0.1 Systems of Linear Equations

Another application of matricies of great interest is representing systems of linear equations.

Definition:
An equation is **linear** in n variables $x_1, x_2, \dots, x_n$ if each term contains at most one x_i and that x_i appears to the first power.

$\therefore$ A "system of m linear equations in n unknowns."

Example (Cutting n by $n-1$ dimensional objects)*:*
Suppose we have two planes which will separate regions in a three dimensional space.

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x + 2y + z &= 0 \end{aligned}$$

Sol'n. $\{(x, y, z) :$ satisfy all equations$\}$

The solution would be the intersection of these planes. Since they extend out infinitely, we would expect them to meet long a line contained in both planes.

Note. Multiplying by a constant, adding a multiple of one to another, and permuting the rows all do not change the solution.

Possible Outcomes: A system of m linear equations in n unknowns has either $0, 1, \text{or inf}$ solutions.

The technique to solving systems of linear equations is to develop a faster, simple method using matrices. We represent the system as a matrix, preform a series of operations to obtain a simpler matrix, and get our solution from the simpler solution.

Example:

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x + 2y + z &= 0 \end{aligned}$$

We take the coefficients and constants and put them into a matrix:

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \vec{r_1} \to \vec{r_1} - \vec{r_2}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \vec{r_2} \to \vec{r_2} - 2\vec{r_1}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 5 & 2 \end{bmatrix}$$

$$\Big\downarrow \vec{r_2} \to \frac{1}{2}\vec{r_2}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{5}{2} & 1 \end{bmatrix}$$

Thus, we find:

$$\begin{aligned} x - 2z &= 1 \\ y + \frac{5}{2}z &= -1 \end{aligned}$$

Sol'n. $z = t$ as a free variable

$$\begin{aligned} x &= 1 + 2t \\ y &= -1 - \frac{5}{2}t \end{aligned}$$

It's a line as expected!

$$L(t) = \left(1 + 2t, -1 - \frac{5}{2}t, t\right)$$

0.2 Vector Form

Given:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The **vector form** of the system is: $A\vec{x} = b$ The **coefficient matrix** is $A = (a_{ij})$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

The **augmented matrix** is $A^{\#} = \begin{bmatrix} A \; \vec{b} \end{bmatrix}$.

Definition (Elementary Row Operations)*:*
Elementary row operations (EROs) are defined as follows:

- $P_{ij} : \vec{r_i} \leftrightarrow \vec{r_j}$
- $M_i(c) : \vec{r_i} \leftarrow c\vec{r_i}$
- $A_{ij}(c) : \vec{r_j} \leftarrow \vec{r_j} + c\vec{r_i}$

These elementary row operations let us solve systems of linear equations in the same way we learned in early math classes. Adding equations together is the same as the row sum operation etc. Using EROs, the process to solve these systems is to write the augmented matrix of the system, perform EROs to get a reduced matrix and obtain the simpler linear system.

Example:
Solve if possible.

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 5 \\ x - y - 2z &= -5 \end{aligned}$$

We write the augmented matrix of the system and preform EROs:

$$A^{\#} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix} \xrightarrow[\text{2. } A_{13}(-1)]{\text{1. } A_{12}(-2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

$$\xrightarrow{\text{3. } A_{23}(2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \operatorname{ref} A^{\#}$$

$$\xrightarrow[\text{6. } A_{31}(-1)]{\text{5. } A_{32}(1)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{7. } A_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \operatorname{rref} A^{\#}$$

We obtain two matrices, the row echelon form and the reduced row echlon form. From ref $A^{\#}$, we find:

$$\begin{aligned} x + y + z &= 3 \\ y - z &= -1 \\ z &= 2 \end{aligned}$$

From rref $A^{\#}$, we find:

$$\begin{aligned} x &= 0 \\ y &= 1 \\ z &= 2 \end{aligned}$$

$$\boxed{(0,\ 1,\ 2)}$$

Thus, (0, 1, 2) solves the system of equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -5 \end{bmatrix}$$

Note. If the number of leading 1's equals the number of unknowns, we obtain one solution.

