0.1 Defining Matrices

Definition (Matrix)*:*
Let n, m be two integers ≥ 1. A **matrix** is an array of numbers with m rows and n columns (called a $m \times n$ matrix).

We call a_{ij} the **ij-entry** which is the entry in the ith row and the jth column. We write a matrix often as $A = (a_{ij})$ and define a_{ij}.

Each column of an $m \times n$ matrix is a **column vector**. Each row of an $m \times n$ matrix is a **row vector**.

Example (Identity Matrix)*:*
The Kronecker delta is defined as follows:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then we define the **identity matrix** as:

$$I_n = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example:
If we have a matrix, $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, the second column vector of A is $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and the second row vector of A is $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

We can describe matrices using their column or row vectors. For example:

$$A = \begin{bmatrix} \vec{r} \\ \vec{s} \\ \vec{t} \end{bmatrix}$$

where

$$\vec{r} = \begin{bmatrix} 3 & 4 \end{bmatrix}, \vec{s} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \vec{t} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

Or:

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

where

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

0.2 Matrix Operations

We treat matricies the same way as numbers. Let A be an $m \times n$ matrix and B be an $p \times q$ matrix.

- We can add A and B. If $m = p$ and $n = q$, then $A + B = (a_{ij} + b_{ij})$
- We can multiply by a scalar c: $c \cdot A = (c \cdot a_{ij})$
- We can multiply A and B. If $n = p$, then $A \cdot B = (c_{ij})$ where $c_{ij} = \vec{r_i}(A) \cdot \vec{c_i}(B)$. Caution: In general, $A \cdot B \neq B \cdot A$.
- We also define the transpose of a matrix. The transpose of A is $A^t = (d_{ij})$ where $d_{ij} = a_{ji}$. When we take a transpose, we switch the columns into rows and vice versa.

Certain special matrices can be described with other terminology. Suppose we have a matrix, $A = (a_{ij}), i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- If $m = n$, then A is a **square matrix**.
- If $A^t = A$, then A is a **symmetric matrix**. Note: this means A must also be square.
- If $A^t = -A$, then A is said to be **skew-symmetric**.
- If for all i, j such that $i \neq j$, $a_{ij} = 0$, then A is called **diagonal**.

0.2.1 Determinants

The determinant is a property of a square matrix, A.

Geomtric Def'n.

Let $A = \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (vectors in $\mathbb{R}^n$). Let Π be the parallelotope defined by basing them all at the same point. Then $V^\sigma(\Pi) = \det \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (signed n volume).

Example:

$$A = \begin{bmatrix} \vec{c_1} & \vec{c_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det A = -1$$

Algebraic Def'n.

Let $A = (a_{ij})$, $n \times n$. Let A_{ij} = submatrix obtained from A by eliminating row i and column j, $(n-1) \times (n-1)$ matrix.

The minor of A is $M_{ij} = \det A_{ij}$. The cofactor of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Then $\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ (for any i, $1 \leq i \leq n$). We call this the cofactor expansion along the ith row.

Example:

$$A = \begin{bmatrix} 7 & 3 & 12 \\ 2 & 5 & 8 \\ 1 & 5 & 2 \end{bmatrix}$$

Find the determinant.

Use row 3.

$$\det A = \sum_{j=1}^{3} a_{3j} C_{3j} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$\left.\begin{aligned} C_{31} &= (-1)^{3+1} \begin{vmatrix} 3 & 12 \\ 5 & 8 \end{vmatrix} = -36 \\ C_{32} &= (-1)^{3+2} \begin{vmatrix} 7 & 12 \\ 2 & 8 \end{vmatrix} = -32 \\ C_{33} &= (-1)^{3+3} \begin{vmatrix} 7 & 3 \\ 2 & 5 \end{vmatrix} = 29 \end{aligned}\right\} \det A = 1(-36) + 5(-32) + 2(29) = \boxed{-138}$$

<u>Properties of Determinants</u>

1. $\det A = \det A^t$

   (Therefore, we can also use the cofactor expansion along any column as well.)

