0.1 Defining Vector Spaces and Subspaces

We've seen the sets $\mathbb{R}^2$ and $\mathbb{R}^3$ where the elements are vectors as translations in plane and 3-space. These sets have an algebraic structure.

Definition (Vector Space)*:*
Any set following axioms A1-A10 is called a **vector space**.

Given a vector space A, vectors $u, v, w \in A$ and scalars r, s:

A1. We can define addition (and the space is closed under addition).

A2. We can define scalar multiplication (and the space is closed under scalar multiplication).

A3. $u + v = v + u$

A4. $(u + v) + w = u + (v + w)$

A5. There exists an additive identity in A.

A6. There exists an additive inverse for any $u \in A$.

A7. There exists a multiplicative identity in A.

A8. $(rs)v = r(sv)$

A9. $r(u + v) = ru + rv$

A10. $(r + s)v = rv + sv$

By studying vector spaces, we can develop algebraic machinery to apply to all kinds of other objects. For example, this idea will be explored for Fourier series.

Example:
$Q = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ (Polynomials of degree 2 or less)

This is a vector space!

A1. Adding doesn't increase the degree. ✓

A2. Scalar multiplication similarly doesn't increase the degree. ✓

A3. Addition in $\mathbb{R}$ is commutative. ✓

A4. By the associative property of $\mathbb{R}$,

$$
\begin{aligned}
&\Big((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\Big) + (c_0 + c_1x + c_2x^2)\\
&= \Big((a_0 + b_0) + c_0\Big) + \Big((a_1 + b_1) + c_1\Big)x + \Big((a_2 + b_2) + c_2\Big)x^2\\
&= \Big(a_0 + (b_0 + c_0)\Big) + \Big(a_1 + (b_1 + c_1)\Big)x + \Big(a_2 + (b_2 + c_2)\Big)x^2\\
&= a_0 + a_1x + a_2x^2 + (b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 \checkmark
\end{aligned}
$$

A5. 0 polynomial ✓

A6. $-(a_0 + a_1 x + a_2 x^2) = -a_0 - a_1 x - a_2 x^2$ ✓

A7. $1 \cdot (a_0 + a_1 x + a_2 x^2) = a_0 + a_1 x + a_2 x^2$ ✓

A8. ✓

A9. ✓

A10. ✓

Example:
$V = \mathbb{R}^2$
$S = \{(x, y) \mid x^2 - y^2 = 0\}$
Is S a vector space?

No, since addition is not closed.

Example:

$$V = M_{3\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \middle| \, a, b, c, d, e, f \in \mathbb{R} \right\}$$

$S = \{A \in V \mid \text{columns each sum to } 0\}$
Is S a vector space?

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -2 \end{bmatrix} \in S$. Now, $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Closed under scalar mult? ✓
Consider $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ s.t. $a + b + c = 0$ and $d + e + f = 0$.
Then $m \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} ma & md \\ mb & me \\ mc & mf \end{bmatrix}$.
So $ma + mb + mc = m(a + b + c) = m(0)$ and $md + me + mf = m(d + e + f) = m(0)$. ✓

Closed under addition? ✓
Consider $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}, \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} \in S$.
Now $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} = \begin{bmatrix} a + a' & d + d' \\ b + b' & e + e' \\ c + c' & f + f' \end{bmatrix}$
Check $(a + a') + (b + b') + (c + c') = (a + b + c) + (a' + b' + c') = 0$.

Note. Since the above examples of S are subsets of known vector spaces, we just have to check:

1. Closed under $+, -$
2. Closed under scalar multiplication

These are called **subspaces**.

Vectors spaces do not just include $\mathbb{R}^n$ and $\mathbb{C}^n$. We have:

- $C^k(I)$: function on an internal I with k continuous derivatives
- $P_n(\mathbb{R})$: polynomials with degree up to n
- $M_{m\times n}(\mathbb{R})$

Example:
Consider $y'' + a_1(x)y' + a_2(x) = 0$ on an interval I. Let S be the set of solutions to this LODE. Is S a vector space?

- Addition: If $y_1, y_2 \in S$, is $y_1 + y_2 \in S$?

  $$(y_1+y_2)''+a_1(x)(y_1+y_2)'+a_2(x)(y_1+y_2) = y_1''+y_2''+a_1(x)(y_1'+y_2')+a_2(x)(y_1+y_2) = 0+0$$

  Therefore, S is closed under addition.

- Scalar Multiplication: If $y \in S$, is $cy \in S$? Yes since $(cy)'' = cy''$ and $(cy)' = cy'$.

