only if each state ~~sta~~ satisfies it, X is valid if and only if each state satisfies it, X is invalid if and only if some state falsifies it.

Unit-4

Axiom schemes of first order calculus

The axiom scheme of first order calculus are $A1, A2, A3$ of propositional calculus two for the quantifies $\forall$ and two for the equality predicat $(\approx)$

This means that the propositional constants, $\top, \bot$ and the connectives $(\wedge, \vee, \leftrightarrow)$ and the quantifies $\exists$ will be defined in terms of $\neg$ $\rightarrow$ and $\forall$

⇒ The axiom schems of FC are :-

A1) $X \rightarrow (Y \rightarrow X)$

A2) $(X \rightarrow (Y \rightarrow Z)) \rightarrow ((X \rightarrow Y) \rightarrow (X \rightarrow Z))$

A3) $(\neg X \rightarrow \neg Y) \rightarrow ((\neg X \rightarrow Y) \rightarrow X)$

V. imp A4) $\forall x Y \rightarrow Y[x/t]$, provided t is free in Y

not very imp.

A5) $\forall x (Y \rightarrow Z) \rightarrow (Y \rightarrow \forall x Z)$, provided x is not free in Y

A6) $t \approx t$

A7) $(y \approx t) \rightarrow (X[x/t] \rightarrow X[x/t])$, provided y, t are free in X.

MP: $\dfrac{X, \; X \rightarrow Y}{Y}$

UG (Universal generalization) : $\dfrac{X}{\forall x X}$

Theorem → $\boxed{(X \to Y) \to (\neg Y \to \neg X)}$

Q- Using axiom schemes of FC prove the following:

1) $\forall y Z \vdash \forall x Z[y/x]$, given that x ~~doesnt~~ dosen't occur in Z.

→

1. $\forall y Z$ (premise)
2. $\forall y Z \to Z[y/x]$ (by A_4)
3. $Z[y/x]$ (by MP)
4. $\forall x Z[y/x]$ (by UG)

2) $\forall x \forall y Z \vdash \forall y \forall x Z$

→

1. $\forall x \boxed{\forall y Z}$ (premise)
2. $\forall x (\forall y Z) \to \forall y Z$ (A_4) ∵ Here free condition not given so use, $\boxed{\forall x Y \Rightarrow Y}$
3. $\forall y Z$ (by M.P.)
4. $\forall y Z \to Z$ (A_4)
5. Z (by M.P.)
6. $\forall x Z$ [by UG]
7. $\forall y \forall x Z$ [by UG]

3) $\forall x \neg Y \vdash \neg \forall x Y$

→ 1. $\forall x \neg Y$ (premise)

2. $\forall x \neg Y \to \neg Y$ (by A_4)
3. $\neg Y$ (by M.P.)
4. $\forall x Y \to Y$ [by A_4]
5. $(\forall x Y \to Y) \to (\neg Y \to \neg \forall x Y)$ (Theorem)

Here, we directly introduce law & use for this Q.

6. $\neg Y \to \neg\forall x Y$ (MP 4,5)

7. $\neg\forall x Y$ (by M.P. 3,6)

V. Imp.

4) $\vdash \forall x(X \to Y) \to (\forall x \neg Y \to \forall x \neg X)$

If $\vdash X \to Y$ is there then replace by $X \vdash Y$

~~$\forall x(X \to Y) \vdash (\forall x \neg Y \to \forall x \neg X)$~~

~~$\vdash \forall x(X \to Y)$ (Premise)~~

1. $\vdash \forall x(X \to Y) \to (\forall x \neg Y \to \forall x \neg X)$
2. $\forall x(X \to Y) \vdash (\forall x \neg Y \to \forall x \neg x)$
3. $\forall x(X \to Y), \forall x \neg Y \vdash \forall x \neg X$
4. $\forall x(X \to Y)$ (premise)
5. $\forall x(X \to Y) \to (X \to Y)$ (by A4)
6. $(X \to Y)$ (by MP)
7. $(X \to Y) \to (\neg Y \to \neg X)$ (Theorem)
8. $\neg Y \to \neg X$ (by MP)
9. $\forall x \neg Y$ (Premise)
10. $\forall x \neg Y \to \neg Y$ (by A4)
11. $\neg Y$ (by MP 8,11)
12. $\neg X$ (by MP)
13. $\forall x \neg X$ (by UG)

Imp

5) $\vdash \neg(\forall x X \to Y) \to \forall x \neg(X \to Y)$, x is not free in Y.