Definition (Echelon Forms)*:*
The **row echelon form** (ref) of any matrix is one such that:

(a) The first nonzero entry of each nonzero row is 1. ("Leading one")

(b) If a row has a leading one, each row above it has a leading 1 further to the left.

The **reduced row echlon form** (rref) of any matrix also has the conditon:

(c) Each entry above a leading 1 is 0.

To put a matrix into its ref or rref form:

1. Use P_{ij} so row i has the left most nonzero entry among rows $\geq i$.
2. Use $M_i(k)$ to create a leading 1 in $\vec{r_i}$.
3. Use $A_{ij}(k)$ to zero out column below leading 1 in $\vec{r_i}$. For rref, zero out columns above as well.
4. If there is no nonzero row below i, stop.

   Else:
5. Replace $i = i + 1$, repeat on submatrix with rows $\geq i + 1$.

0.3 Theorem of Rank

Definition (Rank)*:*
The number of leading 1's in ref(A) is the rank of A, usually written as "rk(A)."

Theorem 1. *Let A, $A^{\#}$ be the coefficient and augmented matrix for a system of linear equations with m equations and n unknowns.*

Then, $\exists$

1. *No solutions if $\mathit{rk}\, A < \mathit{rk}\, A^{\#}$.*
2. *Unique solutions if $\mathit{rk}\, A = \mathit{rk}\, A^{\#} = n$.*
3. *∞ solutions if $\mathit{rk}\, A = \mathit{rk}\, A^{\#} < n$.*

Note. There are unique solutions if exactly all variables are bound and infinitely many if there are $n - \text{rk}A$ free variables.

Example:
Here is an illustration of this. Suppose we have:

$$\text{rref}A^{\#} = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \begin{bmatrix} 1 & * & 0 & * & 0 & a \\ 0 & 0 & 1 & * & 0 & b \\ 0 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 0 & d \end{bmatrix} \end{array}$$

If $d \neq 0$, there are no solutions. If $d = 0$, there are ∞ solutions. There are no unique solutions possible since there are two free variables: $x_2 = s$ and $x_4 = t$. The bound variables are x_1, x_2, and x_5.

From this illustration, we can see two important cases:

1. If we have a homogenous system, $A\vec{x} = 0$, then we always have the trivial solution $\vec{x} = 0$. $\exists$ nontrivial solutions $\iff$ rk$(A) < n$.
2. If A is $m \times n$ and $m < n$, then since rk$(A) \leq m$, rk$(A) < n$

   $\therefore$ have either 0 or ∞ solns

Example:
Fix $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{w}$ (all) such that $\vec{v} \times \vec{w} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\vec{v} \times \vec{w} = \begin{bmatrix} w_3 - w_2 \\ w_1 - w_3 \\ w_2 - w_1 \end{bmatrix}$$

Now we have a system of linear equations with $\vec{w}$ as the unknown.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^{\#} = \begin{bmatrix} 0 & -1 & 1 & b_1 \\ 1 & 0 & -1 & b_2 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow{1.\ P_{12}} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & -1 & 1 & b_1 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow[3.\ M_2(-1)]{2.\ A_{13}(1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 1 & -1 & b_2+b_3 \end{bmatrix}$$
$$\xrightarrow{4.\ A_{23}(-1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 0 & 0 & b_1+b_2+b_3 \end{bmatrix}$$

We have solutions $\iff b_1 + b_2 + b_3 = 0 = \vec{v} \cdot \vec{b}$. Then we have one free variable

$$\left.\begin{aligned} w_3 &= t \\ w_2 &= -b_1 + t \\ w_1 &= b_2 + t \end{aligned}\right\} = \boxed{(b_2 + t,\ -b_1 + t,\ t)}$$