A theorem is a collection of statements where is premises must be an axiom.
e.g - If, x is even, y is odd, then x+y is odd.

Unit-2

~~Natural Deduction~~

Natural Deduction -

↳ In this we have a collection of proof rules. By applying this rules successfully, we may infer a conclusion from a given collection of premises.

Rules ① Rule of Conjunction.

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q} \quad (\wedge i)$$

↳ (and - introduction)

$$\frac{\begin{array}{c} p \wedge q \\ (q) \end{array}}{\therefore p} \; (\wedge e_1) \qquad \frac{\begin{array}{c} p \wedge q \\ (p) \end{array}}{\therefore q} \; (\wedge e_2)$$

(and - elimination)

Q - Using the rules of conjunction prove that
$p \wedge q, r \underset{(F)}{\vdash} (q \wedge r)$.

Sol^n

$$\left.\begin{array}{l} p\wedge q \\ r \end{array}\right\} \text{premises}$$

$$\frac{\dfrac{p\wedge q \qquad q\ (\wedge e_2)}{r\ (\wedge i)}}{(q\wedge r)}$$

$$\left.\begin{array}{l} p\wedge q \\ r \end{array}\right\} \text{premises}$$

$q\ (\wedge e_2)$

$r\wedge q\ (\wedge i\ 2,3)$

② **Rules of double negation :-**

$$\boxed{\frac{p}{\neg\neg p\ (\neg\neg i)} \qquad \frac{\neg\neg p}{\therefore\ p\ (\neg\neg p)}}$$

(introduction) (elimination)

Q:- using the rules of double negation prove that (i) $p,\ \neg\neg(q\wedge r) \vdash (\neg\neg p)\wedge r$.

(ii) $(p\wedge q)\wedge r,\ u\wedge t \vdash q\wedge u$ -

Sol^n (i)

$$\left.\begin{array}{c} p \\ \neg\neg(q\wedge r) \end{array}\right\} \text{premises.}$$

~~[illegible]~~

$q\wedge r\ (\neg\neg p\ 2)$

$r\ (\wedge e_2)$

$\neg\neg p\ (\neg\neg i\ 1)$

$(\neg\neg p)\wedge r\ (\wedge i\ 4,5)$.

(iii) $(p \wedge q) \wedge r$, $u \wedge t$ } premises.

u $(\wedge e_1)$

$p \wedge q$ $(\wedge e_1)$

q $(\wedge e_2)$

$q \wedge u$ $(\wedge i\ 3, 5)$

③ **Rules of eliminating implication.**

$$\boxed{\begin{array}{c} p \\ p \to q \\ \hline q \quad (\to e) \end{array}}$$

$$\boxed{\begin{array}{c} \text{Modus Tollen} \\ p \to q \\ \sim q \\ \hline \therefore \sim p \quad (M.T) \end{array}}$$

Application

Q- Using the rule of eliminating implication verify that $p \to (q \to r), p, \neg r \vdash \neg q$.

Sol^n

$p \to (q \to r)$
p
$\neg r$
} premises.

$q \to r$ $(\to e\ 1, 2)$

$\neg q$ $(M.T.\ 3, 4)$

④ **Rules implies introduction.**

To prove $p \to q$, assume that p is the assumption. $\boxed{(\to i)}$

Q- Verify that $\neg q \to \neg p \vdash p \to \neg\neg q$

Solⁿ

$\neg q \to \neg p$ (premise)

p (assumption)

$p \to q$ ($\to i$ 2)

q ($\to e$ 2,3)

$\neg\neg q$ ($\neg\neg i$ 4)

Q - $p \wedge q \to r \vdash p \to (q \to r)$

$p \wedge q \to r$ (premises)

p (assumption)

q (assumption)

$p \wedge q$ ($\wedge i$ 2,3)

r ($\to e$ 1,4)

$q \to r$ ($\to i$ 3)

$p \to (q \to r)$ ($\to i$ 2)

