0.1 Introduction to LODEs

Definition (Linear Ordinary Differential Equation (LODE) of Order n)**:**
A LODE of order n is an equation:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

where $y^{(i)}$ is the ith derivative.

Example (Hooke's Law)*:*
Hooke's Law states the force exerted by a spring is proportional to its displacement from equilibrium:

$$F = -ky$$

This applies when displacement is small compared to the total range of the spring. (Fun Fact: Hooke stated this as an anagram riddle before officially publishing his discovery.)

From Newton,

$$F = m\frac{d^2y}{dt^2}$$
$$\therefore \text{ By substitution, } y^{(2)} + \omega^2 y = 0$$
$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

This describes the system. The solution is some $y(t)$: what actually happens. Finding $y(t)$ is our goal.

Check: $y = c_0 \cos \omega t + c_1 \sin \omega t$ is a solution where c_0, c_1 are constants.

$$\begin{aligned} y^{(1)} &= -c_0\omega \sin \omega t + c_1\omega \cos \omega t \\ y^{(2)} &= -c_0\omega^2 \cos \omega t - c_1\omega^2 \sin \omega t \\ &= -\omega y \end{aligned}$$

Simple harmonic motion!

Definition (General Solution of a LODE)**:**
A general solution of a LODE is a solution of the form:

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

where the y_i are independent solutions and c_j are unknowns.

Note. Any solution is some specialization of the c_j's

Example (Newton's Law of Cooling)*:*
Newton's Law of Cooling is as follows:

$$\frac{dT}{dt} = -k(T - T_m)$$

where T = temperature at time t, t = time, k = constant, and T_m = ambient temp

Check: $T = T_m + ce^{-kt}$ is a solution. Witness:

$$\begin{aligned} \frac{dT}{dt} &= -kce^{-kt} \\ &= -k(T - T_m) \end{aligned}$$

0.2 Other Examples of Differential Equations

Here are additional examples of differential equations explored more throughly.

0.2.1 Orthogonal Trajectory Problem

Given a function $f(x, y)$, there is a family of level curves $f(x, y) = f(p)$, $p = (a, b)$. We write the gradient of $f(x, y)$ as $< f_x, f_y >$.

Note. The gradient has the following important properties:

- The direction is in the greatest rate of change of f
- $\vec{\nabla} f \perp f(x, y) = f(p)$

The problem is thus to find a curve that follows the gradient i.e. an **orthogonal trajectory curve**.

Potential Applications:

- Heat seaking missiles, f = temperature
- Charged particle, f = electric potential
- Bear at scout jamboree, f = peanut butter scent intensity

Theory: Let $g(x, y) = g(p)$ be an orthogonal trajectory curve (in xy-plane).

Goal: Find g.

Require: $\vec{\nabla} f(p) \perp \vec{\nabla} g(p)$

Then, it follows that $f_x g_x + f_y g_y = 0$

$\therefore$ The slope of D at p is the slope of the vector $< -g_y, g_x > = -\frac{g_x}{g_y} = \frac{f_y}{f_x}$.
$\therefore \frac{dy}{dx} = \frac{f_y}{f_x}$ is a 1st order LODE for g.

To find g, use the relation $dy = \frac{f_y}{f_x} dx$ then integrate.

Example:

$$f(x, y) = x^2 + y^2$$

Level curves are circles

$$\begin{gathered}
\frac{f_y}{f_x} = \frac{y}{x} \\
\therefore dy = \frac{y}{x} dx \\
\int \frac{1}{y}\, dy = \int \frac{1}{x}\, dx \\
\therefore \ln y = \ln x + C \\
y = kx
\end{gathered}$$

These are all lines through (0,0). Compute these through $p = (a, b), k = \frac{b}{a}$.

0.2.2 Initial Value Problems

Most general LODE of order n is:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

where $F(x)$ is the driving term.

An **initial value problem (IVP)** is an equation, $(*)$ together with initial conditions, $(**)$:

$$\begin{gathered}
y(x_0) = y_0 \\
y^{(1)}(x_0) = y_1 \\
y^{(n-1)}(x_0) = y_{n-1}
\end{gathered}$$

Example:
Solve the IVP $y'' + \omega^2 y = 0, y(0) = 1, y'(0) = 0$

$$\begin{gathered}
\text{The general solution is: } y = c_0 \cos(\omega t) + c_1 \sin(\omega t) \\
\text{Initial conditions impose:} \\
c_0 \cos 0 + \overset{0}{\cancel{c_1 \sin 0}} = 1 \implies c_0 = 1 \\
\overset{0}{\cancel{-c_0 \omega \sin 0}} + c_1 \omega \cos 0 = 0 \implies c_1 = 0 \\
\therefore \boxed{y = \cos \omega t}
\end{gathered}$$

In general, $c_0 = y(0)$, $c_1 = \frac{y'(0)}{\omega}$.

