

A theorem is a collection of statements where \vdash is premises must be an axiom.

e.g. - If x is even, y is odd, then $x+y$ is odd.

Unit-2

Natural Deduction

In this we have a collection of proof rules. By applying these rules successfully, we may infer a conclusion from a given collection of premises.

Rules ①. Rule of Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q} \quad (\wedge i) \quad (\text{and-introduction})$$

$$\frac{\begin{array}{c} p \wedge q \\ (q) \end{array}}{\therefore p \quad (\wedge e_1)} \quad \frac{\begin{array}{c} p \wedge q \\ (p) \end{array}}{\therefore q \quad (\wedge e_2)} \quad (\text{and-elimination})$$

8 - Using the rules of conjunction prove that $p \wedge q, \vdash \vdash (q \wedge p)$.

Solⁿ

$p \wedge q$ } premises
 $\neg r$ } premises

$$\frac{\begin{array}{c} p \wedge q \\ \neg r \end{array}}{\frac{\neg \neg r \text{ (neg)}}{\frac{\neg \neg r \text{ (ini)}}{(p \wedge q)}}}$$

$$\frac{\begin{array}{c} p \wedge q \\ \neg r \end{array}}{\frac{\neg \neg r \text{ (neg)}}{\frac{\neg \neg r \text{ (12,3)}}{p \wedge q \text{ (1; 2,3)}}}}$$

② Rules of double negation :-

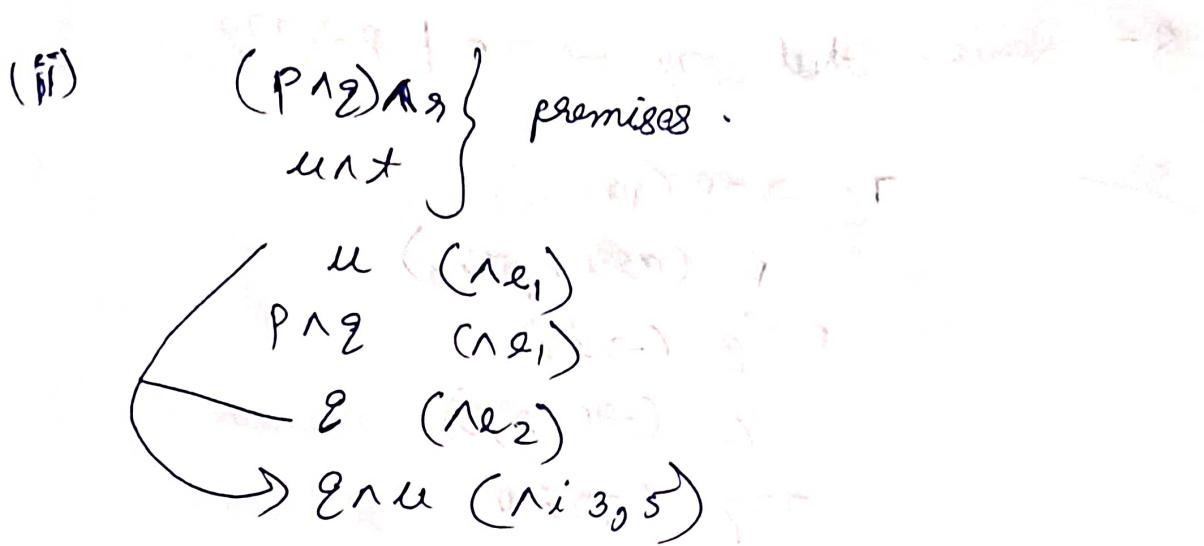
$\frac{p}{\neg \neg p \text{ (11i)}}$ (introduction)	$\frac{\neg \neg p}{\neg p \text{ (11p)}}$ (elimination)
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Q- using the rules of double negation prove that (i) $p, \neg \neg (q \wedge r) \vdash \neg \neg p \wedge r$.
(ii) $(p \wedge q) \wedge r, \neg p \vdash q \wedge r$.

