quadratic into turning point form, as well as to solve the quadratic. We will tackle both in one question in the next example.

Example 2.7

Write the equation $y = 6x^2 + 4x - 2$ in turning point form by completing the square, and hence solve the quadratic for x.

To complete the square: make sure the value of a is 1 (which is the coefficient of x^2). Since it is currently 6, we can factor out 6 in the right-hand side to make the coefficient equal to 1.

$$y = 6x^2 + 4x - 2$$
$$y = 6\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$$

Take the value of b (which is $\frac{2}{3}$), divide that by 2, and square it. Since we are squaring it, the sign of b does not matter.

$$\frac{2}{3} \longrightarrow \left(\frac{2}{6}\right)^2 = \left(\frac{1}{3}\right)^2 \quad \text{(do not expand the brackets)}$$

Now, we add $\left(\frac{1}{3}\right)^2$ into the equation, after the $\frac{2}{3}x$. However, to ensure this remains equal, we must also take away $\left(\frac{1}{3}\right)^2$ at the end:

$$y = 6\left[x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \frac{1}{3} - \left(\frac{1}{3}\right)^2\right]$$

Now we can really easily factorise this first part: $x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2$. This is a quadratic, so as in our previous examples, we look for two numbers that multiply to give $\left(\frac{1}{3}\right)^2$ and add to give $\frac{2}{3}$, which are obviously $\frac{1}{3}$ and $\frac{1}{3}$. Remember, this was only obvious because we didn't expand the brackets of $\left(\frac{1}{3}\right)^2$ in the previous step.

Factorising and simplifying:

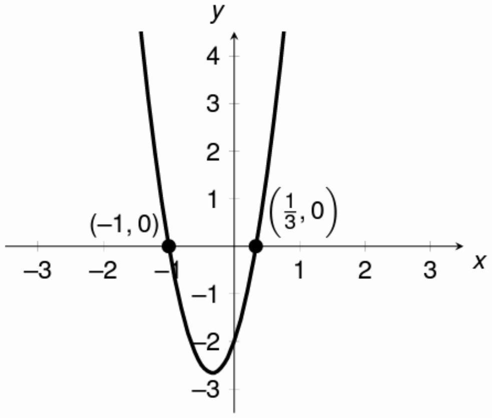
$$\begin{aligned} y &= 6\left[\left(x+\frac{1}{3}\right)\left(x+\frac{1}{3}\right)-\frac{1}{3}-\left(\frac{1}{3}\right)^2\right] \\ &= 6\left[\left(x+\frac{1}{3}\right)^2-\frac{1}{3}-\frac{1}{9}\right] \\ &= 6\left[\left(x+\frac{1}{3}\right)^2-\frac{4}{9}\right] \end{aligned}$$

We now have turned the quadratic into turning point form. Now, let's continue by solving the quadratic.

When we solve quadratics, we make y equal to 0.

$$\begin{aligned} 0 &= 6\left[\left(x+\frac{1}{3}\right)^2-\frac{4}{9}\right] \\ 0 &= \left(x+\frac{1}{3}\right)^2-\frac{4}{9} \\ \frac{4}{9} &= \left(x+\frac{1}{3}\right)^2 \\ \pm\frac{2}{3} &= x+\frac{1}{3} \text{ (remember the } \pm\text{)} \\ x &= \pm\frac{2}{3}-\frac{1}{3} \\ x &= \frac{2}{3}-\frac{1}{3} \text{ or } x = -\frac{2}{3}-\frac{1}{3} \\ x &= \frac{1}{3} \text{ or } x = -1 \end{aligned}$$

Let's look at a graph of the equation:



As you can see, our x-intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$. But what about our other key point, the turning point of the graph? Let's examine the turning point form of the equation: $y = 6\left[\left(x+\frac{1}{3}\right)^2-\frac{4}{9}\right]$. We can expand the brackets to make things clearer: $y = 6\left(x+\frac{1}{3}\right)^2-\frac{8}{3}$. Now, remembering our general turning point form: $y = a(x-h)^2+k$, what are our values of a, h, and k?

$$a = 6, \quad h = -\frac{1}{3}, \quad k = -\frac{8}{3}$$

Remember that the general form has $x - h$, so we need a h-value of $-\frac{1}{3}$.

To find our turning point, remember that the graph has been translated $-\frac{1}{3}$ in the x-axis direction and $-\frac{8}{3}$ in the y-axis direction. Since the turning point of a regular parabola that has not been translated $\left(y = x^2\right)$ is always $(0, 0)$, that means our turning point is $\left(-\frac{1}{3}, -\frac{8}{3}\right)$. Essentially, that means the turning point is (h, k). Now you are able to graph this quadratic independently.

2.3.4 Using a calculator

Our final method of solving quadratics is by using technology. As I've said before, when we solve quadratics, we are looking for the x-intercepts of the graph. So, on a TI-84 Plus graphing calculator:

1. Firstly, graph the equation by pressing the "`y =`" button and entering your equation.
2. Press the "`Graph`" button to view your graph.
3. Press the blue "`2ND`" button and then "`TRACE`" to bring up the "`Calculate`" menu.
4. To find x-intercepts, select "`2: zero`" (remember that x-intercepts are also called 'zeros')
5. It will prompt you to find the "`Left Bound`" of the x-intercept. Do this by moving your cursor near the intercept. You will probably not be able to find the intercept exactly, so you select the left bound, which will be slightly to the left of the intercept and therefore slightly above or below the y-axis. Press "`ENTER`".
6. Now select the "`Right Bound`" of the intercept and press "`ENTER`".
7. Press "`ENTER`" again, and the calculator will give you the coordinates of the intercept.

Also keep in mind that the solver function on the TI-84 (accessed by pressing `MATH`$\longrightarrow$`9`), where you input an equation and it attempts to solve it for you, only gives you one answer. Since we know quadratics can have more than one answer, it is best to **not** use this function for quadratics. Alternatively, your calculator may have a polynomial equation solver program.

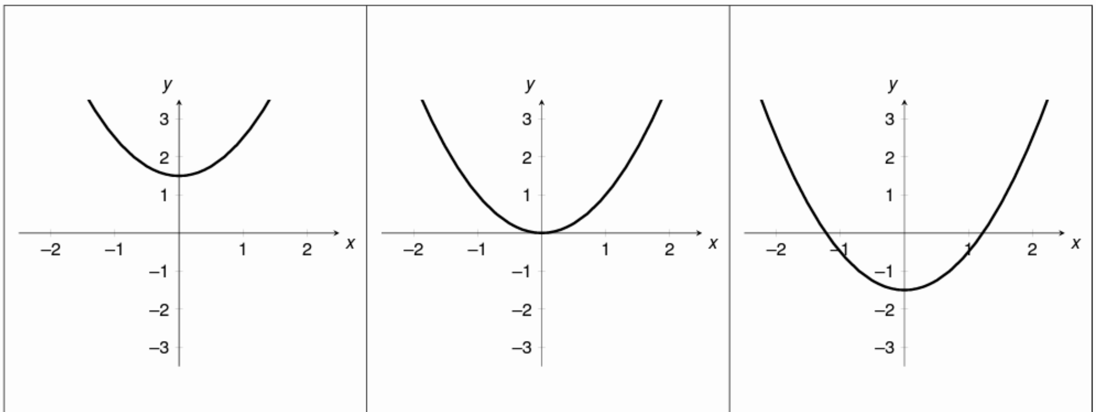
You may also need to find the turning point of a quadratic using a calculator. To do this:

1. Instead of pressing "`2: zero`" in the "`Calculate`" menu, press either "`3: minimum`" or "`4: maximum`" depending on whether the turning point is the highest point on the graph (maximum) or the lowest point on the graph (minimum).
2. Select the left and right bound like above, and the calculate will give you the coordinates of the turning point.

You can try to solve some of the quadratics in the previous examples by using a calculator; you should end up with the same answers.

2.3.5 The discriminant

Now, let's go back to something I said at the beginning of this section, which was that quadratics usually have two answers... but not all the time! Remember that solving the quadratic means finding the x-intercepts. But what if the graph only intersects the x-axis once, or never intersects it at all? Shown below are the three possibilities for solutions to a quadratic equation:



Does not touch the x-axis $\rightarrow$ **no solutions**	Touches the x-axis once $\rightarrow$ **one solution**	Crosses the x-axis twice $\rightarrow$ **two solutions**

So, what happens when we try to solve a quadratic equation that has no solution? Let's say we try to use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve $y = x^2 + 3$. If we plug in the numbers:

$$x = \frac{\pm\sqrt{-4 \times 1 \times 3}}{2}$$
$$x = \frac{\pm\sqrt{-12}}{2}$$

We run into a problem: we cannot take the square root of a negative number (unless you do Specialist Maths!), so it is impossible for us solve this equation now, and your calculator will tell you so as well. As you can see, the problem occurred because of a certain part of the quadratic formula: $b^2 - 4ac$, which is called the **discriminant.**

Three things can happen with the discriminant:

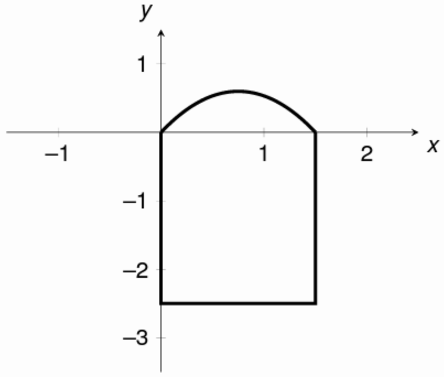
- Like we just saw, if $\boldsymbol{b^2 - 4ac < 0}$, there is **no solution.**
- However, the discriminant can end up to equal zero: $\boldsymbol{b^2 - 4ac = 0}$. If this happens, then this occurs: $x = \dfrac{-b \pm \sqrt{0}}{2a} = \dfrac{-b}{2a}$. This eliminates the $\pm$ sign as well, resulting in only **one solution.**
- Lastly, like in all the earlier examples we went through, the discriminant can be greater than 0: $\boldsymbol{b^2 - 4ac > 0}$ and so **two solutions** are found.

We know how to solve and graph quadratics now, so let's look at how quadratics can be used in worded problems.

2.3.6 Problem solving questions involving quadratics

Example 2.8

The graph below shows a door, where the arch at the top of the door is parabolic in shape. This parabola can be described by the equation: $y = ax^2 + bx + c$. The top of the arch is 3.1 m above the bottom of the door. The door is 1.5 m wide and height of the door not including the arch is 2.5 m. Find the values of a, b, and c.



To find out the equation of the parabola, we need some information about the parabola, such as any point that the parabola runs through. As we already know, the x-intercepts and turning point are the most important points in a parabola, so let's try to find them here.

The x-intercepts are $(0, 0)$ and $(1.5, 0)$, as we know the door is 1.5 m wide. Let's pause for a second and see what happens when we put the point $(0, 0)$ into our equation:

$$y = ax^2 + bx + c$$
$$0 = a \times 0^2 + b \times 0 + c$$
$$0 = c$$

This gives us our value of c, so we can leave our equation as $y = ax^2 + bx$. Now let's see what happens when we try the same with (1.5, 0), or $\left(\frac{3}{2}, 0\right)$:

$$y = ax^2 + bx$$
$$0 = a \times \left(\frac{3}{2}\right)^2 + b \times \left(\frac{3}{2}\right)$$
$$0 = \frac{9}{4}a + \frac{3}{2}b$$

Here, we have two unknowns, which is not enough information to solve the equation. Let's change course and go back to finding our turning point now:

The question stated that the top of the arch (the turning point) is 3.1 m above the bottom. However, we want to find the top of the arch from the x-axis (to find the y-value of its coordinates). This means we have to subtract the height of the door without the arch:

$$3.1 - 2.5 = 0.6 = \frac{3}{5}$$

We know the y-value of the turning point now, but what about the x-value? We can find it by finding the point halfway between the x-intercepts, using the symmetry of the parabola (We used this trick before). If you don't know how to do this, it is the same as taking the average of two numbers.

$$\frac{0 + 1.5}{2} = \frac{3}{4}$$

We now know the turning point is $\left(\frac{3}{4}, \frac{3}{5}\right)$. Let's try to put it into the equation:

$$y = ax^2 + bx$$
$$\frac{3}{5} = a\left(\frac{3}{4}\right)^2 + b\left(\frac{3}{4}\right)$$
$$\frac{3}{5} = \frac{9}{16}a + \frac{3}{4}b$$

This equation also has two unknowns; however, we can solve these now with simultaneous equations!

$$\text{Equation 1: } 0 = \frac{9}{4}a + \frac{3}{2}b$$
$$\text{Equation 2: } \frac{3}{5} = \frac{9}{16}a + \frac{3}{4}b$$

Now, we need to eliminate one variable—what is the best way? Let's try to make a have the same coefficient in both equations, so that we can subtract one equation from the other and eliminate a. We'll start by dividing both sides of Equation 1 by 4:

$$\frac{0}{4} = \frac{\frac{9}{4}a + \frac{3}{2}b}{4}$$
$$0 = \frac{\frac{9}{4}a}{4} + \frac{\frac{3}{2}b}{4}$$
$$0 = \frac{9}{16}a + \frac{3}{8}b$$

Now we can subtract Equation 1 from 2 and find b:

$$\begin{aligned}\frac{3}{5} - 0 &= \frac{9}{16}a + \frac{3}{4}b - \left(\frac{9}{16}a + \frac{3}{8}b\right)\\ \frac{3}{5} &= \frac{3}{4}b - \frac{3}{8}b\\ \frac{3}{5} &= \frac{3}{8}b\\ b &= \frac{8}{5}\end{aligned}$$

We can find a now by substituting the value of b into Equation 1:

$$\begin{aligned}0 &= \frac{9}{4}a + \frac{3}{2} \times \frac{8}{5}\\ 0 &= \frac{9}{4}a + \frac{24}{10}\\ -\frac{24}{10} &= \frac{9}{4}a\\ a &= -\frac{16}{15}\end{aligned}$$

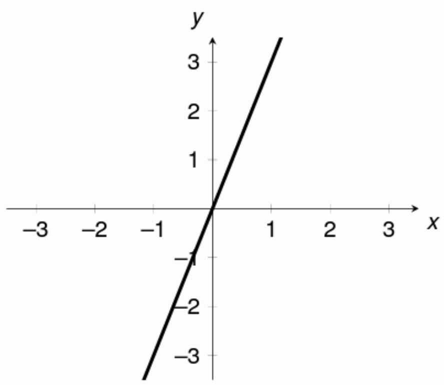
Therefore, $a = -\frac{16}{15}$, $b = \frac{8}{5}$, and $c = 0$.

Note that there are multiple different approaches to solving with simultaneous equations: you could choose to eliminate b first or subtract Equation 2 from 1 instead – you should still reach the same answer in the end. However, you may end up doing more algebra than necessary, so try to look for the most efficient approach.

That concludes the section about quadratics; however, we will return to polynomials after we look at the next short topic: inverse proportions.

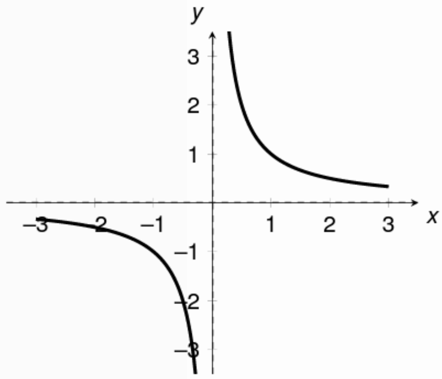
2.4 Inverse proportions

Consider the equation: $y = 3x$, which is shown on graph like:



We can say that as the value of x increases, the value of y also increases, which is decided by multiplying the x value by 3. This means that the variable y is proportional to x, which can be written as $y \propto x$. The 3 in the equation is called the constant of variation – it does not affect whether the variables are proportional or not.

Now consider this graph of $y = \frac{1}{x}$, which is called a hyperbola.



Let's just look at the first quadrant for now: we can see that as x increases, y decreases. We say that y is **inversely proportional** to x, which can be expressed as: $y \propto \frac{1}{x}$.

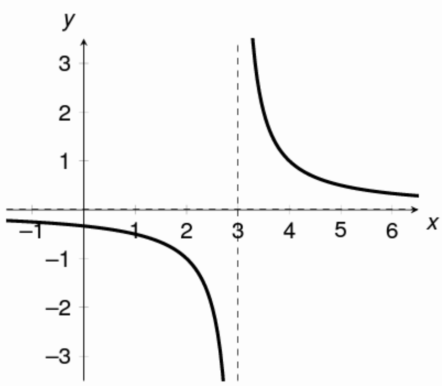
This is also true for the other half of the graph, in the third quadrant: as x decreases, y increases. This equation can also have a constant of variation like before, such as $y = \frac{3}{x}$. The 3 is the constant of variation, which affects the shape of the graph, but the variables are still inversely related.

That aside, what about the overall shape of the graph? Why does it look so different to previous graphs we have encountered? It's because this graph has **asymptotes,** which are parts of the graph where the curve cannot cross.

To explain what an asymptote is, let me ask you something: can you divide by zero? The answer is obviously no. Even our calculators get confused if we try and do that. So, what happens when our x value is 0 on the graph of $y = \frac{1}{x}$. Well... that's where our asymptote is: a vertical line at $x = 0$ (the y-axis). The graph appears to get pretty close to that asymptote, but it will never actually reach it.

However, you might have noticed that there is also a horizontal asymptote at $y = 0$ (the x-axis). Why is this? Let's see what happens if our y value is 0 in equation: $0 = \frac{1}{x}$. You'll find it is impossible to solve this equation, no matter what we do. Therefore, we have a total of two asymptotes: one vertical and one horizontal.

The asymptotes of a hyperbola depend on the equation. Take, for example: $y = \frac{1}{x-3}$. Our vertical asymptote occurs when $x - 3$ (the denominator of the fraction) equals zero. That means our asymptote is now a vertical line at $x = 3$ instead.



Now that we've got a grasp of that, let's jump right back into polynomials!

2.5 Powers and polynomials

We looked at quadratics previously, which were polynomials of degree 2. Now, it's time to discover polynomials of degree 3 and higher! As explained before, polynomials of degree 3 have an x to the power of 3, such as: $P(x) = x^3 + x^2 + 2x + 1$, which are called cubic polynomials. Keep in mind they do not need have an x raised to every power below 3 either; for example, this is still a cubic polynomial: $P(x) = 2x^3 + 1$.

