Unit-5

Topic: Modular logic

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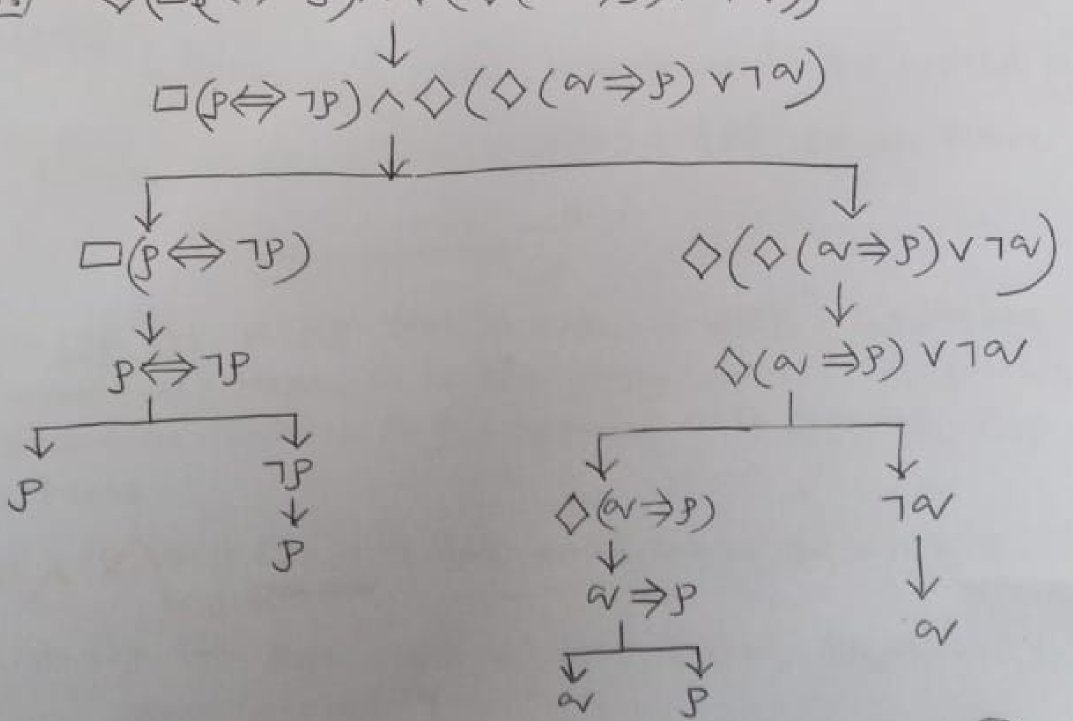
#Introduction: A modal is an expression that is used to quantify the truth of a judgement. The modal logic (or K logic) is an extension of propositional logic and it has two more connectives, namely – $\Box$ (necessity) and $\Diamond$ (possibility). Thus K logic has the connectives $\neg, \Box, \Diamond, \wedge, \vee, \Rightarrow, \Leftrightarrow$. The propositional constants $\top$ (top), $\bot$ (bottom) and the propositional variables (p_0, p_1, p_2, p, q, r etc) together are known as atomic modal propositions or atomic modal formulas. We use the abbreviation $\boxed{\text{'mp'}}$ for a modal proposition.

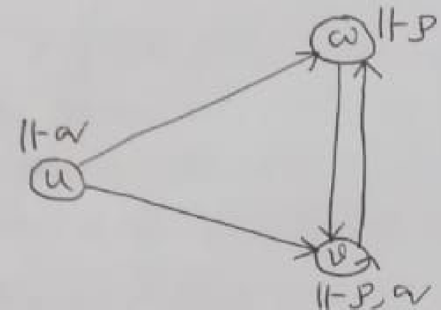
Q) Obtain a parse tree for $\Diamond(\Box(p \Leftrightarrow \neg p) \wedge \Diamond(\Diamond(q \to p) \vee \neg q))$.

