ExW10: Hofstadter's butterfly

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1 The problem

Consider a 2D square lattice with 3×3 sites. Assume that the nearest neighbours hopping amplitude is c. Furthermore, assume that the magnetic flux through each plaque is Φ.

Examine the following cases:

1. open boundary conditions.
2. periodic boundary condition in one axis and open boundary condition in the other.

Write the Hamiltonian explicitly for both cases. Plot the eigenenergies as a function of the flux for a 15×15 system.

2 The solution

The problem consists of a 2D lattice formed of identical sites. The distance between two adjacent sites is a. The problem considers a case in which a particle can move between adjacent sites only, with hopping amplitude c. Let us use the position basis $|n, m>$ where n, m denotes the x and y position of the site respectively.

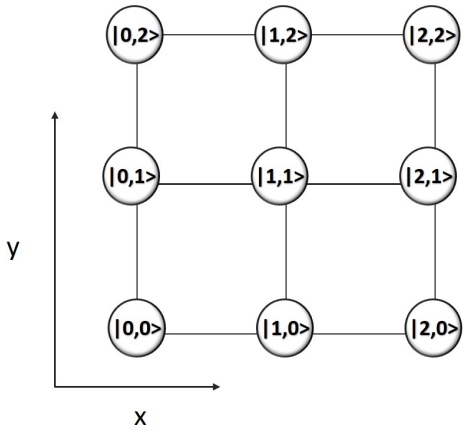

Figure 1: A schematic description of the system

2.1 Open boundary conditions

In the absence of a magnetic field the Hamiltonian of a 3×3 site system, represented in the position basis, has the form:

$$H = \begin{array}{c} \begin{array}{ccccccccc} |0,0> & |1,0> & |2,0> & |0,1> & |1,1> & |2,1> & |0,2> & |1,2> & |2,2> \end{array} \\ \begin{pmatrix} 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ c & 0 & c & 0 & c & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & c & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & 0 & c & 0 & 0 \\ 0 & c & 0 & c & 0 & c & 0 & c & 0 \\ 0 & 0 & c & 0 & c & 0 & 0 & 0 & c \\ 0 & 0 & 0 & c & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & c & 0 & c & 0 & c \\ 0 & 0 & 0 & 0 & 0 & c & 0 & c & 0 \end{pmatrix} \end{array}$$

When a homogeneous magnetic field, $\vec{B}$, is applied in the $\hat{z}$ direction, there is a magnetic flux Φ through each plaque.
Choosing the landau gauge for the vector potential:

$$\vec{A} = B(0, x, 0) \tag{1}$$

The accumulated phase between sites for motion in the x direction is zero while the accumulated phase for motion between sites in the y direction, $|n, m>, |n, m+1>$ is:

$$\phi = \frac{e}{\hbar} \int \vec{A} d\vec{r} = \frac{e}{\hbar} \int_{am}^{a(m+1)} Bxdy = \frac{e}{\hbar} Bax = \frac{e}{\hbar} Ba^2 n = \phi_0 n \tag{2}$$

defining $\phi_0 \equiv \frac{e}{\hbar} Ba^2$.

The x and y positions of the sites are discrete, where $x = na$, $y = ma$,
$0 \leq n, m \leq N$, and N is the number of sites in each axis. Note that a circulation around any plaque contributes the same phase ϕ_0. In addition, the magnetic flux through each plaque is $\Phi = Ba^2$.

The Hamiltonian of a 3×3 sites represented in the position basis with the conditions described above is of the form:

$$H = \begin{array}{c} \begin{array}{ccccccccc} |0,0> & |1,0> & |2,0> & |0,1> & |1,1> & |2,1> & |0,2> & |1,2> & |2,2> \end{array} \\ \begin{pmatrix} 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 \\ c & 0 & c & 0 & ce^{i\phi_0} & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & ce^{2i\phi_0} & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & 0 & c & 0 & 0 \\ 0 & ce^{-i\phi_0} & 0 & c & 0 & c & 0 & ce^{i\phi_0} & 0 \\ 0 & 0 & ce^{-2i\phi_0} & 0 & c & 0 & 0 & 0 & ce^{2i\phi_0} \\ 0 & 0 & 0 & c & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & ce^{-i\phi_0} & 0 & c & 0 & c \\ 0 & 0 & 0 & 0 & 0 & ce^{-2i\phi_0} & 0 & c & 0 \end{pmatrix} \end{array}$$

By numerically diagonalizing the Hamiltonian of a 15×15 system, we obtain the energy spectrum as a function of Φ:

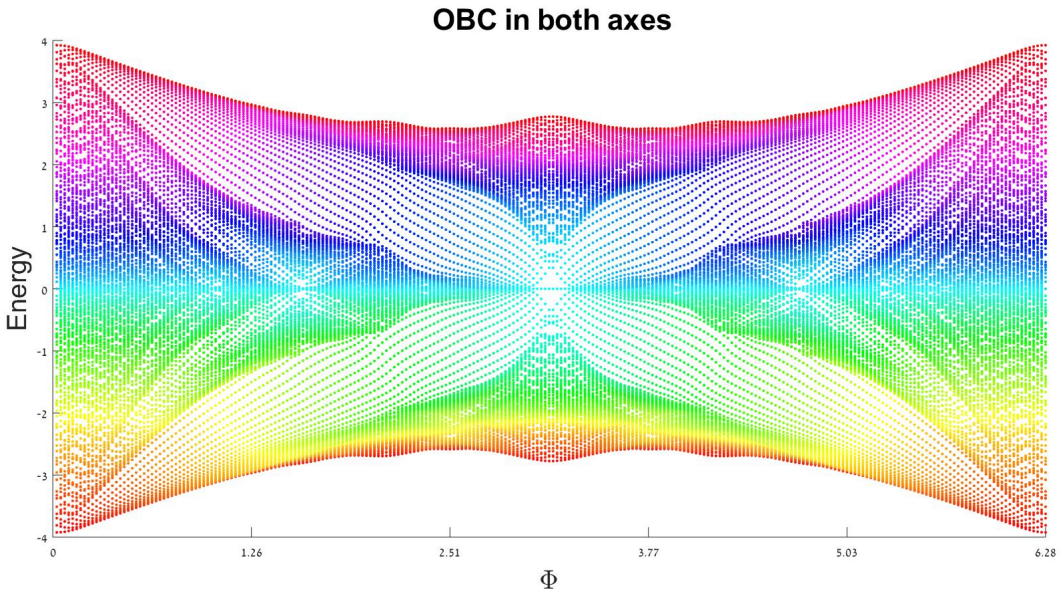

Figure 2: The energy spectrum of a 15×15 site system with OBC in both axes.

2.2 Periodic boundary conditions in the x axis

Upon adding periodic boundary condition in the x axis and using the same gauge as used before, there will be no accumulated phase in the x axis. The calculation of the phase accumulated in the y axis is as presented before.
Under these conditions the Hamiltonian is:

$$H = \begin{array}{c} \begin{array}{ccccccccc} |0,0> & |1,0> & |2,0> & |0,1> & |1,1> & |2,1> & |0,2> & |1,2> & |2,2> \end{array} \\ \begin{pmatrix} 0 & c & c & c & 0 & 0 & 0 & 0 & 0 \\ c & 0 & c & 0 & ce^{i\phi_0} & 0 & 0 & 0 & 0 \\ c & c & 0 & 0 & 0 & ce^{2i\phi_0} & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & c & c & 0 & 0 \\ 0 & ce^{-i\phi_0} & 0 & c & 0 & c & 0 & ce^{i\phi_0} & 0 \\ 0 & 0 & ce^{-2i\phi_0} & c & c & 0 & 0 & 0 & ce^{2i\phi_0} \\ 0 & 0 & 0 & c & 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & 0 & ce^{-i\phi_0} & 0 & c & 0 & c \\ 0 & 0 & 0 & 0 & 0 & ce^{-2i\phi_0} & c & c & 0 \end{pmatrix} \end{array}$$

As before, diagonalizing the hamiltonian of a 15×15 system yields the following energy spectrum as a function of Φ:

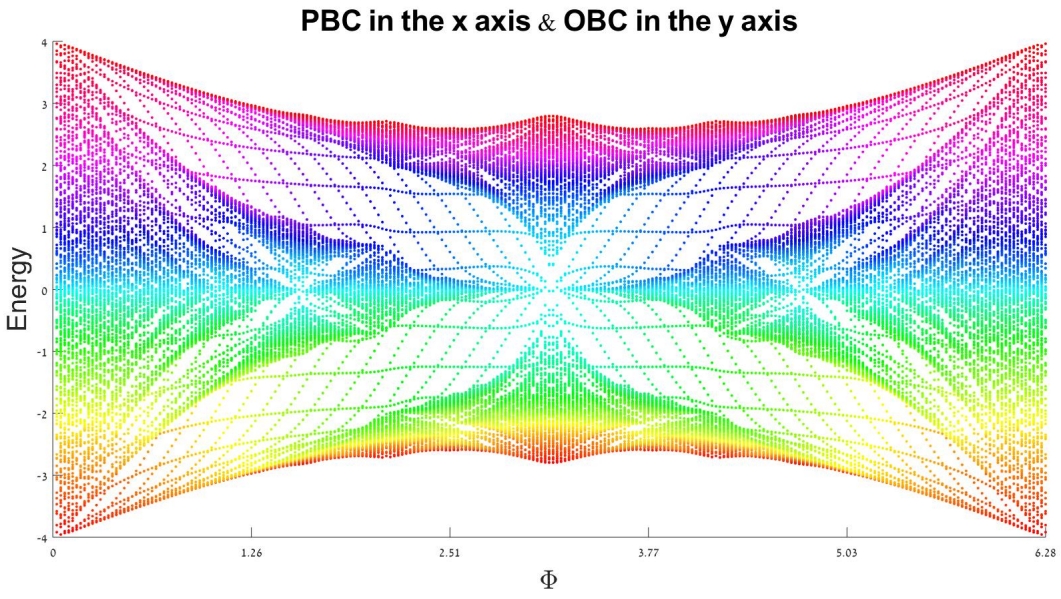

Figure 3: The energy spectrum of a 15×15 sites system with PBC in the x axis and OBC in the y axis