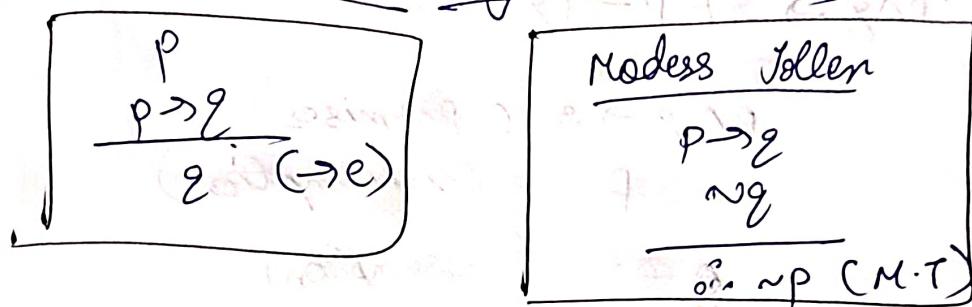
Solⁿ

(i) $\frac{\begin{array}{c} p \\ \neg \neg (q \wedge r) \end{array}}{\frac{\text{premises}}{}}$

$$\frac{\begin{array}{c} \text{over over elimination} \\ \neg \neg r \text{ (11p 2)} \\ \neg r \text{ (12)} \\ \neg \neg p \text{ (11i 1)} \\ \neg (\neg \neg p) \wedge r \text{ (1; 4,5)} \end{array}}{\frac{\text{premises}}{}}$$

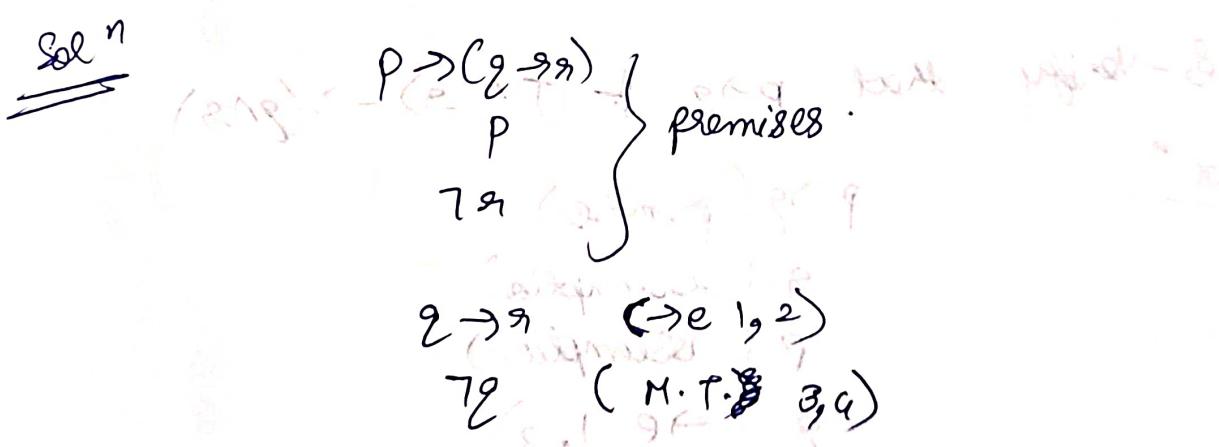


③ Rules of eliminating implication



Application

Using the rule of eliminating implication
 verify that $P \rightarrow (Q \rightarrow R), P, \neg R \vdash \neg Q$.



④ Rule implies (introduction)

To prove $P \rightarrow Q$, assume that P is the assumption, $\boxed{(\rightarrow i)}$

8 - Verify that $\neg q \rightarrow \neg p \vdash p \rightarrow \neg q$

Solⁿ

$\neg q \rightarrow \neg p$ (premise)

p (assumption)

$p \rightarrow q$ ($\rightarrow i_2$)

$\neg q$ ($\rightarrow e, 2, 3$)

$\neg q$ ($\neg q, i_4$)

8 - $p \wedge q \rightarrow \neg p \vdash p \rightarrow (q \rightarrow \neg p)$

$p \wedge q \rightarrow \neg p$ (premises)

p (assumption)

$\neg p$ ($\neg p, i_1$)

$p \wedge q$ ($\wedge i_2, 3$)

$\neg p$ ($\rightarrow e, 1, 4$)

$q \rightarrow \neg p$ ($\rightarrow i_3$)

$p \rightarrow (q \rightarrow \neg p)$ ($\rightarrow i_1, 2$)

8 - Verify that $p \rightarrow q \vdash (p \wedge q) \rightarrow (q \wedge p)$

Solⁿ

$p \rightarrow q$ (premises)

q (assumption)

p (assumption)

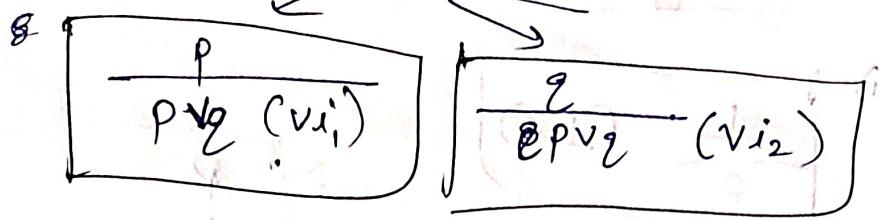
q ($\rightarrow e, 1, 3$)