Solution: $\Diamond(\Box(p \Leftrightarrow \neg p) \wedge \Diamond(\Diamond(q \Rightarrow p) \vee \neg q))$



(a) Let $M = (\{u, v, w\}, \{(u,v), (u,w), (v,v), (v,w), (w,v)\}, \phi)$ be the model with $\phi(u) = \{q\}$, $\phi(v) = \{p, q\}$ and $\phi(w) = \{p\}$. Determine whether $M \Vdash \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ and $M \vdash \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

<u>Solution:</u> Here $W = \{u, v, w\}$ and
$R = \{(u,v), (u,w), (v,v), (v,w), (w,v)\}$.
Therefore the model M can be represented by:



The worlds accessible from u are v and w and both $v \Vdash p$ and $w \Vdash p$ hold. Hence $u \Vdash \Box p$. But $u \nVdash \Box q$ as the world w is accessible from u but $w \nVdash q$.

Next, the worlds accessible from v are v and w. Both $v \Vdash p$ and $w \Vdash p$. So $v \Vdash \Box p$. But $v \nVdash \Box q$ as w is accessible from v but $w \nVdash p$.

Finally, the only world accessible from w is v and $v \Vdash p$ and $v \Vdash q$. So $w \Vdash \Box p$ and $w \Vdash \Box q$. Since w is accessible from u, $u \Vdash \Box(p \wedge q)$. But $w \nVdash p \wedge q$. Further $v \nVdash \Box(p \wedge q)$ since $w \nVdash p \wedge q$ and $w \Vdash \Box(p \wedge q)$ as $v \Vdash p \wedge q$.

All these can be summarized below.
($r : \Box(p \wedge q)$, $t : \Box p \wedge \Box q$)

	p	q	$p \wedge q$	$\Box p$	$\Box q$	r	t	$r \rightarrow t$	$t \rightarrow r$
u	$\nVdash$	$\Vdash$	$\nVdash$	$\Vdash$	$\nVdash$	$\nVdash$	$\nVdash$	$\Vdash$	$\Vdash$
v	$\Vdash$	$\Vdash$	$\Vdash$	$\Vdash$	$\nVdash$	$\nVdash$	$\nVdash$	$\Vdash$	$\Vdash$
w	$\Vdash$	$\nVdash$	$\nVdash$	$\Vdash$	$\Vdash$	$\Vdash$	$\Vdash$	$\Vdash$	$\Vdash$

From the last two columns we have,
$M \vdash \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ and $M \vdash \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

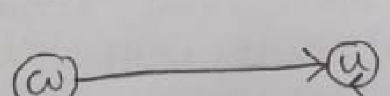
Accessibility relation:

A frame is an ordered pair (W, R), where W is a set of worlds (interpretations), and R is a binary relation on W, called the accessibility relation. For worlds $u, w \in W$, 'wRu' can be read as 'u is accessible from w'.

Model: A model of K is a triple $(W, R, \emptyset)$, where the pair (W, R) is a frame and $\emptyset$ is a mapping from W to the power set associating each world $w \in W$ to a subset $\emptyset(w)$ of atomic propositions.

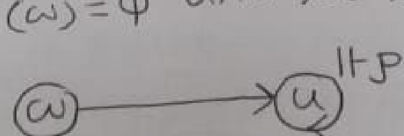
(a) Let $M = (W, R, \emptyset)$ be the model with $W = \{u, w\}$, $R = \{(w,u), (u,u)\}$, $\emptyset(w) = \phi$ (empty set) and $\emptyset(u) = \{p\}$ for atomic p. Then check whether the following mps are true at the world w.
(i) $\Box p$, (ii) $\Box\Box p$, (iii) $\Box p \rightarrow \Box\Box p$, (iv) $\Box\Box p \cdot \rightarrow p$.

