0.1 Defining Vector Spaces and Subspaces

We've seen the sets $\mathbb{R}^2$ and $\mathbb{R}^3$ where the elements are vectors as translations in plane and 3-space. These sets have an algebraic structure.

Definition (Vector Space)*:*
Any set following axioms A1-A10 is called a **vector space**.

Given a vector space A, vectors $u, v, w \in A$ and scalars r, s:

A1. We can define addition (and the space is closed under addition).

A2. We can define scalar multiplication (and the space is closed under scalar multiplication).

A3. $u + v = v + u$

A4. $(u + v) + w = u + (v + w)$

A5. There exists an additive identity in A.

A6. There exists an additive inverse for any $u \in A$.

A7. There exists a multiplicative identity in A.

A8. $(rs)v = r(sv)$

A9. $r(u + v) = ru + rv$

A10. $(r + s)v = rv + sv$

By studying vector spaces, we can develop algebraic machinery to apply to all kinds of other objects.

Example:
$Q = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ (Polynomials of degree 2 or less)

This is a vector space!

A1. Adding doesn't increase the degree. ✓

A2. Scalar multiplication similarly doesn't increase the degree. ✓

A3. Addition in $\mathbb{R}$ is commutative. ✓

A4. By the associative property of $\mathbb{R}$,

$$\begin{aligned}
&\Big((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\Big) + (c_0 + c_1x + c_2x^2) \\
&= \Big((a_0 + b_0) + c_0\Big) + \Big((a_1 + b_1) + c_1\Big)x + \Big((a_2 + b_2) + c_2\Big)x^2 \\
&= \Big(a_0 + (b_0 + c_0)\Big) + \Big(a_1 + (b_1 + c_1)\Big)x + \Big(a_2 + (b_2 + c_2)\Big)x^2 \\
&= a_0 + a_1x + a_2x^2 + (b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 \checkmark
\end{aligned}$$

A5. 0 polynomial ✓

A6. $-(a_0 + a_1x + a_2x^2) = -a_0 - a_1x - a_2x^2$ ✓

A7. $1 \cdot (a_0 + a_1x + a_2x^2) = a_0 + a_1x + a_2x^2$ ✓

A8. ✓

A9. ✓

A10. ✓

Example:
$V = \mathbb{R}^2$
$S = \{(x, y) \mid x^2 - y^2 = 0\}$
Is S a vector space?

No, since addition is not closed.

Example:

$$V = M_{3\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \middle| \; a, b, c, d, e, f \in \mathbb{R} \right\}$$

$S = \{A \in V \mid \text{columns each sum to } 0\}$
Is S a vector space?

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -2 \end{bmatrix} \in S$. Now, $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Closed under scalar mult? ✓
Consider $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ s.t. $a + b + c = 0$ and $d + e + f = 0$.
Then $m \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} ma & md \\ mb & me \\ mc & mf \end{bmatrix}$.
So $ma + mb + mc = m(a + b + c) = m(0)$ and $md + me + mf = m(d + e + f) = m(0)$. ✓

Closed under addition? ✓
Consider $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}, \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} \in S$.
Now $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} = \begin{bmatrix} a + a' & d + d' \\ b + b' & e + e' \\ c + c' & f + f' \end{bmatrix}$
Check $(a + a') + (b + b') + (c + c') = (a + b + c) + (a' + b' + c') = 0$.

Note. Since the above examples of S are subsets of known vector spaces, we just have to check:

1. Closed under $+, -$
2. Closed under scalar multiplication

These are called **subspaces**.

Vectors spaces do not just include $\mathbb{R}^n$ and $\mathbb{C}^n$. We have:

- $C^k(I)$: function on an internal I with k continuous derivatives
- $P_n(\mathbb{R})$: polynomials with degree up to n
- $M_{m\times n}(\mathbb{R})$

Example:
Consider $y'' + a_1(x)y' + a_2(x) = 0$ on an interval I. Let S be the set of solutions to this LODE. Is S a vector space?

