

## 0.1 Defining Matrices

Let  $n, m$  be two integers  $\geq 1$ . A **matrix** is an array of numbers with  $m$  rows and  $n$  columns (called a  $m \times n$  matrix).

We call  $a_{ij}$  the **ij-entry** which is the entry in the  $i$ th row and the  $j$ th column. We write a matrix often as  $A = (a_{ij})$  and define  $a_{ij}$ .

Each column of an  $m \times n$  matrix is a **column vector**. Each row of an  $m \times n$  matrix is a **row vector**.

*Example (Identity Matrix):*

The Kronecker delta is defined as follows:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then we define the **identity matrix** as:

$$I_n = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

*Example:*

If we have a matrix,  $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$ , the second column vector of  $A$  is  $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$  and the second row vector of  $A$  is  $[1 \quad -1]$ .

We can describe matrices using their column or row vectors. For example:

$$A = \begin{bmatrix} \vec{r} \\ \vec{s} \\ \vec{t} \end{bmatrix}$$

where

$$\vec{r} = [3 \quad 4], \vec{s} = [1 \quad -1], \vec{t} = [2 \quad 2]$$

Or:

$$A = [\vec{a} \quad \vec{b}]$$

where

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

### 0.1.1 Matrix Operations

We treat matrices the same way as numbers. Let  $A$  be an  $m \times n$  matrix and  $B$  be an  $p \times q$  matrix.

- We can add  $A$  and  $B$ . If  $m = p$  and  $n = q$ , then  $A + B = (a_{ij} + b_{ij})$
- We can multiply by a scalar  $c$ :  $c \cdot A = (c \cdot a_{ij})$
- We can multiply  $A$  and  $B$ . If  $n = p$ , then  $A \cdot B = (c_{ij})$  where  $c_{ij} = \vec{r}_i(A) \cdot \vec{c}_j(B)$ .  
Caution: In general,  $A \cdot B \neq B \cdot A$ .
- We also define the transpose of a matrix. The transpose of  $A$  is  $A^t = (d_{ij})$  where  $d_{ij} = a_{ji}$ . When we take a transpose, we switch the columns into rows and vice versa.

Certain special matrices can be described with other terminology. Suppose we have a matrix,  $A = (a_{ij}), i = 1, \dots, m$  and  $j = 1, \dots, n$ .

- If  $m = n$ , then  $A$  is a **square matrix**.
- If  $A^t = A$ , then  $A$  is a **symmetric matrix**. Note: this means  $A$  must also be square.
- If  $A^t = -A$ , then  $A$  is said to be **skew-symmetric**.
- If for all  $i, j$  such that  $i \neq j$ ,  $a_{ij} = 0$ , then  $A$  is called **diagonal**.

### Determinants

The determinant is a property of a square matrix,  $A$ .

#### Geometric Def'n.

Let  $A = [\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n]$  (vectors in  $\mathbb{R}^n$ ). Let  $\Pi$  = parallelotope defined by basing them all at the same point. Then  $V^\sigma(\Pi) = \det [\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n]$  (signed  $n$  volume).

*Example:*

$$A = [\vec{c}_1 \quad \vec{c}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det A = -1$$

#### Algebraic Def'n.

Let  $A = (a_{ij}), n \times n$ . Let  $A_{ij}$  = submatrix obtained from  $A$  by eliminating row  $i$  and column  $j$ ,  $(n-1) \times (n-1)$  matrix.

The minor of  $A$  is  $M_{ij} = \det A_{ij}$ . The cofactor of  $A$  is  $C_{ij} = (-1)^{i+j} \det A_{ij}$ .

Then  $\det A = \sum_{j=1}^n a_{ij} C_{ij}$  (for any  $i, 1 \leq i \leq n$ ). We call this the cofactor expansion along the  $i$ th row.

*Example:*

$$A = \begin{bmatrix} 7 & 3 & 12 \\ 2 & 5 & 8 \\ 1 & 5 & 2 \end{bmatrix}$$

Find the determinant.

Use row 3.

$$\det A = \sum_{j=1}^3 a_{3j} C_{3j} = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$$

$$\left. \begin{aligned} C_{31} &= (-1)^{3+1} \begin{vmatrix} 3 & 12 \\ 5 & 8 \end{vmatrix} = -36 \\ C_{32} &= (-1)^{3+2} \begin{vmatrix} 7 & 12 \\ 2 & 8 \end{vmatrix} = -32 \\ C_{33} &= (-1)^{3+3} \begin{vmatrix} 7 & 3 \\ 2 & 5 \end{vmatrix} = 29 \end{aligned} \right\} \det A = 1(-36) + 5(-32) + 2(29) = \boxed{-138}$$

### Properties of Determinants

1.  $\det A = \det A^t$

(Therefore, we can also use the cofactor expansion along any column as well.)

