0.1 Determinants and Cross Products

0.1.1 Determinants

Example (Computing areas)*:*
Suppose we have a paralellogram made by the vectors $(1,4)$ and $(8,3)$. Compute the area.

We could compute the area of the paralellogram using geometry, but we can easily see the area as 29 using the determinant.

$$A = \left| \det \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \right| = |1 \cdot 3 - 8 \cdot 4| = |-29| = 29$$

We can interpret this matrix as a linear transformation from the standard basis in $\mathbb{R}^2$, a change of basis.

$$e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto f_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto f_2 = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

0.1.2 Cross Product

With the standard basis in $\mathbb{R}^3$, let $u = (u_1, u_2, u_3) = u_1e_1 + u_2e_2 + u_3e_3$ and $v = (v_1, v_2, v_3) = v_1e_1 + v_2e_2 + v_3e_3$.

We define the cross product of $u \wedge v$ as:

$$u \wedge v = \left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}, -\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}, \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right)$$

This also has a geometric interpretation: the three components of the cross product are the areas of the "shadows" of the the paralellogram formed by u and v in yz, xz, xy -planes respectively.

Properties:

1. The area of the paralellogram u, v is equal to $|u \wedge v|$.
2. The direction of $u \wedge v$ (if non-zero) is perpendicular to the paralellogram u, v using the right hand rule.
3. The volume of the paralellopiped u, v, w is equal to $(u \wedge v) \cdot w = \det \begin{bmatrix} u & v & w \end{bmatrix} = \det \begin{bmatrix} u \\ v \\ w \end{bmatrix}$
4. $u \wedge v = -v \wedge u$

5. $u \wedge (v + w) = u \wedge v + u \wedge w$
6. $(u(t) \wedge v(t))' = u'(t) \wedge v(t) + u(t) \wedge v'(t)$