0.1 Theory of Curves

Our main tool to understand curves will be vector methods. When we consider $\mathbb{R}^3$, we cannot use equations to describe curves since we obtain surfaces.

How to Describe Curves:

- Solutions to (x, y) to eqn $f(x, y) = 0$.

$$x^2 + y^2 - 1 = 0$$

- Image of vector valued function, a parameterization.

$$\begin{aligned} \gamma : I &\to \mathbb{R}^2 \text{ (or } \mathbb{R}^n) \\ t &\mapsto (x(t), y(t)) \end{aligned}$$

$$\begin{aligned} e.g.\ \gamma(t) &= (\cos t, \sin t),\ 0 \leq t \leq 2\pi \\ \gamma :\ &[0, 2\pi] \to \mathbb{R}^2 \end{aligned}$$

Example (Parameterization)*:*
Line: Euclid said there exists a unique line between any two points $p, q \in \mathbb{R}^n$ $(p \neq q)$. Parameterize by $\lambda(t)$: $p + t(q - p),\ t \in \mathbb{R}$

Helix: $\gamma(t) = (\cos t, \sin t, -t)$

Basic Fact: Every parameterized curve, $\gamma(t) :\ I \to \mathbb{R}^n$ has a **velocity**, $\gamma'(t)$. $\gamma'(t)$ is the vector tangent to (the trace of) $\gamma(t)$ pointing in the traveling direction.

In $\mathbb{R}^3$, $\gamma'(t) = (x'(t), y'(t), z'(t))$.

$|\gamma'(t)|$ is the speed.

The distance along $\gamma(t)$ is computed using:

$$s = \int_{t_1}^{t_2} |\gamma'(t)|\, dt$$

Another basic fact is that we can apply derivatives to the different (vector) products. We begin by examining how we can use the dot product.

$$(\gamma(t) \cdot \gamma(t))' = \gamma'(t) \cdot \gamma(t) + \gamma(t) \cdot \gamma'(t) = 2\gamma'(t) \cdot \gamma(t)$$