- Addition: If $y_1, y_2 \in S$, is $y_1 + y_2 \in S$?

  $$(y_1+y_2)''+a_1(x)(y_1+y_2)'+a_2(x)(y_1+y_2) = y_1''+y_2''+a_1(x)(y_1'+y_2')+a_2(x)(y_1+y_2) = 0+0$$

  Therefore, S is closed under addition.

- Scalar Multiplication: If $y \in S$, is $cy \in S$? Yes since $(cy)'' = cy''$ and $(cy)' = cy'$.

Therefore S is a vector space, a subspace of $C^2(I)$.

Example:
Let $V = \mathbb{R}^3$ and S be the solutions to the linear system:

$$\begin{aligned} 2x + 3y + z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

Then $S \subset V$ is a subspace. All vectors in the subspace lie on a line through the origin contained on both planes.

Using this concept of subspaces, we can find solutions to certain differential equations or systems of linear equations.

Definition (Null Space)*:*
The solutions of a homogeneous linear system $A\vec{x} = 0$ is called the **null space** of A, any $A_{m\times n}$. It is a subspace of $\mathbb{R}^n$ and is also known as the **kernel** of A.

For $\mathbb{R}^3$, subspaces consist of either a point $\vec{0}$, a line through $\vec{0}$, or a plane through $\vec{0}$.

0.2 Spanning Sets

Definition (Span)*:*
A linear combination of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ is a vector $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + n\vec{v_n}$ in V.

The **span** of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is the set of all linear combinations. The span is a subspace of the vector space V.

Example:
Compute the span of $\{(-4,1,3),\ (5,1,6),\ (6,0,2)\}$ in $\mathbb{R}^3$.

We want the set $\{(a,b,c)\}$ that are linear combinations of the three vectors. In other words, the solutions of the linear system:

$$\begin{bmatrix} -4 & 5 & 6 & a \\ 1 & 1 & 0 & b \\ 3 & 6 & 2 & c \end{bmatrix}$$

$$\xrightarrow{1.\ 2.\ 3.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 9 & 6 & a+4b \\ 0 & 3 & 2 & c-3b \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 3 & 2 & c-3b \\ 0 & 0 & 0 & a+13b-3c \end{bmatrix}$$

1. P_{12}
2. $A_{12}(4)$
3. $A_{13}(-3)$
4. P_{23}
5. $A_{23}(-3)$

There is a solution if and only if $\mathrm{rk}A = \mathrm{rk}A^{\#}$. So $a+13b-3c=0$. We conclude the span is on a plane $\boxed{x+13y-3z=0}$.

Note. One of the three vectors is redundant.

Example:
Write $(-4,1,3)$ as a linear combination of $(5,1,6)$ and $(6,0,2)$.

We want constants c_1, c_2 such that:

$$c_1 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

We need to solve $\begin{bmatrix} 5 & 6 & -4 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}$

$$\xrightarrow[2.\ 3.]{1.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 6 & -9 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{6.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} - \dfrac{3}{2} \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$.

The moral of this example is that sometimes a spanning set has redundant vectors. We would like a spanning set to have no redundant vectors or a "minimally spanning set."

Definition (Linear Dependence)*:*
A set $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ in a vector space V is **linearly dependent** if there exists some $c_1, c_2, \dots, c_n$ not all zero such that: $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$. (This is called the dependence relation, and we would say one of the vectors is redundant.)

Otherwise, the set is **linearly independent** and the relation $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$.

Example:
The vectors $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ are linearly dependent.
Find the dependence relation.

We need to solve $A\vec{x} = \vec{0}$ and $\text{rk}A \leq 2$ for there to be nontrivial solutions.

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Parameterizing, we find that $c_3 = t,\ c_2 = t,\ c_1 = 2t - t = t$.

Therefore $\boxed{\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}}$.

