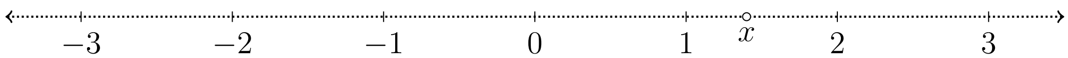
0.1 Constructing the Real Numbers

0.1.1 Upper Bounds

Now, we seen in the previous section that $\mathbb{Q}$ has "gaps". $x^2 = 2$ has no solution in $\mathbb{Q}$.



We need to fill in these gaps somehow while not knowing where the gaps and holes are.

Definition (Upper Bound)*:*
Let $E \subset S$ ordered. If there exists $\beta \in S$ such that for all $x \in E$, $x \leq \beta$, then β is an **upper bound (u.b.)** for E. We say E is bounded above.

A lower bound can be defined similarly with "greater than or equal to."

Example:
Consider the set $A = \{x \mid x^2 < 2\}$. 2 is an u.b. for A. $\dfrac{2}{3}$ is also an u.b. for A.

Definition (Least Upper Bound)*:*
If there exists an $\alpha \in S$ such that:

1. α is an upper bound of E
2. If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called a **least upper bound (lub)** of E or the **suprenum** of E. Write $\alpha = \sup E$.

Example:
Let $S = \mathbb{Q}$.

1. $E = \left\{\frac{1}{2}, 1, 2\right\}$ $\boxed{\sup E = 2}$
2. $E = \{x \in \mathbb{Q} \mid x < 0\}$ $\boxed{\sup E = 0}$
3. $E = \mathbb{Q}$ $\boxed{\sup E \text{ does not exist}}$
4. $E = A$ (as defined above) $\boxed{\sup E \text{ does not exist}}$

Definition (Least Upper Bound Property)*:*
A set S has the **least upper bound property** if every nonempty subset of S that has an upper bound has a least upper bound.

0.1.2 Dedekind Cuts

Definition (Dedekind Cut)*:*
A **Dedekind cut** α is a subset of $\mathbb{Q}$ such that:

1. $\alpha \neq \emptyset, \mathbb{Q}$

2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$. (Closed downward)
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. (No largest number)

Example:
$\alpha = \{x \in \mathbb{Q} \mid x < 0\}$ is a cut.

1. $\alpha \neq 0, \mathbb{Q}$ ✓
2. Let $p \in \alpha$, $q \in \mathbb{Q}$. Assume $q < p$. By the transitivity property of order, $q < 0$. Thus, $p \in \alpha$. ✓
3. Let $p \in \alpha$ and $r \in \alpha$ such that $r = \dfrac{q}{2}$. Since $q < 0$, $q < \dfrac{q}{2}$. Thus $q < r$. ✓

Example:
$\gamma = \{r \mid r \leq 2\}$ is not a cut. This set does have a largest element, 2.

Definition (Rational Numbers)*:*
Let $\mathbb{R} = \{\alpha \mid \alpha \text{ is a cut }\}$.

We also define the following:

- $\alpha < \beta$ to mean $\alpha \subsetneqq \beta$. This is an order.
- $\alpha + \beta = \{r + s \mid r \in \alpha \text{ and } s \in \beta\}$. (This means $\mathbb{R}$ is a field.)
- $\alpha \cdot \beta$
    1. For positive cuts $\{\alpha \mid \alpha > 0^*\} = \mathbb{R}_+$:
       If $\alpha, \beta \in \mathbb{R}_+$, let $\alpha \cdot \beta = \{p \mid p < rs \,\text{for some}\; r \in \alpha, s \in \beta,\, and r, s > 0\}$.
    2. For cases with negative cuts, $\alpha \cdot \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*,\ \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*,\ \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*,\ \beta < 0^* \end{cases}$
       where the products are the same as defined for postive cuts and $-\alpha$, $-\beta$ are the additive inverses of α, β respectively. (The additive inverses are defined below.)

Theorem 1. *$\mathbb{R}$ is an ordered field with the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

Proofs Showing $\mathbb{R}$ is an Ordered Field

The following section proves Theorem 1.

Proof.

Step 1: We must show there is order on $\mathbb{R}$.
Let $\alpha, \beta, \gamma \in \mathbb{R}$. We must show that they demonstrate both trichotomy and transitivity.

1. Trichotomy:
   It is clear that at most one of the following can be true: $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$. For example, if $\alpha < \beta$, then $\alpha \subsetneqq \beta$ and by the definition of a proper subset, $\alpha \neq \beta$ and β is not a proper subset of α.

To show at least one of them must be true, suppose the first two statements are false. Then α is not a subset of β. By definition of a proper subset, there exists some $a \in \alpha$ such that $a \notin \beta$. Consider some $b \in \beta$. Since β is closed downward, $b < a$. This also means $b \in \alpha$ since α is also closed downward. This shows that $\beta \subsetneqq \alpha$. Thus $\beta < \alpha$. We conclude that at least one of these statements must be true.

2. Transitivity:
   We assume that $\alpha < \beta$ and $\beta < \gamma$. By definition of $<$, $\alpha \subsetneqq \beta$ and $\beta \subsetneqq \gamma$. By definition of a proper subset, $\alpha \subsetneq \gamma$ and we conclude that $\alpha < \gamma$.

We have shown the cuts demonstrate order.

<u>Step 2:</u> Next, we must show that addition is closed (**A1**).
Let $\alpha, \beta \in \mathbb{R}$ and $\gamma = \{r + s \,|\, r \in \alpha \text{ and } b \in \beta\}$. To show addition is closed, we must show that γ is a cut.

