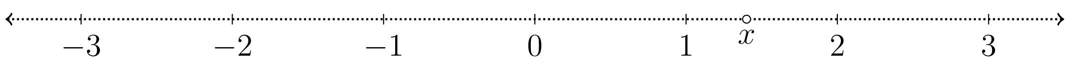
# 0.1 Constructing the Real Numbers

## 0.1.1 Upper Bounds

Now, we seen in the previous section that $\mathbb{Q}$ has "gaps". $x^2 = 2$ has no solution in $\mathbb{Q}$.




We need to fill in these gaps somehow while not knowing where the gaps and holes are.

*Definition* (Upper Bound)*:*
Let $E \subset S$ ordered. If there exists $\beta \in S$ such that for all $x \in E$, $x \leq \beta$, then $\beta$ is an **upper bound (u.b.)** for $E$. We say $E$ is bounded above.

A lower bound can be defined similarly with "greater than or equal to."

*Example:*
Consider the set $A = \{x \mid x^2 < 2\}$. 2 is an u.b. for A. $\dfrac{2}{3}$ is also an u.b. for $A$.

*Definition* (Least Upper Bound)*:*
If there exists an $\alpha \in S$ such that:

1. $\alpha$ is an upper bound of $E$
2. If $\gamma < \alpha$, then $\gamma$ is not an upper bound of E.

Then $\alpha$ is called a **least upper bound (lub)** of $E$ or the **suprenum** of $E$. Write $\alpha = \sup E$.

*Example:*
Let $S = \mathbb{Q}$.

1. $E = \left\{\dfrac{1}{2}, 1, 2\right\}$ $\boxed{\sup E = 2}$
2. $E = \{x \in \mathbb{Q} \mid x < 0\}$ $\boxed{\sup E = 0}$
3. $E = \mathbb{Q}$ $\boxed{\sup E \text{ does not exist}}$
4. $E = A$ (as defined above) $\boxed{\sup E \text{ does not exist}}$

*Definition* (Least Upper Bound Property)*:*
A set $S$ has the **least upper bound property** if every nonempty subset of $S$ that has an upper bound has a least upper bound.

## 0.1.2 Dedekind Cuts

*Definition* (Dedekind Cut)*:*
A **Dedekind cut** $\alpha$ is a subset of $\mathbb{Q}$ such that:

1. $\alpha \neq \emptyset, \mathbb{Q}$

2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$. (Closed downward)
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. (No largest number)

*Example:*
$\alpha = \{x \in \mathbb{Q} \mid x < 0\}$ is a cut.

1. $\alpha \neq 0, \mathbb{Q}$ ✓
2. Let $p \in \alpha$, $q \in \mathbb{Q}$. Assume $q < p$. By the transitivity property of order, $q < 0$. Thus, $p \in \alpha$. ✓
3. Let $p \in \alpha$ and $r \in \alpha$ such that $r = \frac{q}{2}$. Since $q < 0$, $q < \frac{q}{2}$. Thus $q < r$. ✓

*Example:*
$\gamma = \{r \mid r \leq 2\}$ is not a cut. This set does have a largest element, 2.

*Definition* (Rational Numbers)*:*
Let $\mathbb{R} = \{\alpha \mid \alpha \text{ is a cut }\}$.

We also define the following:

- $\alpha < \beta$ to mean $\alpha \subsetneqq \beta$. This is an order.
- $\alpha + \beta = \{r + s \mid r \in \alpha \text{ and } s \in \beta\}$. (This means $\mathbb{R}$ is a field.)
- $\alpha \cdot \beta$
    1. For positive cuts $\{\alpha \mid \alpha > 0^*\} = \mathbb{R}_+$:
       If $\alpha, \beta \in \mathbb{R}_+$, let $\alpha \cdot \beta = \{p \mid p \leq rs \, \text{for some } r \in \alpha, s \in \beta, \, andr, s > 0\}$.
    2. For cases with negative cuts, $\alpha \cdot \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*,\ \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*,\ \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*,\ \beta < 0^* \end{cases}$
       where the products are the same as defined for postive cuts and $-\alpha$, $-\beta$ are the additive inverses of $\alpha$, $\beta$ respectively. (The additive inverses are defined below.)