Q - Verify that $p \to q \vdash (p \wedge r) \to (q \wedge r)$

Solⁿ

$p \to q$ (premises)

r (assumption)

p (assumption)

q ($\to e$ 1,3)

$p \wedge r$ (assumption)

$q \wedge r$ ($\wedge i$ 2,4)

$(p \wedge r) \to (q \wedge r)$ ($\to i$ 5,6)

⑤ Rules of disjunction.

$$\boxed{\frac{p}{p \vee q}\ (\vee i_1)} \quad \boxed{\frac{q}{p \vee q}\ (\vee i_2)}$$

⑥ Elimination of disjunction.

$$\boxed{\frac{p \vee q}{q}\ (\vee e)}$$

Q— Verify that (i) $q \to r \vdash (p \vee q) \to (p \vee r)$

Solⁿ

$q \to r$ (premises)

$p \vee q$ (assumption)

q ($\vee e_2$)

r ($\to e$ 1, 3)

$p \vee r$ ($\vee i_2$ 4)

$p \vee q \to p \vee r$ ($\to i$ 2, 5)

(ii) $p \vee q \vdash q \vee p$

Solⁿ

$p \vee q$ (premises)

q ($\vee e_1$)

$q \vee p$ ($\vee i_1$ 2)

(iii) $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

(iv) $p \to q,\ p \to \neg q \vdash \neg q$

Imp (v) $p \to (q \to r),\ p,\ \neg r \vdash \neg q$

(vi) $(p \wedge \neg q) \to r,\ \neg r,\ p \vdash q$

i)
$$\frac{\perp}{p}\ (\perp e)$$

2)
$$\frac{p \quad \neg p}{\perp}\ (\neg e)$$

3)
$$\frac{\boxed{\begin{matrix} p \\ \vdots \\ \perp \end{matrix}}}{\neg p}\ (\neg i)$$

Q - Verify $\neg p \vee q \vdash p \rightarrow q$

$\neg p \vee q$ (premises)

q ($\vee e$ 1)

p (assumption)

$\therefore p \rightarrow q$ ($\rightarrow i$ 2,3)

i) Modus Tollen (MT)

$$\frac{p \rightarrow q \quad \neg q}{\therefore \neg p}\ (MT)$$

Proof :-

$p \rightarrow q$ } premises.

$\neg q$

p (assumption)

q ($\rightarrow e$ 1,3)

$\perp$ ($\neg e$ 2,4)

$\boxed{\neg p}$ ($\neg i$ 3,5).

(proved)

2) $\boxed{\dfrac{p}{\therefore \neg\neg p} \ (\neg\neg i)}$

Proof:

1. p (premises)
2. $\neg p$ (assumption)
3. $\perp$ ($\neg e$ 1,2)
4. $\boxed{\neg\neg p}$ ($\neg i$ 2, 3)

(proved)

V.Imp

3) Proof By Contradiction (PBC) [Reduction to absurdity]

→ If from $\neg p$, we obtain a contradiction, then we have ~~entitled~~ entitled to deduce p.

$$\boxed{\dfrac{\neg p \rightarrow \perp}{\therefore \ p} \ (PBC)}$$

whenever → sign so assume LHS always

Proof:

1. $\neg p \rightarrow \perp$ (premises)
2. $\neg p$ (assumption)
3. $\perp$ ($\rightarrow e$ 1, 2)
4. $\neg\neg p$ ($\neg i$ 2,3)
5. $\boxed{p}$ ($\neg\neg e$ 4)

V.Imp

4) Law of the excluded middle (LEM)

→ According to this law $p \vee \neg p$ is always true, whatever p is. There is no 3rd possibility and hence excluded middle.

$\boxed{p \vee \neg p \Rightarrow \text{always true}}$

Proof:-

$\neg(p \vee \neg p)$ is true (assumption)

p (assumption)

$p \vee \neg p$ ($\vee i$, 2)

$\perp$ ($\neg e$ 1, 3)

$\neg\neg(p \vee \neg p)$ (e)

$p \vee \neg p$ ($\neg\neg e$ 5)

* Prepositional calculus (PC)

→ In prepositional calculus, we have all the prepositional variables (p, q, r, etc) and only two connectives $\sim$, $\rightarrow$.