2. $\det A \neq 0 \iff A$ is invertible

3. $\det AB = \det A \det B$

4. $\det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i} + \vec{r_i}' \\ \vdots \\ \vec{r_n} \end{bmatrix} = \det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i} \\ \vdots \\ \vec{r_n} \end{bmatrix} + \det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i}' \\ \vdots \\ \vec{r_n} \end{bmatrix}$

Note. By (3), $\det(A^{-1}) = (\det A)^{-1}$

(Additionally, elementary row operations affect the determinant in specific ways and are discussed in **??**.)

0.3 Inverse Matricies

An $n \times n$ matrix A may or may not have an **inverse**: A matrix B such that

$$AB = BA = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We write $B = A^{-1}$.

For a linear system, $A\vec{x} = \vec{b}$ with $A,\ n \times n$, if A is invertible:

$$\underbrace{A^{-1}A}_{I_n}\vec{x} = A^{-1}\vec{b}$$
$$\therefore \text{ Sol'n. is } \vec{x} = A^{-1}\vec{b}$$

We can vary $\vec{b}$ and the solutions are immediate.

For the 2×2 case, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A has an inverse $\iff |A| = ad - bc \neq 0$.

Then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$$

Invertible?

$18 - 35 = -17 \neq 0$ ✓

$$A^{-1} = \frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}$$

<u>Check</u>

$$\frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 & 0 \\ 0 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 1. *A $n \times n$ matrix A has an inverse $\iff rkA = n$. If A has an inverse, then A^{-1} is given by $rref[A\, I_n] = [I_n\, A^-1]$*

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Is it invertible?

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[2.\ A_{13}(-1)]{1.\ A_{12}(-2)} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[4.\ A_{31}(1)]{3.\ A_{32}(-1)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{5.\ A_{21}(-2)} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of A^{-1}

1. If A is invertible, so is A^{-1}, $(A^{-1})^{-1} = A$.
2. If A, B are invertible, $n \times n$, then so is $A \cdot B$ and $(BA)^{-1} = B^{-1}A^{-1}$.

   Witness $A\left(B \cdot B^{-}1\right)A^{-1} = AI_nA^{-}1 = I_n$.
3. If A is invertible, so is A^t, $(A^t)^{-1} = (A^{-1})^t$.

Theorem 2. *Suppose A is $n \times n$. Then the following are equivalent*

1. *A is invertible*
2. *$A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$*
3. *$rk(A) = n$*
4. *$rref(A) = I_n$*
5. $\det(A) \neq 0$

Computing Inverses

Recall if $A = a_{ij}$ is a 2×2 matrix, then if A is invertible,

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 3. *If A is invertible then*

$$A^{-1} = \frac{1}{\det A}(b_{ij})$$

where b_{ij} = the ji-th cofactor of $A = (-1)^{i+j} \det A_{ij}$.

Example:
Is A invertible? If so, find A^{-1}.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 5 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 2 - 1 + 1$$

Therefore, A is invertible.

Then $\frac{1}{\det A} \left.\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\right\} (C_{ij})^t$

$C_{11} = (-1)^2 \cdot 2 = 2$ $\quad C_{21} = -1$ $\quad C_{31} = 0$
$C_{12} = (-1)^3 \cdot 2 = -2$ $\quad C_{22} = 1$ $\quad C_{32} = -1$
$C_{13} = 3$ $\quad C_{23} = 1$ $\quad C_{33} = 1$

Then $A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

0.4 Matricies as Linear Transformations

Matricies can be used to model transformations of vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. This is accomplished by having an $m \times n$ matrix, A, written as:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

Example ($\mathbb{R}^n$ to $\mathbb{R}^m$)*:*
Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, $A : \mathbb{R}^2$ to $\mathbb{R}^3$ and:

$$A\vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$$

Note. We are getting a linear combination of the column vectors of A. In other words, $A\vec{v} = x\vec{a} + y\vec{b}$.