Project 1

LF CH

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$$
\begin{aligned}
u_{tt} = c^2 D_2 u \qquad & t \geq 0, \\
Lw = 0 \qquad & t \geq 0, \\
u = f, \quad u_t = 0 \quad & t = 0,
\end{aligned}
\tag{1}
$$

$$
\begin{array}{r}
u^T H u_{tt} = c^2 u^T H P D_2 P u = c^2 P^T H P D_2 P U = c^2 (Pu)^T H D_2 P u = \\
c^2 (Pu)^T H H^{-1} (-M + BD) P u = c^2 (Pu)^T (-M + BD) P u = \\
-c^2 (Pu)^T M P u + c^2 (Pu)^T B D P u \\
+ u_{tt}^T H u = -c^2 (Pu)^T M^T P u + c^2 (Pu)^T B D^T p u \\
\hline
= u^T H u_{tt} + u_{tt}^T H u = -c^2 (Pu)^T (M + M^T) P u + c^2 (Pu)^T (BD + BD^T) P u \\
= \dfrac{d}{dt} \|u_t\|_H^2 = -2c^2 (Pu)^T M P u + c^2 (Pu)^2 B D P u + c^2 B D^T P u
\end{array}
$$

$B = e_m d_m - e_1 d_1$ and let $w = PV$.

$$
\begin{aligned}
\frac{d}{dt} \|v_t\|_H^2 = -2c^2 w^T M w + c^2 w^T (e_m d_m - e_1 d_1) w + c^2 w^T (e_m d_m^T - e_1 d_1^T) w = \\
-2c^2 w^T M w + c^2 w^T e_m d_m w - c^2 w^T e_1 d_1 w + c^2 w^T e_m d_m^T w - c^2 w^T e_1 d_1^T w
\end{aligned}
$$

1 The system

When introducing u_t to the boundary operator we cant use the previous form of semi-discrete approximation. We therefore introducing the variable substitution $v = u_t$. By creating a system of first order equations we can solve it with RK4.

$$
\begin{aligned}
\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ c^2 D_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
L \begin{bmatrix} u \\ v \end{bmatrix} = 0 \\
\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}
\end{aligned}
\tag{2}
$$

We introduce $e^{(1)}$, $e^{(2)}$ and the Kronecker-operator to form the boundary operator

$$
e^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\tag{3}
$$

$$L \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha_l(e^{(2)} \otimes e_1) + \beta_l(e^{(1)} \otimes e_1) + \gamma_l i(e^{(1)} \otimes d_1) \\ \alpha_r(e^{(2)} \otimes e_m) + \beta_r(e^{(1)} \otimes e_m) + \gamma_r(e^{(1)} \otimes d_m) \end{bmatrix} \tag{4}$$

Apply the projection and energy method to verify stability. We multiply with $\begin{bmatrix} u \\ v \end{bmatrix}^T \overline{H}$ and let: $\overline{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} H$

$$\begin{bmatrix} u \\ v \end{bmatrix}^T \overline{H} \begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} u \\ v \end{bmatrix}^T \overline{H} P \begin{bmatrix} 0 & 1 \\ c^2 D_1 & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix} = \left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \overline{H} \begin{bmatrix} 0 & 1 \\ c^2 D_2 & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix} =$$

$$\left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \begin{bmatrix} 0 & H \\ c^2 D_2 H & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix} = \left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \begin{bmatrix} 0 & H \\ c^2 H H^{-1}(-M + BD) & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \begin{bmatrix} 0 & H \\ c^2(-M + BD) & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix}$$

$$+ \left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \begin{bmatrix} 0 & H \\ c^2(-M + BD^T) & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix} 2$$

$$\frac{d}{dt} \| \begin{bmatrix} u \\ v \end{bmatrix} \|_H = \left(P \begin{bmatrix} u \\ v \end{bmatrix} \right)^T \begin{bmatrix} 0 & H \\ c^2(-2M - BD - BD^T) & 0 \end{bmatrix} P \begin{bmatrix} u \\ v \end{bmatrix} =$$

$$PHuPv + c^2 Pv(-2M - BD - BD^T)Pu =$$

$$PHuPv - c^2 Pv2Mpu + Pc^2 vBDPu + pc^2 vBDPu =$$

$$PHuPv - c^2 Pv2MPu + Pc^2 v(e_m d_m - e_1 d_1)Pu + c^2 Pv(e_m d_m^T - e_1 d_1^T)Pu =$$

$$PHuPv - c^2 Pv2MPu + Pc^2 ve_m d_m Pu - Pc^2 ve_1 d_1 Pu + Pc^2 ve_m d_m^T Pu - Pc^2 ve_1 d_1^T Pu$$

Using that $d_1 u = \frac{1}{c} e_1^T v$ and $d_m = -\frac{1}{c} e_m^T v$

$$PHuvP - c^2 Pv2MuP + cPve_m e_m^T vP - cPve_1 e_1^T vP + c^2 Pve_m d_m^T uP - c^2 Pve_1 d_1^T uP =$$