0.1 Introduction to Linear Transformations

Let A be an $m \times n$ matrix. We can interpret it as a function or transformation between vector spaces, where $A : \mathbb{R}^n \to \mathbb{R}^m$.

Note that A is a linear transformation since $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$ and $A(c\vec{v}) = cA\vec{v}$.

Example:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

What does it do?

Let

$$\vec{e_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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Example:
Find a matrix for $\frac{\Pi}{2}$ rotation.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

0.2 Eigenvalues and Eigenvectors

Most linear transformations can be understood with eigenvalues and eigenvectors.

Definition:
Let A be $n \times n$.

An **eigenvalue** of A is a scalar λ such that $A\vec{v} = \lambda\vec{v}$ has a nonzero solution $\vec{v}$.

An **eigenvector** $\vec{v}$ for λ is a nonzero $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenspace** for λ is the set of all $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenbasis** for λ is a basis λ's eigenspace.

We will look at a simple matrix to give concrete examples for all of these definitions.

Example:
Given A $n \times n$, find its eigenvalues, eigenvectors, eigenspace, and eigenbasis.

There are two eigenvalues, $\lambda = 2, 1$.

For $\lambda = 2$, $\vec{e_1}$ is one possible eigenvector. Another possible eigenvector is $3\vec{e_1}$. The eigenspace for $\lambda = 2$ is the x-axis. The eigenbasis is simply $\{\vec{e_1}\}$.

For $\lambda = 1$, the eigenbasis is simply $\{\vec{e_2}\}$.

We can determine this intutitively by considering some vectors and applying the linear transformation A.

To determine the eigenvalues and eigenvectors analytically, note that $A\vec{v} = \lambda\vec{v}$ for nonzero $\vec{v}$ is the same as $(A - \lambda I)\vec{v} = \vec{0}$. Thus all λ satisfy $\det(A - \lambda I) = 0$. This is a polynomial in λ with degree n.

The eigenvalues are the roots of $P_A(x)$ and the eigenvectors are $\ker A - \lambda I = \{\vec{v} \neq 0 : (A - \lambda I)\vec{v} = \vec{0}\}$

Definition:
The **algebraic multiplicity** of λ is the multiplicity of the factor $(x - \lambda)^m$ in $P_A(x)$.

Example:
For the matrix A, find its eigenvalues and a basis for the corresponding eigenspaces.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix}$$

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1. 1.

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 3 & 0 \\ -1 & 0 - \lambda & 1 \\ -2 & -1 & 4 - \lambda \end{bmatrix}$$

   $\det A - \lambda I = -(\lambda - 2)^3$ $\lambda = 2$ is an eigenvalue.

2. Want basis for

$$\ker \begin{bmatrix} 0 & 3 & 0 \\ -1 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$\operatorname{rref} A - \lambda I = \begin{array}{c} \begin{array}{ccc} 1 & 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} & & \uparrow \end{array} \end{array}$$

   Then, a basis for $\ker A - \lambda I$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   The eigenspace (dimension 1) is then:

$$\left\{ c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

Example:
Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

$< x, y > \to < x + 2y, 2x + 4y >$

Look A takes all input vectors to the line $y = 2x$.

Insert diagram

Computing the characteristic polynomial,

$$\begin{aligned} p_A(x) = \det(A - xI) &= \begin{vmatrix} 1 - x & 2 \\ 2 & 4 - x \end{vmatrix} \\ &= x^2 - 5x + 0 \\ &= x(x - 5) \end{aligned}$$

Thus, A has two eigenvalues: $\lambda = 0$, 5 with algebraic multiplicity of $m = 1$ for both.

Definition:
A is **nondefective** if its algebraic and geometric multiplicities are equal.

0.3 Diagonalization