

# Real Analysis

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# Preface

I was originally going to spending the summer of 2024 typing up my old math notes. Then I realized that I was going to need Real Analysis for Complex Analysis. Makes sense based on the name. I just assumed that I would be fine since Real Analysis wasn't a prerequisite for the course I was taking. Whoops.

- Christopher

*"God made the integers; all else is the work of man." - Leopold Kronecker*

# Chapter 1

## Rational and Real Numbers

### 1.1 Set Theory Foundation

This is a review of some introductory proof concepts that are important for laying the foundations of both rational numbers and real analysis.

*Definition:*

A **set** is a collection of objects called elements of the set.

*Example:*

1.  $S = \{1, 2, 3\}$  ( $= \{1, 2, 3, 3\}$ )
2.  $E = \{\text{Even integers}\}$
3.  $\{\text{College students}\}$

*Notation:*

- $x \in S$  means  $x$  is in  $S$ .
- $x \notin S$  means  $x$  is not in  $S$ .
- The empty set  $\emptyset$  is the set with no elements.
- $A \subseteq B$  means  $A$  is a subset of  $B$  (i.e. if  $x \in A$ , then  $x \in B$ ).
- If  $A \subseteq B$  but  $B \not\subseteq A$   $A$  is a proper subset.

If  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ . Otherwise  $A \neq B$ .

We can define more sets in terms of other sets. *Set Operations:* Let  $A$  and  $B$  be sets.

- Union:  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Intersection:  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Compliment:  $B - A = \{x \mid x \in B \text{ and } x \notin A\}$

- Product:  $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

If  $U$  is a universal set (set of everything in context), we write  $\bar{A} = U - A = \{x \mid x \in U \text{ and } x \notin A\}$ .

## 1.2 Functions and Relations

It is also important to define some types of relations and functions.

*Definition* (Relation):

A (binary) **relation**  $R$  on a set  $S$  is a subset of  $S \times S$ . If  $(a, b) \in R$ , we write  $aRb$ .

*Example* (of relations): 1.  $L$  "loves" is a relation on  $P \times P$  (where  $P$  is a set of all people).

2. The set  $R = \{(0, 0), (0, 1), (2, 2), (7, 18)\}$  is a relation on  $\mathbb{Z}^+$ . We would write  $0R0$ ,  $0R1$ ,  $2R2$ , and  $7R18$ .

*Definition* (Equivalence Relation):

An equivalence relation on a set  $S$  is a relation s.t.:

1. Reflexive: For each  $a \in S$ ,  $a \sim a$ .
2. Symmetric: For  $a, b \in S$ , if  $a \sim b$ , then  $b \sim a$ .
3. Transitive: For  $a, b, c \in S$ , if  $a \sim b$  and  $b \sim a$

Functions in the general sense are also a type of relation.

*Definition* (Function):

A **function**,  $F$  from a set  $A$  to a set  $B$  is a relation s.t.: if  $aFb$  and  $aFb'$  then  $b = b'$ .

This is a rule that assigns a unique  $a \in A$  to a unique  $b \in B$ . Write  $F(a) = b$ .

## 1.3 The Rational Numbers

Assume  $\mathbb{Z}$ , the integers, have arithmetic order. What is  $\mathbb{Q}$ ? Perhaps it's the set:

$$\left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

However, what does that fraction notation actually mean? When we first begin teaching fractions to children we talk about splitting things like cake into smaller pieces. If we have a whole cake made of 3 slices, we can give one person a slice so they have  $\frac{1}{3}$  of the cake. If we have a cake of 6 slices, we could give them 2 slices instead. They would have  $\frac{2}{6}$ . These two fractions are equivalent though! We need more rigor (this is mathematics of course).

We say that the fractions are equivalent ordered pairs  $(1, 3) \sim (2, 6)$ . These belong to the same **equivalence class**,  $\left[ \frac{1}{3} \right]$ .

*Definition* (Rational Numbers):

The **rational numbers**,  $\mathbb{Q}$ , is the set  $\left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$  where  $\frac{m}{n}$  is an equivalence class of  $(m, n)$  with the relation  $(m, n) \sim (p, q)$  if  $mq = np$  and  $q, n \neq 0$

*Proof.* Is  $\sim$  an equivalence relation? Need to show  $\sim$  reflexive, symmetric, and transitive.