0.2.1 Method for Testing Spans

Given $\{\vec{v_1}, \dots, \vec{v_n}\}$, to determine if they are linearly dependent, form a matrix

$$A = \begin{bmatrix} \vec{v_1} \ \vec{v_2} \ \dots \ \vec{v_n} \end{bmatrix}$$

Then the set is linearly dependent if and only if $A\vec{c} = \vec{0}$ has nontrivial solutions. In other words, there exists some non-zero scalars $c_1, \dots, c_k$ such that $c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}$ if and only if $\text{rk}A < k$.

The technique is to commit ERO's and find rref(A).

For example, suppose we have a span and rref($A^{\#}$) is:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & \uparrow & & \uparrow & & \end{matrix} \end{array}$$

We have two free variables.

<u>Claim:</u> Each free variable gives a nontrivial dependence with bound variables.

So for this example, there are two dependence relations where $\vec{v_2}$ and $\vec{v_4}$ are redundant and $\vec{v_1}$, $\vec{v_3}$, $\vec{v_5}$ form a linearly independent span.

In general, we can conclude that $v_1, \ldots, \vec{v_k}$ are linearly independent if and only if there are no free variables in rref$(A^{\#})$ or equivalently rk$A = k$.

Theorem 1. *In $\mathbb{R}^n$, a set $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is linearly independent if and only if, for the matrix $A = [\vec{v_1}, \ldots, \vec{v_n}]$, $\det A \neq 0$ or rref$A = I_n$.*

Example:
Find all values of k such that: $\{(1, 1, 0, -1), (1, k, 1, 1), (2, 1, k, 1), (-1, 1, 1, k)\}$ are linearly independent.

Applying the previous theorem, the solution form is:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & k & 1 & 1 \\ 0 & 1 & k & 1 \\ -1 & 1 & 1 & k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then $\det A = k^3 - 2k^2 - 5k + 6 = (k-1)(k+2)(k-3)$. We conclude $\boxed{k \neq -2, 1, -3}$.

0.3 Linear Independence of Functions

Definition (Wronskian)*:*
Let $f_1, f_2, \ldots, f_n$ be in $C^{n=1}(I)$. The **Wronskian** is:

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1 & f_2 & \ldots & f_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1 & f_2 & \ldots & f_n \end{vmatrix}$$

Then if $W[f_1, f_2, \ldots, f_n](x) \neq 0$ for some $x \in I$, then the functions are linearly independent.

Sketch (of Proof)

The dependence relation has the form:

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \text{ for all } x$$

This implies:

$$c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' = 0 \text{ for all } x$$

and so on, ending with

$$c_1 f_1^{n-1} + c_2 f_2^{n-1} + \cdots + c_n f_n^{n-1} = 0 \text{ for all } x$$

.

We conclude that $W[f_1, f_2, \ldots, f_n](x) = 0$ for all x.

Therefore if $W[f_1, f_2, \ldots, f_n](x) \neq 0$, then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \neq 0$ for some x.

Example (Exponentials)*:*
Determine whether e^x and e^{-x} are linearly independent on $(-\infty, \infty)$.

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

We conclude they are linearly independent on $(-\infty, \infty)$.

What about e^x and e^{2x}?

$$W[e^x, e^{2x}] = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

We also conclude they are linearly independent.

Example ($P_2(x)$)*:*
Determine the linear independence of $\{1, x, x^2\}$.

$$W[1, x, x^2] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

Therefore, $\{1, x, x^2\}$ are linearly independent.

What about $\{1, x, x^2\}$?

$$W[1, x, 2x] = \begin{vmatrix} 1 & x & 2x \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

We cannnot draw a conclusion soley on the Wronskian, but look:

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot 2x = 0$$

$$c_1 = 0,\ c_2 = -2,\ c_3 = 1$$

Therefore, $\{1, x, x^2\}$ are linearly dependent.

0.4 Bases and Dimensions

Given any number of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ in a vector space V, we can define a subspace of V as $W = \text{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ and write $W \subset V$. Equivalently we can write W as $W = \{\sum a_i \vec{i} : a_i \in \mathbb{R}\}$.

Definition (Basis)*:*
A **basis** for a vector space V is a set of vectors that:

1. span V.
2. are linearly independent.