Solution: Since $R = \{(w,u), (u,u)\}$, so w is accessible from u and u is accessible from u itself. So we have the following graph:



[Remember: $u \Vdash p$ means that p is true in the world u]

Since $\emptyset(w) = \phi$ and $\emptyset(u) = \{p\}$, so we have



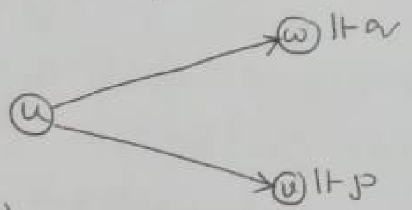
(i) $w \Vdash \Box p$ iff $x \Vdash p$ for each world x accessible from w. Here, u is the only world accessible from w and $u \Vdash p$. Hence, $w \Vdash \Box p$ i.e; $\Box p$ is true.

[Remark: (A) $w \Vdash \Box x$ iff for each $u \in W$, $(w,u) \in R$ and $u \Vdash x$.

(B) $w \Vdash \Diamond x$ iff for some world $u \in W$, $(w,u) \in R$ and $u \Vdash x$.]

Q. Determine whether the following model M satisfies the mps.

M:



(a) $\Box(p \vee q) \rightarrow \Box p \vee \Box q$ (b) $\Box(p \wedge q) \longrightarrow \Box p \wedge \Box q$.

<u>Solution</u>: Here $W = \{u, v, w\}$,
$\emptyset(u) =$ null set, $\emptyset(v) = \{p\}$, $\emptyset(w) = \{q\}$.

(a) Here $u \Vdash \Box(p \vee q)$ as both the worlds w, v accessible from u satisfy $p \vee q$. But $u \nVdash \Box p$ as w is accessible from u, but $w \nVdash p$. Again, $u \nVdash \Box q$ as v is accessible from u, but $v \nVdash q$. Therefore, $u \nVdash (\Box p \vee \Box q)$ and conseqeuntly, $M \nVdash \Box(p \vee q) \rightarrow \Box p \vee \Box q$.

(b) $u \nVdash \Box p$, $u \nVdash \Box q$ but $v \Vdash \Box p$, $v \Vdash \Box q$ vacuously as there are no world accessible from v. Also, $w \Vdash \Box p$ and $w \Vdash \Box q$. Thus $u \nVdash \Box p \wedge \Box q$, $v \Vdash \Box p \wedge \Box q$, and $w \Vdash \Box p \wedge \Box q$. Since v is a world accessible from u, and $v \nVdash p \wedge q$, $u \nVdash \Box(p \wedge q)$. Again vacuously $v \Vdash \Box(p \wedge q)$ and $w \Vdash \Box(p \wedge q)$. So, all the worlds satisfy $\Box(p \wedge q) \rightarrow \Box p \wedge \Box q$. Hence, $M \Vdash \Box(p \wedge q) \rightarrow \Box p \wedge \Box q$.

— x —

Axiomatic system for modal logic (K-calculus)

(A1) $A \to (B \to A)$

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(A3) $(\neg A \to \neg B) \longrightarrow ((\neg A \to B) \to A)$

(K) $\Box(X \to Y) \longrightarrow (\Box X \to \Box Y)$

(MP)

$$\frac{\begin{array}{c} X \\ X \longrightarrow Y \end{array}}{\therefore\ Y}$$

(N) $$\frac{X}{\therefore \Box X}$$

(R) $$\frac{X \longrightarrow Y}{\therefore \Box X \longrightarrow \Box Y}$$

Q. Show that $\vdash \Box(p \wedge q) \longrightarrow (\Box p \wedge \Box q)$.

Proof:

1. $p \wedge q \longrightarrow p$ (propositional calculus)
2. $\Box(p \wedge q) \to \Box p$ (R)
3. $p \wedge q \longrightarrow q$ (propositional calculus)
4. $\Box(p \wedge q) \to \Box q$ (R)
5. $(\Box(p \wedge q) \to \Box p) \to ((\Box(p \wedge q) \to \Box q) \to (\Box(p \wedge q) \longrightarrow (\Box p \wedge \Box q)))$ (propositional calculus)
6. $(\Box(p \wedge q) \to \Box q) \to (\Box(p \wedge q) \to (\Box p \wedge \Box q))$ (MP 2, 5)
7. $\Box(p \wedge q) \longrightarrow (\Box p \wedge \Box q)$ (MP 4, 6)

Q. Show that $\vdash (\Box p \wedge \Box q) \longrightarrow \Box(p \wedge q)$.

Proof:

1. $p \to (q \to (p \wedge q))$ (propositional calculus)
2. $\Box p \to \Box(q \to (p \wedge q))$ (R)
3. $\Box(q \to (p \wedge q)) \to (\Box q \to \Box(p \wedge q))$ (K)
4. $\Box p \longrightarrow (\Box q \longrightarrow \Box(p \wedge q))$ (HS 2, 3)
5. $(\Box p \wedge \Box q) \to \Box(p \wedge q)$.

(ii) $w \Vdash \Box\Box p$ iff $x \Vdash \Box p$ for each world x accessible from w. Again, $x \Vdash \Box p$ iff $y \Vdash p$ for each world y accessible from x. Since $R = \{(w,u),(u,u)\}$, so $w \Vdash \Box\Box p$ iff $u \Vdash \Box p$ iff $u \Vdash p$. As $u \Vdash p$ (see the 2nd diagram), so $w \Vdash \Box\Box p$ i.e; $\Box\Box p$ is true.

(iii) Since $w \Vdash \Box^m p \to \Box^n p$ $(m,n \in N)$ and $w \Vdash p \to p$, so in particular $(m=1, n=2)$, we have, $w \Vdash \Box p \to \Box^2 p$ i.e; $w \Vdash \Box p \to \Box\Box p$. Hence $\Box p \to \Box\Box p$ is true.

(iv) Since $w \nVdash \Box p \to p$, so $w \nVdash \Box(\Box p) \to p$ i.e; $w \nVdash \Box\Box p \to p$. Hence $\Box\Box p \to p$ is false.

(Q) Which of the following mps are true at the world w in M? $\quad M: \; (w) \longrightarrow (u) \Vdash p$

(a) $\Box p \to \Box\Diamond p$, (b) $\Diamond p \to \Box\Diamond p$, (c) $\Box p \to \Diamond\Box p$, (d) $\Diamond p \to \Diamond\Box p$

$\boxed{\text{Solution:}}$ Clearly, $R = \{(w,u)\}$.

(a) $w \Vdash \Box p \to \Box\Diamond p$ iff $w \nVdash \Box p$ or $w \Vdash \Box\Diamond p$. Now $w \nVdash \Box p$ iff for some world accessible from w, p is false at that world. Here, there is only one world, namely u which is accessible from w and $u \Vdash p$. Hence $w \Vdash \Box p$. On the other hand, $w \Vdash \Box\Diamond p$ iff $u \Vdash \Diamond p$. But there is no world accessible from u. Thus, $u \nVdash \Diamond p$ and so $w \nVdash \Box\Diamond p$. Hence, $w \nVdash \Box p \to \Box\Diamond p$. So $\Box p \to \Box\Diamond p$ is false.

(b) $w \Vdash \Diamond p$ $(\because (w,u) \in R, u \Vdash p)$. From (a) it follows that $w \nVdash \Box\Diamond p$. Hence $w \nVdash \Diamond p \to \Box\Diamond p$.

(c) $\because (w,u) \in R$ and $u \Vdash p$, so $w \Vdash \Box p$. ATQ, each world accessible from u satisfies p. But there is not world accessible from u which violates $w \Vdash \Diamond\Box p$. Hence $\Box p \to \Diamond\Box p$ is true.

(d) Similar to (c).