6) $\vdash \forall x((x \approx f(Y)) \to Qx) \to Q f(y)$

6) $\rightarrow$ 1. $\vdash \forall x((x \approx f(y)) \rightarrow \mathcal{Q}x) \rightarrow \mathcal{Q}f(y)$

2. $\forall x((x \approx f(y)) \rightarrow \mathcal{Q}x) \vdash \mathcal{Q}f(y)$

3. $\forall x((x \approx f(y)) \rightarrow \mathcal{Q}x)$ (Premise)

4. $\forall x((x \approx f(y)) \rightarrow \mathcal{Q}x) \rightarrow$

$(\underbrace{f(y)}_{\text{replace } x = f(y)} \approx f(y) \rightarrow \mathcal{Q}f(y))$ (by A4)

5. $(f(y) \approx f(y)) \rightarrow \mathcal{Q}f(y)$ (by MP)

$f(y) \approx f(y)$

6. $f(y) \approx f(y)$ (A6)

7. $\mathcal{Q}f(y)$ (by MP)

~~Q-Show that $\vdash \forall x((x \approx f(y)) \rightarrow \mathcal{Q}x) \rightarrow \mathcal{Q}f(y)$~~

~~Proof:- 1) $\vdash \forall x((x \approx f(y)) \rightarrow \mathcal{Q}x) \rightarrow \mathcal{Q}f(y)$~~

~~2) $\forall x((x \approx f(y)) \rightarrow \mathcal{Q}x$~~

* Inconsistency of a formula

$\rightarrow$ A set | collection of formulas Σ is said to be inconsistent in first order logic (FL) iff their exist a formula Ψ such that $\Sigma \vdash \Psi$ and $\Sigma \vdash \neg\Psi$.

(i) $\Sigma \vdash \Psi$ iff collection of $\neg\Psi$ is inconsistent.

(ii) $\Sigma \vdash \neg\Psi$ iff collection of Ψ is inconsistent.

$\boxed{\text{If } X \text{ is given, } X \to (\neg Y \to \neg(X \to Y)) \text{ (Theorem)}}$

Q- If x is not free in Y, then show that $\vdash \neg(\forall x \, X \to Y) \to \forall x \, \neg(X \to Y)$

<u>solⁿ</u> $\vdash \neg(\forall x \, X \to Y) \to \forall x \, \neg(X \to Y)$

iff $\neg(\forall x \, X \to Y) \vdash \underline{\forall x \neg(X \to Y)}$

Y

iff $\{\underline{\neg(\forall x \, X \to Y)}, \underline{\neg\forall x \, \neg(X \to Y)}\}$ is inconsistent

so consider this in r.h.s of $\vdash$ without $\neg$ — considered before (Y)

iff $\neg\forall x \, \neg(X \to Y) \vdash (\forall x \, X \to Y)$

iff $\underline{\neg\forall x \, \neg(X \to Y), \forall x \, X} \vdash Y$

Σ

iff $\{\neg\forall x \, \neg(X \to Y), \forall x \, X, \neg Y\}$ is inconsistent

$\Sigma \vdash Y \Rightarrow \neg Y$ in collection.

iff $\forall x \, X, \neg Y \vdash \forall x \, \neg(X \to Y)$

1. $\forall x \, X$ (premise)
2. $\forall x \, X \to X$ (A4)
3. X (M.P)
4. $X \to (\neg Y \to \neg(X \to Y))$ (Theorem)
5. $\neg Y \to \neg(X \to Y)$ (M.P)
6. $\neg Y$ (premise)
7. $\neg(X \to Y)$ (M.P)
8. $\forall x \, \neg(X \to Y)$ (UG)

* Adequacy and compactness

(i) $\Sigma \vdash X$ iff $\Sigma \vdash_{FC} X$

(ii) Σ is satisfiable iff Σ is consistent

let Σ be an non-empty set of formula's and X be any other formula. Then,

(i) If $\Sigma \vdash X$, then Σ has a finite subset Δ such that $\Delta \vdash X$.

(ii) If Σ is unsatisfiable then Σ has a finite subset Δ which is unsatisfiable

(iii) If all finite non-empty subsets of Σ are satisfiable, then Σ is satisfiable.

Some inference rules.