Step 1 Reflexive: Let  $(p, q) \in \mathbb{Q}$ . Show  $(p, q) \sim (p, q)$

Since  $ab = ba$ ,  $(p, q) \sim (p, q)$  ✓

Step 2 Symmetry: Let  $(p, q), (m, n) \in \mathbb{Q}$ . Assume  $(p, q) \sim (m, n)$ . Show  $(m, n) \sim (p, q)$ .

$$\begin{aligned} (p, q) \sim (m, n) &\implies pn = qm \\ &\implies qm = pn \\ &\implies mq = np \\ &\implies (m, n) \sim (p, q) \quad \checkmark \end{aligned}$$

Step 3 Transitive: Let  $(p, q), (m, n), (a, b) \in \mathbb{Q}$ . Assume  $(p, q) \sim (m, n)$  and  $(m, n) \sim (a, b)$ . Show  $(p, q) \sim (a, b)$ .

Need cancellation law on  $\mathbb{Z}$ : if  $ab = ac$  and  $a \neq 0$  then  $b = c$ .

$$(p, q) \sim (m, n) \implies pn = qm \text{ and } (m, n) \sim (a, b) \implies mb = na$$

*Case 1:*  $p = 0$

$$\begin{aligned} p = 0 &\implies pn = qm = 0 \\ &\implies m = 0 \text{ since } q \neq 0 \\ &\implies mb = na = 0 \\ &\implies a = 0 \text{ since } n \neq 0 \\ &\implies pb = qa = 0 \\ &\implies (p, q) \sim (a, b) \quad \checkmark \end{aligned}$$

*Case 2:*  $m = 0$

Similar to Case 1. ✓

*Case 3:*  $p, m \neq 0$

Multiplying  $pn = qm$  by  $ab$ :  $ab(pn) = ab(qm)$ .

$$\implies na(pb) = mb(qa)$$

$$\implies pb = qa \text{ by cancellation law } (m \neq 0 \text{ and } mb = na) \implies (p, q) \sim (a, b) \quad \checkmark$$

□

### 1.3.1 Arithmetic (of Rationals)

Our definitions of arithmetic on  $\mathbb{Q}$  be well-defined. For example, we could define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$$

However,

$$\begin{aligned}\frac{1}{2} + \frac{1}{3} &= \frac{2}{5} \\ \frac{2}{4} + \frac{3}{7} &= \frac{3}{7}\end{aligned}$$

$\frac{1}{2}$  and  $\frac{2}{4}$  are in the same equivalent class, but  $\frac{2}{5}$  and  $\frac{3}{7}$  are not. This is not well-defined. We want a definition of addition not dependent on our representatives chosen.

Now,  $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$ . This is well-defined but not helpful.

*Definition* (Addition in  $\mathbb{Q}$ ):

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

If this well-defined?

*Proof.* Assume  $(a, b) \sim (a', b')$  and  $(c, d) \sim (c', d')$ . Show  $(ad + bc, bd) \sim (a'd' + b'c', b'd')$ .

$$(a, b) \sim (a', b') \implies ab' = ba'$$

$$(c, d) \sim (c', d') \implies cd' = dc'$$

$$\begin{aligned}b'd'(ad + bc) &= b'd'ad + b'd'bc \\ &= (d'd)(ab') + (b'b)(cd') \\ &= (d'd)(ba') + (b'b)(dc') \\ &= (bd)(a'd') + (bd)(c'b') \\ &= bd(a'd' + c'b')\end{aligned}$$

$$\implies (ad + bc, bd) \sim (a'd' + b'c', b'd') \checkmark$$

□

*Definition* (Multiplication in  $\mathbb{Q}$ ):

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

If this well-defined?

*Proof.* Assume  $(a, b) \sim (a', b')$  and  $(c, d) \sim (c', d')$ . Show  $(ac, bd) \sim (a'c', b'd')$ .

$$(a, b) \sim (a', b') \implies ab' = ba'$$

$$(c, d) \sim (c', d') \implies cd' = dc'$$

$$\begin{aligned}acb'd' &= (ab')(cd') \\ &= (ba')(dc') \\ &= (a'c')(bd)\end{aligned}$$

$$\implies (ac, bd) \sim (a'c', b'd') \checkmark$$

□

In what way does  $\mathbb{Q}$  extend  $\mathbb{Z}$ ?

