Linear Algebra[1]

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Contents

Preface

This document compiles all my notes on linear algebra including any self-study (e.g. advanced applications in modern physics). This also serves as a foundation for the other analysis topics covered in my notes series.

For most engineers, linear algebra is simply the study of matrices and vectors in $\mathbb{R}^n$. For a mathematician, linear algebra is the study of vector spaces and linear transformations. This definition means linear algebra is more abstract than vectors describing physical quantities or systems of linear equations. They can be functions for example, and thinking of functions in a vector space is surprisingly useful for advanced differential equations. In modern or quantum physics, all the behaviors of particles can be modeled using complex vector spaces (see sections on Hilbert spaces and complex vector spaces). Thus, there is a strong motivation in science to understand linear algebra hollistically.

If you need this for school and industry, I hope that you are able to do whatever you are trying to accomplish. If you are here because this interests you, I hope you find this as entertaining as I did.

- Christopher

Chapter 1

Vectors and Vector Spaces

Definition (Vector)*:*

Chapter 2

Matricies

2.1 Defining Matrices

Let n, m be two integers ≥ 1. A **matrix** is an array of numbers with m rows and n columns (called a $m \times n$ matrix).

We call a_{ij} the **ij-entry** which is the entry in the ith row and the jth column. We write a matrix often as $A = (a_{ij})$ and define a_{ij}.

Each column of an $m \times n$ matrix is a **column vector**. Each row of an $m \times n$ matrix is a **row vector**.

Example (Identity Matrix)*:*
The Kronecker delta is defined as follows:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 2, & i \neq j \end{cases}$$

Then we define the **identity matrix** as:

$$I_n = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example:
If we have a matrix, $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, the second column vector of A is $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and the second row vector of A is $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

We can describe matrices using their column or row vectors. For example:

$$A = \begin{bmatrix} \vec{r} \\ \vec{s} \\ \vec{t} \end{bmatrix}$$

where

$$\vec{r} = \begin{bmatrix} 3 & 4 \end{bmatrix}, \vec{s} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \vec{t} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

Or:

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

where

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

2.1.1 Matrix Operations

We treat matricies the same way as numbers. Let A be an $m \times n$ matrix and B be an $p \times q$ matrix.

- We can add A and B. If $m = p$ and $n = q$, then $A + B = (a_{ij} + b_{ij})$
- We can multiply by a scalar c: $c \cdot A = (c \cdot a_{ij})$
- We can multiply A and B. If $n = p$, then $A \cdot B = (c_{ij})$ where $c_{ij} = \vec{r_i}(A) \cdot \vec{c_i}(B)$. Caution: In general, $A \cdot B \neq B \cdot A$.
- We also define the transpose of a matrix. The transpose of A is $A^t = (d_{ij})$ where $d_{ij} = a_{ji}$. When we take a transpose, we switch the columns into rows and vice versa.

Certain special matrices can be described with other terminology. Suppose we have a matrix, $A = (a_{ij}), i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- If $m = n$, then A is a **square matrix**.
- If $A^t = A$, then A is a **symmetric matrix**. Note: this means A must also be square.
- If $A^t = -A$, then A is said to be **skew-symmetric**.
- If for all i, j such that $i \neq j$, $a_{ij} = 0$, then A is called **diagonal**.

Determinants

The determinant is a property of a square matrix, A.

Geomtric Def'n.

Let $A = \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (vectors in $\mathbb{R}^n$). Let Π = parallelotope defined by basing them all at the same point. Then $V^\sigma(\Pi) = \det \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (signed n volume).

Example:

$$A = \begin{bmatrix} \vec{c_1} & \vec{c_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\det A = -1$$

Algebraic Def'n.

Let $A = (a_{ij})$, $n \times n$. Let A_{ij} = submatrix obtained from A by eliminating row i and column j, $(n-1) \times (n-1)$ matrix.

The minor of A is $M_{ij} = \det A_{ij}$. The cofactor of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Then $\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ (for any i, $1 \le i \le n$). We call this the cofactor expansion along the ith row.

Example:

$$A = \begin{bmatrix} 7 & 3 & 12 \\ 2 & 5 & 8 \\ 1 & 5 & 2 \end{bmatrix}$$

Find the determinant.

Use row 3.

$$\det A = \sum_{j=1}^{3} a_{3j} C_{3j} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

2.1.2 Inverse Matricies

An $n \times n$ matrix A may or may not have an **inverse**: A matrix B such that

$$AB = BA = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We write $B = A^{-1}$.

For a linear system, $A\vec{x} = \vec{b}$ with A, $n \times n$, if A is invertible:

$$\underbrace{A^{-1}A}_{I_n} \vec{x} = A^{-1}\vec{b}$$
$$\therefore \text{ Sol'n. is } \vec{x} = A^{-1}\vec{b}$$

We can vary $\vec{b}$ and the solutions are immediate.