$p \wedge q$ (assumption)

$p \wedge q$ ($\wedge i_2, 4$)

$(p \wedge q) \rightarrow (q \wedge p)$ ($\rightarrow i_5, 6$)

⑤ Rules of disjunction



⑥ Elimination of disjunction



8 Verify that (i) $q \rightarrow \neg \vdash (p \vee q) \rightarrow (p \vee \neg)$

Solⁿ

$q \rightarrow \neg$ (premises)

$p \vee q$ (assumption)

q (v_{e_2})

\neg ($\rightarrow q, 1, 3$)

$p \vee \neg$ (v_{i_2}, q)

$p \vee q \rightarrow p \vee \neg$ ($\neg i, 2, 5$)

(ii) $p \vee q \vdash q \vee p$

Solⁿ

$p \vee q$ (premises)

q (v_{e_1})

$q \vee p$ ($v_{i_1}, 2$)

(iii) $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

(iv) $p \rightarrow q, p \rightarrow r \vdash q \wedge r$

~~(v)~~ $q \rightarrow (q \rightarrow r), p, q \vdash r$

(vi) $(p \wedge q) \rightarrow r, q, r, p \vdash q$

$$\boxed{1) \frac{P}{\perp} (1e)}$$

$$\boxed{2) \frac{P}{\frac{\perp}{\perp}} (\neg e)}$$

$$\boxed{3) \frac{P}{\frac{\perp}{\perp}} (\neg i)}$$

8 - Verify $\neg P \vee Q \vdash P \rightarrow Q$

$\neg P \vee Q$ (premises)

2 (ve 1)

P (Assumption)

$P \rightarrow Q$ ($\rightarrow i$ 2, 3)

9) Modus Tollens (MT)

$$\boxed{\frac{P \rightarrow Q}{\neg Q} \therefore \neg P \text{ (MT)}}$$

Proof :-

$P \rightarrow Q$ { premises. }
 $\neg Q$

P (Assumption)
 $\neg Q$ ($\rightarrow e$ 1, 3)
 \perp ($\neg e$ 2, 4)

$\neg P$ ($\neg i$ 3, 5)
(proved)

$$2) \boxed{\frac{P}{\therefore \neg\neg P}} (77i)$$

Proof:

$$\begin{aligned} & P \text{ (premises)} \\ \rightarrow & \neg P \text{ (assumption)} \\ \rightarrow & \perp \text{ (7e 1, 2)} \\ \boxed{\neg\neg P} & \text{ (7i 2, 3)} \\ & \text{(proved)} \end{aligned}$$

3) Proof By Contradiction (PBC) [Reduction to absurdity]

→ If from $\neg P$, we obtain a contradiction, then we have ~~contradiction~~ to deduce P .

$$\boxed{\frac{\neg P \rightarrow \perp}{\therefore P}} \text{ (PBC)}$$

Whenever \rightarrow sign so assume LHS always

Proof:

$$\begin{aligned} & \neg P \rightarrow \perp \text{ (premises)} \\ \rightarrow & \neg P \text{ (assumption)} \\ \rightarrow & \perp \text{ (7e 1, 2)} \\ \rightarrow & \neg\neg P \text{ (7i 2, 3)} \\ \rightarrow & \boxed{P} \text{ (77e 4)} \end{aligned}$$

4) Law of the excluded middle (LEM)

→ According to this law $P \vee \neg P$ is always true, whatever P is. There is no 3rd possibility and hence excluded middle.

$$\boxed{P \vee \neg P \rightarrow \text{always true}}$$

Proof :-

$\neg(p \vee \neg p)$ is true (assumption)
 p (assumption)
 $\rightarrow p \vee \neg p$ (V1, 2)
 \perp (T2, 1, 3)

$\neg(\neg(p \vee \neg p))$ @
 $p \vee \neg p$ (T1, 5)

* Propositional calculus (PC)

→ In propositional calculus, we have all the propositional variables (p, q, r, \dots , etc) and only two connectives \rightarrow and \neg .

→ Axiom schemes in PC -

$$A1) A \rightarrow (B \rightarrow A)$$

$$A2) (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$A3) (\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow B) \rightarrow A)$$

Modus Ponens (MP):

$$\frac{P \rightarrow Q}{Q} \quad (MP)$$

→ Proof and theorem -

A proof in PC is defined to be finite sequence of proposition, where each proposition is either an axiom or can be derived from the earlier two propositions by an application of MP.

