

# Differential Equations<sup>1</sup>

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# Preface

This document compiles all my notes on differential equations including any self-study. Some sections require content from other math subjects to be completely understood (such as linear algebra or other analysis topics). The corresponding topic and section will be referred to in-text.

If you need this for school and industry, I hope that you are able to do whatever you are trying to accomplish. If you are here because this interests you, I hope you find this as entertaining as I did.

- Christopher

# Chapter 1

## Linear Ordinary Differential Equations

### 1.1 Introduction to LODEs

*Definition* (Linear Ordinary Differential Equation (LODE) of Order  $n$ ):  
A **linear ordinary differential equation** of order  $n$  is an equation:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

where  $y^{(i)}$  is the  $i$ th derivative.

*Example* (Hooke's Law):

Hooke's Law states the force exerted by a spring is proportional to its displacement from equilibrium:

$$F = -ky$$

This applies when displacement is small compared to the total range of the spring. (Fun Fact: Hooke stated this as an anagram riddle before officially publishing his discovery.)

From Newton,

$$F = m \frac{d^2y}{dt^2}$$

$$\therefore \text{By substitution, } y^{(2)} + \omega^2 y = 0$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

This describes the system. The solution is some  $y(t)$ : what actually happens. Finding  $y(t)$  is our goal.

*Check:*  $y = c_0 \cos \omega t + c_1 \sin \omega t$  is a solution where  $c_0, c_1$  are constants.

$$\begin{aligned} y^{(1)} &= -c_0 \omega \sin \omega t + c_1 \omega \cos \omega t \\ y^{(2)} &= -c_0 \omega^2 \cos \omega t - c_1 \omega^2 \sin \omega t \\ &= -\omega^2 y \end{aligned}$$

Simple harmonic motion!

*Definition* (General Solution of a LODE):

A **general solution of a LODE** is a solution of the form:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where the  $y_i$  are independent solutions and  $c_j$  are unknowns.

*Note.* Any solution is some specialization of the  $c_j$ 's

*Example* (Newton's Law of Cooling):

Newton's Law of Cooling is as follows:

$$\frac{dT}{dt} = -k(T - T_m)$$

where  $T$  = temperature at time  $t$ ,  $t$  = time,  $k$  = constant, and  $T_m$  = ambient temp

*Check:*  $T = T_m + ce^{-kt}$  is a solution. Witness:

$$\begin{aligned} \frac{dT}{dt} &= -kce^{-kt} \\ &= -k(T - T_m) \end{aligned}$$

## 1.2 Other Examples of Differential Equations

Here are additional examples of differential equations explored more thoroughly.

### 1.2.1 Orthogonal Trajectory Problem

Given a function  $f(x, y)$ , there is a family of level curves  $f(x, y) = f(p)$ ,  $p = (a, b)$ . We write the gradient of  $f(x, y)$  as  $\langle f_x, f_y \rangle$ .

*Note.* The gradient has the following important properties:

- The direction is in the greatest rate of change of  $f$
- $\vec{\nabla} f \perp f(x, y) = f(p)$

The problem is thus to find a curve that follows the gradient i.e. an **orthogonal trajectory curve**.

**Potential Applications:**

- Heat seeking missiles,  $f$  = temperature
- Charged particle,  $f$  = electric potential
- Bear at scout jamboree,  $f$  = peanut butter scent intensity

**Theory:** Let  $g(x, y) = g(p)$  be an **orthogonal trajectory curve** (in xy-plane).

*Goal:* Find  $g$ .

*Require:*  $\vec{\nabla} f(p) \perp \vec{\nabla} g(p)$

Then, it follows that  $f_x g_x + f_y g_y = 0$

Therefore the slope of  $D$  at  $p$  is the slope of the vector  $\langle -g_y, g_x \rangle = -\frac{g_x}{g_y} = \frac{f_y}{f_x}$ .

Therefore  $\frac{dy}{dx} = \frac{f_y}{f_x}$  is a 1st order LODE for  $g$ .

To find  $g$ , use the relation  $dy = \frac{f_y}{f_x} dx$  then integrate.

*Example:*

$$f(x, y) = x^2 + y^2$$

Level curves are circles

$$\begin{aligned} \frac{f_y}{f_x} &= \frac{y}{x} \\ \therefore dy &= \frac{y}{x} dx \\ \int \frac{1}{y} dy &= \int \frac{1}{x} dx \\ \therefore \ln y &= \ln x + C \\ y &= kx \end{aligned}$$

These are all lines through  $(0,0)$ . Compute these through  $p = (a, b)$ ,  $k = \frac{b}{a}$ .

**1.2.2 Initial Value Problems**

Most general LODE of order  $n$  is:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

where  $F(x)$  is the driving term.