For the 2×2 case, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A has an inverse $\iff |A| = ad - bc \neq 0.$

Then

$$A^{-}1 = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$$

Invertible?

$18 - 35 = -17 \neq 0$ ✓

$$A^{-1} = \frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}$$

<u>Check</u>

$$\frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 & 0 \\ 0 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 1. *A $n \times n$ matrix A has an inverse $\iff rkA = n$. If A has an inverse, then A^{-1} is given by $rref[A\, I_n] = [I_n\, A^{-}1]$*

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Is it invertible?

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[2.\, A_{13}(-1)]{1.\, A_{12}(-2)} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[4.\, A_{31}(1)]{3.\, A_{32}(-1)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{5.\, A_{21}(-2)} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of A^{-1}

1. If A is invertible, so is A^{-1}, $\left(A^{-1}\right)^{-1} = A$.
2. If A, B are invertible, $n \times n$, then so is $A \cdot B$ and $(BA)^{-1} = B^{-1}A^{-1}$.

   Witness $A\left(B \cdot B^{-}1\right) A^{-1} = AI_nA^{-}1 = I_n$.
3. If A is invertible, so is A^t, $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$.

Theorem 2. *Suppose A is $n \times n$. Then the following are equivalent*

1. *A is invertible*
2. *$A\vec{x} = \vec{b}$ has a unique sol'n, $\forall \vec{b}$*
3. *$rk(A) = n$*
4. *$rref(A) = I_n$*
5. *$det(A) \neq 0$*

2.2 Matricies as Linear Transformations

Matricies can be used to model transformations of vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. This is accomplished by having an $m \times n$ matrix, A, written as:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

Example ($\mathbb{R}^n$ to $\mathbb{R}^m$)*:*
Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, $A : \mathbb{R}^2$ to $\mathbb{R}^3$ and:

$$A\vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$$

Note. We are getting a linear combination of the column vectors of A. In other words, $A\vec{v} = x\vec{a} + y\vec{b}$.

Chapter 3

Systems of Linear Equations

3.1 Systems of Linear Equations

Another application of matricies of great interest is representing systems of linear equations.

Definition:
An equation is **linear** in n variables x_1, x_2, ... , x_n if each term contains at most one x_i and that x_i appears to the first power.

A "system of m linear equations in n unknowns."

Example (Cutting n by $n-1$ dimensional objects)*:*
Suppose we have two planes which will separate regions in a three dimensional space.

$$3x + 2y - z = 1$$
$$2x + 2y + z = 0$$

Sol'n. $\{(x, y, z) : \text{satisfy all equations}\}$

The solution would be the intersection of these planes. Since they extend out infinitely, we would expect them to meet long a line contained in both planes.

Note. Multiplying by a constant, adding a multiple of one to another, and permuting the rows all do not change the solution.

Possible Outcomes: A system of m linear equations in n unknowns has either $0, 1, \text{or}\inf$ solutions.

The technique to solving systems of linear equations is to develop a faster, simple method using matrices. We represent the system as a matrix, preform a series of operations to obtain a simpler matrix, and get our solution from the simpler solution.

Example:

$$
\begin{aligned}
3x + 2y - z &= 1 \\
2x + 2y + z &= 0
\end{aligned}
$$

We take the coefficients and constants and put them into a matrix:

$$
\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}
\Bigg\downarrow \vec{r_1} \to \vec{r_1} - \vec{r_2}
\begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}
\Bigg\downarrow \vec{r_2} \to \vec{r_2} - 2\vec{r_1}
\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 5 & 2 \end{bmatrix}
\Bigg\downarrow \vec{r_2} \to \frac{1}{2}\vec{r_2}
\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{5}{2} & 1 \end{bmatrix}
$$

Thus, we find:

$$
\begin{aligned}
x - 2z &= 1 \\
y + \frac{5}{2}z &= -1
\end{aligned}
$$

Sol'n. $z = t$ as a free variable

$$
\begin{aligned}
x &= 1 + 2t \\
y &= -1 - \frac{5}{2}t
\end{aligned}
$$

It's a line as expected!

$$
L(t) = \left(1 + 2t, -1 - \frac{5}{2}t, t\right)
$$

3.2 Vector Form

Given:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The **vector form** of the system is: $A\vec{x} = b$ The **coefficient matrix** is $A = (a_{ij})$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

The **augmented matrix** is $A^{\#} = \begin{bmatrix} A \ \vec{b} \end{bmatrix}$.

Definition (Elementary Row Operations)*:*
Elementary row operations (EROs) are defined as follows:

- $P_{ij} : \vec{r_i} \leftrightarrow \vec{r_j}$
- $M_i(c) : \vec{r_i} \leftarrow c\vec{r_i}$
- $A_{ij}(c) : \vec{r_j} \leftarrow \vec{r_j} + c\vec{r_i}$

These elementary row operations let us solve systems of linear equations in the same way we learned in early math classes. Adding equations together is the same as the row sum operation etc. Using EROs, the process to solve these systems is to write the augmented matrix of the system, perform EROs to get a reduced matrix and obtain the simpler linear system.