The last proposition of a proof is called a theorem of PC. The fact that A is a theorem of PC is written as $\vdash_{PC} A$. (A is provable in PC).

Application

1) Proof of $\vdash_{PC} \neg \neg p \rightarrow (p \rightarrow (\neg p \rightarrow p))$

$$\vdash_{PC} \neg \neg p \rightarrow (p \rightarrow (\neg p \rightarrow p))$$

Proof:

$$(A1) \left[\begin{array}{l} A: p \rightarrow (\neg p \rightarrow p) \\ B: \neg \neg p \end{array} \right]$$

$$(\neg p \rightarrow (p \rightarrow (\neg p \rightarrow p))) \rightarrow (\neg \neg p \rightarrow (p \rightarrow (\neg p \rightarrow p)))$$

$$\neg p \rightarrow (p \rightarrow (\neg p \rightarrow p)) \quad (A1)$$

$$\neg \neg p \rightarrow (p \rightarrow (\neg p \rightarrow p)) \quad (MP, 1, 2)$$

2) Proof of

$$\vdash \neg q \rightarrow (p \rightarrow p)$$

Proof:

$$\left[\begin{array}{l} A: p \\ B: \neg q \rightarrow p \end{array} \right]$$

$$1) \neg q \rightarrow ((\neg q \rightarrow p) \rightarrow p) \quad (A1)$$

$$\left[\begin{array}{l} A: p \\ B: \neg q \rightarrow p \\ C: p \end{array} \right]$$

$$((p \rightarrow ((\neg q \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (\neg q \rightarrow p)) \rightarrow (p \rightarrow p))) \rightarrow$$

$$\begin{array}{c} ((p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow (q \rightarrow p))) \rightarrow ((p \rightarrow ((q \rightarrow p))) \\ \rightarrow (p \rightarrow p)) \quad (A_2) \end{array}$$

$$\begin{array}{c} 2) \underbrace{(p \rightarrow ((q \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (q \rightarrow p)) \rightarrow} \\ (p \rightarrow p)) \quad (A_2) \end{array}$$

$$3) (p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p) \quad (MP \ 1,2)$$

$$4) p \rightarrow (q \rightarrow p) \quad (A_1)$$

$$5) p \rightarrow p \quad (MP \ 3,4)$$

$$\textcircled{3} \quad \begin{cases} A : p \rightarrow p \\ B : q \end{cases}$$

$$\textcircled{4} \quad \cancel{q \rightarrow (p \rightarrow p)} \quad (MP)$$

$$6) (p \rightarrow p) \rightarrow (q \rightarrow (p \rightarrow p)) \quad (A_1)$$

$$7) q \rightarrow (p \rightarrow p) \quad (MP \ 5,6)$$

Q - Verify that $p \rightarrow q \wedge q \rightarrow r \vdash p \rightarrow r$.

Soln

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \end{array} \quad \begin{array}{c} \{ \text{premises} \\ \text{assumption} \\ \text{closed} \end{array}$$

* Some Notations

P_1, P_2, \dots, P_n

Premises

Σ
↓
conclusion

If P_1, \dots, P_n verifies Σ , then $P_1, P_2, \dots, P_n \vdash \Sigma$.

If P_1, \dots, P_n doesn't verify Σ , then $P_1, P_2, \dots, P_n \not\vdash \Sigma$

In general, $\Sigma = \{P_1, P_2, P_3, \dots, P_n\}$ • $\Sigma \models \Sigma$

Indicates that the given set of propositions (Σ) may or may not verify Σ .

* Soundness of propositional logic (Imp)

If, for all valuations (Models) in which $P_1, P_2, P_3, \dots, P_n$ evaluate to \top , the conclusion Σ evaluates to \top as well, we say that

$P_1, P_2, \dots, P_n \models \Sigma$ holds

(semantic entailment relation)

Q - check whether $P \wedge Q \models \Sigma$, $P \vee Q \models \Sigma$ holds.

Solⁿ

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

[$P \vee Q : T$, $Q : F$ is

(not permissible)]

∴ $P \wedge Q \models \Sigma$ holds

∴ $P \vee Q \models \Sigma$ does not hold.

* Soundness theorem (S)

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ and β be propositional formulas.
If $\alpha_1, \alpha_2, \dots, \alpha_n$ verifies β is valid then,
 $\alpha_1, \alpha_2, \dots, \alpha_n \models \beta$ holds.

* Monotonicity theorem / Monotone theorem.