The general form of a cubic is: $P(x) = ax^3 + bx^2 + cx + d$. Like the quadratics, it can also be in another form: $P(x) = a(x-h)^3 + k$. This form isn't called 'turning point form' however... you'll see why later.

Solving cubics is a little bit trickier than quadratics. However, before we tackle that, let's first quickly touch on how to **expand factorised cubics.**

You should already know how to expand quadratics like this: $P(x) = (x-1)(x+2)$, using the **FOIL** method **(First, Inner, Outer, Last).** For a factorised cubic, like this: $P(x) = (x-1)(x-3)(x+4)$, expand the first two pairs of brackets first, then think about splitting the terms in the third set of brackets:

$$\begin{aligned}(x-1)(x-3)(x+4) &= \left(x^2-4x+3\right)(x+4)\\ &= x \times \left(x^2-4x+3\right) + 4 \times \left(x^2-4x+3\right)\\ &= x^3-4x^2+3x+4x^2-16x+12\\ &= x^3-13x+12\end{aligned}$$

Now let's try to solve some cubics!

Our only approaches to solving cubics is either by factorisation, or by using a calculator (meaning there is no 'cubic' formula like the quadratic formula). Back when we factorised quadratics, we learned that something like $x^2 + x - 2$ can be factorised into $(x-1)(x+2)$. Note that each factor has an x to the power of only 1 – this means it is a **linear** factor. Quadratics can only be factorised into two linear factors (think: two x's to the power of 1 multiply to give x^2, our quadratic).

For cubics, there are more possibilities: it can be factorised into three linear factors **or** one quadratic factor and one linear factor (again, think: x^2 from the quadratic factor and x from the linear factor still multiply to give x^3, a cubic). Generally, our plan of approach to factorising cubics is factorise it into one linear and quadratic factor first. From there, we can factorise the quadratic factor into two linear factors since we already know how. Ultimately, we end up with three linear factors, which we can then solve.

Firstly, we need to know how to find the linear factor of a cubic, which I will show through an example:

Example 2.9

Find the linear factor of $P(x) = x^3 - 4x^2 + x + 6$.

We need to use something called the **Factor Theorem.** We need to find out what value of x goes into the equation to make $P(x)$ equal to 0. Fortunately, our options are limited; this value is always going to be a factor of the constant in the polynomial: in this case, 6. Let's try 1.

$$\begin{aligned}P(1) &= (1)^3 - 4(1)^2 + 1 + 6\\ P(1) &= 4\end{aligned}$$

Since this does not equal 0, we know 1 was not our number! Let's try –1 this time (don't forget the factors of 6 can be negative as well).

$$\begin{aligned}P(-1) &= (-1)^3 - 4(-1)^2 - 1 + 6\\ P(-1) &= 0\end{aligned}$$

Therefore, $x = -1$ is what we're after. This process is just trial and error.

To find the linear factor, change the sign of the number (i.e. –1 becomes +1) and add x, so here, our factor is $(x + 1)$.

Now, we've found one linear factor, and we need to factorise the cubic into a linear factor and a quadratic factor. We do this by dividing the cubic by the linear factor, resulting in a quadratic factor. Remember that whenever you divide by a number by a factor, you get another factor. For example, 6 divided by 3 (a factor of 6) is 2, which is another factor of 6.

Let's continue the example:

Example 2.10

Given that $(x+1)$ *is a linear factor of* $P(x) = x^3 - 4x^2 + x + 6$*, factorise* $P(x)$ *into a linear and quadratic factor.*

To divide the cubic, we need to do **long division** (don't worry, I'll explain the steps!). Let's divide the cubic by the linear factor:

$$x+1\overline{)x^3 - 4x^2 + x + 6}$$

Now, look at the first term of $x^3 - 4x^2 + x + 6$, which is x^3, and consider the first term of $x + 1$, which is x. What is x^3 divided by x? The answer is x^2, which we write above the long division symbol like so:

$$\begin{array}{r l} & x^2 \\ x+1) & \overline{x^3 - 4x^2 + x + 6} \end{array}$$

Now, multiply x^2 by both x and 1 to get x^3 and x^2. Write them underneath like:

$$\begin{array}{r l} & x^2 \\ x+1) & \overline{x^3 - 4x^2 + x + 6} \\ & x^3 + x \end{array}$$

Now, subtract the bottom row from the top row:

$$\begin{array}{r l} & x^2 \\ x+1) & \overline{x^3 - 4x^2 + x + 6} \\ & -x^3 + x \\ & 0 - 5x^2 \end{array}$$

Notice that the x^3 is eliminated: the term in this position should disappear every time we do long division.

Bring the next available term in $x^3 - 4x^2 + x + 6$ down:

$$\begin{array}{r l} & x^2 \\ x+1) & \overline{x^3 - 4x^2 + x + 6} \\ & -x^3 + x \\ & \quad - 5x^2 + x \end{array}$$

I've removed the 0 here because it is not needed (and makes things clearer). You do not need to write this when working out; I've only included it before for demonstrative purposes.

We now repeat the process again, except with the bottom row. Take the first term of $-5x^2 + x$ and divide it by the first term of $x + 1$, so $-5x^2 \div x = -5x$. Write this above the long division symbol.

$$
\begin{array}{r l}
 & x^2 - 5x \\
x + 1) & \overline{x^3 - 4x^2 + x + 6} \\
 & -x^3 + x \\
 & \quad -5x^2 + x
\end{array}
$$

This time, multiply $-5x$ by x and 1 and do the subtraction:

$$
\begin{array}{r l}
 & x^2 - 5x \\
x + 1) & \overline{x^3 - 4x^2 + x + 6} \\
 & -x^3 + x \\
 & \quad -5x^2 + x \\
 & \quad - -5x^2 - 5x \\
 & \qquad 6x
\end{array}
$$

Bring down the next term and repeat the process again:

$$
\begin{array}{r l}
 & x^2 - 5x + 6 \\
x + 1) & \overline{x^3 - 4x^2 + x + 6} \\
 & -x^3 + x \\
 & \quad -5x^2 + x \\
 & \quad - -5x^2 - 5x \\
 & \qquad 6x + 6 \\
 & \qquad -6x + 6 \\
 & \qquad\quad 0
\end{array}
$$

The final subtraction is the remainder. If we divide the polynomial by a factor (which we know we did), then the remainder should be 0. You can use this to check if you've done the division correctly. However, if you don't divide by a factor, then you will have a remainder.

The equation on the top is our quadratic factor: $x^2 - 5x + 6$. Therefore, with our two factors, we can write the polynomial as: $P(x) = (x + 1)\left(x^2 - 5x + 6\right)$.

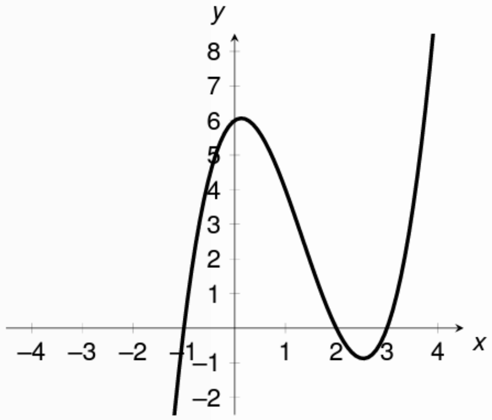
Now, we already know how to factorise $\left(x^2 - 5x + 6\right)$ as it is a quadratic, which becomes $(x - 3)(x - 2)$. We have successfully factorised $P(x) = x^3 - 4x^2 + x + 6$ into three linear factors: $P(x) = (x + 1)(x - 3)(x - 2)$. We can solve it now easily by making it equal to 0: $(x + 1)(x - 3)(x - 2) = 0$. Using the null factor theorem, x can equal -1, 3, or 2, which are our three solutions!

You may also encounter a quadratic factor that cannot be factorised, due to the value of its discriminant. That means the cubic would only have one solution, from the linear factor.

> **Key Point :**
> Remember that our ultimate goal is to factorise a polynomial so we can solve it. Before you begin using the factor theorem and so on, consider if there is an easier way to factorise the polynomial, such as extracting a common factor.

The other way you could solve a cubic is by using a calculator, a process which is exactly the same as solving a quadratic.

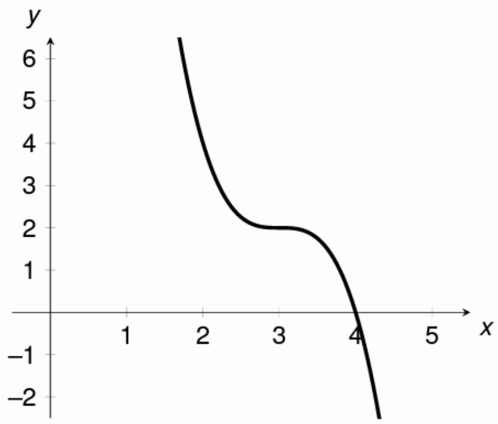
Now, let's look at a graph of the polynomial we just factorised: $P(x) = x^3 - 4x^2 + x + 6$, or in factorised form, $P(x) = (x+1)(x-3)(x-2)$.



As we already know from solving the equation, the x-intercepts are $(-1, 0)$, $(2, 0)$, and $(3, 0)$. Cubics can also have two turning points as shown; however, don't worry about trying to find those points for now, as you'll need to use calculus, which you'll learn later on.

Also notice how as x increases, y increases to infinity. Similarly, when x decreases, y decreases to negative infinity. If we recall our general form for cubics: $P(x) = ax^3 + bx^2 + cx + d$, or $P(x) = a(x-h)^3 + k$, this property will always occur if a is positive. If it is negative, the opposite happens: as x decreases, y increases to infinity and as x increases, y decreases to negative infinity.

Let's look at another example of a cubic, this time with the function: $y = -2(x-3)^3 + 2$.



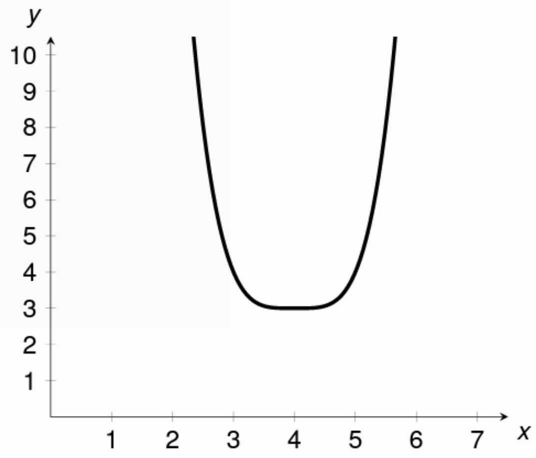
This graph has a **point of inflection:** the graph stops curving, straightens out, then starts curving again. Also note that this cubic does not have any turning points like the previous one does.

Key Point :
If we compare our equation to the general form: $P(x) = a(x-h)^3 + k$, the values of h and k are 3 and 2. This gives us our point of inflection on the graph, $(3, 2)$. The value of h will move the point of inflection horizontally and k will move it vertically. Therefore, the point of inflection is (h, k).

Now, let's look at some polynomials to degree 4 – **quartic functions!**

You can see quartic functions in the expanded form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, or $P(x) = a(x-h)^4 + k$, which is turning point form for quartics. Fortunately, any quartics that you'll be asked to solve will be easy to factorise (i.e. you won't have to use factor theorem or long division – just the null factor law!).

Let's break down a quartic with the equation: $P(x) = (x-4)^4 + 3$.



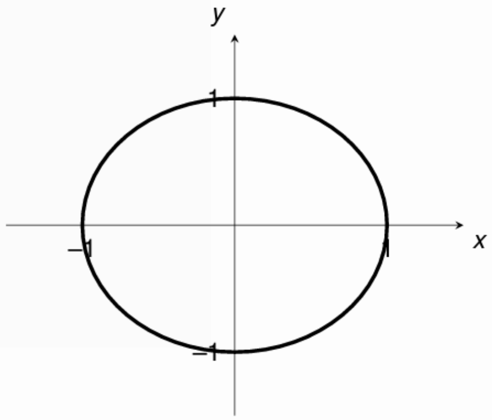
We have a turning point on the graph with the coordinates $(4, 3)$, which is found from the h and k values from the turning point form of the equation $P(x) = (x-4)^4 + 3$. There is not too much you need to know about quartics since you'll probably only encounter simple ones like this. Just note that this looks different to the quadratic $P(x) = (x-4)^2 + 3$, as the quadratic turning point will be less flat.

Before we end the topic, we'll quickly look at some graphs of other **relations,** which are *not functions.*

2.6 Graphs of relations

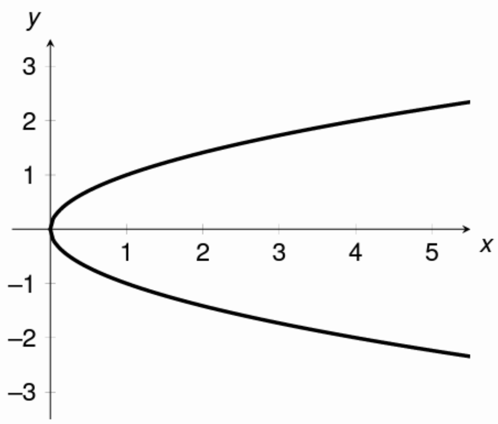
One of our other graphs that we look at is the circle, which has the general equation $(x-h)^2+(y-k)^2=r^2$. The centre of the circle is at (h, k) and the radius of the circle is r.

Shown below is a basic circle: $x^2+y^2=1$.



Note that this relation **is not** a function because it fails the vertical line test, which is also why the equation of a circle usually isn't in that '$y=$' form that functions are expressed in. As h and k both equal 0 here, the centre of the circle is the origin, and the radius is 1.

The other relation we'll look at is a slightly different type of parabola, with the equation: $y^2=x$, which looks like this:



It looks like the quadratic $y=x^2$ but instead flipped on its side. Again, this is not a function as it fails the vertical line test. The turning point here is $(0, 0)$, and note how the axis of symmetry is the x-axis.

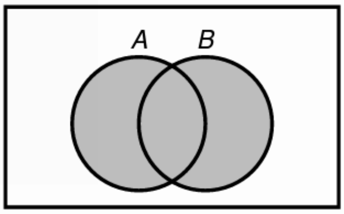
We are now going to change the focus to probability – this tends to be a more challenging topic that most students don't like, but hopefully you'll feel more confident after we go through it here!

Topic 3

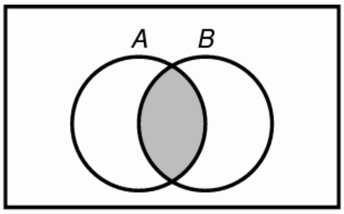
Counting and probability

Before we get to solving any probability questions, we first have to review some probability notation.

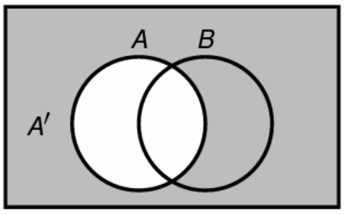
- A **set** is a collection of objects, numbers, etc. which is written as, for example, $A = \{1, 2, 3, 4\}$.
- Objects in that set are called **elements.** Here, the element 1 is in the set of A, which we write as $1 \in A$ (or 'one is an element of A').
- The **union** of two different sets, A and B, is the set of all elements in A and B, written as $A \cup B$.



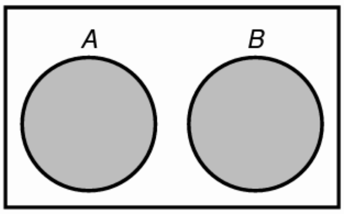
  – Elements in *both* A and B are only listed *once* in $A \cup B$.
  – For example: let $A = \{1, 2\}$ and $B = \{2, 3\}$. This would mean that $A \cup B = \{1, 2, 3\}$. Even though the 2 is in both A and B, we only write it once in the union of the two sets.
- The **intersection** of two different sets, A and B, is the set of elements that are in both A and B, written as $A \cap B$. For example, given $A = \{1, 2\}$ and $B = \{2, 3\}$, $A \cap B = \{2\}$.



- The **complement** of a set, denoted A', includes all elements except the elements in the set A.



- If two sets do not have any elements in common, they are **mutually exclusive,** shown as $A \cap B = \emptyset$.



Now, let's review sample spaces and probability.

Topic 3 – Counting and probability

3.1 Sample spaces and probability

A sample space, shown as ε, lists all the possible outcomes of an experiment using set notation. For example, flipping a coin has the sample space: $\varepsilon = \{H, T\}$.

An event is a subset of the sample space. Say we roll a die, which has the sample space: $\{1, 2, 3, 4, 5, 6\}$. An event could be rolling an even number, which would be the set $\{2, 4, 6\}$.

Remember that probability is the likelihood of something occurring. All probabilities are between 0 and 1, where 0 is impossible and 1 is certain.

Each outcome in a sample space has a probability of occurring, denoted as $Pr\,(A)$, for example. Remember that the probability of all the outcomes in the sample space must sum to 1.

Going back to the dice example: since we know that there is an equal chance of rolling any of the 6 numbers on the die, the probability of rolling a single outcome, such as 3, is:

$$\mathrm{Pr}\,(3) = \frac{1}{\text{total number of outcomes}} = \frac{1}{6}$$

We can also find the probability of an event that is equal to the sum of the probabilities of the outcomes in that event. Let's say A is the event to roll an even number on a die.

$$\begin{aligned} A &= \{2, 4, 6\} \\ Pr(2) &= \frac{1}{6} \\ Pr(4) &= \frac{1}{6} \\ Pr(6) &= \frac{1}{6} \\ Pr(A) &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

When the outcomes are equally likely, the probability of an event can more easily be found by calculating:

$$Pr\,(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Now, say event A is rolling a die with a number less than 3, so our outcomes are $A = \{1, 2\}$. The complementary event, A', is all the outcomes in the sample space not included in A which we write as: $A' = \{3, 4, 5, 6\}$. To find the probability of A', we use $\mathrm{Pr}\left(A'\right) = 1 - \mathrm{Pr}\,(A)$. Here, 1 is the probability that something will happen for certain, so we take away the probabilities of the outcomes in A to find $Pr\left(A'\right)$.

Example 3.1

You perform an experiment, which results in values of 1, 2, 3, or 4. The chance of either 1, 2, or 3 occurring is equally likely. 4 is twice as likely to occur as 3. Find the probability of each outcome:

First, let's establish our sample space: $\varepsilon = \{1, 2, 3, 4\}$. Now, we'll try write down the information given in the question mathematically. 1, 2, and 3 all have the same probability, so $Pr\,(1) = Pr\,(2) = Pr\,(3)$. Meanwhile, 4 is twice as likely to occur as 3, so $Pr\,(4) = 2 \times Pr\,(3)$.