→ Axiom schemes in PC -

A1) $A \rightarrow (B \rightarrow A)$

A2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

A3) $(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow B) \rightarrow A)$

Modus Ponen (MP):

$$\frac{\begin{array}{c} p \rightarrow q \\ p \end{array}}{q} \text{ (MP)}$$

→ Proof and theorem-

A proof in PC is defined to be finite sequence of proposition, where each proposition is either and axiom or can be derived from the earlier two propositions by an application of MP.

The last proposition of a proof is called a theorem of PC. The fact that A is a theorem of PC is written as "$\vdash_{PC} A$". (A is provable in PC).

<u>Application</u>

1) Proof of

$$\vdash_{PC} r \to (p \to (q \to p))$$

<u>Proof:</u>

$$(A1)\begin{bmatrix} A : p \to (q \to p) \\ B : r \end{bmatrix}$$

$$(p \to (q \to p)) \to (r \to (p \to (q \to p)))$$

$$p \to (q \to p) \quad (A1)$$

$$r \to (p \to (q \to p)) \quad (MP\ 1,2)$$

2) Proof of

$$\vdash q \to (p \to p)$$

<u>Proof:</u>

$$\begin{bmatrix} A : p \\ B : q \to p \end{bmatrix}$$

1) $p \to ((q \to p) \to p)$ (A1)

$$\begin{bmatrix} A : p \\ B : (q \to p) \\ C : p \end{bmatrix}$$

$$\cancel{((p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))) \to}$$

$((p \to ((q\to p) \to p)) \to (p \to (q \to p))) \to ((p \to ((q \to p) \to$
$\to (p \to p))$ (A2)

2) $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (A2)

3) $(p \to (q \to p)) \to (p \to p)$ (MP 1,2)

4) $p \to (q \to p)$ (A1)

5) $p \to p$ (MP 3,4)

$\left[\begin{array}{l} A : p \to p \\ B : q \end{array}\right]$

~~$q \to (p \to p)$ (MP~~

6) $(p \to p) \to (q \to (p \to p))$ (A1)

7) $q \to (p \to p)$ (MP 5,6)

Q- Verify that $p \to q,\ q \to r \vdash p \to r$.

Soln

$p \to q$
$q \to r$ } premises

q (assumptions)

* Some Notations

$\underbrace{P_1, P_2, \ldots P_n}_{\text{Premises}} \quad . \quad \underset{\text{conclusion}}{Q}$

If $P_1, \ldots P_n$ verifies Q, then $P_1, P_2 \ldots P_n \vdash Q$.

If $P_1, \ldots P_n$ doesn't verify Q, then $P_1, P_2 \ldots P_n \nvdash Q$

In general, $\Sigma = \{P_1, P_2, P_3, \ldots P_n\}$. $\Sigma \, ! \vdash Q$

Indicates that the given set of preposition (Σ) may or may not verify Q.

* Soundness of Prepositional logic (Imp)

If, for all valuations (Models) in which $P_1, P_2, P_3, \ldots P_n$ evaluate to T, the conclusion Q evaluates to T as well, we say that

$P_1, P_2, \ldots P_n \models Q$ holds.

(semantic entailment relation)

Q- check whether $P \wedge Q \models Q$, $P \vee Q \models Q$ holds.

Soln

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	$\boxed{F}$	F	$\boxed{T}$ X
F	T	F	T ✓
F	F	F	F

[$P \vee Q$: T, Q : F is not permisible]

$\therefore P \wedge Q \models Q$ holds

$\therefore P \vee Q \models Q$ does not hold.