Let $\Sigma \subseteq \Delta$ be 2 collections of propositions and ~~and~~ Σ is a subset of Δ [$\Sigma \subseteq \Delta$]

of φ be any other proposition then,

1) if $\Sigma \vdash \varphi$, then $\Delta \vdash \varphi$

2) if Σ is inconsistent, then Δ is also inconsistent.

[Inconsistent collection :- The collection of propositions Σ is said to be inconsistent if and only if there exist propositions φ such that Σ verifies φ and Σ verifies $\neg \varphi$]

$\varphi \vdash \varphi$ and $\Sigma \vdash \neg \varphi$
 $\Sigma \vdash (\varphi \wedge \neg \varphi)$

* Paradox of Material implication

Let Σ be a collection of propositions,
then Σ is inconsistent if and only if
 Σ verifies φ for each proposition φ

* A particular derived rule

hypothetical Syllogism (HS)

$$\begin{array}{c} A \rightarrow B \\ B \rightarrow C \\ \hline \therefore A \rightarrow C \quad (\text{HS}) \end{array}$$

1) $(\neg q \rightarrow \neg p) \vdash (p \rightarrow q)$

Proof :- $\neg q \rightarrow \neg p$ (premise)

$$(\neg q \rightarrow \neg p) \rightarrow ((\neg q \rightarrow p) \rightarrow p) \quad (\text{A3}) \quad \begin{bmatrix} A: \neg q \\ B: \neg p \end{bmatrix}$$

$$(\neg q \rightarrow p) \rightarrow p \quad (\text{MP } 1, 2)$$

$$p \rightarrow (\neg q \rightarrow p) \quad (\text{A}) \quad \begin{bmatrix} A: p \\ B: \neg q \end{bmatrix}$$

$$p \rightarrow q \quad (\text{HS } 4, 3)$$

Deriv 8 -

$$8 - \vdash (\neg \neg p \rightarrow p)$$

$$1) \neg \neg p \rightarrow (\neg p \rightarrow \neg \neg p) \quad (\text{A1}) \quad \begin{bmatrix} A: \neg \neg p \\ B: \neg p \end{bmatrix}$$

$$2) (\neg p \rightarrow \neg \neg p) \rightarrow ((\neg p \rightarrow \neg p) \rightarrow p) \quad (\text{A3}) \quad \begin{bmatrix} A: p \\ B: \neg \neg p \end{bmatrix}$$

$$3) \neg \neg p \rightarrow ((\neg p \rightarrow \neg p) \rightarrow p) \quad (\text{HS } 1, 2)$$

$$4) (\neg \neg p \rightarrow ((\neg p \rightarrow \neg p) \rightarrow p)) \rightarrow (\neg \neg p \rightarrow p) \quad (\text{HS } 1, 2)$$

$$5) (\neg \neg p \rightarrow (\neg p \rightarrow \neg p)) \rightarrow (\neg \neg p \rightarrow p) \quad (\text{MP } 3, 4) \quad \begin{bmatrix} A: \neg \neg p \\ B: \neg p \rightarrow p \end{bmatrix}$$

$$6) \neg p \rightarrow \neg p \quad (\text{Theorem})$$

$$7) (\neg p \rightarrow \neg p) \rightarrow (\neg \neg p \rightarrow (\neg p \rightarrow \neg p)) \quad (\text{A1}) \quad \begin{bmatrix} A: \neg p \rightarrow \neg p \\ B: \neg \neg p \end{bmatrix}$$

$$8) \neg \neg p \rightarrow (\neg p \rightarrow \neg p) \quad (\text{MP } 6, 7)$$

$$9) \neg \neg p \rightarrow p \quad (\text{MP } 5, 8)$$

~~Imp~~ 8- $(\neg p \rightarrow q) \rightarrow ((c q \rightarrow \neg p) \rightarrow p)$,

$(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg p)$, $\neg p \rightarrow q \vdash p$

1) $(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg p)$ { premises

2) $\neg p \rightarrow q$

3) $q \rightarrow \neg p$ (MP 1, 2)

4) $(\neg p \rightarrow q) \rightarrow ((q \rightarrow \neg p) \rightarrow p)$ (premises)

5) $(q \rightarrow \neg p) \rightarrow p$ (MP 2, 4)

6) $\neg p \rightarrow q$ (MP 3, 5)

7) $\neg p \rightarrow ((q \rightarrow \neg p) \rightarrow p)$ (A1) [A: $\neg p \rightarrow$]
[B: $q \rightarrow \neg p$]

* Provable Equivalence

→ Pnd proof

→ GPC proof