2.  $\det A \neq 0 \iff A$  is invertible

3.  $\det AB = \det A \det B$

4.  $\det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_i + \vec{r}_i' \\ \vec{r}_n \end{bmatrix} = \det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_i \\ \vec{r}_n \end{bmatrix} + \det \begin{bmatrix} \vec{r}_1 \\ \vec{r}_i' \\ \vec{r}_n \end{bmatrix}$

Moreover, suppose  $A \xrightarrow{\vec{1}} B$  is an ERO. Then,

5. If  $1.P_{ij}$ , then  $\det B = -\det A$ .

6. If  $1.M_i(k)$ , then  $\det B = k \cdot \det A$ .

7. If  $1.A_{ij}(k)$ , then  $\det B = \det A$ .

*Note.* By (3),  $\det(A^{-1}) = (\det A)^{-1}$

### 0.1.2 Inverse Matrices

An  $n \times n$  matrix  $A$  may or may not have an **inverse**: A matrix  $B$  such that

$$AB = BA = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We write  $B = A^{-1}$ .

For a linear system,  $A\vec{x} = \vec{b}$  with  $A$ ,  $n \times n$ , if  $A$  is invertible:

$$\underbrace{A^{-1}A}_{I_n} \vec{x} = A^{-1}\vec{b}$$

$$\therefore \text{Sol'n. is } \vec{x} = A^{-1}\vec{b}$$

We can vary  $\vec{b}$  and the solutions are immediate.

For the  $2 \times 2$  case,  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A$  has an inverse  $\iff |A| = ad - bc \neq 0$ .

Then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*Example:*

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$$

Invertible?

$$18 - 35 = -17 \neq 0 \checkmark$$

$$A^{-1} = \frac{-1}{17} \begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}$$

Check

$$\frac{-1}{17} \begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix} = \frac{-1}{17} \begin{bmatrix} -17 & 0 \\ 0 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Theorem 1.**  $A$   $n \times n$  matrix  $A$  has an inverse  $\iff \text{rk}A = n$ . If  $A$  has an inverse, then  $A^{-1}$  is given by  $\text{rref}[A \ I_n] = [I_n \ A^{-1}]$

*Example:*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Is it invertible?

$$\begin{aligned}
 & \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 & \xrightarrow[2. \ A_{13}(-1)]{1. \ A_{12}(-2)} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \\
 & \xrightarrow[4. \ A_{31}(1)]{3. \ A_{32}(-1)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \\
 & \xrightarrow{5. \ A_{21}(-2)} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \\
 & A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \\
 & AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

### Properties of $A^{-1}$

1. If  $A$  is invertible, so is  $A^{-1}$ ,  $(A^{-1})^{-1} = A$ .
2. If  $A, B$  are invertible,  $n \times n$ , then so is  $A \cdot B$  and  $(BA)^{-1} = B^{-1}A^{-1}$ .  
Witness  $A(B \cdot B^{-1})A^{-1} = AI_nA^{-1} = I_n$ .
3. If  $A$  is invertible, so is  $A^t$ ,  $(A^t)^{-1} = (A^{-1})^t$ .

**Theorem 2.** Suppose  $A$  is  $n \times n$ . Then the following are equivalent

1.  $A$  is invertible
2.  $A\vec{x} = \vec{b}$  has a unique solution for all  $\vec{b}$
3.  $rk(A) = n$
4.  $rref(A) = I_n$
5.  $\det(A) \neq 0$

### Computing Inverses

Recall if  $A = a_{ij}$  is a  $2 \times 2$  matrix, then if  $A$  is invertible,

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Theorem 3.** *If  $A$  is invertible then*

$$A^{-1} = \frac{1}{\det A} (b_{ij})$$

where  $b_{ij}$  = the  $ji$ -th cofactor of  $A = (-1)^{i+j} \det A_{ij}$ .

*Example:*

Is  $A$  invertible? If so, find  $A^{-1}$ .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 5 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 2 - 1 + 1$$

Therefore,  $A$  is invertible.

$$\text{Then } \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \Bigg\} (C_{ij})^t$$

$$\begin{aligned} C_{11} &= (-1)^2 \cdot 2 = 2 & C_{21} &= -1 & C_{31} &= 0 \\ C_{12} &= (-1)^3 \cdot 2 = -2 & C_{22} &= 1 & C_{32} &= -1 \\ C_{13} &= 3 & C_{23} &= 1 & C_{33} &= 1 \end{aligned}$$

$$\text{Then } A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

## 0.2 Matrices as Linear Transformations

Matrices can be used to model transformations of vectors from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . This is accomplished by having an  $m \times n$  matrix,  $A$ , written as:

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

*Example* ( $\mathbb{R}^n$  to  $\mathbb{R}^m$ ):

Let  $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$ , and  $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Thus,  $A : \mathbb{R}^2$  to  $\mathbb{R}^3$  and:

$$A\vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$$

*Note.* We are getting a linear combination of the column vectors of  $A$ . In other words,  $A\vec{v} = x\vec{a} + y\vec{b}$ .