**Theorem 1.** *$\mathbb{R}$ is an ordered field with the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

### Proofs Showing $\mathbb{R}$ is an Ordered Field

The following section proves Theorem 1.

*Proof.*

Step 1: We must show there is order on $\mathbb{R}$.
Let $\alpha, \beta, \gamma \in \mathbb{R}$. We must show that they demonstrate both trichotomy and transitivity.

1. Trichotomy:
   It is clear that at most one of the following can be true: $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$. For example, if $\alpha < \beta$, then $\alpha \subsetneqq \beta$ and by the definition of a proper subset, $\alpha \neq \beta$ and $\beta$ is not a proper subset of $\alpha$.

To show at least one of them must be true, suppose the first two statements are false. Then $\alpha$ is not a subset of $\beta$. By definition of a proper subset, there exists some $a \in \alpha$ such that $a \notin \beta$. Consider some $b \in \beta$. Since $\beta$ is closed downward, $b < a$. This also means $b \in \alpha$ since $\alpha$ is also closed downward. This shows that $\beta \subsetneqq \alpha$. Thus $\beta < \alpha$. We conclude that at least one of these statements must be true.

2. Transitivity:
   We assume that $\alpha < \beta$ and $\beta < \gamma$. By definition of $<$, $\alpha \subsetneqq \beta$ and $\beta \subsetneqq \gamma$. By definition of a proper subset, $\alpha \subsetneq \gamma$ and we conclude that $\alpha < \gamma$.

We have shown the cuts demonstrate order.

Step 2: Next, we must show that addition is closed (**A1**).
Let $\alpha, \beta \in \mathbb{R}$ and $\gamma = \{r + s \mid r \in \alpha \text{ and } b \in \beta\}$. To show addition is closed, we must show that $\gamma$ is a cut.

1. First, we must show that $\gamma \neq \emptyset$, $\mathbb{Q}$.
   It should be clear that $\gamma$ cannot be the empty set. Since $\alpha, \beta \neq \mathbb{Q}$, there exists $c \notin \alpha$, $d \notin \beta$. Now $a < c$ and $b < d$ for all $a \in \alpha$, $b \in \beta$. Thus $a + b < c + d$. Therefore $c + d \notin \gamma$. We conclude that $\gamma \neq \emptyset, \mathbb{Q}$.

2. Second, we must show that $\gamma$ is closed downward.
   Let $p \in \gamma, q \in \mathbb{Q}$. Assume $q < p$. Since $p \in \gamma$, there exists $r \in \alpha$ and $s \in \beta$ such that $p = r + s$ so $q < r + s$. This means $q - s < r$. Since $\alpha$ is closed downward, $q - s \in \alpha$. Then $q = q - s + s$ where $q - s \in \alpha$ and $s \in \beta$. We have shown that $\gamma$ is closed downward.

3. Third, we must show that $\gamma$ has no largest number.
   Let $p \in \gamma$. Then there exists some $a \in \alpha$ and $b \in \beta$ such that $p = a + b$. Since both cuts $\alpha, \beta$ have no largest number, there exists $c \in \alpha$ and $d \in \beta$ where $a < c$ and $b < d$. Thus $c + d \in \gamma$ and $p < c + d$. We have shown that $\gamma$ has no largest member.

We have shown that $\gamma$ meets the definition of a cut.

Step 3: **A2** and **A3** follows since addition in $\mathbb{Q}$ is commutative and associative.

Step 4: We must show that $0^* = \{q \in \mathbb{Q} \mid q < 0\}$ is the additive identity for $\mathbb{R}$ (**A4**).
In other words, we must show that $\alpha + 0^* = \alpha$. Let $\alpha \in \mathbb{R}$.