While it's not stated explicitly, we know that the probability of all outcomes adds up to 1:

$$Pr\,(1) + Pr\,(2) + Pr\,(3) + Pr\,(4) = 1$$

We can use some algebra to solve this. Let x equal the probability of either 1, 2, or 3:

$$Pr\,(1) = Pr\,(2) = Pr\,(3) = x$$

Since we established that $Pr\,(3) = x$, we can substitute it into $Pr\,(4) = 2 \times Pr\,(3)$:

$$Pr\,(4) = 2x$$

Let's go back to $Pr\,(1) + Pr\,(2) + Pr\,(3) + Pr\,(4) = 1$, and substitute in what we've established before:

$$\begin{aligned} x + x + x + 2x &= 1 \\ 5x &= 1 \\ x &= \frac{1}{5} \end{aligned}$$

Now we know that probability of either 1, 2, or 3 is $\frac{1}{5}$. To find the probability of 4:

$$\begin{aligned} Pr(4) &= 2x \\ &= 2 \times \frac{1}{5} \\ &= \frac{2}{5} \end{aligned}$$

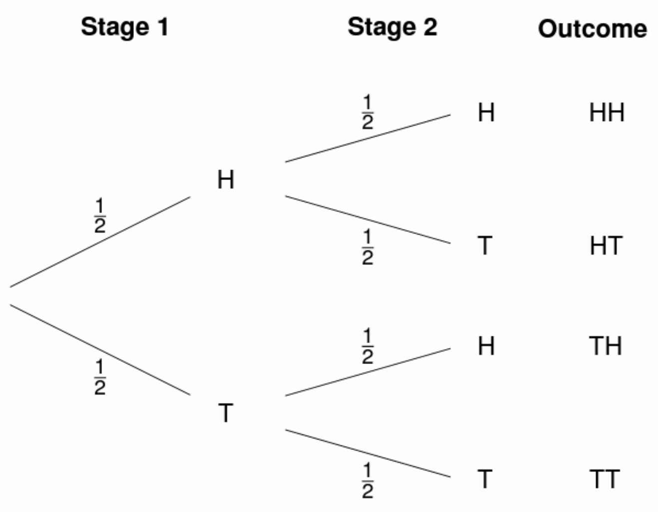
Therefore, the probabilities of the outcomes are: $Pr\,(1) = \frac{1}{5}$, $Pr\,(2) = \frac{1}{5}$, $Pr\,(3) = \frac{1}{5}$, and $Pr\,(4) = \frac{2}{5}$.

Experiments like this have only one step; now let's look at experiments which have multiple steps, each with their own outcomes.

3.2 Multi-stage experiments

Before, we considered the experiment of tossing a coin, which results in either a head or tail. What if we change our experiment to toss a coin twice? How do we keep track of our outcomes?

We do this by using a tree diagram:



Note that our outcomes for Stage 1 and 2 are the same (heads or tails). The sample space for this experiment is: $\varepsilon = \{HH, HT, TH, TT\}$, which we find by starting from the left and following each branch until the end.

To find the probability of each outcome, we first write the probability for outcomes within each stage along the branches, as shown. Since the probability of getting a heads or tails is $\frac{1}{2}$, all the probabilities are the same. Now, to find the probability for getting two heads (HH), **multiply** the relevant probabilities together.

At Stage 1 we got heads, which has a probability of $\frac{1}{2}$. At Stage 2 we also got heads with a probability of $\frac{1}{2}$. Multiplying these together gives us:

$$Pr(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

KEY POINT :
Drawing a diagram is always a useful tip when solving questions, as it allows you to see connections and properties that you might not have seen if you were trying to keep track of everything in your head. In probability, it is especially useful to draw tree diagrams.

3.3 The addition rule

If we want to find the probability of the union of two sets ($A \cup B$), we find the probability of the outcomes in A and B and add them together. However, if these sets are not mutually exclusive, meaning that some outcomes exist in both set A and B, then we will have to subtract the probability of these outcomes as we will have counted them twice. The outcomes that are in both A and B is the intersection: $A \cap B$.

The formula is:

$$Pr\,(A \cup B) = Pr\,(A) + Pr\,(B) - Pr\,(A \cap B)$$

Example 3.2

In a university, the probability a student plays table tennis is 0.18 and the probability a student is a part of the book club is 0.25. The probability that a student both plays table tennis and attends the book club is 0.11. What is the probability that a student either plays table tennis or attends the book club?

First, we translate our information into equations:

$$Pr(TT) = 0.18$$
$$Pr(BC) = 0.25$$

I've used the abbreviations TT for table tennis and BC for book club – you should also do this in your working to save time!

The probability that a student does both activities is the probability of the intersection of TT and BC:

$$Pr(TT \cap BC) = 0.11$$

When the question asks '**either** plays TT **or** in BC', it is asking you for the probability of the **union** of TT and BC. Now we use our formula and substitute in values:

$$\begin{aligned} Pr(TT \cup BC) &= Pr(TT) + Pr(BC) - Pr(TT \cap BC) \\ &= 0.18 + 0.25 - 0.11 \\ &= 0.32 \end{aligned}$$

Therefore, the probability that a student either plays table tennis or attends the book club is 0.32.

3.4 Conditional probability

We use conditional probability to calculate the probability of an event, given that another event has already occurred. The probability of event A given event B has occurred is written as $Pr\,(A|B)$.

To demonstrate how conditional probability is used, let's consider rolling a die again and define event A as 'rolling a six' and event B as 'rolling an even number'. If we say that event B has already occurred, we know our outcomes are $\{2, 4, 6\}$. This limits down our number of outcomes from 6 to 3; when we find the probability of rolling a six as well, it is now $\frac{1}{3}$. That is, $Pr\,(A|B) = \frac{1}{3}$.

Our general formula for conditional probability is:

$$Pr\,(A|B) = \frac{Pr\,(A \cap B)}{Pr(B)}$$

Example 3.3

100 students in a school were questioned about whether they were going to buy their teacher a present at the end of the school year.

	Male	***Female***	***Total***
Yes	*35*	*30*	*65*
No	*25*	*10*	*35*
Total	*60*	*40*	*100*

One person was chosen at random. Given this person is male, what the probability they intend to buy their teacher a present this year?

Let's state what the question is asking:

$$Pr\,(\text{Yes}|\text{Male})$$

Now put this into our formula to figure out what we need to find:

$$Pr\,(\text{Yes}|\text{Male}) = \frac{Pr(\text{Yes}|\text{Male})}{Pr(\text{Male})}$$

To find $Pr(\text{Yes}|\text{Male})$: we look at the table to find the square which is in the 'male' column and the 'yes' row. There are 35 people in this category. As 100 people were surveyed, the probability is $\frac{35}{100} = \frac{7}{20}$.

To find $Pr(\text{Male})$: the table states 60 males were surveyed, making the probability equal to $\frac{60}{100} = \frac{3}{5}$.

Now simply plug in the numbers:

$$\begin{aligned} Pr\,(\text{Yes}|\text{Male}) &= \frac{Pr(\text{Yes}|\text{Male})}{Pr(\text{Male})} \\ &= \frac{\frac{7}{20}}{\frac{3}{5}} \\ &= \frac{7}{12} \end{aligned}$$

Therefore, given that a random male is chosen, there is a $\frac{7}{12}$ chance they will buy their teacher a present.

3.5 Independent events

When we talked about conditional probability, the event that was 'given' always affected the probability of another event. However, if two events are independent, one occurrence of an event has no effect on the occurrence of another event.

To show you what if mean, let A be the event that the second toss of a coin is heads, and B be the event that the first toss is heads. What is $Pr(A|B)$? Logically, you can already tell that flipping a coin once changes nothing about the possible outcomes. The second time you flip the coin, the coin has not changed, and the outcomes are the same.

We can also show this mathematically:

$$\begin{aligned} Pr(A|B) &= \frac{Pr(A \cap B)}{Pr(B)} \\ &= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

As you can see, $Pr\,(A|B) = \frac{1}{2}$ is the same probability as $Pr\,(A) = \frac{1}{2}$ (which is just tossing heads). Therefore, $Pr\,(A|B) = \mathrm{Pr}\,(A)$. This leads us to a formula for **independence:**

$$\begin{aligned} Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} &= Pr(A) \\ \frac{Pr(A \cap B)}{Pr(B)} &= Pr(A) \\ Pr(A \cap B) &= Pr(A) \times Pr(B) \end{aligned}$$

Topic 3 – Counting and probability

Therefore, two events are only independent if:

$$Pr(A \cap B) = Pr(A) \times Pr(B)$$

Example 3.4

An experiment is performed where a number is drawn at random from: $\{1, 2, 3, 4, 5, 6, 7, 8\}$. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{1, 3, 5, 7\}$. *Are events A and B independent?*

We know that if A and B are independent, then the following is true: $Pr\,(A \cap B) = Pr\,(A) \times Pr\,(B)$. First, let's find the value of $Pr\,(A \cap B)$. To do this, we need to figure out the set of $A \cap B$, which is equal to $\{1, 3\}$. So, $Pr\,(A \cap B)$ is the probability that either 1 or 3 gets picked from the 8 total numbers:

$$\begin{aligned} Pr(A \cap B) &= \frac{\text{number of outcomes in } A \cap B}{\text{total number of outcomes}} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

Then let's find $Pr(A)$ and $Pr(B)$:

$$\begin{aligned}Pr(A) &= \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}\\ &= \frac{4}{8}\\ &= \frac{1}{2}\\ Pr(B) &= \frac{\text{number of outcomes in } B}{\text{total number of outcomes}}\\ &= \frac{4}{8}\\ &= \frac{1}{2}\end{aligned}$$

Let's see if the equation is true:

$$\begin{aligned}Pr(A \cap B) &= Pr(A) \times Pr(B)\\ \frac{1}{4} &= \frac{1}{2} \times \frac{1}{2}\\ \frac{1}{4} &= \frac{1}{4}\end{aligned}$$

Therefore, events A and B are independent.

3.6 Counting methods and binomial expansion

Here we will look at more efficient ways of counting the number of outcomes possible, which is needed when we get to more complicated probability problems. Previously, it was easy to know the outcomes of rolling a die, and we also drew a tree diagram to find the outcomes of flipping a coin twice. However, we will get to a point where it is too tedious to list out all the outcomes.

Let's think about our example of flipping a coin twice – we know that there are four outcomes. We can also calculate the outcomes by multiplying the number of options at each stage (see the tree diagram on page 31). Since there are two options at Stage 1 (head or tails), and again two options at Stage 2, the number of outcomes is $2 \times 2 = 4$.

Something called **arrangements** and **combinations** can also help us with counting outcomes when we're dealing with more complex events.

3.6.1 Arrangements

If we have a set of objects $\{A, B, C\}$, we can arrange these objects in different orders, such as BAC. For an **arrangement**, the order of the objects is important. That is, ABC and BAC are considered two different arrangements because – even though they use the same letters, the order is different. Our arrangement does not need to have 3 letters either.

We use this formula to find the number of ways to arrange n objects in groups of size r:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

For the set of $\{A, B, C\}$, let's say we want to find the number of ways to arrange into groups of 2 (i.e. $r = 2$).

There are 3 letters, so $n = 3$.

$$\begin{aligned} {}^nP_r &= \frac{n!}{(n-r)!} \\ {}^3P_2 &= \frac{3!}{(3-2)!} \\ &= \frac{3 \times 2 \times 1}{1} \\ &= 6 \end{aligned}$$

Note that there should be a "`nPr`" button on your calculator, so you don't have to use the formula.

Since the set is small, we can also arrange the letters ourselves to see if we get the same number, which we do:

$$AB,\ \ AC,\ \ BA,\ \ BC,\ \ CA,\ \ CB\ = 6 \text{ outcomes}$$

3.6.2 Combinations

Now let's look at **combinations.** This is where we are grouping objects of a set, but this time the order of objects is not important.

Previously, we arranged $\{A, B, C\}$ into the six arrangements listed above. However, if we want to find the number of combinations, AB and BA are considered the same because they use the same letters; the order doesn't matter. Therefore, the combinations from the set $\{A, B, C\}$ are: AB, AC, BC.

We can also use a formula to do this. To find the number of combinations of n objects in groups of size r:

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Note that ${}^nC_r$ can also be written as $\begin{pmatrix} n \\ r \end{pmatrix}$. There is also a "`nCr`" button on your calculator.

$$\begin{aligned} {}^nC_r &= \frac{n!}{r!\,(n-r)!} \\ {}^3C_2 &= \frac{3!}{2!(3-2)!} \\ &= \frac{6}{2 \times 1} \\ &= 3 \end{aligned}$$

Combinations are also important for the **binomial theorem,** which we use to expand binomials. To expand $(a + n)^n$, which is a binomial ('bi-' meaning two) as it has two terms, a and n , we use this formula, which is related to Pascal's Triangle:

$$(a+b)^n = a^n + \begin{pmatrix} n \\ 1 \end{pmatrix} a^{n-1}b + \begin{pmatrix} n \\ 2 \end{pmatrix} a^{n-2}b^2 + \cdots + \begin{pmatrix} n \\ r \end{pmatrix} a^{n-r}b^r + \cdots + \begin{pmatrix} n \\ n-1 \end{pmatrix} ab^{n-1} + b^n$$

This looks confusing, so it is best to explain it with an easy example.

Example 3.5

Use the binomial theorem to expand $(a+b)^3$.

Here, $n = 3$. Now we'll draw up Pascal's Triangle. We start at the top by writing 1. Then, below we write two 1s on the outside. For the row below, the two 1s add to give 2, so we write that in the middle. We continue writing 1s on the outside and adding the values below. The pattern continues on, and each level of the triangle relates to the value of n. Since $n = 3$, we look at the bottom row: 1 3 3 1.

$$\begin{array}{ccccccc|c} & & & 1 & & & & n=0 \\ & & 1 & & 1 & & & n=1 \\ & 1 & & 2 & & 1 & & n=2 \\ 1 & & 3 & & 3 & & 1 & n=3 \end{array}$$

Now, take our two terms in the binomial, a and b, and multiply together, ab. Now multiply ab to each number in 1, 3, 3, 1 and add together like:

$$1ab + 3ab + 3ab + 1ab$$

The numbers 1, 3, 3, and 1 are the coefficients to the terms. Now, take the first term of the binomial $(a + b)$, which is a. For our expansion, we'll change its index so the leftmost a starts to power of n. For each a afterwards, the power decreases by 1 until we reach 0:

$$1a^3b + 3a^2b + 3a^1b + 1a^0b$$

For the second term of the binomial, b, we do the opposite. We'll change it so it starts at the power of 0 and increases by 1 until it reaches n, which is 3:

$$1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

This is how we expand the binomial. However, there is a better way to find the coefficients (1, 3, 3, 1) than drawing up the triangle every time. We can instead use combinations!

Our first coefficient was 1. To find 1, we do the combination $\begin{pmatrix} n \\ 0 \end{pmatrix}$. There is a 0 because it is the first coefficient (we start from 0):

$$\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 1$$

Our next number is found by calculating $\begin{pmatrix} n \\ 1 \end{pmatrix}$. The 0 changes to a 1... you should be seeing the pattern!

$$\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$$

We continue on:

$$\binom{n}{2} = \binom{3}{2} = 3$$
$$\binom{n}{3} = \binom{3}{3} = 1$$

At the point where the top and bottom numbers are the same, we stop. As you can see, we found the coefficients (1, 3, 3, 1) using combinations instead. The expansion would look like:

$$\binom{3}{0} a^3b^0 + \binom{3}{1} a^2b^1 + \binom{3}{2} a^1b^2 + \binom{3}{3} a^0b^3$$
$$=1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

Just try to keep track of all the patterns in your head as we try another example.

KEY POINT :
This is the type of thing where the formula isn't of much use since it is long and confusing. Understanding the explanations here are more important, and then you can practise expanding some binomials to make the process stick in your memory.

Example 3.6

Use binomial theorem to expand $(2-3x)^4$.

First, we'll find the coefficients of the expanded terms. Remember how the pattern goes?

We start with $\binom{n}{0}$, which is $\binom{4}{0}$ as $n = 4$. Then keep increasing the bottom number to n:

$$\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$$

Now, multiply the two terms in the binomial together: $2 \times -3x$ and multiply it to each coefficient:

$$\binom{4}{0} \times 2 \times (-3x) + \binom{4}{1} \times 2 \times (-3x) + \binom{4}{2} \times 2 \times (-3x) + \binom{4}{3} \times 2 \times (-3x) + \binom{4}{4} \times 2 \times (-3x)$$

Now, remember we have to change the powers of 2 and $-3x^4$. Remember the pattern:

- For the first term (2) the power starts at n and decreases until 0.
- For the second term $(-3x)$ the power starts at 0 and increases until n.

$$\binom{4}{0} \times 2^4 \times (-3x)^0 + \binom{4}{1} \times 2^3 \times (-3x)^1 + \binom{4}{2} \times 2^2 \times (-3x)^2 + \binom{4}{3} \times 2^1 \times (-3x)^3 + \binom{4}{4} \times 2^0 \times (-3x)^4$$

Now just simplify:

$$16 - 96x + 216x^2 - 216x^3 + 81x^4$$

This can be kind of confusing to do at first, but once you do a few practice questions you'll get the hang of the process.

Topic 4

Exponential functions 1

In this very short topic, we'll look at indices and the index laws. This is an important foundation for later topics covered in Unit 2, as well as the Units 3&4 course next year.

4.1 Indices and the index laws

The expression x^2 is called a power, read as 'x the power of 2', which you probably know by now. The x is called the **base,** and 2 is called the **exponent** or **index**. Indices (the plural of index) can be positive, negative, or in fraction form.

The index laws cover calculations involving powers, and you should learn all of these.

Index Law 1: Multiplying Powers	$a^m \times a^n = a^{m+n}$
Index Law 2: Raising Powers	$\left(a^m\right)^n = a^{m \times n}$
Index Law 3: Dividing Powers	$a^m \div a^n = a^{m-n}$
Index Law 4: Products	$(ab)^m = a^m \times b^m$
Index Law 5: Quotients	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
Index Law 6: The Zero Index	$a^0 = 1$
Index Law 7: Negative Integer Indices	$a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$

4.2 Rational indices

A radical or root is the opposite operation of applying a power. If we raise 2 to the power of 2, we get $2^2 = 4$. Now if we take the square root of 4, $\sqrt{4} = 2$, we get back to 2. However, we can write roots as powers:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

For example, the square root of 4 is $\sqrt[2]{4}$ (typically we don't include the two when writing square roots, however you must remember it's still there), can be written as $4^{\frac{1}{2}}$. This will get a little more complicated when we look at practical applications of finding the square root, but for now you just have to understand this notation.

