Real Analysis

Christopher K. Walsh

January 6, 2025

Contents

"God made the integers; all else is the work of man." - Leopold Kronecker

Chapter 1

Rational and Real Numbers

1.1 Set Theory

Definition (Set)*:*
A **set** is a collection of objects called elements of the set.

Example:

1. $S = \{1, 2, 3\}$ $(= \{1, 2, 3, 3\})$
2. $E = \{$Even integers $\}$
3. $\{$College students$\}$

Notation:

- $x \in S$ means x is in S.
- $x \notin S$ means x is not in S.
- The empty set $\emptyset$ is the set with no elements.
- $A \subseteq B$ means A is a subset of B (i.e. if $x \in A$, then $x \in B$).
- If $A \subseteq B$ but $B \subsetneq A$ A is a proper subset.

If $A \subseteq B$ and $B \subseteq A$ then $A = B$. Otherwise $A \neq B$.

We can define more sets in terms of other sets. *Set Operations:* Let A and B be sets.

- Union: $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$
- Compliment: $B - A = \{x \,|\, x \in B \text{ and } x \notin A\}$
- Product: $A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}$

If U is a universal set (set of everything in context), we write $\bar{A} = U - A$
$= \{x \mid x \in U \text{ and } x \notin A\}$.

1.2 Functions and Relations

It is also important to define some types of relations and functions.

1.2.1 Relations

Definition (Relation)*:*
A (binary) **relation** R on a set S is a subset of $S \times S$. If $(a, b) \in R$, we write aRb.

Example (of relations)*:* 1. L "loves" is a relation on $P \times P$ (where P is a set of all people).

2. The set $R = \{(0, 0), (0, 1), (2, 2), (7, 18)\}$ is a relation on $\mathbb{Z}^+$. We would write $0R0$, $0R1$, $2R2$, and $7R18$.

Definition (Equivalence Relation)*:*
An equivalence relation on a set S is a relation s.t.:

1. Reflexive: For each $a \in S$, $a \sim a$.
2. Symmetric: For $a, b \in S$, if $a \sim b$, then $b \sim a$.
3. Transitive: For $a, b, c \in S$, if $a \sim b$ and $b \sim a$

1.2.2 Functions

Functions in the general sense are also a type of relation.

Definition (Function)*:*
A **function**, F from a set A to a set B is a relation s.t.: if aFb and aFb' then $b = b'$.
This is a rule that assigns a unique $a \in A$ to a unique $b \in B$. Write $f : A \to B$ and $f(a) = b$.

1.3 The Rational Numbers

Assume Z, the integers, have arithmetic order. What is $\mathbb{Q}$? Perhaps it's the set:
$\left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z}, n \neq 0 \right\}$.

However, what does that fraction notation actually mean? When we first begin teaching fractions to children we talk about splitting things like cake into smaller pieces. If we have a whole cake made of 3 slices, we can give one person a slice so they have $\frac{1}{3}$ of the cake. If we have a cake of 6 slices, we could give them 2 slices instead. They would have $\frac{2}{6}$. These two fractions are equivalent though! We need more rigor (this is mathmematics of course).

We describe the equivalent fractions as equivalent ordered pairs $(1,3) \sim (2,6)$. These belong to the same **equivalence class**, $\left[\frac{1}{3}\right]$.

Definition (Rational Numbers)*:*
The **rational numbers**, $\mathbb{Q}$, is the set $\left\{\frac{m}{n} \,\middle|\, m,n \in \mathbb{Z}, n \neq 0\right\}$ where $\frac{m}{n}$ is an equivalence class of (m,n) with the relation $(m,n) \sim (p,q)$ if $mq = np$ and $q,n \neq 0$

Proof. Is $\sim$ an equivalence relation? Need to show $\sim$ reflextive, symmetric, and transitive.