Theorem 1. *Assume in $(*)$ that the $a_i(x)$ and $F(x)$ are continuous on some interval I. Then $\exists!$ solution to IVP $(*)$ + $(**)$ on I.*

Note. This establishes existence and uniqueness. (The symbol $\exists!$ means "there exists a unique-".)

0.3 First Order DE's

The form of a first order differential equation is: $y' = f(x, y)$. Does it have a solution? If so, is it unique?

Theorem 2. *Given $y' = f(x, y)$. Assume $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and f_y is continuous in $(a, b) \times (c, d)$. Then for each $(x_0, y_0) \in (a, b) \times (c, d)$, $\exists!$ solution to y=y(x) running through (x_0, y_0) and this solution holds for some interval I around x_0*

Note. The notation $[a, b] \times [c, d]$ describes the domain and range using the Cartesian product of the two sets. In other words, $a \leq x \geq b$ and all $c \leq y \geq d$.

Example:
Is there a unique solution to: $y' = xy^{\frac{1}{2}}$, $y(0) = 0$?

Sol'n. $f(x, y) = xy^{\frac{1}{2}}$ is continuous on $(-\infty, \infty) \times [0, \infty]$ $f_y = \frac{1}{2}xy^{-\frac{1}{2}}$ is not continuous at y = 0.

$\therefore$ Thm is useless

There are solutions for the IVP where y(0) = 0. Check it out:

- $y = 0$ is a solution. $y(0) = 0$
- $y = \frac{1}{16}x^4$ is also a solution.

$$y(0) = 0$$
$$y' = \frac{1}{4}x^3 = xy^{\frac{1}{2}} \checkmark$$

Thus the solutions are not unique as expected from the theorem.

Note. Slope fields can be used to visualize solutions to a differential equation.

Definition (Equilibrium Solutions)**:**
Equilibrium solutions are solutions to a differential equation that have a derivative of zero everywhere i.e. equal to a constant value.

In terms of a first order linear differential equation, the equilibrium solutions is $\{y = y_0 \mid f(x, y_0) = 0\}$ On slope field diagrams, they are where a horizontal line fits as a solution.

0.3.1 Analytical Techniques for First Order ODEs

1. **Seperable:** If the first order ODE is one of the form $p(y)y' = q(x)$ (or $y' = r(y)q(x)$, ...) Then since $dy = y'dx$, we get $p(y)dy = q(x)dx$ and can integrate (in principle).

   Definition (Autonomous ODE)**:**
   An autonoumous ODE is one with no independent variable. It is usually seperable.

2. **Integrating Factor:** For LODE's in the form, $y' + p(x)y = q(x)$, let $I(x) = e^{\int p(x)\,dx}$ be the integrating factor.

Claim. $y(x) = \frac{1}{I(x)} \int q(x)I(x)\,dx$ is the general solution. Let's test this claim using the chain rule.

$$\begin{aligned} I(x)y(x) &= \int q(x)I(x)\,dx \\ \frac{d}{dx}I(x)y(x) &= I'(x)y(x) + I(x)y'(x) \\ &= p(x)I(x)y(x) + I(x)y'(x) \\ &= I(x)(p(x)y(x) + y'(x))) \\ &= I(x)q(x)\text{by original LODE} \end{aligned}$$

$\therefore I(x)y(x) = \int I(x)q(x)\,dx \implies$ claim is true

Note. The indefinite integral and the lack of a integration constant is not formally correct. However, the constant is removed upon the simplification. We can always divide both sides of the equation by e^C. :)

Example:
Find the general solution for $y' + (\tan x)y = \cos^2 x$ on $-\pi/2$, $\pi/2$.

Sol'n. Recognize this is the form for the integrating factor with $p(x) = \tan x$ and $q(x) = \cos^2 x$.

$$\begin{aligned} I(x) &= e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x \\ \therefore (\sec x)y &= \int \cos(x)\,dx = \sin x + C \\ \therefore y &= \sin x \cdot \cos x + C \cdot \cos x \end{aligned}$$

0.3.2 Application: Revisiting Newton's Law of Cooling

Newton defined his Law of Cooling as:

$$T' = -k(T - T_m)$$

. In its LODE form, $dT = -k(T - T_m)dt$. (Now, T_m could be time-dependent.)