Therefore S is a vector space, a subspace of $C^2(I)$.

Example:
Let $V = \mathbb{R}^3$ and S be the solutions to the linear system:

$$\begin{aligned} 2x + 3y + z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

Then $S \subset V$ is a subspace. All vectors in the subspace lie on a line through the origin contained on both planes.

Using this concept of subspaces, we can find solutions to certain differential equations or systems of linear equations.

Definition (Null Space)*:*
The solutions of a homogeneous linear system $A\vec{x} = 0$ is called the **null space** of A, any $A_{m\times n}$. It is a subspace of $\mathbb{R}^n$ and is also known as the **kernel** of A.

For $\mathbb{R}^3$, subspaces consist of either a point $\vec{0}$, a line through $\vec{0}$, or a plane through $\vec{0}$.

0.2 Spanning Sets

Definition (Span)*:*
A linear combination of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is a vector $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + n\vec{v_n}$ in V.

The **span** of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is the set of all linear combinations. The span is a subspace of the vector space V.

Example:
Compute the span of $\{(-4,1,3),\ (5,1,6),\ (6,0,2)\}$ in $\mathbb{R}^3$.

We want the set $\{(a,b,c)\}$ that are linear combinations of the three vectors. In other words, the solutions of the linear system:

$$\begin{bmatrix} -4 & 5 & 6 & a \\ 1 & 1 & 0 & b \\ 3 & 6 & 2 & c \end{bmatrix}$$

$$\xrightarrow{1.\ 2.\ 3.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 9 & 6 & a+4b \\ 0 & 3 & 2 & c-3b \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 3 & 2 & c-3b \\ 0 & 0 & 0 & a+13b-3c \end{bmatrix}$$

1. P_{12}
2. $A_{12}(4)$
3. $A_{13}(-3)$
4. P_{23}
5. $A_{23}(-3)$

There is a solution if and only if $\text{rk}A = \text{rk}A^{\#}$. So $a + 13b - 3c = 0$. We conclude the span is on a plane $\boxed{x + 13y - 3z = 0}$.

Note. One of the three vectors is redundant.

Example:
Write $(-4,1,3)$ as a linear combination of $(5,1,6)$ and $(6,0,2)$.

We want constants c_1, c_2 such that:

$$c_1 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

We need to solve $\begin{bmatrix} 5 & 6 & -4 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}$

$$\xrightarrow[2.\ 3.]{1.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 6 & -9 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{6.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} - \dfrac{3}{2} \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$.

The moral of this example is that sometimes a spanning set has redundant vectors. We would like a spanning set to have no redundant vectors or a "minimally spanning set."

Definition (Linear Dependence)*:*
A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ in a vector space V is **linearly dependent** if there exists some $c_1, c_2, \ldots, c_n$ not all zero such that: $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$. (This is called the dependence relation, and we would say one of the vectors is redundant.)

Otherwise, the set is **linearly independent** and the relation $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$.

Example:
The vectors $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ are linearly dependent.
Find the dependence relation.

We need to solve $A\vec{x} = \vec{0}$ and $\text{rk}A \leq 2$ for there to be nontrivial solutions.

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Parameterizing, we find that $c_3 = t,\ c_2 = t,\ c_1 = 2t - t = t$.

Therefore $\boxed{\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}}$.

0.2.1 Method for Testing Spans

Given $\{\vec{v_1}, \ldots, \vec{v_n}\}$, to determine if they are linearly dependent, form a matrix $A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \ldots & \vec{v_n} \end{bmatrix}$. Then the set is linearly dependent if and only if $A\vec{c} = \vec{0}$ has nontrivial solutions. In other words, there exists some non-zero scalars $c_1, \ldots, c_k$ such that $c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}$ if and only if $\text{rk}A < k$.

The technique is to commit ERO's and find rref(A).

For example, suppose we have a span and rref($A^{\#}$) is:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & \uparrow & & \uparrow & & \end{matrix} \end{array}$$

We have two free variables.

<u>Claim:</u> Each free variable gives a nontrivial dependence with bound variables.

So for this example, there are two dependence relations where $\vec{v_2}$ and $\vec{v_4}$ are redundant and $\vec{v_1},\ \vec{v_3},\ \vec{v_5}$ form a linearly independent span.

In general, we can conclude that $v_1, \ldots, \vec{v_k}$ are linearly independent if and only if there are no free variables in $\mathrm{rref}(A^{\#})$ or equivalently $\mathrm{rk} A = k$.