0.1 Introduction to Linear Transformations

Let A be an $m \times n$ matrix. We can interpret it as a function or transformation between vector spaces, where $A : \mathbb{R}^n \to \mathbb{R}^m$.

Note that A is a linear transformation since $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$ and $A(c\vec{v}) = cA\vec{v}$.

Example:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

What does it do?

Let

$$\vec{e_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

.

Example:
Find a matrix for $\frac{\Pi}{2}$ rotation.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

0.2 Eigenvalues and Eigenvectors

Most linear transformations can be understood with eigenvalues and eigenvectors.

Definition:
Let A be $n \times n$.

An **eigenvalue** of A is a scalar λ such that $A\vec{v} = \lambda\vec{v}$ has a nonzero solution $\vec{v}$.

An **eigenvector** $\vec{v}$ for λ is a nonzero $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenspace** for λ is the set of all $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenbasis** for λ is a basis λ's eigenspace.

We will look at a simple matrix to give concrete examples for all of these definitions.

Example:
Given A $n \times n$, find its eigenvalues, eigenvectors, eigenspace, and eigenbasis.

There are two eigenvalues, $\lambda = 2, 1$.

For $\lambda = 2$, $\vec{e_1}$ is one possible eigenvector. Another possible eigenvector is $3\vec{e_1}$. The eigenspace for $\lambda = 2$ is the x-axis. The eigenbasis is simply $\{\vec{e_1}\}$.

For $\lambda = 1$, the eigenbasis is simply $\{\vec{e_2}\}$.

We can determine this intutitively by considering some vectors and applying the linear transformation A.

To determine the eigenvalues and eigenvectors analytically, note that $A\vec{v} = \lambda\vec{v}$ for nonzero $\vec{v}$ is the same as $(A - \lambda I)\vec{v} = \vec{0}$. Thus all λ satisfy $\det(A - \lambda I) = 0$.