The correspondence is  $\frac{n}{1} \longleftrightarrow n$ . Addition and multiplication is the same in  $\mathbb{Q}$  as in  $\mathbb{Z}$ .

*Note.* We can define subtraction by adding the negative of a number (multiply by  $-1$ ).

### 1.3.2 Order

*Definition* (Order):

An **order** on a set  $S$  is a relation  $<$  satisfying:

1. (Trichotomy) If  $x, y \in S$ , exactly one is true:  $x < y$ ,  $x = y$ ,  $y < x$ .
2. (Transitivity) If  $x, y, z \in S$ ,  $x < y$ , and  $y < z$ ,  $x < z$ .

*Example:*

In  $\mathbb{Z}$ , say  $m < n$  if  $n - m$  is positive, i.e. in  $\mathbb{N}$ .

*Example:*

In  $\mathbb{Z} \times \mathbb{Z}$ , say  $(a, b) < (c, d)$  if  $a < c$  or ( $a = c$  and  $b < d$ ). This is called the dictionary order.

*Example:*

In  $\mathbb{Q}$ , say  $\frac{m}{n}$  is positive if  $mn > 0$ . This is well-defined.

*Proof.* Assume  $(m, n) \sim (p, q)$  and  $mn > 0$ . Show  $pq > 0$ .

Suppose, to the contrary,  $pq < 0$ .

$$\begin{aligned} (m, n) \sim (p, q) &\implies mq = np \\ &\implies (mq)^2 = mqn timerp \end{aligned}$$

By assumption,  $mnpq < 0$ , a contradiction since  $mn > 0$ . Thus,  $pq > 0$ . □

So  $\frac{a}{b} < \frac{c}{d}$  if  $\frac{c}{d} + \frac{-a}{b}$  is positive.

Write  $y > x$  for  $x < y$  and  $x \leq y$  for  $x < y$  or  $x = y$ .

**Theorem 1.**  $x^2 = 2$  has no solution in  $\mathbb{Q}$ .

*Proof (by contradiction).* Assume  $x^2$  has a solution in  $\mathbb{Q}$ , i.e.  $x = \frac{p}{q}$  where  $p, q \in \mathbb{Z}$ .

Also assume  $p, q$  are in “lowest terms,” i.e. they have no common factors. (We can do this using elements in the equivalence classes of  $\mathbb{Q}$ .)

So  $\left(\frac{p}{q}\right)^2 = 2$ , hence  $p^2 = 2q^2$ .

Then  $p^2$  is even (divisible by 2).

Then  $p$  is even. (If  $p$  was odd,  $p^2$  would be odd.)

So  $p = 2m$  for some  $m \in \mathbb{Z}$ , hence  $p^2 = 4m^2 = 2q^2$ .

Then  $2m^2 = q^2$ .

Then  $q^2$  is even, hence  $q$  is even.



This contradicts the fact that  $p, q$  are in “lowest terms.” So,  $x^2 = 2$  must have no solution in  $\mathbb{Q}$ .  $\square$

### 1.3.3 Fields

*Definition* (Field):

A **field** is a set  $F$  with two operations  $+$ ,  $\times$  satisfying axioms:

- A1.**  $F$  is closed under  $+$ . (Adding two things in the set gives you something in the set.)
- A2.**  $+$  is commutative.
- A3.**  $+$  is associative.
- A4.**  $F$  has an additive identity, call it 0.
- A5.** Every element has an additive inverse.
- M1.**  $F$  is closed under  $\times$ .
- M2.**  $\times$  is commutative.
- M3.**  $\times$  is associative.
- M4.**  $F$  has an additive identity, call it 1.
- M5.** Every element except 0 has an additive inverse.
- D1.**  $\times$  distributes over  $+$ .

*Example:*

In  $\mathbb{Q}$ , the 0 element is  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and the 1 element is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

*Definition* (Ordered Field):

An **ordered field** is a field with an order s.t. order is preserved by field operations.

- 1. If  $y < z$ , then  $x + y < x + z$ .
- 2. If  $y < z$  and  $x > 0$ , then  $xy < xz$ .

*Note.*  $\mathbb{Z}$  is a ring not a field. There are no multiplicative inverses.  
 $\mathbb{Q}$  is an ordered field!