1. First, we must show that $\alpha + 0^* \subset \alpha$.
   Let $a \in \alpha$ and $b \in 0^*$. Since $b < 0$, $a + b < a$. Thus $a + b \in \alpha$ (since $\alpha$ is closed downward). We conclude that $\alpha + 0^* \subset \alpha$.

2. Second, we must show that $\alpha \subset \alpha + 0^*$.
   Let $a, b \in \alpha$. Since $\alpha$ has no largest member, we can pick $a, b$ such that $b > a$. Then $a - b \in 0^*$ and similar to before $a = b + (a - b) \in \alpha + 0^*$. We conclude that $\alpha \subset \alpha + 0^*$.

Thus we conclude that $\alpha + 0^* = \alpha$.

Step 5: Next, we must show that the additive inverse for $\alpha \in \mathbb{R}$ is $\beta = \{p \in \mathbb{Q} \mid \text{ there exists } r > 0 \text{ s.t. } -p - r \notin \alpha\}$ (**A5**).

Let $\alpha \in \mathbb{R}$.

1. First, we must show that $\beta$ is a cut.

   i. We must show that $\beta$ is non-trivial.
   Since $\alpha \neq \emptyset$, there exists some $a \in \alpha$. Since $\alpha$ is closed downward, $a - b \in \alpha$ for all $b > 0$. Thus $-a \notin \beta$ so $\beta \neq \mathbb{Q}$.

   Since $\alpha \in \mathbb{Q}$, there exists some $c \notin \alpha$. Consider $d = -c - 1$. Then $-d - 1 \notin \alpha$ (since $c = -d - 1$). Thus $d \in \beta$ so $\beta \neq \emptyset$.

   ii. We must show that $\beta$ is closed downward.
   Let $p \in \beta$, $q \in \mathbb{Q}$. We assume that $q < p$. There exists some $r > 0$ s.t. $-p-r \notin \alpha$. Since $q < p$, $-q - r > -p - r$. Thus $-q - r \notin \alpha$ (since $\alpha$ is closed downward), so $q \in \beta$. We conclude that $\beta$ is closed downward.

   iii. We must show that $\beta$ has no largest member.
   Consider $j = f + (h/2)$. Then $j > f$, and $-j - (h/2) = -f - h \notin \alpha$. Then $j \in \beta$. Thus we find $\beta$ also has no largest member.

   We conclude that $\beta$ satisfies the definition of a cut.

2. Second, we must show that $\alpha + \beta = 0^*$.

   i. We need to show that $\alpha + \beta \subset 0^*$.
   Let $s \in \alpha$ and $t \in \beta$. By definition of $\beta$, there exists $u > 0$ s.t. $-t - u \notin \alpha$. Then $-t - u > s$ and it follows that $s + t < -u < 0$. Therefore $s + t \in 0^*$.

   ii. We also need to show that $0^* \subset \alpha + \beta$.
   Let $v \in 0^*$ and $w = -v/2$. Then $w > 0$. By the Archimedean property of $\mathbb{Q}$, there exists $n \in \mathbb{Z}$ s.t. $nw \in \alpha$ and $(n + 1)w \notin \alpha$. Consider $x = -(n + 2)w$. Now $-x = nw + 2w \implies -x - w = nw + w = (n+1)w \notin \alpha$. By definition of $\beta$, $x \in \beta$. We find $v = -2w = -(n + 2)w + nw = p + x$ and conclude that $v \in \alpha + \beta$.

   This shows that $\alpha + \beta = 0^*$.

We have proved that $\beta$ is the additive inverse for $\alpha$ and that $\mathbb{R}$ satisfies all the addition axioms. Next we will show that $\mathbb{R}$ satisfies the multiplication axioms.