4.3 Scientific notation

Finally, when we deal with large numbers, it is inconvenient to fully write out that number, so we use scientific notation (also known as standard form).

For instance, the number 33,335 can be written as 3.3335×10^4. As you can see, the number is changed so it is a product of a number between 1 and 10 (in this case, 3.3335) and a power of 10 (in this case, 10^4).

To do this, start from the decimal on the right of 33,335.0 and move it left until it is beside the leftmost number:

$$33,335.0 \rightarrow 3,333.5 \rightarrow 333.35 \rightarrow 33.335 \rightarrow 3.3335$$

Since we moved the decimal 4 times, 10 is raised to the power of 4 (10^4). Take 3.3335 and 10^4 and multiply to get the scientific notation: 3.3335×10^4.

Our next topic links back to the first topic: sequences and series, so you may want to go back and refresh your knowledge of arithmetic sequences and series before we jump into the geometric ones.

Topic 5

Geometric sequences and series

5.1 Geometric sequences

If you recall back to our arithmetic sequences, such as 1, 2, 3, 4, each number is called a term and we add or subtract a fixed value to one term to get to the next.

For geometric sequences, we instead **multiply** one term by a fixed amount to get to the next, such as the sequence: 1, 2, 4, 8, where each term is multiplied by 2.

To find the n^{th} term in a geometric sequence, we use the formula:

$$t_n = t_1 r^{n-1}$$

- t_1 is the first term
- r is the fixed amount we are multiplying by (also called the common ratio between terms)
- n is the number representing a term in the sequence

Note that this formula is **exponential.** We have r raised to the power of $n - 1$. As we find successive terms in a sequence, the exponent will increase.

Example 5.1

Find the 10^{th} *term in the geometric sequence:* 25, 37.5, 56.25...

Looking at the relevant formula: $t_n = t_1 r^{n-1}$, we need to find the 10^{th} term, so $n = 10$. We know that $t_1 = 25$; now we need to find r. The question is: what number do we multiply 25 by to get to 37.5? We can figure it out by finding the common ratio between successive terms by calculating:

$$r = \frac{t_2}{t_1} = \frac{37.5}{25} = \frac{3}{2}$$

You can also use any two successive terms:

$$\frac{t_3}{t_2} = \frac{56.25}{37.5} = \frac{3}{2}$$

Now we use our formula:

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ t_{10} &= 25 \times \left(\frac{3}{2}\right)^{10-1} \\ &= \frac{492075}{512} \end{aligned}$$

Therefore, the 10^{th} term is $\frac{492075}{512}$.

Key Point :
Now that we know two types of sequences – arithmetic and geometric – if the question doesn't tell you what type a sequence is, you will have to identify it yourself by looking for either a **common difference** or a **common ratio.**

5.2 Geometric series

Geometric series is the sum of the terms in a geometric sequence up to and including a certain term. The formula we use is:

$$S_n = t_1 \frac{(r^n - 1)}{(r - 1)}$$

This formula starts from the beginning of the sequence and finds the sum of the first n terms.

Example 5.2

Find the sum of the first eleven terms in the sequence: $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}...$

Here, $n = 11$ and $t_1 = \frac{1}{3}$. To find r, remember we find the common ratio between two successive terms: $r = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$. Using the formula:

$$\begin{aligned} S_n &= t_1 \frac{(r^n - 1)}{(r - 1)} \\ S_{11} &= \frac{1}{3} \times \frac{\left(\left(\frac{1}{3}\right)^{11} - 1\right)}{\frac{1}{3} - 1} \\ &\approx 0.4999 \end{aligned}$$

We can apply geometric sequences and series in problem solving questions such as compound interest or depreciation.

5.3 Applications of compound interest

If we invest a certain amount of money ($100), and it is compounded annually at an interest rate of 10%:

1. After one year we will have $100 plus 10% of 100, which is: $100 × 1.1 = $110.
2. After two years: $100 × 1.1 × 1.1 = $121, or $100 × 1.1^2 = $121.
3. After three years: $100 × 1.1^3 = $133.1.

As you can see, this looks a lot like the formula for finding the term in geometric sequence: $t_n = t_1 r^{n-1}$, which is exactly what is happening. At the end of every year, we multiply the money by a fixed amount, creating a geometric sequence. The sequence we found was: 100, 110, 121, 133.1. However, keep in mind that r is not the interest rate (0.1 or 10%) itself but includes the money present as well (1.1 or 110%). Also, the first value in the sequence is the original amount of money, $100. We found the value after one year: $100 × 1.1 = $110, and the exponent of 1.1 is 1, so the value of n is 2 (2 – 1 = 1).

Example 5.3

$5,000 is invested at a rate of 6% which is compounded annually. What is the value after 6 years?

First, we know that we need the geometric sequence formula: $t_n = t_1 r^{n-1}$. Here, t_1 is $5,000 and r is 1.06. We're asked for the value after 6 years: this means the 7th term in the sequence, as we must remember the first term is $5,000, so $n = 7$. Let's plug the numbers in and solve:

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ t_7 &= 200 \times (1.06)^{7-1} \\ &= 7092.5 \end{aligned}$$

Therefore, after 6 years, the value of the investment is $7092.5.

You might also find questions about **depreciation** – this is the same concept as compound interest except that an amount of money is going down instead of up.

Part II

Unit 2: Calculus and further functions

Topic 1

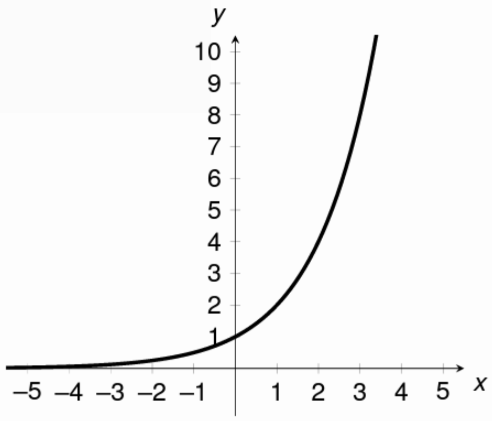
Exponential functions 2

1.1 Graphs and transformations of exponentials

In our first topic of Unit 2, we continue our study of exponential functions.

We are going to look at graphs of different types of exponential functions, which are just functions where x is an exponent or index.

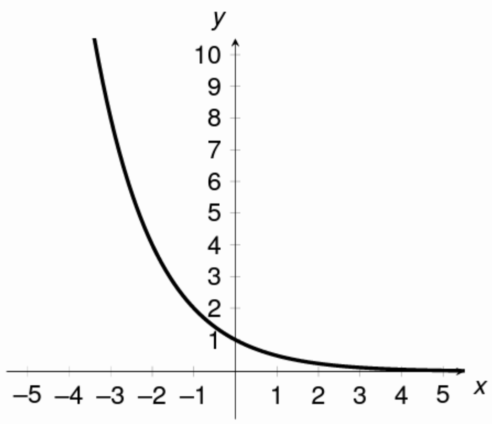
Our first function is in the form $y = a^x$ when $a > 1$. We'll look at a graph of $y = 2^x$.



The first thing to note is that we have a horizontal asymptote at $y = 0$ (the x-axis); as the x value keeps decreasing, the y-values decrease closer and closer until zero, but they never reach 0. This can be explained with logarithms (which we'll cover later!). The main point is that if we make the y value 0 and try to solve for x, the equation is impossible to solve: $0 = 2^x$.

As the x-value increases, the y-value increases as well, and the y-intercept is at $(0, 1)$.

Now, we'll see what happens when $0 < a < 1$ for $y = a^x$. Let's look at $y = \left(\frac{1}{2}\right)^x$.



Again, the x-axis is the asymptote as $0 = \left(\frac{1}{2}\right)^x$ is impossible to solve. The y-intercept is also at $(0, 1)$. However, the graph looks like it has been mirrored: as x decreases, y increases.

Also note that we didn't look at a graph of $y = 1^x$ because 1 to the power of anything is always 1, so the graph would just be the straight line: $y = 1$.

We can also translate exponential functions like we did to other functions before.

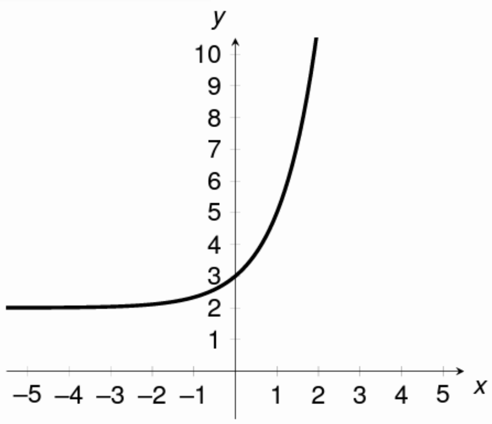
For exponential functions in the form: $y = a^{x+c} + b$

- c translates in the positive x-axis direction.
- b translates in the positive y-axis direction.
- a dilates the curve, making it steeper or flatter, and negative a values will reflect the graph in the x-axis.

You may find equations in the form $y = a^{kx}$. Changing the value of k has the same result as changing the value of a. For example, if we change $y = 2^x$ to $y = 4^x$, we can also make that same change by doing $y = 2^{2x}$.

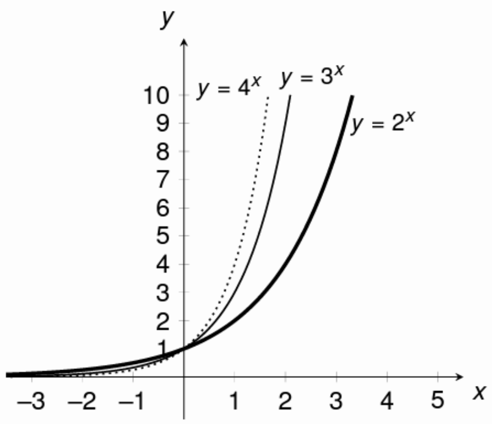
(Remember the index laws: 2^{2x} is the same as $(2^2)^x$, which is 4^x).

Now let's look at the b constant. For example, below is a graph of: $y = 3^x + 2$.



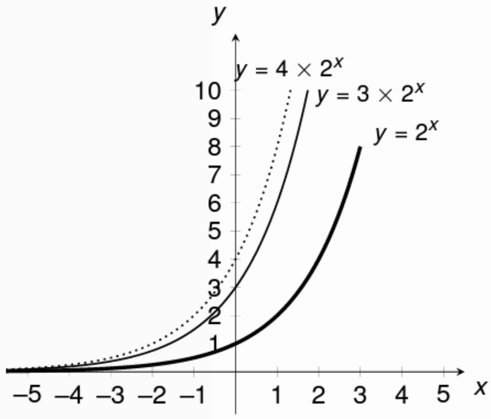
Since we moved the graph upwards with a b value of +2, the asymptote that was originally at $y = 0$ was moved up to $y = 2$.

Next, let's consider the influence of a dilating the curve. Below are the equations of $y = 2^x$, $y = 3^x$ and $y = 4^x$ all graphed.



As the value of a increases, the incline of the curve gets steeper. However, notice that the y-intercept of all the equations is the same, at (0, 1).

You also may find exponentials as: $y = d \times a^x$. The value of d also dilates the curve, but it also changes the y-intercept, unlike changing the value of a. Below, the functions shown are: $y = 2^x$, $y = 3 \times 2^x$, and $y = 4 \times 2^x$.



Like changing the value of a, as we increase the value of d, the curve becomes steeper. The key difference is that the y-intercepts of each equation are different: $(0, 1)$, $(0, 3)$ and $(0, 4)$ for the respective functions $y = 2^x$, $y = 3 \times 2^x$ and $y = 4 \times 2^x$.

1.2 Solving exponential functions

The index laws we looked at earlier mostly cover operations involving powers that have the same base. When we are solving exponentials, if we want to use the index laws, we have to make sure this is the case, by manipulating the equation. We can equate the indices of two powers if their bases are the same to solve exponentials as well, which is shown below.

Example 1.1

Solve $49^x = 7$ *for* x.

We can say that the 7 on the right-hand side is a power with base 7 and index 1, which is 7^1. On the left-hand side, the power has a base of 49, so we'll try to change the base to 7.

49 can be rewritten as 7^2. But we have to raise that to the power of x as well:

$$\left(7^2\right)^x = 7^{2x}$$

Our equation is now:

$$7^{2x} = 7^1$$

Since the bases of both powers are the same, we can equate the indices. That is, we can say for sure that the index of one power must be equal to the index of the other power since nothing else is different. Then, we simply solve for x:

$$\begin{aligned} 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

We will cover questions involving modelling using exponentials after we cover logarithms, a topic closely related to exponentials.

Topic 2

The logarithmic function 1

2.1 Introduction to logs

To explain what logarithms (logs) are, consider this equation:

$$10^2 = 100$$

We have a power with a base (10) and index (2), giving us a final value of 100.

What if we have the final value (100), the base (10) and want to find out what the index is? We use a logarithm, which looks like this:

$$\log_a(b) = x$$

The a is the value of the base, and b is the final value. When we put our numbers in, $\log_{10}(100) = 2$. This is read as 'log of 100 to the base 10 equals 2.'

With this information, the equation $10^2 = 100$ can be rewritten as $\log_{10}(100) = 2$. More generally:

$$a^x = b \text{ is equivalent to } \log_a b = x$$

To evaluate $\log_a b$, ask yourself: "a to the power of what number gives b?"

To do operations with logarithms, we have Logarithm Laws that are derived from the Index Laws.

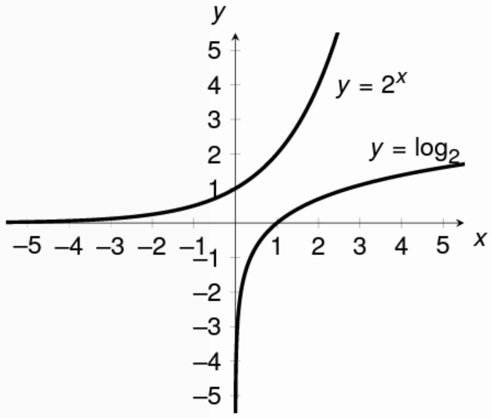
Log Law 1: Logarithm of a Product	$\log_a(mn) = \log_a m + \log_a n$
Log Law 2: Logarithm of a Quotient	$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
Log Law 3: Logarithm of a Power	$\log_a(m^p) = p \times \log_a m$
Log Law 4: Logarithms of 1	$\log_a 1 = 0$ and $\log_a a = 1$

Additionally, the log of a negative number (to any base) is impossible.

The next one is not a law, but handy to know: if we have $y = a^{\log_a b}$ and convert the power of a to a logarithm expression: $\log_a y = \log_a b$. We can see that $y = b$, which means:

$$a^{\log_a b} = b$$

But what do logarithms look like on a graph? It turns out that $y = a^x$ and $y = \log_a x$ have an inverse relationship which you can see below. Let's graph $y = 2^x$ and $y = \log_2 x$:



You can see the inverse relationship here. The function of $y = \log_2 x$ is a reflection of the function $y = 2^x$ in the line $y = x$. More generally the graph $y = \log_a x$ is a reflection of the graph $y = a^x$ in the line $y = x$.

Now, let's try to solve some equations using logs.

Example 2.1

Solve $\log_3 (x-1) + \log_3 (x+1) = 1$ *for* x.

We need to somehow get the x terms on one side to solve this, so let's start by combining the two logs using Log Law 1. Note we can do this because the bases of the logs are the same:

$$\begin{aligned} \log_3(x-1) + \log_3(x+1) &= 1 \\ \log_3[(x-1)(x+1)] &= 1 \\ \log_3\left[x^2-1\right] &= 1 \end{aligned}$$

Now, we can rewrite this in our equivalent form to get rid of the log and solve:

$$\begin{aligned} \log_3\left[x^2-1\right] &= 1 \\ x^2-1 &= 3 \\ x^2 &= 4 \\ x &= \pm 2 \text{ (don't forget the } \pm \text{ sign)} \end{aligned}$$

We have found that $x = 2$ or $x = -2$. However, there is one thing you have to remember with logs... you can **never** take the log of a negative number! Let's test that by assuming $x = -2$:

$$\begin{aligned} \log_3(-2-1) + \log_3(-2+1) &= 1 \\ \log_3(-3) + \log_3(-1) &= 1 \end{aligned}$$

We cannot take the log of a negative number, so that means $x = -2$ is **not a valid solution.**

Whereas, if we take the $x = 2$ solution:

$$\begin{aligned} \log_3(2-1) + \log_3(2+1) &= 1 \\ \log_3(1) + \log_3(3) &= 1 \end{aligned}$$

Everything is fine here, and if we plug this into our calculator, we'll see that this is correct. Therefore, we have confirmed that our solution is $x = 2$.

Key Point :
For logarithms, the main thing to remember when solving equations is that one of your solutions may be invalid since you can't take the log of a negative number. So, be sure to check by putting your solution back into the equation; your answer won't be marked as correct if you mistakenly state an invalid solution.

Now let's put our logarithm and exponential knowledge to use by solving from modelling problems.

Example 2.2

A rat-infested sewer in a city contains 500,000 rats at 12:00 p.m. on Sunday. The number of rats increases by 10% every hour. At what time will the sewer exceed 4 million rats and overrun the city?

We have to create an exponential model that connects the variable of time (t) and the number of rats (R). First, let's calculate the population of rats each hour to see if we can find any patterns:

$$\begin{aligned}
&t = 0, \ \ R = 500,000 \\
&t = 1, \ \ R = 500,000 \times 1.1 \\
&t = 2, \ \ R = (500,000 \times 1.1) \times 1.1 = 500,000 \times 1.1^2 \\
&t = 3, \ \ R = 500,000 \times 1.1^3
\end{aligned}$$

You can see that 1.1 is raised to the power of t to find the number of rats. Now, we can establish that:

$$R = 500\,000 \times 1.1^t$$

Once we have our model, we can go about solving the question: we want to find what value of t gives $R = 4,000,000$:

$$\begin{aligned}
4,000,000 &= 500,000 \times 1.1^t \\
\frac{4,000,000}{500,000} &= \frac{500,000}{500,000} \times 1.1^t \\
8 &= 1.1^t
\end{aligned}$$

You might not know how to solve this at first glance, but all we need to do now is use our logarithm knowledge!