Step 1 Reflective: Let $(p,q) \in \mathbb{Q}$. Show $(p,q) \sim (p,q)$
Since $ab = ba$, $(p,q) \sim (p,q)$ ✓

Step 2 Symmetry: Let $(p,q),(m,n) \in \mathbb{Q}$. Assume $(p,q) \sim (m,n)$. Show $(m,n) \sim (p,q)$.

$$\begin{aligned}(p,q) \sim (m,n) &\implies pn = qm \\ &\implies qm = pn \\ &\implies mq = np \\ &\implies (m,n) \sim (p,q) \checkmark\end{aligned}$$

Step 3 Transitive: Let $(p,q),(m,n),(a,b) \in \mathbb{Q}$. Assume $(p,q) \sim (m,n)$ and $(m,n) \sim (a,b)$. Show $(p,q) \sim (a,b)$.
Need cancellation law on $\mathbb{Z}$: if $ab = ac$ and $a \neq 0$ then $b = c$.
$(p,q) \sim (m,n) \implies pn = qm$ and $(m,n) \sim (a,b) \implies mb = na$

Case 1: $p = 0$

$$\begin{aligned}p = 0 &\implies pn = qm = 0 \\ &\implies m = 0 \text{ since } q \neq 0 \\ &\implies mb = na = 0 \\ &\implies a = 0 \text{ since } n \neq 0 \\ &\implies pb = qa = 0 \\ &\implies (p,q) \sim (a,b) \checkmark\end{aligned}$$

Case 2: $m = 0$
Similar to Case 1. ✓

Case 3: $p,m \neq 0$
Multiplying $pn = qm$ by ab: $ab(pn) = ab(qm)$.
$\implies na(pb) = mb(qa)$
$\implies pb = qa$ by cancellation law ($m \neq 0$ and $mb = na$) $\implies (p,q) \sim (a,b)$ ✓ □

1.3.1 Arithmetic (of Rationals)

Our definitions of arithmetic on $\mathbb{Q}$ be well-defined. For example, we could define addition as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

However,

$$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$$
$$\frac{2}{4} + \frac{3}{7} = \frac{3}{7}$$

$\frac{1}{2}$ and $\frac{2}{4}$ are in the same equivalent class, but $\frac{2}{5}$ and $\frac{3}{7}$ are not. This is not well-defined. We want a definition of addition <u>not dependent on our representatives chosen</u>.

Now, $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$. This is well-defined but not helpful.

Definition (Addition in $\mathbb{Q}$)*:*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

If this well-defined?

Proof. Assume $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Show $(ad + bc, bd) \sim (a'd' + b'c', b'd')$.
$(a, b) \sim (a', b') \implies ab' = ba'$
$(c, d) \sim (c', d') \implies cd' = dc'$

$$\begin{aligned} b'd'(ad + bc) &= b'd'ad + b'd'bc \\ &= (d'd)(ab') + (b'b)(cd') \\ &= (d'd)(ba') + (b'b)(dc') \\ &= (bd)(a'd') + (bd)(c'b') \\ &= bd(a'd' + c'b') \end{aligned}$$

$\implies (ad + bc, bd) \sim (a'd' + b'c', b'd')$ ✓ □

Definition (Multiplication in $\mathbb{Q}$)*:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

If this well-defined?

Proof. Assume $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Show $(ac, bd) \sim (a'c', b'd')$.

$(a, b) \sim (a', b') \implies ab' = ba'$

$(c, d) \sim (c', d') \implies cd' = dc'$

$$\begin{aligned} acb'd' &= (ab')(cd') \\ &= (ba')(dc') \\ &= (a'c')(bd) \end{aligned}$$

$\implies (ac, bd) \sim (a'c', b'd')$ ✓ □

In what way does $\mathbb{Q}$ extend $\mathbb{Z}$?

The correspondence is $\dfrac{n}{1} \longleftrightarrow n$. Addition and multiplication is the same in $\mathbb{Q}$ as in $\mathbb{Z}$.

Note. We can define subtraction by adding the negative of a number (multiply by -1).

1.3.2 Order

Definition (Order)*:*
An **order** on a set S is a relation $<$ satisfying:

1. (Trichotomy) If $x, y \in S$, exactly one is true: $x < y$, $x = y$, $y < x$.
2. (Transitivity) If $x, y, z \in S$, $x < y$ and $y < z, x < z$.

Example:
In $\mathbb{Z}$, say $m < n$ if $n - m$ is positive, i.e. in $\mathbb{N}$.

Example:
In $\mathbb{Z} \times \mathbb{Z}$, say $(a, b) < (c, d)$ if $a < c$ or ($a = c$ or $b < d$). This is called the dictionary order.

Example:
In $\mathbb{Q}$, say $\dfrac{m}{n}$ is positive if $mn > 0$. This is well-defined.