Soundness Theorem (imp)

Let $\alpha_1, \alpha_2, \dots \alpha_n$ and β be prepositional formulas

If $\alpha_1, \alpha_2, \dots \alpha_n$ verifies β is valid then,

$\alpha_1, \alpha_2, \dots \alpha_n \models \beta$ holds.

Monotonicity theorem / Monoton theorem.

Let Σ & Δ be 2 collections of prepositions and ~~let~~ Σ is a subset of Δ $[\Sigma \subseteq \Delta]$

If q be any other preposition then,

1) if $\Sigma \vdash q$, then $\Delta \vdash q$.

2) if Σ is inconsistent, then Δ is also inconsistent.

[Inconsistent collection :- The collection of prepositions Σ is said to be inconsistent if and only if their exist a preposition q such that Σ verifies q and Σ verifies $\sim q$]

$\Sigma \vdash q$, $\Sigma \vdash (\neg q)$

Paradox of Material Implication

Let Σ be a collection of preposition, then Σ is inconsistent if and only if Σ verifies q for each preposition q

* A particular derived rule

Hypothetical Syllogism: (HS)

$$\begin{array}{c} A \to B \\ B \to C \\ \hline \therefore A \to C \text{ (HS)} \end{array}$$

Imp (i) $(\neg q \to \neg p) \vdash (p \to q)$

Proof:- $\neg q \to \neg p$ (premise)

$(\neg q \to \neg p) \to ((\neg q \to p) \to q)$ (A3) $\begin{bmatrix} A : q \\ B : p \end{bmatrix}$

$(\neg q \to p) \to q$ [MP 1,2]

$p \to (\neg q \to p)$ (A) $\begin{bmatrix} A : p \\ B : \neg q \end{bmatrix}$

$p \to q$ (HS 4, 3)

V. Imp Q.

Q- $\vdash (\neg\neg p \to p)$

1) $\neg\neg p \to (\neg p \to \neg\neg p)$ (A1) $\begin{bmatrix} A : \neg\neg p \\ B : \neg p \end{bmatrix}$

2) $(\neg p \to \neg\neg p) \to ((\neg p \to \neg p) \to p)$ (A3) $\begin{bmatrix} A : p \\ B : \neg p \end{bmatrix}$

3) $\neg\neg p \to ((\neg p \to \neg p) \to p)$ (HS 1, 2)

4) $(\neg\neg p \to ((\neg p \to \neg p) \to p)) \to$
$(\neg\neg p \to (\neg p \to \neg p)) \to (\neg\neg p \to p)$ (A2) $\begin{bmatrix} A : \neg\neg p \\ B : \neg p \to \neg p \\ C : p \end{bmatrix}$

5) $(\neg\neg p \to (\neg p \to \neg p)) \to (\neg\neg p \to p)$ (MP 3, 4)

6) $\neg p \to \neg p$ (Theorem)

7) $(\neg p \to \neg p) \to (\neg\neg p \to (\neg p \to \neg p))$ (A1) $\begin{bmatrix} A : \neg p \to \neg p \\ B : \neg\neg p \end{bmatrix}$

8) $\neg\neg p \to (\neg p \to \neg p)$ (MP 6, 7)

9) $\neg\neg p \to p$ (MP 5, 8)

Imp :- $(\neg p \to q) \to ((q \to \neg p) \to p)$,
$(\neg p \to q) \to (q \to \neg p)$, $\neg p \to q \vdash p$

1) $(\neg p \to q) \to (q \to \neg p)$ } Premises
2) $\neg p \to q$

3) $q \to \neg p$ (MP 1, 2)

4) $(\neg p \to q) \to ((q \to \neg p) \to p)$ (Premises)

5) $(q \to \neg p) \to p$ (MP 2, 4)

6) p ~~(MP 3, 5)~~ (MP 3,5)

~~6) $p \to ((q \to \neg p) \to p)$ (A1)~~ ~~$\left[\begin{array}{l} A : p \\ B : q \to \neg p \end{array}\right]$~~

* Provable Equivalence

→ Pnd proof
→ GPC proof