Step 6: We must show that $\mathbb{R}$ is closed under multiplication (**M1**). We will only consider the positve case for multiplication (because I am lazy and the whole thing is tedious. Most books only do the addition axioms.)
Let $\alpha, \beta \in \mathbb{R}_+$ and $\alpha\beta = \{p \mid p \leq ab, a \in \alpha, b \in \beta, \text{ and } a, b > 0\}$. We must prove $\alpha\beta \in \mathbb{R}$ as before.

i. Since $\alpha, \beta$ are non-trivial, there exists some $a \in \alpha, b \in \beta$ s.t. $a, b > 0$. Now $ab \leq ab$. Thus $ab \in \alpha\beta$ so $\alpha\beta \neq \emptyset$.

   There also exists some $c \notin \alpha, d \notin \beta$ s.t. $c > a$ and $d > b$ for all $a \in \alpha$ and $b \in \beta$. Now $cd > ab$ for all $a \in \alpha, b \in \beta$. Thus $cd \notin \alpha\beta$ so $\alpha\beta \neq \mathbb{Q}$.

ii. Let $p \in \alpha\beta$, $q \in \mathbb{Q}$, and $q < p$. Now $p < ab$ for some $a \in \alpha$, $b \in \beta$. By order of $\mathbb{Q}$, $q < p < ab$ so $q \in \alpha\beta$.

iii. Since $\alpha, \beta$ have no largest member, we can pick some $c > a$ and $d > b$ s.t. $c \in \alpha$ and $d \in \beta$. Thus $p \leq ab < cd$, so $ab \in \alpha\beta$.

We have shown that $\alpha\beta$ satisfy all three properties of a cut.

Step 6: We must show that multiplication in $\mathbb{R}$ is commutative (**M2**) and associative (**M3**). Let $p \in \alpha\beta$. Now $p \leq ab$ for some $a \in \alpha$, $b \in \beta$, $a, b > 0$. Since multiplication in $\mathbb{Q}$ is commutative, $p \leq ba$. Thus, $p \in \beta\alpha$. Clearly $\alpha\beta \subset \beta\alpha$ and $\beta\alpha \subset \alpha\beta$. We conclude that $\alpha\beta = \beta\alpha$ and multiplication in $\mathbb{R}$ is commutative.

The proof that multiplication in $\mathbb{R}$ is associative is similar relying on multiplication in $\mathbb{Q}$ is associative.

Step 7: We must show that there is a multiplicative identity in $\mathbb{R}$ (**M4**).
We will define it to be $1^* = p \in \mathbb{Q} \mid p < 1$. Let $\alpha \in \mathbb{R}$. We must demonstrate that $\alpha \cdot 1^* = \alpha$.

1. We must show $\alpha \cdot 1^* \subset \alpha$.
   Let $p \in \alpha \cdot 1^*$. Now $p \leq a \cdot b$ for some $a \in \alpha$, $b \in 1^*$. Since $b < 1$, $p \leq ab < a$ so $p \in \alpha$.

2. We also must show $\alpha \subset \alpha \cdot 1^*$.
   We can choose some $c \in \alpha$ s.t. $c > a$. This implies $\dfrac{a}{c} < 1$ so $\dfrac{a}{c} \in 1^*$. Thus $a = c \cdot \dfrac{a}{c} \in \alpha \cdot 1^*$.

We conclude that $\alpha \cdot 1^* = \alpha$.

Step 8: We must show that there is a multiplicative inverse in $\mathbb{R}$ (**M5**).
We will define the multiplicative inverse for $\alpha \in \mathbb{R}_+$ as:
$\beta = 0^* \cup \{0\} \cup \{p \in \mathbb{Q} \mid \text{there is an } r \in \mathbb{Q} \text{ with } r > 1 \text{ and } \frac{1}{rp} \notin \alpha\}$. Let $\alpha \in \mathbb{R}$.

1. We must show $\beta \in \mathbb{R}$.

   i. Clearly $\beta \neq \emptyset$. Since $\alpha \neq \mathbb{Q}$ and $\alpha \in \mathbb{R}_+$, there exists some $a \in \alpha$ where $a > 0$. Then $1/a \notin \beta$ since there is no

The cases for multiplication with negative cuts are also similar using the identity: $\gamma = -(-\gamma)$.