(i) $\forall e$: $\dfrac{\forall x X}{X[x/t]}$

(ii) $\exists e$: $\dfrac{\exists x X \quad c}{X[x/c]}$

(iii) $\forall i$: $\dfrac{\begin{matrix} y \\ X[x/y] \end{matrix}}{\forall x X}$ or $\dfrac{\begin{matrix} t \\ X[x/t] \end{matrix}}{\forall x X}$

(iv) $\exists i$: $\dfrac{X[x/t]}{\exists x X}$

Rule :- $\dfrac{a \approx b, \; X[x/a]}{X[x/b]}$ or $\dfrac{a \approx b, \; X[x/b]}{X[x/a]}$

Q-1) show that $\{\forall x (P_{xy} \to Q_x), \forall z P_{zy}\} \vdash \forall x Q_x$

Proof :- ① $\forall x (P_{xy} \to Q_x)$ (Premise)
② $\forall z P_{zy}$ (Premise)
③ $P_{ty} \to Q_t$ ($\forall e$, 1)
④ P_{ty} ($\forall e$, 2)
⑤ Q_t (MP 3,4)
⑥ $\forall x Q_x$ ($\forall i$, 5)

Q-2) Show that $\forall x (P_{xy} \to Q_x)$ $\exists z P_{zy} \vdash \exists x Q_x$

Proof :- 1) $\forall x (P_{xy} \to Q_x)$ (Premise)
2) $\exists z P_{zy}$ (Premise)
3) $P_{ty} \to Q_t$ ($\forall e$, 1)
4) P_{ty} ($\exists e$, 2)
5) Q_t (MP 3,4)
6) $\exists x Q_x$ ($\exists i$, 5)

8-3) show that $\vdash \forall x (X \to Y) \to (X \to \forall x Y)$ (x is not free in X)

Proof:- 1) $\forall x (X \to Y)$ (CP)

2) X (CP)

3) $X \to Y[x/y]$ ($\forall e$, 1)

4) $Y[x/y]$ (MP 2, 3)

5) $\forall x Y$ ($\forall i$, 4)

6) $X \to \forall x Y$ ($\to i$)

7) $\forall x (X \to Y) \to (X \to \forall x Y)$ ($\to i$, 1, 6)

8-4) show that $\{Pa, \forall x (Px \to Qx), \forall x (Rx \to \neg Qx), Rb\} \vdash \neg(a \approx b)$.

Proof:- 1) $\forall x (Px \to Qx)$

2) $Pa \to Qa$ ($\forall e$, 1)

3) Pa (Premise)

4) Qa (MP 2, 3)

5) Rb (Premise)

6) $\forall x (Rx \to \neg Qx)$ (Premise)

7) $Rb \to \neg Qb$ ($\forall e$, 6)

8) $\neg Qb$ (MP 5, 7)

9) $a \approx b$ (CP)

10) $\neg Q a$ (Since $\dfrac{a \approx b, X[x/b]}{X[x/a]}$)

11) $\bot$ ($\bot$ 4, 10)

12) $\neg(a \approx b)$ ($\neg i$)

$\bot$ appears means our assumption is wrong, $\therefore$ $a \approx b$ (CP) is wrong, so right one is $\neg(a \approx b)$

Rule

$$\frac{a \approx b, X[x/a]}{X[x/b]}$$

can be inter-changed

Q5) Show that $\forall x(Lx \rightarrow Fx) \vdash \forall x(\exists y(Ly \wedge Qxy) \rightarrow \exists y(Fy \wedge Qxy))$

Proof:-

1) $\forall x(Lx \rightarrow Fx)$
2) $Lc \rightarrow Fc$ $(\forall e, 1)$
3) $\exists y(Ly \wedge Qxy)$ (CP)
4) $Lc \wedge Qxc$ $(\exists e, 3)$

~~5) Lc (MP 2, 4)~~

5) Lc $(\wedge e_1, 4)$
6) Fc (MP 2, 5)
7) Qxc $(\wedge e_2, 4)$
8) $Fc \wedge Qxc$ $(\wedge i\ 6, 7)$
9) $\exists y(Fy \wedge Qxy)$ $(\exists i, 8)$
10) $\exists y(Ly \wedge Qxy) \rightarrow \exists y(Fy \wedge Qxy)$ $(\rightarrow i\ 3, 9)$
11) $\forall x(\exists y(Ly \wedge Qxy) \rightarrow \exists y(Fy \wedge Qxy))$ $(\forall i, 10)$