1. First, we must show that $\gamma \neq \emptyset$, $\mathbb{Q}$.
   It should be clear that γ cannot be the empty set. Since $\alpha, \beta \neq \mathbb{Q}$, there exists $a' \notin \alpha$, $b' \notin \beta$. Now $a < a'$ and $b < b'$ for all $a \in \alpha$, $b \in \beta$. Thus $a + b < a' + b'$. Therefore $a' + b' \notin \gamma$. We conclude that $\gamma \neq \emptyset, \mathbb{Q}$.

2. Second, we must show that γ is closed downward.
   Let $p \in \gamma, q \in \mathbb{Q}$. Assume $q < p$. Since $p \in \gamma$, there exists $r \in \alpha$ and $s \in \beta$ such that $p = r + s$ so $q < r + s$. This means $q - s < r$. Since α is closed downward, $q - s \in \alpha$. Then $q = q - s + s$ where $q - s \in \alpha$ and $s \in \beta$. We have shown that γ is closed downward.

3. Third, we must show that γ has no largest number.
   Let $t \in \gamma$. Then there exists $u \in \alpha$ and $v \in \beta$ such that $t = u + v$. Since both cuts α, β have no largest number, there exists $x \in \alpha$ where $u < x$ and $y \in \beta$ where $v < y$. Thus $x + y \in \gamma$ and $t < x + y$. We have shown that γ has no largest member.

We have shown that γ meets the definition of a cut.

<u>Step 3:</u> **A2** and **A3** follows since addition in $\mathbb{Q}$ is commutative and associative.

<u>Step 4:</u> We must show that $0^* = \{q \in \mathbb{Q} \mid q < 0\}$ is the additive identity for $\mathbb{R}$ (**A4**).
In other words, we must show that $\alpha + 0^* = \alpha$. Let $\alpha \in \mathbb{R}$ and $0^* = \{q \in \mathbb{Q} \mid q < 0\}$.

1. First, we must show that $\alpha + 0^* \subset \alpha$.
   Let $a \in \alpha$ and $b \in 0^*$. Since $b < 0$, $a + b < a$. Thus $a + b \in \alpha$ (since α is closed downward). We conclude that $\alpha + 0^* \subset \alpha$.

2. Second, we must show that $\alpha \subset \alpha + 0^*$.
   Let $x, y \in \alpha$ and $z \in 0^*$. Since α has no largest member, we can pick x, y such that $y > x$. Then $x - y \in 0^*$ and similar to before $x = y + (x - y) \in \alpha + 0^*$. We conclude that $\alpha \subset \alpha + 0^*$.

Thus we conclude that $\alpha + 0^* = \alpha$.

<u>Step 5:</u> Next, we must show that the additive inverse for $\alpha \in \mathbb{R}$ is $\beta = \{p \in \mathbb{Q} \,|$ there exists

$r > 0$ s.t. $-p - r \notin \alpha\}$ (**A5**).
Let $\alpha \in \mathbb{R}$ and $\beta = \{p \in \mathbb{Q} \,|$ there exists $r > 0$ s.t. $-p - r \notin \alpha\}$.

1. First, we must show that β is a cut.

   i. We must show that β is non-trivial.
   Since $\alpha \neq \emptyset$, there exists some $a \in \alpha$. Since α is closed downward, $a - b \in \alpha$ for all $b > 0$. Thus $-a \notin \beta$ so $\beta \neq \mathbb{Q}$.

   Since $\alpha \in \mathbb{Q}$, there exists some $c \notin \alpha$. Consider $d = -c - 1$. Then $-d - 1 \notin \alpha$ (since $c = -d - 1$). Thus $d \in \beta$ so $\beta \neq \emptyset$.

   ii. We must show that β is closed downward.
   Let $f \in \beta$, $g \in \mathbb{Q}$. We assume that $g < f$. There exists some $h > 0$ s.t. $-f - h \notin \alpha$. Since $g < f$, $-g - h > -f - h$. Thus since α is closed downward, $-g - h \notin \alpha$. We conclude that β is closed downward.

   iii. We must show that β has no largest member.
   Consider $j = f + (h/2)$. Then $j > f$, and $-j - (h/2) = -f - h \notin \alpha$. Then $j \in \beta$. Thus we find β also has no largest member.

   We conclude that β satisfies the definition of a cut.

2. Second, we must show that $\alpha + \beta = 0^*$.

   i. We need to show that $\alpha + \beta \subset 0^*$.
   Let $s \in \alpha$ and $t \in \beta$. By definition of β, there exists $u > 0$ s.t. $-t - u \notin \alpha$. Then $-t - u > s$ and it follows that $s + t < -u < 0$. Therefore $s + t \in 0^*$.

   ii. We also need to show that $0^* \subset \alpha + \beta$.
   Let $v \in 0^*$ and $w = -v/2$. Then $w > 0$. By the Archimedean property of $\mathbb{Q}$, there exists $n \in \mathbb{Z}$ s.t. $nw \in \alpha$ and $(n+1)w \notin \alpha$. Consider $x = -(n+2)w$. Now $-x = nw + 2w \implies -x - w = nw + w = (n+1)w \notin \alpha$. By definition of β, $x \in \beta$. We find $v = -2w = -(n+2)w + nw = p + x$ and conclude that $v \in \alpha + \beta$.

   This shows that $\alpha + \beta = 0^*$.

We have proved that β is the additive inverse for α and that $\mathbb{R}$ satisfies all the addition axioms. Next we will show that $\mathbb{R}$ satisfies the multiplication axioms.

<u>Step 6:</u> We must show that $\mathbb{R}$ is closed under multiplication.