$8 = 1.1^t$ can be rewritten as the equivalent equation: $\log_{1.1} 8 = t$. We can use a calculator to evaluate this and get the value:

$$t = 21.8176...$$

However, the question asked at what time will the rats *exceed* 4 million; we can just round this number up to $t = 22$. The last step is to add 22 hours to the starting time at 12:00 p.m. on Sunday, which ends up at 10:00 a.m. on Monday.

Therefore, the number of rats will exceed 4 million tomorrow at 10:00 a.m. Monday.

Generally, for this type of question we can use the formula (which we found out ourselves in the question above):

$$A = A_0 b^t$$

- A is the quantity at time t.
- A_0 is the starting quantity.
- b is a constant.
  - If b is positive, the quantity will grow.
  - If b is negative, the quantity will decrease.

Topic 3

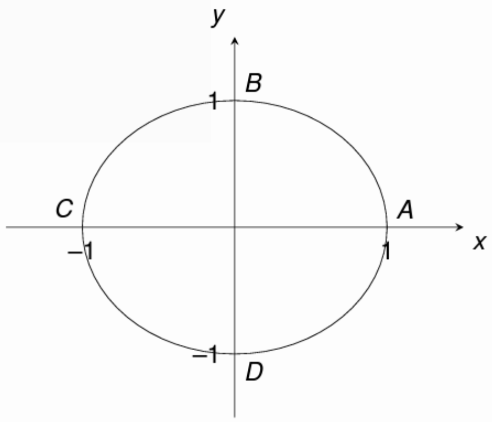
Trigonometric functions 1

So far, we have seen polynomial, exponential, and logarithmic functions. In this topic, we look at a new type of function: trigonometric functions!

However, before that, we must look at some important concepts first.

3.1 Circular measure and radian measure

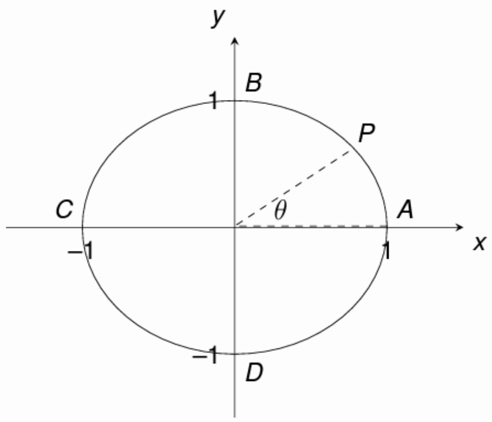
In this graph, we have the unit circle – a circle with a radius of 1.



We know that the formula for the circumference of a circle is $2\pi r$, so the circumference here is $2\pi \times 1 = 2\pi$. If we start from A and travel around the circle, the distance from:

- A to B is a quarter of $2\pi = \frac{2\pi}{4} = \frac{\pi}{4}$
- A to C is a half of $2\pi = \frac{2\pi}{2} = \pi$
- A to D is three-quarters of $2\pi = \frac{3}{4} \times 2\pi = \frac{3\pi}{2}$

Now imagine that we have an angle of AOP, where P is any point on the circle. The point P is determined by the angle.



While you may only be familiar with measuring angles using degrees (e.g. 30°), we can also measure angles with something called **radians.** 1 radian (written like 1^c), is the angle of AOP which gives the arc of AP the length of 1 unit. Generally, x^c is the angle that makes the arc AP have the length x units.

So, if we want the arc of *AP* to sweep around the entire circumference, it will have a length of 2π units, meaning the angle required for this is $2\pi^c$. However, we also know that the angle **in degrees** to sweep out a full revolution of a circle is 360°. Therefore, $360^\circ = 2\pi^c$, which we can simplify to:

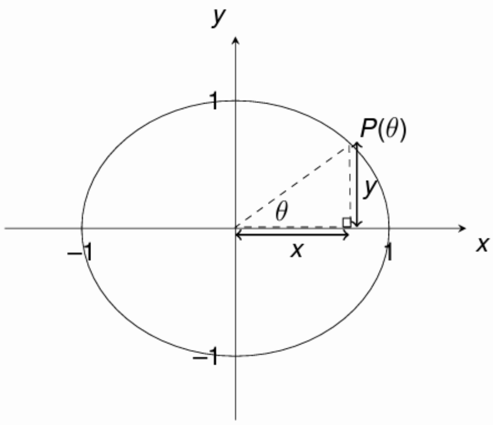
$$180^\circ = \pi^c$$
$$1^\circ = \frac{\pi^c}{180}$$
$$1^c = \frac{180^\circ}{\pi^c}$$

We use this to convert degrees to radians and vice-versa. Note that when we write angles in radians, often the superscript *c* is omitted (i.e. π instead of π^c). It's also worth remembering that if we start at an angle of 0° and we make the point P go anti-clockwise, as shown above, the angle increases positively. However, starting at 0° and going *clockwise* means the angle *decreases*, resulting in a negative angle.

KEY POINT :
When working with radians or degrees, make sure to put your calculator in the correct mode! It is very easy to forget and could be costly when dealing with longer problems.

3.2 Introduction to trigonometric functions

Let's look at the unit circle again. As we know, the point *P* depends on the angle, so we write: $P(\theta)$. Now, what if we want to find the coordinates of point *P*? We can draw a triangle like so:



We'll call the coordinates of point P (x, y): the lengths *x* and *y* are shown on the graph above. So, we need to find out these two lengths, but how do we relate it to the angle of θ?

We can use trigonometry since we have a right-angled triangle: remember SOH CAH TOA? Let's say we want to find the sine of the angle θ. How can we apply our knowledge of sine and cosine to do this?

Sine is equal to **opposite over the hypotenuse.**
The opposite of angle θ is the length *y*.
The hypotenuse is always the longest side in a triangle. In this triangle, the hypotenuse extends from the origin to the edge of the circle – this is the radius of the circle! Since this is a unit circle, the radius is 1, and the hypotenuse is 1.

$$\sin(\theta) = \frac{O}{H} = \frac{y}{1} = y$$

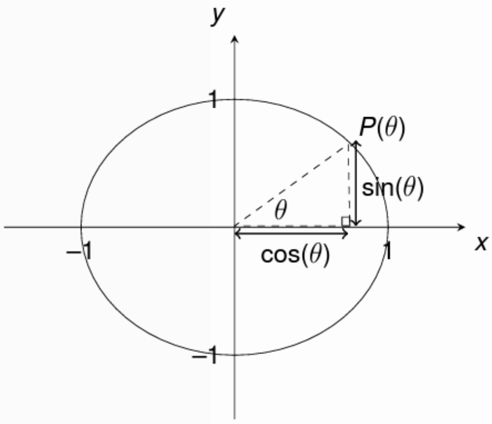
We now have the expression $y = \sin(\theta)$.

We can also find the cosine of θ.
Cosine is equal to the **adjacent over hypotenuse.**
The adjacent is the length *x*, the length right beside angle θ, and the hypotenuse is again equal to 1.

$$\cos(\theta) = \frac{A}{H} = \frac{x}{1} = x$$

We now also have the expression $x = \cos(\theta)$.

We have found that the coordinates of $P(\theta)$ are $(\cos(\theta), \sin(\theta))$. We can rewrite the previous diagram as:

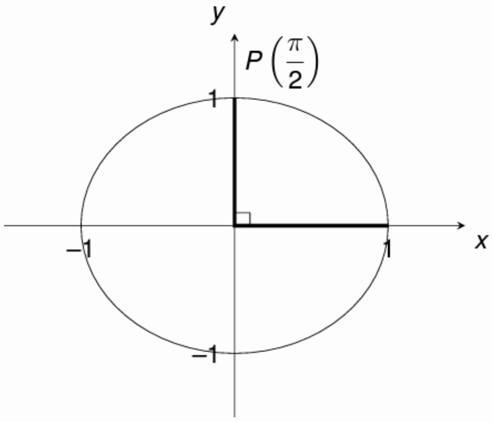


Let's test this out.

Example 3.1

Evaluate $\sin\left(\frac{\pi}{2}\right)$.

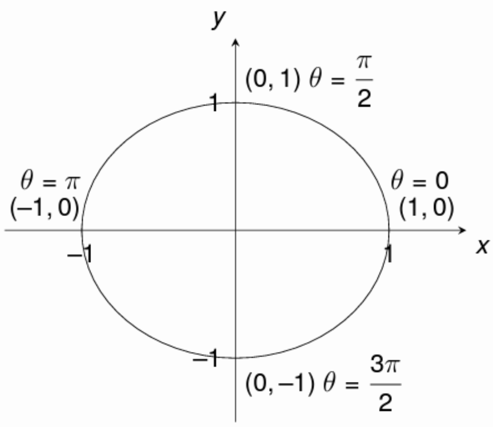
The angle is $\frac{\pi}{2}$, so let's draw that angle on the unit circle:



We know that the sine of $\frac{\pi}{2}$ is the y-coordinate of the point P. We can see in the graph that point P is exactly at (0, 1) because the point P has moved a quarter of the circle's circumference. Therefore:

$$\sin\left(\frac{\pi}{2}\right) = 1$$

You'll find that the sine or cosine of any of these four angles below are easy to evaluate. They are called **boundary angles** as they border the four quadrants of the graph.



Also, remember how an angle of 2π gets you right back to the starting point? That means, if we use sine as an example, $\sin(2\pi) = \sin(0)$. If we go past 2π, it is the same as starting from 0. That means:

$$\sin(2\pi + \theta) = \sin(\theta)$$

and

$$\cos(2\pi + \theta) = \cos(\theta)$$

Example 3.2

Evaluate $\cos\left(\frac{7\pi}{2}\right)$.

First, recognise that $\frac{7\pi}{2}$ is greater than 2π, meaning it goes further than a full circle. So, let's take away 2π, as this won't affect our answer:

$$\frac{7\pi}{2} - 2\pi = \frac{3\pi}{3}$$

Therefore $\cos\left(\frac{7\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right)$. We can now evaluate $\cos\left(\frac{3\pi}{2}\right)$ by using the unit circle. The point will be at $(0, -1)$ and we want the x-value of this coordinate, which is 0. Therefore, our answer is:

$$\cos\left(\frac{7\pi}{2}\right) = 0$$

The other trigonometric function we look at is the tangent.

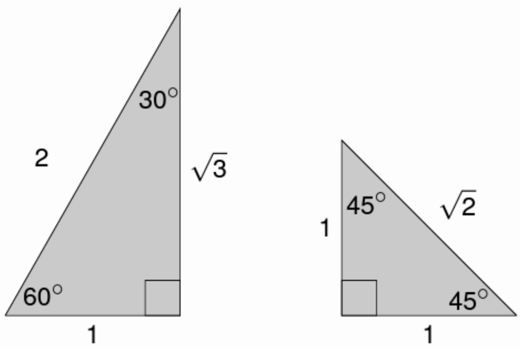
Our tangent function is found by taking the tangent of the angle θ. Using SOH CAH TOA, we know the tangent is equal to the **opposite over the adjacent:**

$$\tan(\theta) = \frac{\mathbf{O}}{\mathbf{A}} = \frac{\sin(\theta)}{\cos(\theta)}$$

You'll see the tan function come up more in the Year 12 course.

3.3 Exact values

We already have four angles that we can easily evaluate the sine or cosine of, without the help of a calculator. Now, using some handy triangles, we can find the exact values of trig functions of some other angles.



You should strive to memorise these two triangles as they will be immensely helpful in evaluating trigonometry questions without a calculator. You can also rewrite these triangles with the angles in radians instead of degrees, if needed.

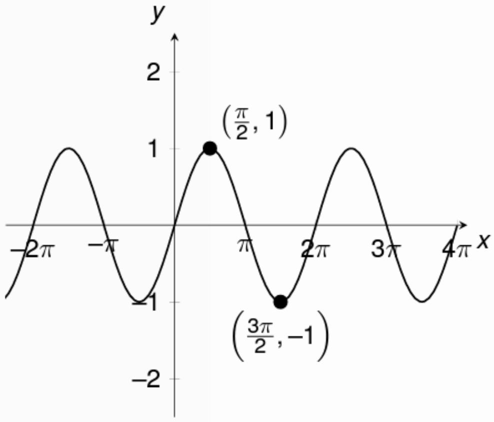
For example, if we want to find $\sin\left(60^\circ\right)$, we use the triangle on the left and find the angle 60°:

$$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\left(60^\circ\right) = \frac{\sqrt{3}}{2}$$

3.4 Graphs of trigonometric functions

The standard sine function is $y = \sin(x)$, which looks like this when graphed:

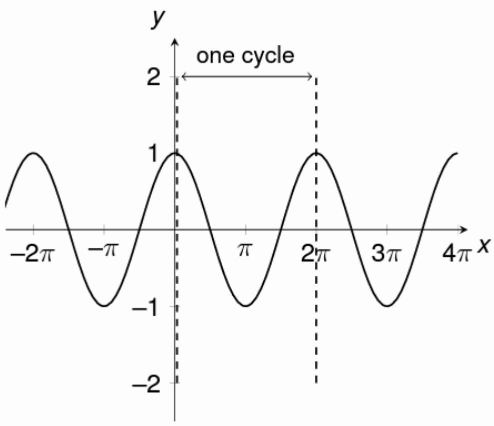


So we're clear, the x-axis is the angle and the y-axis is the sine of the angle. To visualise where this curve comes from, you can think of the unit circle. Since we are looking at the sine function, think about how the y-value of the point P changes as you imagine the angle moving from 0 to 2π.

The sine graph from above, $y = \sin(x)$, has some important features:

- The graph repeats itself after an interval of 2π, which is called the **period.** One period is called a **cycle.** Since the graph repeats, it is called **periodic** function.
- The maximum and minimum values of $\sin(x)$ are 1 and –1. The distance between the mean position ($y = 0$) and the highest position ($y = 1$) or lowest position ($y = -1$) is 1, which is called the **amplitude.**
- Note that at $x = 0$, $y = 0$ as well.

The standard cosine curve: $y = \cos(x)$ is shown below.



It looks similar to the sine curve, with the same amplitude and period. In fact, it is exactly like the sine curve except that it has been translated $\frac{\pi}{2}$ units in the negative x-axis direction. This means that at $x = 0$, $y = 1$.

> **KEY POINT :**
> Using these two graphs are my preferred method of evaluating boundary angles, rather than using the unit circle shown in previous examples. It may just be personal preference, but regardless, you will need to be familiar with the graphs of trig functions. To remember the graphs easier, just focus on one cycle of the graph, as you know it will repeat after that.

However, sine and cosine functions don't always have the same period and amplitude. They can be written in the form:

$$y = a\sin(nx)$$
$$y = a\cos(nt)$$

- a is the amplitude. If a is negative, the graph is reflected in the x-axis.
- The period is found by calculating $\frac{2\pi}{n}$.

We can use this to sketch transformations of sine and cosine graphs.

Example 3.3

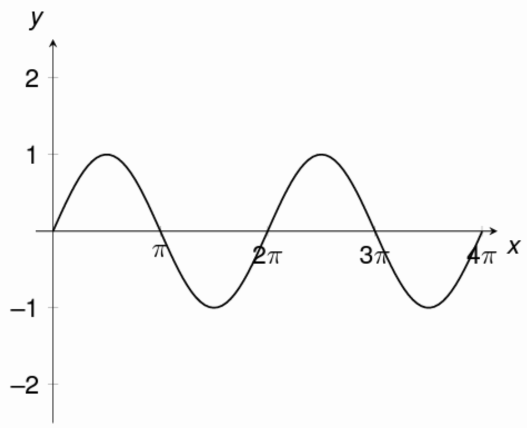
Sketch the graph of $f(x) = -2\sin\left(\frac{x}{2}\right)$ *for* $x \in [0, 4\pi]$.

We first identify that this is a sine function. Now, we find the amplitude and period:

$$\begin{aligned} \text{amplitude} &= 2 \\ \text{period} &= \frac{2\pi}{n} \\ &= \frac{2\pi}{\frac{1}{2}} \\ &= 4\pi \end{aligned}$$

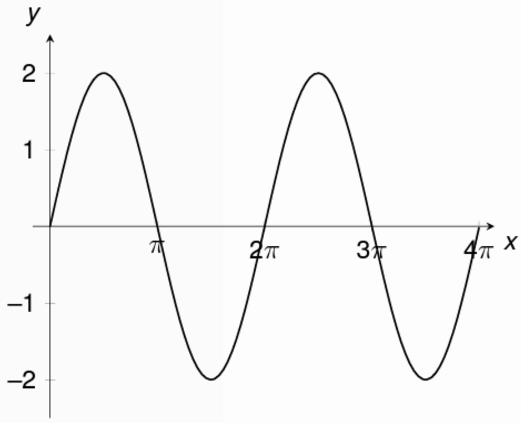
We also have to note that this has been reflected in the x-axis, due to the negative sign in front of the amplitude.

To sketch this, let's draw the standard sine curve (ideally, you should be able to do this from memory). Keep in mind that the question specified $x \in [0, 4\pi]$, meaning the domain is from 0 to 4π.

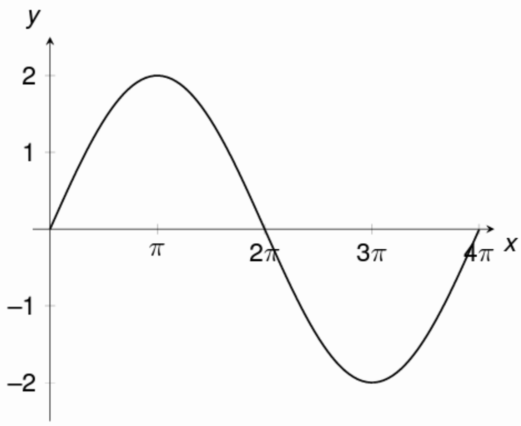


Now, we apply the transformations. We'll do them one at a time.