Proof. Assume $(m, n) \sim (p, q)$ and $mn > 0$. Show $pq > 0$.
Suppose, to the contrary, $pq < 0$.

$$\begin{aligned} (m, n) \sim (p, q) &\implies mq = np \\ &\implies (mq)^2 = mqnp \end{aligned}$$

By assumption, $mnpq < 0$, a contradiction since $mn > 0$. Thus, $pq > 0$. □

So $\dfrac{a}{b} < \dfrac{c}{d}$ if $\dfrac{c}{d} + \dfrac{-a}{b}$ is positive.

Write $y > x$ for $x < y$ and $x \leq y$ for $x < y$ or $x = y$.

Theorem 1. *$x^2 = 2$ has no solution in $\mathbb{Q}$.*

Proof (by contradiction). Suppose, to the contrary, that x^2 has a solution in $\mathbb{Q}$, i.e. $x = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$. Also assume p, q are in "lowest terms," i.e. they have no common factors. (We can do this using elements in the equivalence classes of $\mathbb{Q}$.) So $\left(\dfrac{p}{q}\right)^2 = 2$, hence

$p^2 = 2q^2$. Then p^2 is even (divisible by 2). Then p is even. (If p was odd, p^2 would be odd.) So $p = 2m$ for some $m \in \mathbb{Z}$, hence $p^2 = 4m^2 = 2q^2$. Then $2m^2 = q^2$. Then q^2 is even, hence q is even. This contradicts the fact that p, q are in "lowest terms." So, $x^2 = 2$ must have no solution in $\mathbb{Q}$. $\square$

1.3.3 Fields

Definition (Field)*:*
A **field** is a set F with two operations $+, \times$ satisfying axioms:

A1. F is closed under $+$. (Adding two things in the set gives you something in the set.)

A2. $+$ is commutative.

A3. $+$ is associative.

A4. F has an additive identity, call it 0.

A5. Every element has an additive inverse.

M1. F is closed under $\times$.

M2. $\times$ is commutative.

M3. $\times$ is associative.

M4. F has an multiplicative identity, call it 1, and $1 \neq 0$.

M5. Every element except 0 has an multiplicative inverse.

D1. $\times$ distributes over $+$.

Example:
In $\mathbb{Q}$, the 0 element is $\left[\frac{0}{1}\right]$ and the 1 element is $\left[\frac{1}{1}\right]$.

Definition (Ordered Field)*:*
An **ordered field** is a field with an order s.t. order is preserved by field operations.

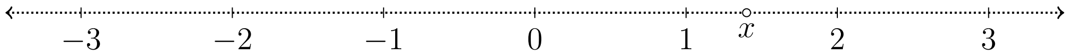
1. If $y < z$, then $x + y < x + z$.
2. If $y < z$ and $x > 0$, then $xy < xz$.

Note. $\mathbb{Z}$ is a ring not a field. There are no multiplicative inverses.
$\mathbb{Q}$ is an ordered field!

1.4 Constructing the Real Numbers

1.4.1 Upper Bounds

Now, we seen in the previous section that $\mathbb{Q}$ has "gaps". $x^2 = 2$ has no solution in $\mathbb{Q}$.



We need to fill in these gaps somehow while not knowing where the gaps and holes are.

Definition (Upper Bound)*:*
Let $E \subset S$ ordered. If there exists $\beta \in S$ such that for all $x \in E$, $x \leq \beta$, then β is an **upper bound (u.b.)** for E. We say E is bounded above.

A lower bound can be defined similarly with "greater than or equal to."

Example:
Consider the set $A = \{x \mid x^2 < 2\}$. 2 is an u.b. for A. $\frac{2}{3}$ is also an u.b. for A.

Definition (Least Upper Bound)*:*
If there exists an $\alpha \in S$ such that:

1. α is an upper bound of E
2. If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called a **least upper bound (lub)** of E or the **suprenum** of E. Write $\alpha = \sup E$.

Example:
Let $S = \mathbb{Q}$.

1. $E = \left\{\frac{1}{2}, 1, 2\right\}$ $\boxed{\sup E = 2}$
2. $E = \{x \in \mathbb{Q} \mid x < 0\}$ $\boxed{\sup E = 0}$
3. $E = \mathbb{Q}$ $\boxed{\sup E \text{ does not exist}}$
4. $E = A$ (as defined above) $\boxed{\sup E \text{ does not exist}}$

Definition (Least Upper Bound Property)*:*
A set S has the **least upper bound property** if every nonempty subset of S that has an upper bound has a least upper bound.