Example:

In $\mathbb{R}^3$, the standard basis is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

In P_2, the standard basis is: $\{1, x, x^2\}$.

Note. These are not unique. (We can write multiple bases.)

Theorem 2. *Any two bases for V have the same number of vectors (assuming they are finite).*

Sketch of Proof

If we had two bases:

$$\{\vec{v_1}, \ldots, \vec{v_m}\},\ \{\vec{w}_1, \ldots \vec{w}_n\}$$

Let say that $m \leq n$ and $A = [\vec{v_1}, \ldots, \vec{v_m}]$. Now $A\vec{c} = \vec{w_j}$ has unique solutions for all j since $\vec{v_i}$ span V. Since A has m columns, By Thm. of Rank, $\text{rk}A = rk(A^{\#}) = n$. *Finish*

Thus, from the previous theorem, we can define the following.

Definition (Dimension)*:*
The **dimension** of a vector space V is the number of elements in any basis.

Example:
Find a basis for the subspace of $\mathbb{R}^4$ spanned by:

$$\underline{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \underline{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}, \underline{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

.

We can simply choose the underlined vectors. The dimension of this subspace is 3.

0.4.1 Secret Trick

Given a $m \times n$ matrix A, it is useful to know:

- A basis for $\text{im}(A)$.
- A basis for $\ker(A) = \text{null}(A)$.
- A basis for the solution space for the linear system, $A\mathbf{c} = \mathbf{0}$.

This section will detail a shorthand method to find all these bases quickly.

Let $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$, an $m \times n$ matrix. First, compute $\text{rref}(A)$ in the usual way. Suppose:

$$\text{rref}(A) = \begin{bmatrix} 0 & 1 & b_{13} & 0 & b_{15} & 0 \\ 0 & 0 & 0 & 1 & b_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The free-variable columns correspond to redundant vectors. The bound-variable columns correspond to a basis of the span. Here, $\mathbf{v}_1, \mathbf{v}_3,$ and $\mathbf{v}_5$ are redundant vectors, and $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$ are a basis for $\text{im}(A)$.

The dependence-relation vector $\mathbf{c}_j$ corresponding to each free-variable column can be constructed by:

1. Put 1 into the j-th spot.
2. If $b_{ij} \neq 0$, and the leading 1 is in the i-th row is in column $k < j$, put $-b_{ij}$ in the k-th spot.
3. Put 0's everywhere else.

Here, $\mathbf{c}_1 = \langle 1, 0, 0, 0, 0, 0 \rangle$, $\mathbf{c}_3 = \langle 0, -b_{13}, 1, 0, 0, 0 \rangle$, $\mathbf{c}_5 = \langle 0, -b_{15}, 0, -b_{25}, 1, 0 \rangle$

Recall that the solution space is equivalent to $\ker(A) = \text{null}(A)$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 3 & 6 & 9 & 6 & 2 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$. Find bases for the $\operatorname{Im} A$ and $\ker A$.

Now $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Applying the secret trick ...

A basis for $\operatorname{Im} A$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$$

Recall $\operatorname{Im} A = \{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_5\vec{v_5} : c_i \in \mathbb{R}\}$.

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ -2 \quad 1 \quad 0 \quad 0 \quad 0 \\ -5 \quad 0 \quad 1 \quad 1 \quad 0 \end{array}$$

A basis for $\ker A$ is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Recall $\ker A = \{\vec{c} \in \mathbb{R}^5 : A\vec{c} = 0\}$.

An important result regarding the kernel and image of a matrix is the rank-nullity theorem.

Theorem 3 (Rank-Nullity Theorem)**.** *Let A be an $m \times n$ matrix. Then*

$$\dim(\ker A) + \dim(\operatorname{Im} A) = n$$

.

Another important result regards the dimension of a basis.

Theorem 4. *Let V be a vector space. If V has a finite basis, then any basis for V has the same number of vectors.*

Corollary 4.1. If $\dim V = n$, then any set of n linearly independent vectors in V is a basis.

The secret trick gives us a method to find a basis and the dimension of the space. Then, any other vector can be expressed using this basis.