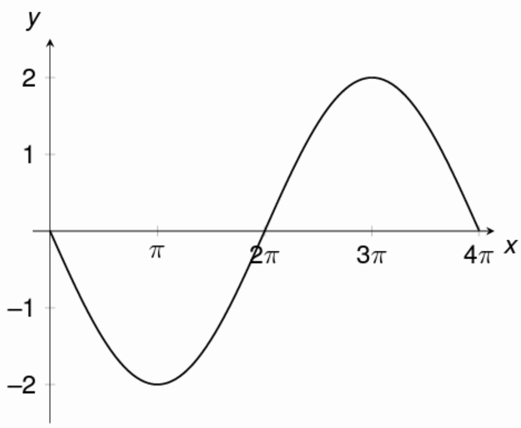
First, we change the amplitude to 2. This means the max or min values are 2 units away from the mean position ($y = 0$).



Then, we change the period to 4π. This means the graph will complete one cycle in 4π units.



Finally, we reflect the graph in the x-axis.

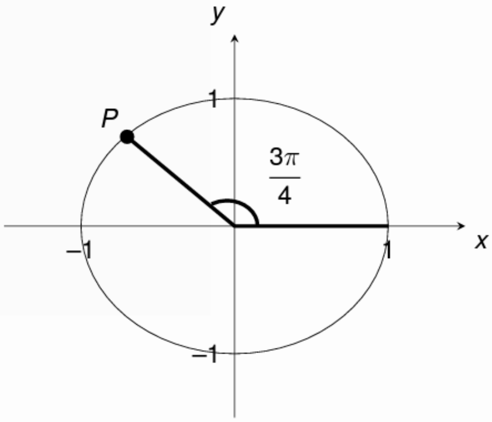


Although I've shown you the steps separately here, you should work towards being able to apply all the transformations in one sketch. Additionally, you can see why it is paramount you memorise the standard trig function curves!

3.5 Symmetry properties of trigonometric functions

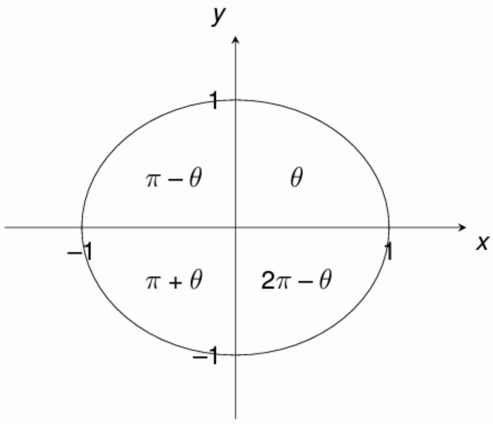
Trigonometric functions have some handy properties that make our lives much easier when we get to solving some trig functions later on.

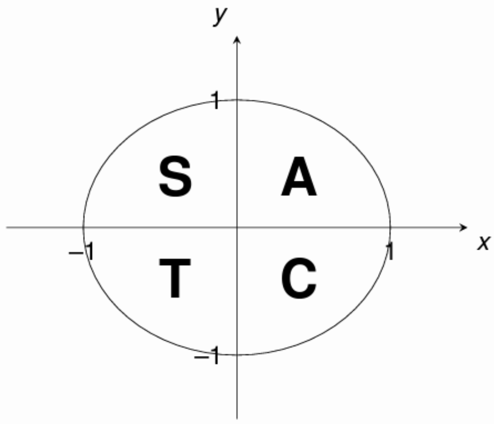
If we think back to the unit circle, the angle θ can result in point P being in either of the four quadrants. Say we have the angle of $\frac{3\pi}{4}$ as shown.



Let's say we want to find the cosine of $\frac{3\pi}{4}$. Currently, we don't know how to evaluate this without a calculator, since it is not a boundary angle or one of the angles in the exact values triangles.

However, with the help of these two pictures, we can work it out.





The picture on the right can be memorised using the mnemonic **A**ll **S**tations **T**o **C**entral, going from the first quadrant to the fourth. It means that:

- **A**ll – **all** trig functions are positive here
- **S**tations – only **sine** functions are positive here
- **T**o – only **tangent** functions are positive here
- **C**entral – only **cosine** functions are positive here

Our angle is in the second quadrant, and the picture on the left tells us that here, it is equal to $\pi - \theta$. That means:

$$\frac{3\pi}{4} = \pi - \theta$$
$$\theta = \pi - \frac{3\pi}{4}$$
$$\theta = \frac{\pi}{4}$$

Now, we can find the cosine of $\frac{\pi}{4}$ because it is one of our angles in the exact values triangles, which equals $\frac{1}{\sqrt{2}}$. However, be careful! This does not mean $\cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$.

There is another step: the sign of $\frac{1}{\sqrt{2}}$ may change depending on which quadrant our original angle was in.

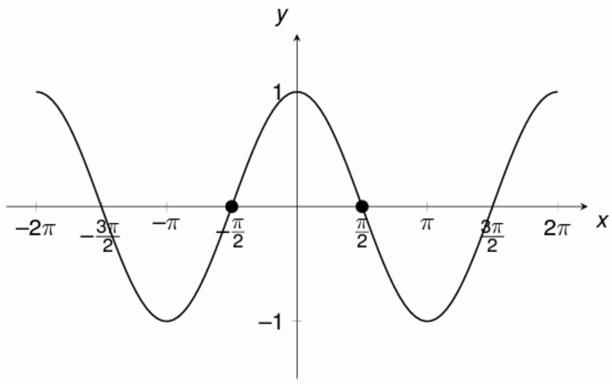
The picture on the right tells us that in the second quadrant, only sine functions are positive. This means that if we take the cosine of $\frac{3\pi}{4}$ (an angle in the second quadrant), we will end up with a **negative** answer.

Therefore, the correct answer is $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$.

KEY POINT :
You should definitely memorise the two pictures above as there is no doubt that you will need them when solving trig functions. Also remember that while the left picture shows angles in radians, you can also convert to degrees if needed: for example, $\pi - \theta \rightarrow 180^\circ - \theta$.

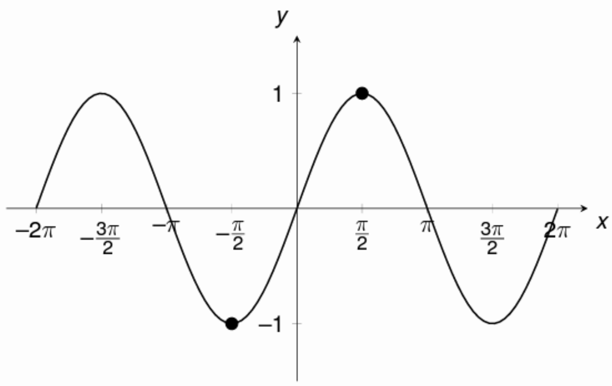
We also have symmetry of trig functions involving negative angles.

For cosine functions: $\cos(-\theta) = \cos(\theta)$. To demonstrate: if you look at the graph below, the angles $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ have the same y-value, which is 0.



Remember that the y-value represents the cosine of the angle. Therefore, the cosine of both angles are the same.

For sine functions: $\sin(-\theta) = -\sin(\theta)$. To demonstrate: the sine of $\frac{\pi}{2}$ is 1, while the sine of $-\frac{\pi}{2}$ is -1. So, they are the same value but have different signs.



For tangent functions:

$$\tan(-\theta) - \frac{\sin(-\theta)}{\cos(\theta)}$$

We know that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, so if you make the angle negative, just apply the two formulas above involving sine and cosine here.

Using these symmetry properties, we can now get to solving some trig functions.

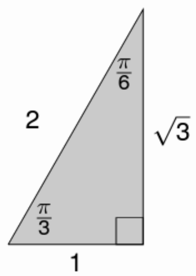
3.6 Solving trigonometric functions

When we solve trig functions, as they are periodic, we will get multiple answers. The question will specify the domain of the graph, which affects how many answers we'll have.

Example 3.4

Solve $\sin(\theta) = \dfrac{1}{2}$ *for* $\theta \in [0, 4\pi]$.

We can use the exact values triangle here; however, we need to rewrite the angles in radians:



Using the triangle, $\sin\frac{\pi}{6} = \frac{1}{2}$, so we know $\frac{\pi}{6}$ is one of our answers, but we will have multiple answers across this domain, and we will use symmetry to find them.

Notice that our answer of $\frac{1}{2}$ is positive and a sine function. So, we look at which quadrant sine functions are positive – the first and second quadrants. Currently, our answer of $\frac{\pi}{6}$ is the angle in the first quadrant, so we need to find the angle for the second quadrant:

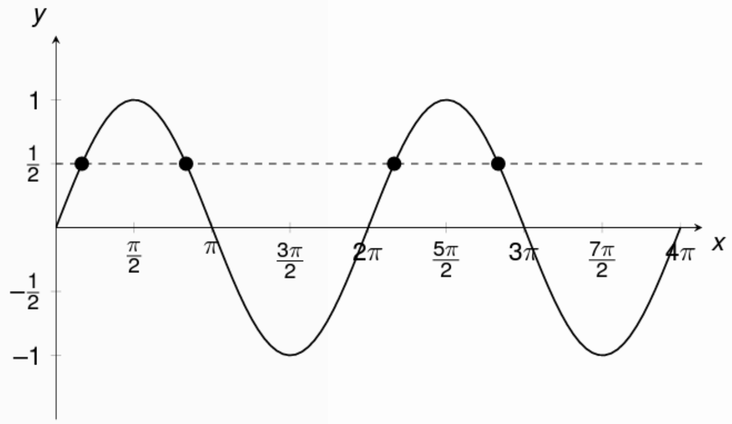
$$\text{first quadrant: } \theta = \frac{\pi}{6}$$
$$\text{second quadrant: } \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

We have two answers now: $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. However, we're not done yet: the domain is $\theta \in [0, 4\pi]$ and the period of $\sin(\theta) = \frac{1}{2}$ is 2π. That means the graph would have two cycles (as shown in the following graph), and currently our answers only cover the first cycle. To find the rest of the answers, simply add the period, 2π, to both our current answers, which gives the angles in the second cycle. Remember that when you add 2π to an angle, taking the sine of that new angle does not change the answer. However, this only applies to standard sine curves which we know to have a period of 2π.

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$
$$\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$$

Therefore, our final answer is that $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$. Note that the answers in the third cycle are greater than 4π, so we don't need to find them.

This is the graph of $y = \sin(\theta)$ and $y = \frac{1}{2}$. The intercepts of these two functions are the answers we found.



It is also possible to solve this equation using a calculator to graph the function $y = \sin(\theta)$ and what you are trying to make y equal to, in this case $y = \frac{1}{2}$. Make sure to create the graph within the stated domain by pressing "`WINDOW`" and entering the "`Xmin`" and "`Xmax`." Then, find the intercepts of the two equations you graphed by accessing the "`CALCULATE`" menu and selecting "`5:intersect`."

The question can be a little more difficult if the equation is in the form: $y = a\sin(nx)$ or $y = a\cos(nx)$, where n is a number other than 1. Let's go through an example like that together.

Example 3.5

Solve $\cos(2\theta) = \frac{\sqrt{3}}{2}$ *for* $\theta \in [0, 2\pi]$.

First, let's take note that $n = 2$. Then, we'll remove it for now. Just think about solving $\cos(\theta) = \frac{\sqrt{3}}{2}$.

Using the exact value triangles: $\theta = \frac{\pi}{6}$. Note that the answer of $\cos(2\theta) = \frac{\sqrt{3}}{2}$ is positive and involves a cosine function. The quadrants where cosine is positive are the first and fourth quadrants. Let's find the angles in these quadrants:

$$\begin{aligned}\text{first quadrant: } \theta &= \frac{\pi}{6}\\ \text{fourth quadrant: } \theta &= 2\pi - \theta\\ &= 2\pi - \frac{\pi}{6}\\ &= \frac{11\pi}{6}\end{aligned}$$

Currently, our answers are $\theta = \frac{\pi}{6}$ and $\frac{11\pi}{6}$.

But remember how $n = 2$? Let's fix that by multiplying θ by 2:

$$2\theta = \frac{\pi}{6}, \frac{11\pi}{6}$$

Now, we divide both sides by 2 in order to get θ on its own:

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}$$

Now, we just have to find the rest of the answers as $\theta \in [0,\ 2\pi]$, which occur in the second cycle. Here's the part you have to remember: to find the answers in the second cycle, **we add the period of the graph.** For standard curves it is 2π, but here it is:

$$\begin{aligned}\text{period} &= \frac{2\pi}{n}\\ &= \frac{2\pi}{2}\\ &= \pi\end{aligned}$$

Then, we add π to our current answers to get our total of four answers:

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

Alternatively, you can choose to find the other answers by adding 2π, since the period of $\cos(2\theta) = \frac{\sqrt{3}}{2}$ is 2π. However, the domain specified is in terms of θ, not 2θ, so you will eventually divide all your answers by 2 in the future to arrive at $\theta = \frac{\pi}{12}, \frac{11\pi}{12}$, by multiplying the domain by *n*. This means $\theta \in [0, 2\pi]$ turns into $\theta \in [0, 4\pi]$. Choose whichever method works for you!

Next, we have some more transformations of trig functions to cover.

3.7 Graphs of trigonometric functions continued

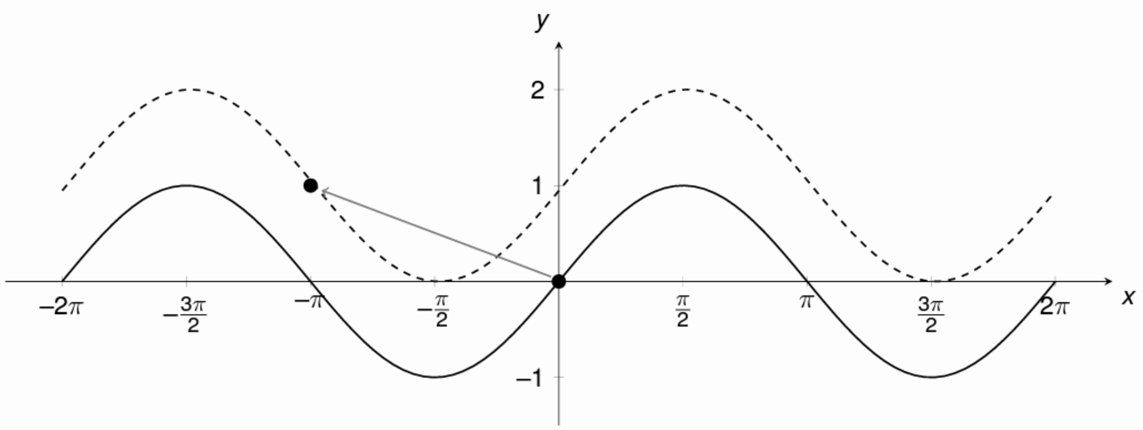
Earlier, we looked at graphs of $y = a\ \sin(nx)$ and $y = a\ \cos(nx)$. Now, let's introduce two more:

$$y = a\sin[n(t \pm \epsilon)] \pm b$$
$$y = a\cos[n(t \pm \epsilon)] \pm b$$

- $\pm\varepsilon$ translates the graph in the t-axis direction (a.k.a. the x-axis)
- $\pm b$ translates the graph in the y-axis direction

Note that we'll start using t instead of x as the most common kinds of questions you'll see in this section are problem solving questions about time (t).

For example, compare the graph of $y = \sin(t)$ and the graph of $y = \sin(t - \pi) + 1$ (the dotted line). Watch where the point I've marked on the curve moves when we apply these translations:



From $y = \sin(t)$, the graph has been moved up 1 unit and to the left π units.

3.8 Applications of trigonometric functions

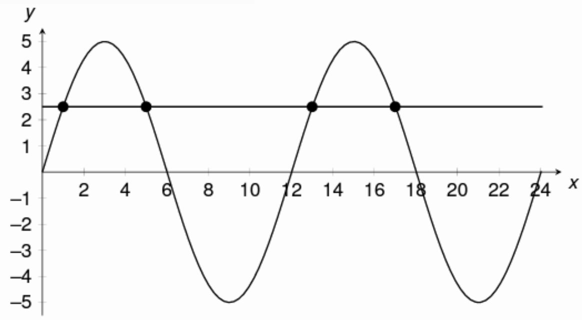
Example 3.6

The height, $h(t)$ metres, of the tide above mean sea level during a day is modelled by: $h(t) = 5\sin\left(\frac{\pi}{6}t\right)$ where $t \in [0, 24]$ and t is the number of hours after midnight. The boat can only cross the harbour when the tide is at least 2.5 metres above mean sea level. Find out when the boat is able to cross the harbour.

First, let's sketch this graph to get a picture of what's happening. To do this we need to know:

$$\text{amplitude} = 5$$
$$\text{period} = \frac{2\pi}{n} = \frac{2\pi}{\frac{\pi}{6}} = 12$$
$$t \in [0, 24] \text{ so there will be two cycles}$$

The boat can cross the harbour when $h(t) = 2.5$ or above. So, let's also sketch $h = 2.5$:



We have four intercepts, so we'll have four answers for t when $h(t) = 2.5$. Whenever $h(t)$ is greater than 2.5, the boat can also cross. Keep this in mind. Our next step is to find at what time $h(t) = 2.5$. That is, to solve $5\sin\left(\frac{\pi}{6}t\right) = 2.5$ or $\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$ First, let's simplify. Like in the previous example, let's focus on solving just $\sin(t) = \frac{1}{2}$. Using exact value triangles, $t = \frac{\pi}{6}$. Our answer of $\frac{1}{2}$ is a positive number, and sine is positive in the first and second quadrants, so our angles are:

$$\text{first quadrant: } t = \frac{\pi}{6}$$
$$\text{second quadrant: } t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Now put back our n value, which is $\frac{\pi}{6}$:

$$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$t = 1, 5$$

Find the other values – remember, **add the period, which is 12.** There should be four answers, as we saw before on the graph:

$$t = 1, 5, 13, 17$$

The final part is to state when the boat can cross the harbour. If we look back at our sketch: for the region where $1 \leq t \leq 5$, we can see that $h(t)$ is at or above 2.5 m. This happens again when $13 \leq t \leq 17$. Then, remember to convert t into the actual time that day. Therefore, the boat can cross the harbour between 1:00 a.m. and 5:00 a.m. and between 1:00 p.m. and 5:00 p.m.

Topic 4

Introduction to differential calculus

We'll first look at the concept of rate of change, which leads into calculus.

4.1 Rates of change and the concept of derivatives

If we have a graph of a linear function, such as $y = x$, what would be the **rate of change** of y with respect to x? In other words, if we change x, what happens to the y-value?

Well, we can tell that the rate of change is positive: as x increases, y also increases. We also know that the gradient of the function is 1: if we increase x by 1 unit, the y will increase by 1 unit also. Essentially, the gradient is the rate of change as it tells us how y changes with x.