1.4.2 Dedekind Cuts

Definition (Dedekind Cut)*:*
A **Dedekind cut** α is a subset of $\mathbb{Q}$ such that:

1. $\alpha \neq \emptyset, \mathbb{Q}$
2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$. (Closed downward)
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. (No largest number)

Example:
$\alpha = \{x \in \mathbb{Q} \mid x < 0\}$ is a cut.

1. $\alpha \neq 0, \mathbb{Q}$ ✓
2. Let $p \in \alpha$, $q \in \mathbb{Q}$. Assume $q < p$. By the transitivity property of order, $q < 0$. Thus, $p \in \alpha$. ✓
3. Let $p \in \alpha$ and $r \in \alpha$ such that $r = \frac{q}{2}$. Since $q < 0$, $q < \frac{q}{2}$. Thus $q < r$. ✓

Example:
$\gamma = \{r \mid r \leq 2\}$ is not a cut. This set does have a largest element, 2.

Definition (Rational Numbers)*:*
Let $\mathbb{R} = \{\alpha \mid \alpha \text{ is a cut }\}$.

We also define the following:

- $\alpha < \beta$ to mean $\alpha \subsetneqq \beta$. This is an order.
- $\alpha + \beta = \{r + s \mid r \in \alpha \text{ and } s \in \beta\}$. (This means $\mathbb{R}$ is a field.)
- $\alpha \cdot \beta$
  1. For positive cuts $\{\alpha \mid \alpha > 0^*\} = \mathbb{R}_+$:
     If $\alpha, \beta \in \mathbb{R}_+$, let $\alpha \cdot \beta = \{p \mid p \leq rs \, \text{for some } r \in \alpha, s \in \beta, \, and r, s > 0\}$.
  2. For cases with negative cuts, $\alpha \cdot \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*,\ \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*,\ \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*,\ \beta < 0^* \end{cases}$
     where the products are the same as defined for postive cuts and $-\alpha$, $-\beta$ are the additive inverses of α, β respectively. (The additive inverses are defined below.)

Theorem 2. $\mathbb{R}$ *is an ordered field with the least upper bound property.* $\mathbb{R}$ *contains* $\mathbb{Q}$ *as a subfield.*

Proofs Showing $\mathbb{R}$ is an Ordered Field

The following section proves Theorem 2.

Proof.

Step 1: We must show there is order on $\mathbb{R}$.
Let $\alpha, \beta, \gamma \in \mathbb{R}$. We must show that they demonstrate both trichotomy and transitivity.

1. Trichotomy:
   It is clear that at most one of the following can be true: $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$. For example, if $\alpha < \beta$, then $\alpha \subsetneqq \beta$ and by the definition of a proper subset, $\alpha \neq \beta$ and β is not a proper subset of α.

   To show at least one of them must be true, suppose the first two statements are false. Then α is not a subset of β. By definition of a proper subset, there exists some $a \in \alpha$ such that $a \notin \beta$. Consider some $b \in \beta$. Since β is closed downward, $b < a$. This also means $b \in \alpha$ since α is also closed downward. This shows that $\beta \subsetneqq \alpha$. Thus $\beta < \alpha$. We conclude that at least one of these statements must be true.

2. Transitivity:
   We assume that $\alpha < \beta$ and $\beta < \gamma$. By definition of $<$, $\alpha \subsetneqq \beta$ and $\beta \subsetneqq \gamma$. By definition of a proper subset, $\alpha \subsetneq \gamma$ and we conclude that $\alpha < \gamma$.

We have shown the cuts demonstrate order.

Step 2: Next, we must show that addition is closed (**A1**).
Let $\alpha, \beta \in \mathbb{R}$ and $\gamma = \{r + s \mid r \in \alpha \text{ and } b \in \beta\}$. To show addition is closed, we must show that γ is a cut.

1. First, we must show that $\gamma \neq \emptyset$, $\mathbb{Q}$.
   It should be clear that γ cannot be the empty set. Since $\alpha, \beta \neq \mathbb{Q}$, there exists $c \notin \alpha$, $d \notin \beta$. Now $a < c$ and $b < d$ for all $a \in \alpha$, $b \in \beta$. Thus $a + b < c + d$. Therefore $c + d \notin \gamma$. We conclude that $\gamma \neq \emptyset, \mathbb{Q}$.

