To define the real numbers and their properties, we need some different ideas.

0.1 Set Theory

Definition:
A **set** is a collection of objects called elements of the set.

Example:

1. $S = \{1, 2, 3\}$ $(= \{1, 2, 3, 3\})$
2. $E = \{\text{Even integers }\}$
3. $\{\text{College students}\}$

Notation:

- $x \in S$ means x is in S.
- $x \notin S$ means x is not in S.
- The empty set $\emptyset$ is the set with no elements.
- $A \subseteq B$ means A is a subset of B (i.e. if $x \in A$, then $x \in B$).
- If $A \subseteq B$ but $B \subsetneq A$ A is a proper subset.

If $A \subseteq B$ and $B \subseteq A$ then $A = B$. Otherwise $A \neq B$.

We can define more sets in terms of other sets. *Set Operations:* Let A and B be sets.

- Union: $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$
- Compliment: $B - A = \{x \,|\, x \in B \text{ and } x \notin A\}$
- Product: $A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}$

If U is a universal set (set of everything in context), we write $\bar{A} = U - A$
$= \{x \,|\, x \in U \text{ and } x \notin A\}$.

0.2 Functions and Relations

It is also important to define some types of relations and functions.

0.2.1 Relations

Definition (Relation)*:*
A (binary) **relation** R on a set S is a subset of $S \times S$. If $(a, b) \in R$, we write aRb.

Example (of relations)*:* 1. L ”loves” is a relation on $P \times P$ (where P is a set of all people).

2. The set $R = \{(0,0), (0,1), (2,2), (7,18)\}$ is a relation on $\mathbb{Z}^+$. We would write $0R0$, $0R1$, $2R2$, and $7R18$.

Definition (Equivalence Relation)*:*
An equivalence relation on a set S is a relation s.t.:

1. Reflexive: For each $a \in S$, $a \sim a$.
2. Symmetric: For $a, b \in S$, if $a \sim b$, then $b \sim a$.
3. Transitive: For $a, b, c \in S$, if $a \sim b$ and $b \sim a$

0.2.2 Functions

Functions in the general sense are also a type of relation.

Definition (Function)*:*
A **function**, F from a set A to a set B is a relation s.t.: if aFb and aFb' then $b = b'$.
This is a rule that assigns a unique $a \in A$ to a unique $b \in B$. Write $f : A \to B$ and $f(a) = b$.