Additionally, the gradient of the graph always stays the same, at 1, since this is a linear function. Therefore, all linear functions have a constant rate of change.

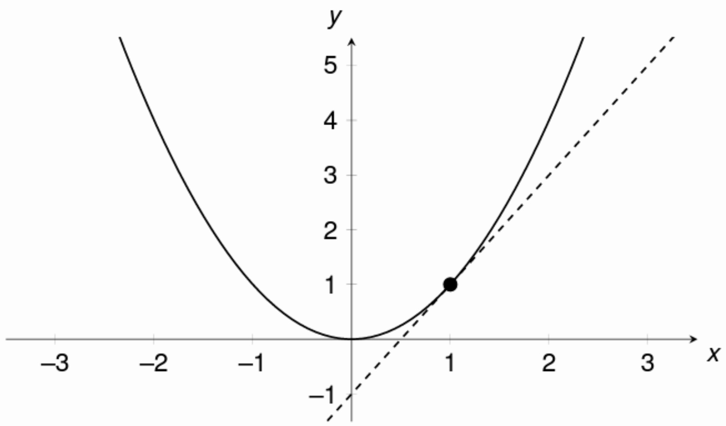
Therefore, the linear function $y = mx + c$ has a constant rate of change, m.

However, not all graphs have a constant rate of change. Generally, if we want to find the **average rate of change** of a function between points a and b on a graph:

$$\text{average rate of change } = \frac{f(b) - f(a)}{b - a}$$

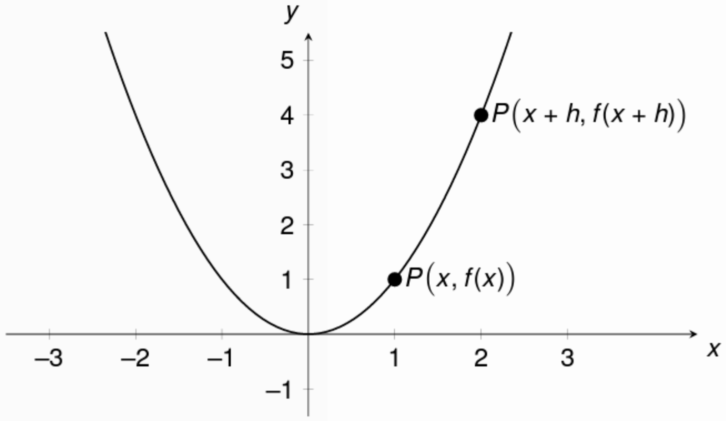
This is the rate of change over a certain interval. What if, instead, we want to find the rate of change at any point of a function, not just over a certain interval? This is called **instantaneous rate of change.**

We can find the instantaneous rate of change at point P in the function $y = x^2$ by drawing the tangent to the curve at that point:



The tangent is the straight line which just touches the curve at point P. That is, the tangent and $y = x^2$ share the common point of P (1, 1).

Then, the gradient of the tangent is the instantaneous rate of change at point P. To figure out how to find the gradient, let's consider finding the average rate of change between these two points:



Note that $P(x, f(x))$ are the coordinates (x, y), but since y is a function of x, we replace the y with $f(x)$.

We have two points, $P(x, f(x))$, and another point somewhere further along in the function, described by $P(x + h,\ f(x + h))$. To find the average rate of change use the formula below, where a and b correspond to our two points.

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- Our first point is $(a,\ f(a)) \to (x,\ f(x))$
- Our second point is $(b, f(b)) \to (x + h, f(x + h))$
- Our average rate of change is $\dfrac{f(x + h) - f(x)}{(x + h) - x}$

Now, imagine that we make the value h smaller and smaller, such that the second point gets closer and closer to the first point. When $h = 0$, the first and second points would be in the same position. This means we aren't finding the average rate of change anymore, but the instantaneous rate of change, as there would be no interval anymore. To see what happens to the average rate of change as the value of h becomes smaller, we use the limit: $\lim\limits_{h \to 0}$, which is read as 'as h tends to 0':

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{(x + h) - x} \\ &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \end{aligned}$$

This is called using **first principles** to find instantaneous rate of change at point $(x,\ f(x))$, which we described before as also the gradient of the tangent at point $(x,\ f(x))$. Once we do this, we have **differentiated** the function to find the **derivative**, which is the gradient of the tangent. The derivative is denoted by $f'(x)$ or y'.

Let's derive the function $f(x) = x^2$ using first principles, and the equation $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{(x + h) - x}$.

We know $f(x) = x^2$. To find $f(x+h)$, we have $f(x+h) = (x+h)^2$. We can now substitute these in, expand, and simplify:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} 2x + h \end{aligned}$$

To evaluate the limit, think about what happens as h tends to 0. Here, the $+h$ would just become $+0$, so we put that in and remove the limit to get:

$$f'(x) = 2x$$

KEY POINT :
You might be thinking that $2x$ doesn't look an awful lot like the gradients we've seen for straight lines before. That's because the gradient of the tangent of a point on $y = x^2$ would be different depending on where the point is. That's why our derivative contains an x, so that we can input the x-value of a point to find the derivative at that point.

Next up, we'll learn how to differentiate by skipping first principles so we can do this faster!

4.2 Properties and computation of derivatives

We'll first look at the rules for deriving functions.

KEY POINT :
To derive $f(x) = x^n$, where n is positive: $f'(x) = n \times x^{n-1}$.

Think: "the index of the power, which is n, is multiplied to the function, then we take 1 off the index".

Note that deriving a constant, where $f(x) = c$, results in $f'(x) = 0$. In other words, a straight vertical line like $f(x) = 2$ has a gradient of 0.

Deriving $f(x) = ax$ results in $f'(x) = a$.

KEY POINT :
If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$.

If $y = f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$.

Alternatively, we can express the derivative of y, as y', or $\frac{dy}{dx}$. Additionally, if we want to derive something, like x^2 for example, we can write: $\frac{d}{dx}\left(x^2\right)$. Let's see how to apply this in an example:

Example 4.1

Derive $f(x) = -x^3 + 2x^2$ with respect to x. Then, find the gradient at $(1, 1)$.

We can derive each term separately: to derive $-x^3$, use the formula $f'(x) = n \times x^{n-1}$.

$$\begin{aligned} g(x) &= -x^3 \\ g'(x) &= 3 \times -x^{3-1} \\ &= -3x^2 \end{aligned}$$

Now, derive $2x^2$ the same way:

$$\begin{aligned} h(x) &= 2x^2 \\ h'(x) &= 2 \times 2x^{2-1} \\ &= 4x \end{aligned}$$

Now, just add our two derivatives together:

$$f'(x) = -3x^2 + 4x$$

Now, to find the gradient at the point $(1, 1)$, just take the x-value of this coordinate, and put it into $f'(x)$:

$$\begin{aligned} f'(1) &= -3(1)^2 + 4(1) \\ &= -3 + 4 \\ &= 1 \end{aligned}$$

Therefore, the gradient at $(1, 1)$ is 1.

4.3 Applications of derivatives

4.3.1 Equation of the tangent

Previously, we've talked about drawing a tangent to a function at a certain point. We know we can find the gradient of the tangent by deriving. Now, we'll take it a step further to find the equation of the tangent. The tangent is a straight line, so the equation will be in the form $y = mx + c$.

We find the equation of the tangent by using $y - y_1 = m(x - x_1)$ where (x_1, y_1) is a point on the curve.

Example 4.2

Find the equation of the tangent for $f(x) = 9x - x^3$ *at* $(1, 8)$.

First, we have to derive the function to find the gradient of the tangent.

$$\begin{aligned} f(x) &= 9x - x^2 \\ f'(x) &= 9 - 3x^2 \end{aligned}$$

Now, the point specified is $(1, 8)$, so we'll find $f'(1)$:

$$\begin{aligned} f'(1) &= 9 - 3(1)^2 \\ &= 6 \end{aligned}$$

The gradient of the tangent is 6; that is, $m = 6$. This gives us:

$$y - y_1 = 6(x - x_1)$$

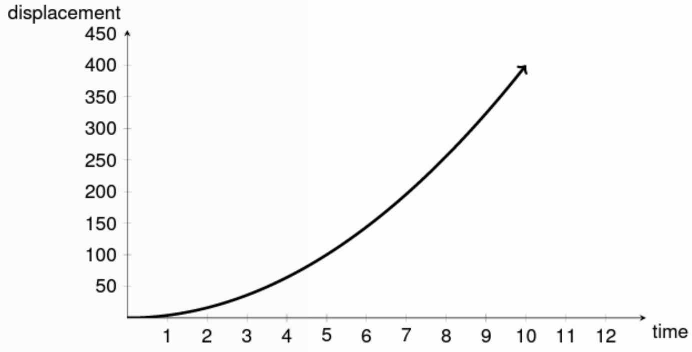
Now, (x_1, y_1) is a point on the graph of $f(x)$. The question already tells us that $(1, 8)$ is a point on that curve, so we'll substitute those numbers in:

$$\begin{aligned} y - 8 &= 6(x - 1) \\ y &= 6x - 6 + 8 \\ y &= 6x + 2 \end{aligned}$$

Therefore, the equation of the tangent is $y = 6x + 2$.

4.3.2 Kinematics

Let's say we have a displacement-time graph, where time is on the x-axis and displacement on the y-axis:

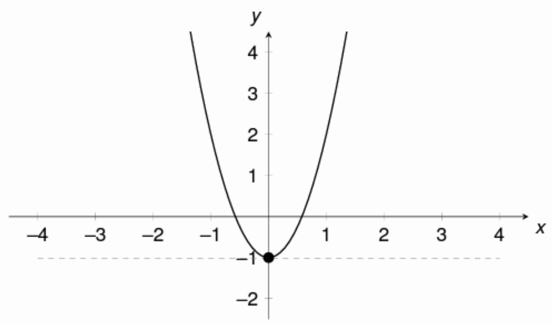


Now let's think about what would happen if we were to differentiate the function above. Remember, if we differentiate $f(x)$ with respect to x, we are finding the rate of change of y with respect to x. In this case, y is the displacement and x is time. So, what is the rate of change of the displacement, with respect to time? This is the same as saying how fast does something move? We know this to be the **velocity** of the object, measured in m/s (metres per second).

Key Point :
When we differentiate a displacement-time graph, the gradient of the tangent is the velocity.

4.3.3 Stationary points

Stationary points are where the gradient of the tangent at a point on the function equal 0. Since a line with a gradient of 0 is a horizontal line, the tangents of stationary points are horizontal lines. The turning point in a parabola is a stationary point, for example:



This is the function $y = 3x^2 - 1$. To find the stationary point, we first derive the function:

$$y' = 6x$$

We want to find the value of x when the gradient is 0. That is, when $y' = 0$:

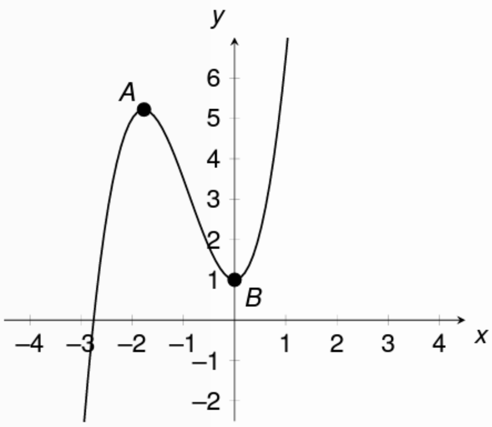
$$0 = 6x$$
$$x = 0$$

We now know that stationary point has an x-value of 0. To find the y-value, we put it back into the function.

$$\text{When } x = 0: \; y = 3 \times (0)^2 - 1$$
$$y = -1$$

Our stationary point is at $(0, -1)$, which you can see on the graph above.

However, we also have different types of turning points, as you can see below.



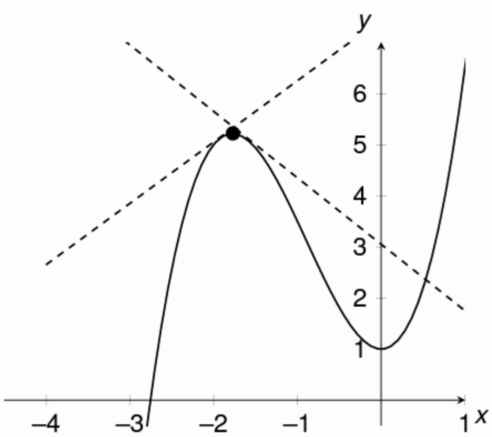
We have two stationary points here: *A* and *B*.

A is called the **local maximum.** We use the term 'local' because this is not the actual maximum value of the graph – as we can see here, as x increases, y increases to infinity, so the *global* maximum would be infinity.

For a local maximum:

- Immediately to the left, the gradient of the tangent is positive: $f'(x) > 0$.
- Immediately to the right, the gradient of the tangent is negative: $f'(x) < 0$.

You can see if we zoom in on *A* and draw some tangents.



You can see that the tangent to the left of A is sloping upwards – a positive gradient. The tangent is the right of A is sloping downwards – a negative gradient.

For a local minimums (e.g. point B), everything is the opposite:

- Immediately to the left, the gradient of the tangent is negative: $f'(x) < 0$.
- Immediately to the right, the gradient of the tangent is positive: $f'(x) > 0$.

Local maximum and minimum points can also be called turning points.

You might have noticed that we've done some questions before that required us to find stationary points – now, we can do it using calculus!

Example 4.3

If $x + y = 2$, calculate the minimum value of $x^2 + y^2$.

This question is a bit abstract, and you might not be sure where to start. It is asking us to find what values of x and y will give us the smallest possible answer of $x^2 + y^2$. It tells us that x and y are related by $x + y = 2$. Notice the question involves 'minimum value' – this implies we will have to differentiate something and find a local minimum point. However, we can't differentiate $x^2 + y^2$ as it has two variables, x and y. When we differentiate, we differentiate with respect to one variable, usually x.

However, we can change $x^2 + y^2$ to have one variable:

$$\begin{aligned} x + y &= 2 \\ y &= 2 - x \end{aligned}$$

Then, we substitute this into $x^2 + y^2$ to get, in function notation:

$$f(x) = x^2 + (2 - x)^2$$

Now, we can differentiate this function:

$$\begin{aligned} f'(x) &= 4x + \left(2x^2 - 4x + 4\right) \\ &= 4x - 4 \end{aligned}$$

We want to find the local minimum, which is a stationary point. Since stationary points are when the gradient equals 0 (be sure to state this in your working):

$$\begin{aligned} \text{Stationary point occurs when } f'(x) &= 0 \\ 0 &= 4x - 4 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

Now, we need to verify that this is a local minimum. Remember that immediately to the left of the minimum is a negative gradient, and immediately to the right is a positive gradient. To do this we can draw the table below.

x	< 1	1	> 1
$f'(x)$			

When $x < 1$, this is to the left of the point. When $x > 1$, this is to the right of the point.

The bottom row is where you put the sign of the gradient at the stated value of x.

1. For $f'(x < 1)$, we can find this by inputting an x-value that is slightly below $x = 1$, such as $x = 0$:

$$\begin{aligned} f'(0) &= 4 \times 0 - 4 \\ &= -4 \text{ (a negative value)} \end{aligned}$$

2. For $f'(1)$, we know this to be the stationary point, so it equals 0 and has no sign.
3. For $f'(x > 1)$, we can input an x-value that is slightly higher than $x = 0$ this time, such as $x = 2$:

$$\begin{aligned} f'(2) &= 4 \times 2 - 4 \\ &= 4 \text{ (a positive value)} \end{aligned}$$

Therefore, our table looks like:

x	< 1	1	> 1
$f'(x)$	negative	0	positive

This confirms that the point where $x = 1$ is a local minimum. We don't actually need to find the y-value of the local minimum, as it is not necessary to answer the question.

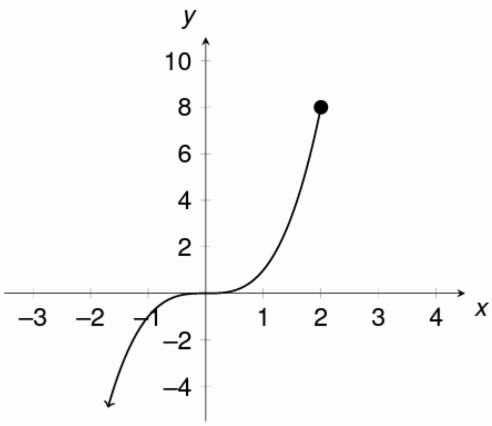
We just need to find the minimum value of $x^2 + y^2$, and we know that this occurs when $x = 1$. To find what y is:

$$\begin{aligned} x + y &= 2 \\ 1 + y &= 2 \\ y &= 1 \end{aligned}$$

Hence, the minimum value of $x^2 + y^2$ is 2.

Key Point :
These types of questions are called **optimisation problems** and require you to find stationary points. Always remember to use the table I showed you above to verify what type of stationary point you have found: either a local minimum or maximum.

Note that the global maximum or minimum may be positive or negative infinity unless the graph is restricted with a specific domain. For example, the graph of $y = x^3$ has a domain of $(-\infty, 2]$. You can see that the global maximum is when $x = 2$, which is the point $(2, 8)$.



We will now move on to more rules for differentiation.

Topic 5

Further differentiation and applications 1

5.1 The chain rule

So far, if we wanted to differentiate an expression such as $(x+3)^2$, we would first have to expand the brackets, then differentiate each term using the rules we learnt earlier. However, there is a quicker way to do this—by using the chain rule.

> **Key Point :**
> **Chain rule:** if $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \times g'(x)$.

It looks more complicated than it actually is. Let's differentiate $(x+3)^2$ as an example:

Here, $h(x) = (x+3)^2$. The first step is to differentiate the power like normal: $(x+3)^2 \longrightarrow 2(x+3)^1$.

Next, we also have to multiply it by the derivative of what's inside the brackets, $(x+3)$, which is written as:

$$2(x+3)^1 \times \frac{d}{dx}(x+3)$$

Then, we evaluate $\frac{d}{dx}(x+3)$ by deriving $(x+3)$:

$$\frac{d}{dx}(x+3) = 1$$

That gives us $2(x+3)^1 \times 1$. Therefore: $h'(x) = 2(x+3)$.

5.2 The product rule

We use the product rule to differentiate functions in the form: $h(x) = f(x) \times g(x)$, such as $h(x) = (x^2+7)(1-x)$. Again, this saves us from having to expand the brackets.

