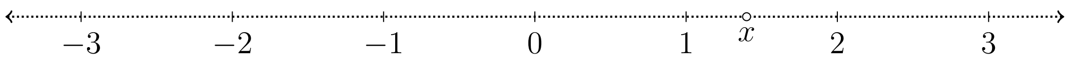
0.1 Constructing the Real Numbers

0.1.1 Upper Bounds

Now, we seen in the previous section that $\mathbb{Q}$ has "gaps". $x^2 = 2$ has no solution in $\mathbb{Q}$.



We need to fill in these gaps somehow while not knowing where the gaps and holes are.

Definition (Upper Bound)*:*
Let $E \subset S$ ordered. If there exists $\beta \in S$ such that for all $x \in E$, $x \leq \beta$, then β is an **upper bound (u.b.)** for E. We say E is bounded above.

A lower bound can be defined similarly with "greater than or equal to."

Example:
Consider the set $A = \{x \mid x^2 < 2\}$. 2 is an u.b. for A. $\dfrac{2}{3}$ is also an u.b. for A.

Definition (Least Upper Bound)*:*
If there exists an $\alpha \in S$ such that:

1. α is an upper bound of E
2. If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called a **least upper bound (lub)** of E or the **suprenum** of E. Write $\alpha = \sup E$.

Example:
Let $S = \mathbb{Q}$.

1. $E = \left\{\frac{1}{2}, 1, 2\right\}$ $\boxed{\sup E = 2}$
2. $E = \{x \in \mathbb{Q} \mid x < 0\}$ $\boxed{\sup E = 0}$
3. $E = \mathbb{Q}$ $\boxed{\sup E \text{ does not exist}}$
4. $E = A$ (as defined above) $\boxed{\sup E \text{ does not exist}}$

Definition (Least Upper Bound Property)*:*
A set S has the **least upper bound property** if every nonempty subset of S that has an upper bound has a least upper bound.

0.1.2 Dedekind Cuts

Definition (Dedekind Cut)*:*
A **Dedekind cut** α is a subset of $\mathbb{Q}$ such that:

1. $\alpha \neq \emptyset, \mathbb{Q}$

2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$. (Closed downward)
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. (No largest number)

Example:
$\alpha = \{x \in \mathbb{Q} \mid x < 0\}$ is a cut.

Proof. Step 1: $\alpha \neq 0, \mathbb{Q}$ ✓
Step 2: Let $p \in \alpha$, $q \in \mathbb{Q}$. Assume $q < p$. By the transitivity property of order, $q < 0$. Thus, $p \in \alpha$. ✓
Step 3: Let $p \in \alpha$ and $r \in \alpha$ such that $r = \dfrac{q}{2}$. Since $q < 0$, $q < \dfrac{q}{2}$. Thus $q < r$. ✓ □

Example:
$\gamma = \{r \mid r \leq 2\}$ is not a cut. This set does have a largest element, 2.

Definition (Rational Numbers)*:*
Let $\mathbb{R} = \{\alpha \mid \alpha \text{ is a cut }\}$.

We also define the following:

- $\alpha < \beta$ to mean $\alpha \subsetneqq \beta$. This is an order.
- $\alpha + \beta = \{r + s \mid r \in \alpha \text{ and } s \in \beta\}$. (This means $\mathbb{R}$ is a field.)
- $\alpha \cdot \beta$
  1. For positive cuts $\{\alpha \mid \alpha > 0^*\} = \mathbb{R}_+$:
     If $\alpha, \beta \in \mathbb{R}_+$, let $\alpha \cdot \beta = \{p \mid p < rs \, \text{for some} r \in \alpha, s \in \beta, \, r, s > 0\}$.
  2. For cases with negative cuts, $\alpha \cdot \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \ \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*, \ \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*, \ \beta < 0^* \end{cases}$
     where the products are the same as defined for postive cuts.

Theorem 1. *$\mathbb{R}$ is an ordered field with the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

Proofs of the Properties of $\mathbb{R}$

Proof. (Show there is order on $\mathbb{R}$.) Let α, β, γ be cuts.

<u>Step 1:</u> (Trichotomy)
It is clear that at most one of the following can be true: $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$. For example, if $\alpha < \beta$, then $\alpha \subsetneqq \beta$ and by the definition of a proper subset, $\alpha \neq \beta$ and β is not a proper subset of α.
Suppose the first two statements are false. Then α is not a subset of β. By definition of a (proper) subset, there exists $a \in \alpha$ such that $a \notin \beta$. If $q \in \beta$, then $q < p$ since $p \notin \beta$. Since cuts are closed downward, $q \in \alpha$ so $\beta \subsetneqq \alpha$. Thus $\beta < \alpha$.

Step 2: (Transitivity)
Assume $\alpha < \beta$ and $\beta < \gamma$. By definition of $<$, $\alpha \subsetneqq \beta$ and $\beta \subsetneqq \gamma$. By definition of a proper subset, $\alpha \subsetneq \gamma$ and we conclude $\alpha < \gamma$. ✓

We have shown the cuts demonstrate order. □

Proof. (**A1:** Show addition is closed) Let α, β be cuts and $\gamma = \alpha + \beta$.

Step 1: (Show that $\gamma \neq \emptyset, \mathbb{Q}$.)
It should be clear that γ cannot be the empty set. Since $\alpha, \beta \neq \mathbb{Q}$, there exists $a' \notin \alpha, b' \notin \beta$. Consider $a \in \alpha$ and $b \in \beta$. Now $a < a'$ and $b < b'$. Thus $a + b < a' + b'$. Therefore $a' + b' \notin \gamma$. We conclude $\gamma \neq \emptyset, \mathbb{Q}$. ✓

Step 2: (Show γ is closed downward.)
Let $p \in \gamma, q \in \mathbb{Q}$. Assume $q < p$. Since $p \in \gamma$, there exists $r \in \alpha$ and $s \in \beta$ such that $p = r + s$ so $q < r + s$. This means $q - s < r$. Since α is closed downward, $q - s \in \alpha$. Then $q = q - s + s$ where $q - s \in \alpha$ and $s \in \beta$ as desired.

Step 3: (Show γ has no largest number.)
Let $t \in \gamma$. Then there exists $u \in \alpha$ and $v \in \beta$ such that $t = u + v$. Since both cuts α, β have no largest number, there exists $x \in \alpha$ where $u < x$ and $y \in \beta$ where $v < y$. Thus $x + y \in \gamma$ and $t < x + y$. ✓

We have shown γ meets the definition of a cut. □

A2 and **A3** follows since addition in $\mathbb{Q}$ is commutative and associative.

Proof. (**A4:** Show $0^* = \{q \in \mathbb{Q} \mid q < 0\}$ is the additive identity for $\mathbb{R}$. In other words, show $\alpha + 0^* = \alpha$.)

Step 1: (Show $\alpha + 0^* \subset \alpha$.)
Let $a \in \alpha$ and $b \in 0^*$. Since $b < 0$, $a + b < a$. Thus $a + b \in \alpha$ (since α is closed downward). We conclude $\alpha + 0^* \subset \alpha$.

Step 2: (Show $\alpha \subset \alpha + 0^*$.)
Let $x, y \in \alpha$ and $z \in 0^*$. We can pick x, y such that $y > x$. Then $x - y \in 0^*$ and similar to before $x = y + (x - y) \in \alpha + 0^*$. We conclude $\alpha \subset \alpha + 0^*$. ✓

We conclude $\alpha + 0^* = \alpha$. □

Proof. (**A5:** Show) □