Example:
Solve if possible.

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 5 \\ x - y - 2z &= -5 \end{aligned}$$

We write the augmented matrix of the system and preform EROs:

$$A^{\#} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix} \xrightarrow[2.\ A_{13}(-1)]{1.\ A_{12}(-2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

$$\xrightarrow{3.\ A_{23}(2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \text{ref}\, A^{\#}$$

$$\xrightarrow[6.\ A_{31}(-1)]{5.\ A_{32}(1)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{7.\ A_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \text{rref}\, A^{\#}$$

We obtain two matrices, the row echelon form and the reduced row echlon form. From ref $A^{\#}$, we find:

$$\begin{aligned} x + y + z &= 3 \\ y - z &= -1 \\ z &= 2 \end{aligned}$$

From rref $A^{\#}$, we find:

$$\begin{aligned} x &= 0 \\ y &= 1 \\ z &= 2 \end{aligned}$$

$$\boxed{(0,\ 1,\ 2)}$$

Thus, (0, 1, 2) solves the system of equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -5 \end{bmatrix}$$

Note. If the number of leading 1's equals the number of unknowns, we obtain one solution.

Definition (Echelon Forms)*:*
The **row echelon form** (ref) of any matrix is one such that:

(a) The first nonzero entry of each nonzero row is 1. ("Leading one")

(b) If a row has a leading one, each row above it has a leading 1 further to the left.

The **reduced row echlon form** (rref) of any matrix also has the conditon:

(c) Each entry above a leading 1 is 0.

To put a matrix into its ref or rref form:

1. Use P_{ij} so row i has the left most nonzero entry among rows $\geq i$.
2. Use $M_i(k)$ to create a leading 1 in $\vec{r_i}$.
3. Use $A_{ij}(k)$ to zero out column below leading 1 in $\vec{r_i}$. For rref, zero out columns above as well.
4. If there is no nonzero row below i, stop.

   Else:
5. Replace $i = i + 1$, repeat on submatrix with rows $\geq i + 1$.

3.3 Theorem of Rank

Definition (Rank)*:*
The number of leading 1's in ref(A) is the rank of A, usually written as "rk(A)."

Theorem 3. *Let A, $A^{\#}$ be the coefficient and augmented matrix for a system of linear equations with m equations and n unknowns.*

Then, there are three cases:

1. *No solutions if $\text{rk}\, A < \text{rk}\, A^{\#}$.*
2. *Unique solutions if $\text{rk}\, A = \text{rk}\, A^{\#} = n$.*
3. *Infinitely many solutions if $\text{rk}\, A = \text{rk}\, A^{\#} < n$.*

Note. There are unique solutions if exactly all variables are bound and infinitely many if there are $n - \text{rk}A$ free variables.

Example:
Here is an illustration of this. Suppose we have:

$$\text{rref}A^{\#} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{matrix} \\ \begin{bmatrix} 1 & * & 0 & * & 0 & a \\ 0 & 0 & 1 & * & 0 & b \\ 0 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 0 & d \end{bmatrix} \end{array}$$

If $d \neq 0$, there are no solutions. If $d = 0$, there are infinitely many solutions. There are no unique solutions possible since there are two free variables: $x_2 = s$ and $x_4 = t$. The bound variables are x_1, x_2, and x_5.

From this illustration, we can see two important cases:

1. If we have a homogenous system, $A\vec{x} = 0$, then we always have the trivial solution $\vec{x} = 0$. There exists nontrivial solutions if and only if $\text{rk}(A) < n$.

2. If A is $m \times n$ and $m < n$, then since $\operatorname{rk}(A) \leq m,\ \operatorname{rk}(A) < n$.
   Therefore, there are either 0 or infinitely many solutions.

Example:

Fix $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{w}$ (all) such that $\vec{v} \times \vec{w} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\vec{v} \times \vec{w} = \begin{bmatrix} w_3 - w_2 \\ w_1 - w_3 \\ w_2 - w_1 \end{bmatrix}$$

Now we have a system of linear equations with $\vec{w}$ as the unknown.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^{\#} = \begin{bmatrix} 0 & -1 & 1 & b_1 \\ 1 & 0 & -1 & b_2 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow{1.\ P_{12}} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & -1 & 1 & b_1 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow[3.\ M_2(-1)]{2.\ A_{13}(1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 1 & -1 & b_2 + b_3 \end{bmatrix}$$
$$\xrightarrow{4.\ A_{23}(-1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{bmatrix}$$

We have solutions $\iff b_1 + b_2 + b_3 = 0 = \vec{v} \cdot \vec{b}$. Then we have one free variable

$$\left.\begin{aligned} w_3 &= t \\ w_2 &= -b_1 + t \\ w_1 &= b_2 + t \end{aligned}\right\} = \boxed{(b_2 + t,\ -b_1 + t,\ t)}$$