> **Key Point :**
> **Product rule:** if $h(x) = f(x)\,g(x)$, then $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$.

Let's try to differentiate $h(x) = (x^2+7)(1-x)$ by using the product rule:

So, we need to first find the derivative of both (x^2+7) and $(1-x)$, which we can do separately. This is how I like to set it up:

$$h(x) = (x^2+7)(1-x)$$

$f(x) = x^2+7$	$g(x) = 1-x$
$f'(x) = 2x$	$g'(x) = -1$

Next, to put the formula into words: we have to multiply the bottom left and the top right boxes together, multiply the top left and bottom boxes together, and then add both products.

$$\begin{aligned}
\text{Using the product rule: } h'(x) &= f'(x)g(x) + f(x)g'(x) \\
&= (2x) \times (1-x) + (-1) \times \left(x^2+7\right) \\
&= 2x - 2x^2 - x^2 - 7 \\
&= -3x^2 + 2x - 7
\end{aligned}$$

Although it doesn't matter whether you choose to multiply $f'(x)\,g(x)$ or $f(x)\,g'(x)$ first, I suggest always start with $f'(x)\,g(x)$, which is multiplying the bottom left and top right boxes together. You'll see why when we talk about the quotient rule next.

5.3 The quotient rule

We use the quotient rule to differentiate functions in the form $h(x) = \dfrac{f(x)}{g(x)}$.

Key Point :
Quotient rule: if $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

Let's try to differentiate $h(x) = \dfrac{x^2 + 4x}{1 - x}$.

Initially, the process is similar to using the product rule. We find the derivative of $x^2 + 4x$ and $1 - x$, again using the setup shown before:

$$h(x) = \frac{x^2 + 4x}{1 - x}$$

$f(x) = x^2 + 4x$	$g(x) = 1 - x$
$f'(x) = 2x + 4$	$g'(x) = -1$

Now, to put the formula in words again: we multiply the bottom left and top right boxes, then *take away* the product of the top left and bottom right boxes. After that, we divide it by $[g(x)]^2$, which is the square of the top right box.

$$\begin{aligned}
\text{Using the quotient rule: } h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x + 4)(1 - x) - (-1)\left(x^2 + 4x\right)}{(1 - x)^2} \\
&= \frac{2x - 2x^2 + 4 - 4x + x^2 + 4x}{(1 - x)^2} \\
&= \frac{-x^2 + 2x + 4}{(1 - x)^2}
\end{aligned}$$

The reason why I told you before to multiply the bottom left and top right boxes first is that here, we are doing a subtraction: $f'(x)\,g(x) - f(x)\,g'(x)$, so the order *does* matter. If you get into the habit of following the process as I've demonstrated it for both the product and quotient rules, you won't get mixed up by the subtraction here.

Key Point :
When using any of these differentiation rules, make sure to state the formula first or state that you are using a certain rule, such as 'using the product rule', in working. Exam markers can be pedantic about this kind of thing, and you don't want to lose marks for an easy step.

In the next topic, we change our focus back to probability.

Topic 6

Discrete random variables 1

6.1 General discrete random variables

Let's consider a probability experiment we've looked at before: tossing a coin twice. As we know, to find the probability of each outcome, we just multiply the probabilities in each branch of a tree diagram. So, for the probabilities of all the outcomes:

$$Pr(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$Pr(HT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$Pr(TH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$Pr(TT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Now, suppose that we are interested in the number of heads in each outcome. We can let the **discrete random variable** X represent the number of heads in an outcome. It is discrete as it only includes countable, distinct values (0, 1, 2) Therefore:

Outcome	Number of heads
HH	$X = 2$
HT	$X = 1$
TH	$X = 1$
TT	$X = 0$

For a random variable, each outcome of the experiment is assigned a value. Now, instead of finding the probability of an outcome occurring, we can find the probability of each value of the random variable occurring. For example, we could find what is the probability of $X = 2$ occurring. This would be written as $Pr(X = 2)$.

- For $Pr(X = 2)$, there is only one outcome with two heads, HH, which has a probability of $\frac{1}{4}$. Therefore, $Pr\,(X = 2) = \frac{1}{4}$.
- For $Pr(X = 1)$, there are two outcomes with one head, HT and TH, which each have a probability of $\frac{1}{4}$. Therefore, we add the two probabilities together to get $Pr\,(X = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
- For $Pr(X = 0)$, there is one outcome with no heads, TT, with a probability of $\frac{1}{4}$. Therefore, $Pr\,(X = 0) = \frac{1}{4}$.

We can express each value of a random variable and its probability in a table:

x	0	1	2
$Pr(X = x)$	1/4	1/2	1/4

The top row (x) are our values of the random variable: 0, 1, 2, and the bottom row are the probabilities when X equals one of the values of the random variable.

What we have created is the **probability distribution** of the discrete random variable X. A probability distribution is also the function $p(x) = Pr(X = x)$. It is a function because, like any other function, we input a value (of the random variable) and the output is the probability of that value of the random variable occurring.

KEY POINT :
The values of $p(x)$, which are probabilities in the bottom row of the table, must be between 0 and 1, like all probabilities. All the values of $p(x)$ must sum to 1 as well. Be sure to remember this, as it is very helpful for questions where it seems like you don't have enough information!

Example 6.1

Below is a probability distribution table for a random variable X. Find $Pr(3 < X < 6)$.

x	1	2	3	4	5	6
$Pr(X = x)$	0.1	0.13	0.17	0.27	0.20	0.13

The question is asking to find the probabilities for all the values of the random variable between 3 and 6.

First, we look at the top row, where the values of the random variable are. Then, find all those are between 3 and 6. Remember that 3 and 6 **are not** included, as the question uses 'less than' symbols (<) rather than 'less than or equal to' symbols ($\leq$). Therefore, the relevant probabilities are $X = 4$ and $X = 5$.

Now, we add these probabilities together to find $Pr(3 < X < 6)$:

$$\begin{aligned} Pr(3 < X < 6) &= 0.27 + 0.20 \\ &= 0.47 \end{aligned}$$

Example 6.2

Using the same probability table, find $Pr(X \geq 4 | X \geq 2)$.

We have to recall back to a previous topic: conditional probability. Let's look at the formula:

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Now, we'll rewrite it with the information from this question:

$$Pr(X \geq 4 | X \geq 2) = \frac{Pr(X \geq 4 \cap X \geq 2)}{Pr(X \geq 2)}$$

We can evaluate the intersection of $Pr\,(X \geq 4 \cap X \geq 2)$ by thinking of where $X \geq 4$ and $X \geq 2$ overlap. A useful tip is to mark this on the probability distribution table:

x	1	2	3	4	5	6
$Pr(X - x)$	0.1	0.13	0.17	0.27	0.20	0.13
				$Pr(X \geq 4)$		
		$Pr(X \geq 2)$				

As we can see, these probabilities overlap in the section $X \geq 4$. Therefore:

$$Pr\,(X \geq 4 \cap X \geq 2) = Pr(X \geq 4)$$

Now we can answer the question:

$$\begin{aligned} Pr(X \geq 4|X \geq 2) &= \frac{Pr(X \geq 4 \cap X \geq 2)}{Pr(X \geq 2)} \\ &= \frac{Pr(X \geq 4)}{Pr(X \geq 2)} \\ &= \frac{0.27 + 0.20 + 0.13}{0.13 + 0.17 + 0.27 + 0.20 + 0.13} \\ &= \frac{0.6}{0.9} \\ &= \frac{2}{3} \end{aligned}$$

6.2 Determining discrete probability distributions

6.2.1 Using a set of data

If we are given a set of data, we can determine the probability distribution for a random variable. Let's say we have some data on the number of televisions in a household.

Number of TVs	**Number of households**
0	121
1	304
2	254
3	199
Total	878

Let X = number of TVs. To find the probability of $Pr(X = 0)$, we first look at the number of households that have no TVs, which is 121. Then, we divide it by the total number of households to find the probability:

$$Pr(X = 0) = \frac{121}{878}$$

We could then create a probability distribution table by repeating this process for all values of the random variable.

x	0	1	2	3
$Pr(X = x)$	$121/878$	$304/878$	$254/878$	$199/878$

6.2.2 Uniform discrete random variables

The discrete random variable of X has a uniform distribution if all the values of X have an equal chance of occurring, such as:

x	0	1	2
$Pr(X = x)$	$1/3$	$1/3$	$1/3$

We can easily find the probabilities of $Pr(X = x)$ by calculating $Pr\,(X = x) = \frac{1}{n}$, where n is the total number of values for X. In the table above, $n = 3$.

6.2.3 Non-uniform discrete random variables

As we have seen before, not all discrete random variables are uniform. When the probabilities are not all equally likely, it is a non-uniform probability distribution.

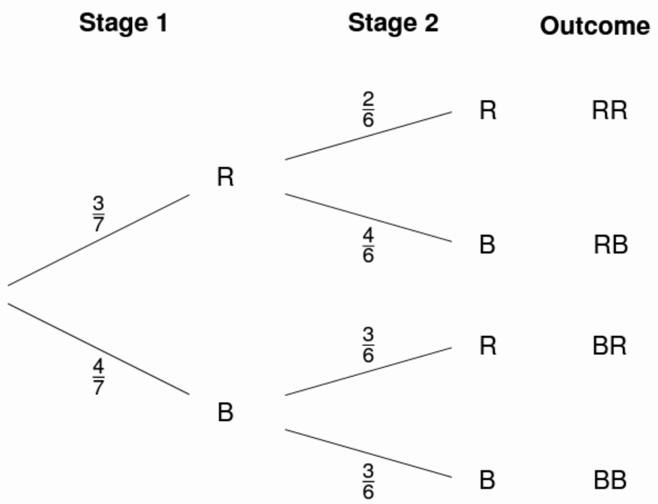
We will look at a question that requires you to create your own probability distribution table from a given scenario.

Example 6.3

Adrian selects two pens at random from a bag containing four blue pens and three red pens. Draw the probability distribution table for the number of blue pens he selects.

Firstly, recognise that this is a **multi-stage** experiment, as he chooses two pens. We can find the outcomes by drawing a tree diagram. Let R represent the red pens and B represent the blue pens. We need to find the probability of each outcome. Remember, to do this we must write the individual probabilities on each branch and multiply the relevant ones together. However, we must also bear in mind an important point: the bag starts with a total of seven pens: four blue and three red. However, after Adrian chooses one pen, these values can change. For example, if Adrian picked a red pen in stage 1, in stage 2 there are six pens total: three red and four blue.

Keep this in mind as you determine your probabilities:



Now, we can work out the probability of each outcome:

$$Pr(RR) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$
$$Pr(RB) = \frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$$
$$Pr(BR) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$
$$Pr(BB) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

Next, we need to make a probability distribution table for the number of blue pens he selects. That means our random variable X will be the number of blue pens, so we need to write:

Let $X =$ number of blue pens selected

X can equal either 0, 1, or 2. To find the probabilities of each value of X, we look at the outcomes and see which ones have our desired number of blue pens. Then, we draw the table:

x	0	1	2
$Pr(X = x)$	$1/7$	$2/7 + 2/7 = 4/7$	$2/7$

We can check that we've done the question correctly by verifying that the sum of the probabilities is 1:

$$\frac{1}{7} + \frac{4}{7} + \frac{2}{7} = \frac{7}{7} = 1$$

6.3 Expected value, variance, and standard deviation

6.3.1 Expected value

The expected value (also known as the mean) of a random variable X, expressed as $E(x)$ or μ, is the long-run average value of X. If we go back to our experiment of tossing a coin twice, where X is the number of heads, the expected value of the number of heads is 1 ($X = 1$). That means, if the experiment is repeated many times, then on average you can expect to get 1 head each time you do the experiment. I will show you how to calculate this expected value, using the formula:

$$E(X) = \mu = \sum_{x} x \times p(x)$$

This looks confusing, but I'll explain the formula in words. First off, for our experiment of tossing two coins, where X is the number of heads, let's bring back our probability distribution table we made earlier:

x	0	1	2
$Pr(X = x)$	$1/4$	$1/2$	$1/4$

To find the expected value, multiply x by $Pr(X = x)$ for each column, then add all the values together:

$$\text{First column: } 0 \times \frac{1}{4} = 0$$
$$\text{Second column: } 1 \times \frac{1}{2} = \frac{1}{2}$$
$$\text{Third column: } 2 \times \frac{1}{4} = \frac{1}{2}$$

Now, add all these values together to get our expected value:

$$E(X) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

This means that, on average, the number of heads you will get is 1 each time you complete the experiment.

6.3.2 Variance and standard deviation

The variance of a random variable, denoted by $Var(X)$, represents the spread of probability distribution around the mean or expected value. A full understanding of variance is not required at this stage, so don't worry too much about what variance actually is. The variance is found by the formula:

$$Var\,(X) = \sum p_i\,(x_i - \mu)^2$$

While this is the formula on the formula sheet, I recommend using the following formula below, which is not on your formula sheet (so you should memorise it):

$$Var\,(X) = E\left(X^2\right) - [E\,(X)]^2$$

This is usually the easiest method of finding variance, so I would suggest using it. You will see how to use the formula in the example below.

The standard deviation, denoted by $sd(X)$ or σ, is also a measure of spread, which is just found by taking the square root of the variance.

$$sd\,(X) = \sigma = \sqrt{Var\,(X)}$$

Since standard deviation is the square root of variance, we can also say that the square of the standard deviation is the variance.

$$\sigma^2 = Var(X)$$

We'll now tackle a question that involves the three concepts that we have talked about.

Example 6.4

Below is a probability distribution table of a random variable X. Find p.

x	0	1	2	4	8
$Pr(X = x)$	p	1/2	1/4	1/8	1/16

If you recall, the sum of the probabilities must be equal to 1. We can use this to find p:

$$p + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1$$
$$p = \frac{1}{16}$$

Example 6.5

Find $E(X)$.

For this, remember that we multiply x and $Pr(X = x)$ for each column and add all the values together:

$$E\,(X) = 0 \times \frac{1}{16} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + 8 \times \frac{1}{16}$$
$$= 2$$

Example 6.6

Find $Var(X)$.

This is the formula that I would use for finding variance:

$$Var\,(X) = E\left(X^2\right) - [E\,(X)]^2$$

So, how we evaluate $E\left(X^2\right)$? This is expected value of X squared. To find this, we follow a similar process to finding expected value; however, instead of multiplying x with $\Pr(X = x)$, we multiply x^2 with $Pr(X = x)$. You can add another row to the probability distribution table like so to make things clearer:

x	0	1	2	4	8
x^2	0	$1^1 = 1$	$2^2 = 4$	$4^2 = 16$	$8^2 = 64$
$Pr(X = x)$	1/16	1/2	1/4	1/8	1/16

As you can see, the second row is x^2, which is just found be squaring x, the top row. Now, we multiply x^2 and $\Pr(X = x)$ for each column and add all values like before.

$$\begin{aligned} E\left(X^2\right) &= 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} + 4^2 \times \frac{1}{8} + 8^2 \times \frac{1}{16} \\ &= \frac{15}{2} \end{aligned}$$

Now, we go back to the variance formula and put the numbers in. Remember that we already know $E(X) = 2$.

$$\begin{aligned} Var(X) &= E\left(X^2\right) - [E(X)]^2 \\ &= \frac{15}{2} - 2^2 \\ &= \frac{7}{2} \\ &= 3.5 \end{aligned}$$

Example 6.7

Find $sd(X)$.

All we have to do is take the square root of the variance.

$$\begin{aligned} sd(X) &= \sqrt{Var(X)} \\ &= \sqrt{\frac{7}{2}} \\ &= 1.8708... \end{aligned}$$

KEY POINT :
When asked to find expected value, variance or standard deviation, you will have to first construct a probability distribution table. Sometimes the question tells you to do this, but other times it may not be so obvious.

That concludes the course content of Maths Methods Units 1&2!

Part III

Exam Revision Tips

Topic 1

Advice for assessments

The best way to revise for Maths Methods is just to work through practice questions. It sounds simple, but although rereading your notes or memorising content might suit other subjects, this ultimately isn't effective for Methods as you need to know how to apply this knowledge. If you keep doing more practice questions, you'll eventually become fluent with the formulas you need to use (and you'll know which formula to use for each question, making you way more efficient!). This way, you'll save time as well by not needing to rely on your formula sheet. And perhaps most importantly, you'll become more competent with your algebra, which is vital for success in Units 3&4 and the assessments you'll complete in test conditions.

Of course, you need to understand the concepts first, which is where this book comes in. However, once you're familiar with the content, your priority should be on practising questions.

Additionally, just try to practise as many simple familiar questions as you can to understand the concepts needed. Don't worry about solving the harder, complex, unfamiliar questions, and definitely don't waste time if you get stuck on one of them. Typically, the exam is split into **60% simple familiar, 20% complex familiar** and **20% complex unfamiliar.** This means you could pass by just answering simple familiar questions only. The complex unfamiliar questions on the test will be difficult, so don't expect to complete them easily, but with enough practice, they should be within your capabilities too.

Before an exam, you will have some **perusal time.** Use this time to look at which questions you can tackle first. Don't try to solve questions in your head during this time, as it is easy to make mistakes. Once writing time begins, you can attempt the easiest questions first if you want to build up your confidence – you don't have to do the questions in the order they are in.

Even if the hard questions on the test look impossible, remember that you can always earn partial marks for a question, so just try something, even if you aren't sure it's right.

Time is a very important factor during the test, and you must ensure that you don't spend too long on a question. Students can get caught up in a single question, on the verge of finding the answer but not quite there yet, and investing way too much time on a few marks – try to avoid this! If you can, during perusal time, work out on average how long you should spend on a question (e.g. if you have 40 questions on a 90 minute paper, you should spend approximately 2.25 minutes per question). And as always, keep an eye on the clock.

Finally, at the end of Year 11 or Year 12, your exams may be all packed in on the same week, so make sure to plan ahead on when to revise, as you will not have much time to revise the night before the test, especially if you have more than one test that day. Also, don't get discouraged about other subjects if you have a bad feeling about an exam you just finished. You don't know what your results are until you get them, so don't worry about them at all.

Most importantly, stay calm during your assessments, and don't give up! It is easy to panic and let your mind wonder, or feel demotivated at times, but try your best to stay focused.

That's it! Remember that Units 1&2 is considered assumed knowledge for Units 3&4 next year, so try to remember as much you can (or you can always come back to this book if you need to revise!).

Good luck with your all your assessments for Maths Methods Units 1&2.