Example:
Suppose $k = \frac{1}{40}$, $T_m(t) = 80e^{-\frac{t}{20}}$, $T(0) = 0$

a) Solve this IVP.

b) Determine the asymptotic behavior of the solution.

c) Find the maximum temperature.

a) Write it out as an ODE:

$$T' = -\frac{1}{40}(T - 80e^{-\frac{t}{20}})$$

Not seperable, 1st order ODE. This is an integrating factor problem. In standard form:

$$T' + \underbrace{\frac{1}{40}}_{p(t)} T = \underbrace{2e^{-\frac{t}{20}}}_{q(t)}$$

$$\implies p(t) = \frac{1}{40} \text{ and } q(t) = 2e^{-\frac{t}{20}}$$

$$I(t) = e^{\int p(t)\,dt}$$
$$I(t) = e^{\int \frac{1}{40}\,dx} = e^{t/40}$$

Multiplying the original LODE by $I(t)$:

$$\underbrace{I(t)T' + I(t)\frac{1}{40}T}_{(IT)'} = \underbrace{I(t)2e^{-\frac{t}{20}}}_{2e^{-\frac{t}{40}}}$$

$$IT = \int 2e^{-\frac{t}{40}}\,dt = -80e^{-\frac{t}{40}} + C$$

$$\therefore T = e^{-t/40}\left[-80e^{-\frac{t}{40}} + C\right]$$

Using IV: $T(0) = 0 \implies 1[-80 + C] = 0$

$$C = 80$$

$$\therefore T = e^{-t/40}\left[-80e^{-\frac{t}{40}} + 80\right]$$

$$\boxed{T = 80\left(-e^{-\frac{t}{20}} + e^{-\frac{t}{40}}\right)}$$

b) $\boxed{\lim_{t\to\infty} T(t) = 0}$

c) Maximum temperature? Occurs when $\frac{dT}{dt} = 0$ or $T(t) = T_m(t)$

$$T(t) = T_m(t)$$
$$80(e^{-\frac{t}{40}} - e^{-\frac{t}{20}}) = 80e^{-\frac{t}{20}}$$
$$e^{-\frac{t}{40}} - e^{-\frac{t}{20}} = e^{-\frac{t}{20}}$$
$$e^{-\frac{t}{40}} = 2e^{-\frac{t}{20}}$$
$$-\frac{t}{40} = \ln 2 - \frac{t}{20}$$
$$\frac{t}{40} = \ln 2$$
$$t = 40\ln 2$$

So $T_{max} = 80\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{20}$

0.3.3 Application: RLC Circuits

The Physics:

- Voltage is the difference ΔV in electric potential V
- Voltage along a wire (usually caused by a battery or EMF) causes charge q to move, causing a current $i = \frac{dq}{dt}$
- **Kirchoff's Second Law:**

$$\sum_{\text{closed loop}} \Delta V = 0$$

  In a circuit, we have components with opposing voltages.

  1. Resistance R causes $\Delta V \overset{\text{(Ohm's Law)}}{=} iR$. Units are ohms Ω.
  2. Capacitance C causes $\Delta V = \frac{1}{C}q$. Units are farads F.
  3. Inductance C causes $\Delta V = L\frac{di}{dt}$. Units are henrys H.
  4. Driver EMF $E(t) = \sum_{\text{circuit components}} \Delta V$.

Aside. The work done to move a unit charge from A to B can be found using multivariable calculus!

$$\begin{aligned} int_C \overrightarrow{E} \cdot \overrightarrow{T}\, ds &= \int_C \overrightarrow{\Delta V} \cdot \overrightarrow{T}\, ds \\ &\overset{FTC}{=} V(b) - V(a) \end{aligned}$$

By Kirchoff's Law, we have an LODE:

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = \frac{1}{L}E(t)$$

Since $i = \frac{dq}{dt}$,

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t)}$$

Generally, either $E(t) = E_0$ (constant as DC) or $E(t) = E_0 \cos \omega t$ (AC, often 60 Hz). To solve this equation, we must restrict it to either an RL case (first order LODE in i) or RC case (first order LODE in q).

Example (RL Circuit)*:*
Given $E(t) = E_0 \cos \omega t$. Find $i(t)$, given $i(0) = 0$.

Sol'n. LODE is $\frac{di}{dt} + \overbrace{\frac{R}{L}}^{p(t)} i = \overbrace{\frac{E_0}{L}}^{q(t)} \cos \omega t$. Not seperable, use integrating factor.

$$I(t) = e^{\int \frac{R}{L}\, dt} = e^{at} \quad \left(a = \frac{R}{L}\right)$$

$$e^{at} i(t) = \int e^{at} E_0 \cos \omega t \, dt$$

Requires integration by parts

$$\boxed{i(t) = \underbrace{\frac{E_0 a}{L(a^2 + \omega^2)} (a \cos \omega t + \omega \sin \omega t)}_{\text{steady state}} + \underbrace{\frac{E_0 a}{L(a^2 + \omega^2)} \left(-a e^{-at}\right)}_{\text{transient}}}$$