2. Second, we must show that γ is closed downward.
   Let $p \in \gamma, q \in \mathbb{Q}$. Assume $q < p$. Since $p \in \gamma$, there exists $r \in \alpha$ and $s \in \beta$ such that $p = r + s$ so $q < r + s$. This means $q - s < r$. Since α is closed downward, $q - s \in \alpha$. Then $q = q - s + s$ where $q - s \in \alpha$ and $s \in \beta$. We have shown that γ is closed downward.

3. Third, we must show that γ has no largest number.
   Let $p \in \gamma$. Then there exists some $a \in \alpha$ and $b \in \beta$ such that $p = a + b$. Since both cuts α, β have no largest number, there exists $c \in \alpha$ and $d \in \beta$ where $a < c$ and $b < d$. Thus $c + d \in \gamma$ and $p < c + d$. We have shown that γ has no largest member.

We have shown that γ meets the definition of a cut.

Step 3: **A2** and **A3** follows since addition in $\mathbb{Q}$ is commutative and associative.

Step 4: We must show that $0^* = \{q \in \mathbb{Q} \mid q < 0\}$ is the additive identity for $\mathbb{R}$ (**A4**).
In other words, we must show that $\alpha + 0^* = \alpha$. Let $\alpha \in \mathbb{R}$.

1. First, we must show that $\alpha + 0^* \subset \alpha$.
   Let $a \in \alpha$ and $b \in 0^*$. Since $b < 0$, $a + b < a$. Thus $a + b \in \alpha$ (since α is closed downward). We conclude that $\alpha + 0^* \subset \alpha$.

2. Second, we must show that $\alpha \subset \alpha + 0^*$.
   Let $a, b \in \alpha$. Since α has no largest member, we can pick a, b such that $b > a$. Then $a - b \in 0^*$ and similar to before $a = b + (a - b) \in \alpha + 0^*$. We conclude that $\alpha \subset \alpha + 0^*$.

Thus we conclude that $\alpha + 0^* = \alpha$.

Step 5: Next, we must show that the additive inverse for $\alpha \in \mathbb{R}$ is $\beta = \{p \in \mathbb{Q} \mid \text{ there exists } r > 0 \text{ s.t. } -p - r \notin \alpha\}$ (**A5**).
Let $\alpha \in \mathbb{R}$.

1. First, we must show that β is a cut.

   i. We must show that β is non-trivial.
      Since $\alpha \neq \emptyset$, there exists some $a \in \alpha$. Since α is closed downward, $a - b \in \alpha$ for all $b > 0$. Thus $-a \notin \beta$ so $\beta \neq \mathbb{Q}$.

Since $\alpha \in \mathbb{Q}$, there exists some $c \notin \alpha$. Consider $d = -c - 1$. Then $-d - 1 \notin \alpha$ (since $c = -d - 1$). Thus $d \in \beta$ so $\beta \neq \emptyset$.

ii. We must show that β is closed downward.
Let $p \in \beta$, $q \in \mathbb{Q}$. We assume that $q < p$. There exists some $r > 0$ s.t. $-p-r \notin \alpha$. Since $q < p$, $-q - r > -p - r$. Thus $-q - r \notin \alpha$ (since α is closed downward), so $q \in \beta$. We conclude that β is closed downward.

iii. We must show that β has no largest member.
Consider $j = f + (h/2)$. Then $j > f$, and $-j - (h/2) = -f - h \notin \alpha$. Then $j \in \beta$. Thus we find β also has no largest member.

We conclude that β satisfies the definition of a cut.

2. Second, we must show that $\alpha + \beta = 0^*$.

i. We need to show that $\alpha + \beta \subset 0^*$.
Let $s \in \alpha$ and $t \in \beta$. By definition of β, there exists $u > 0$ s.t. $-t - u \notin \alpha$. Then $-t - u > s$ and it follows that $s + t < -u < 0$. Therefore $s + t \in 0^*$.

ii. We also need to show that $0^* \subset \alpha + \beta$.
Let $v \in 0^*$ and $w = -v/2$. Then $w > 0$. By the Archimedean property of $\mathbb{Q}$, there exists $n \in \mathbb{Z}$ s.t. $nw \in \alpha$ and $(n + 1)w \notin \alpha$. Consider $x = -(n + 2)w$. Now $-x = nw+2w \implies -x-w = nw+w = (n+1)w \notin \alpha$. By definition of β, $x \in \beta$. We find $v = -2w = -(n + 2)w + nw = p + x$ and conclude that $v \in \alpha + \beta$.

This shows that $\alpha + \beta = 0^*$.

We have proved that β is the additive inverse for α and that $\mathbb{R}$ satisfies all the addition axioms. Next we will show that $\mathbb{R}$ satisfies the multiplication axioms.

Step 6: We must show that $\mathbb{R}$ is closed under multiplication (**M1**). We will only consider the positve case for multiplication (because I am lazy and the whole thing is tedious. Most books only do the addition axioms.)
Let $\alpha, \beta \in \mathbb{R}_+$ and $\alpha\beta = \{p \mid p \leq ab, a \in \alpha, b \in \beta, \text{ and } a, b > 0\}$. We must prove $\alpha\beta \in \mathbb{R}$ as before.

i. Since α, β are non-trivial, there exists some $a \in \alpha, b \in \beta$ s.t. $a, b > 0$. Now $ab \leq ab$. Thus $ab \in \alpha\beta$ so $\alpha\beta \neq \emptyset$.

There also exists some $c \notin \alpha, d \notin \beta$ s.t. $c > a$ and $d > b$ for all $a \in \alpha$ and $b \in \beta$. Now $cd > ab$ for all $a \in \alpha, b \in \beta$. Thus $cd \notin \alpha\beta$ so $\alpha\beta \neq \mathbb{Q}$.

ii. Let $p \in \alpha\beta$, $q \in \mathbb{Q}$, and $q < p$. Now $p < ab$ for some $a \in \alpha$, $b \in \beta$. By order of $\mathbb{Q}$, $q < p < ab$ so $q \in \alpha\beta$.

iii. Since α, β have no largest member, we can pick some $c > a$ and $d > b$ s.t. $c \in \alpha$ and $d \in \beta$. Thus $p \leq ab < cd$, so $ab \in \alpha\beta$.

We have shown that $\alpha\beta$ satisfy all three properties of a cut.

Step 6: We must show that multiplication in $\mathbb{R}$ is commutative (**M2**) and associative (**M3**). Let $p \in \alpha\beta$. Now $p \leq ab$ for some $a \in \alpha$, $b \in \beta$, $a, b > 0$. Since multiplication in $\mathbb{Q}$ is commutative, $p \leq ba$. Thus, $p \in \beta\alpha$. Clearly $\alpha\beta \subset \beta\alpha$ and $\beta\alpha \subset \alpha\beta$. We conclude that $\alpha\beta = \beta\alpha$ and multiplication in $\mathbb{R}$ is commutative.

The proof that multiplication in $\mathbb{R}$ is associative is similar relying on multiplication in $\mathbb{Q}$ is associative.

Step 7: We must show that there is a multiplicative identity in $\mathbb{R}$ (**M4**).
We will define it to be $1^* = p \in \mathbb{Q} \mid p < 1$. Let $\alpha \in \mathbb{R}$. We must demonstrate that $\alpha \cdot 1^* = \alpha$.

1. We must show $\alpha \cdot 1^* \subset \alpha$.
   Let $p \in \alpha \cdot 1^*$. Now $p \leq a \cdot b$ for some $a \in \alpha$, $b \in 1^*$. Since $b < 1$, $p \leq ab < a$ so $p \in \alpha$.

2. We also must show $\alpha \subset \alpha \cdot 1^*$.
   We can choose some $c \in \alpha$ s.t. $c > a$. This implies $\dfrac{a}{c} < 1$ so $\dfrac{a}{c} \in 1^*$. Thus $a = c \cdot \dfrac{a}{c} \in \alpha \cdot 1^*$.

We conclude that $\alpha \cdot 1^* = \alpha$.

Step 8: We must show that there is a multiplicative inverse in $\mathbb{R}$ (**M5**).
We will define the multiplicative inverse for $\alpha \in \mathbb{R}_+$ as:
$\beta = 0^* \cup \{0\} \cup \{p \in \mathbb{Q} \mid \text{there is an } r \in \mathbb{Q} \text{ with } r > 1 \text{ and } \frac{1}{rp} \notin \alpha\}$. Let $\alpha \in \mathbb{R}$.

1. We must show $\beta \in \mathbb{R}$.

   i. Clearly $\beta \neq \emptyset$. Since $\alpha \neq \mathbb{Q}$ and $\alpha \in \mathbb{R}_+$, there exists some $a \in \alpha$ where $a > 0$. Then $1/a \notin \beta$ since there is no

The cases for multiplication with negative cuts are also similar using the identity: $\gamma = -(-\gamma)$.