An **initial value problem (IVP)** is an equation, (\*) together with initial conditions, (\*\*):

$$\begin{aligned}y(x_0) &= y_0 \\y^{(1)}(x_0) &= y_1 \\y^{(n-1)}(x_0) &= y_{n-1}\end{aligned}$$

*Example:*

Solve the IVP  $y'' + \omega^2 y = 0, y(0) = 1, y'(0) = 0$

The general solution is:  $y = c_0 \cos(\omega t) + c_1 \sin(\omega t)$

Initial conditions impose:

$$\begin{aligned}c_0 \cos 0 + c_1 \sin 0 &= 1 \implies c_0 = 1 \\-c_0 \omega \sin 0 + c_1 \omega \cos 0 &= 0 \implies c_1 = 0 \\ \therefore \boxed{y = \cos \omega t}\end{aligned}$$

In general,  $c_0 = y(0), c_1 = \frac{y'(0)}{\omega}$ .

**Theorem 1.** Assume in (\*) that the  $a_i(x)$  and  $F(x)$  are continuous on some interval  $I$ . Then there exists a unique solution to IVP (\*) + (\*\*) on  $I$ .

## 1.3 First Order DE's

The form of a **first order differential equation** is:  $y' = f(x, y)$ . Does it have a solution? If so, is it unique?

**Theorem 2.** Given  $y' = f(x, y)$ . Assume  $f(x, y)$  is continuous on  $[a, b] \times [c, d]$  and  $f_y$  is continuous in  $(a, b) \times (c, d)$ . Then for each  $(x_0, y_0) \in (a, b) \times (c, d)$ , there exists a unique solution to  $y = y(x)$  running through  $(x_0, y_0)$  and this solution holds for some interval  $I$  around  $x_0$ .

*Note.* The notation  $[a, b] \times [c, d]$  describes the domain and range using the Cartesian product of the two sets. In other words,  $a \leq x \leq b$  and  $c \leq y \leq d$ .

*Example:*

Is there a unique solution to:  $y' = xy^{\frac{1}{2}}, y(0) = 0$ ?

*Sol'n.*  $f(x, y) = xy^{\frac{1}{2}}$  is continuous on  $(-\infty, \infty) \times [0, \infty]$   $f_y = \frac{1}{2}xy^{-\frac{1}{2}}$  is not continuous at  $y = 0$ .

Therefore the theorem is useless.

There are solutions for the IVP where  $y(0) = 0$ . Check it out:

- $y = 0$  is a solution.  $y(0) = 0$
- $y = \frac{1}{16}x^4$  is also a solution.

$$\begin{aligned} y(0) &= 0 \\ y' &= \frac{1}{4}x^3 = xy^{\frac{1}{2}} \checkmark \end{aligned}$$

Thus the solutions are not unique as expected from the theorem.

*Note.* Slope fields can be used to visualize solutions to a differential equation.

*Definition* (Equilibrium Solutions):

**Equilibrium solutions** are solutions to a differential equation that have a derivative of zero everywhere i.e. equal to a constant value.

In terms of a first order linear differential equation, the equilibrium solutions are the set  $\{y = y_0 \mid f(x, y_0) = 0\}$  On slope field diagrams, they are where a horizontal line fits as a solution.

### 1.3.1 Analytical Techniques for First Order ODEs

1. **Seperable:** If the first order ODE is one of the form  $p(y)y' = q(x)$  (or  $y' = r(y)q(x)$ , ...) then since  $dy = y'dx$ , we get  $p(y)dy = q(x)dx$  and can integrate (in principle).
2. **Integrating Factor:** For LODE's in the form,  $y' + p(x)y = q(x)$ , let  $I(x) = e^{\int p(x) dx}$  be the integrating factor.

*Claim.*  $y(x) = \frac{1}{I(x)} \int q(x)I(x) dx$  is the general solution. Let's test this claim using the chain rule.

$$\begin{aligned} I(x)y(x) &= \int q(x)I(x) dx \\ \frac{d}{dx}I(x)y(x) &= I'(x)y(x) + I(x)y'(x) \\ &= p(x)I(x)y(x) + I(x)y'(x) \\ &= I(x)(p(x)y(x) + y'(x)) \\ &= I(x)q(x) \text{ by original LODE} \end{aligned}$$

$$\therefore I(x)y(x) = \int I(x)q(x) dx \implies \text{claim is true}$$

*Note.* The indefinite integral and the lack of a integration constant is not formally correct. However, the constant is removed upon the simplification. We can always divide both sides of the equation by  $e^C$ .

An **autonomous ODE** is one with no independent variable. It is usually seperable.

*Example:*

Find the general solution for  $y' + (\tan x)y = \cos^2 x$  on  $-\pi/2, \pi/2$ .



*Sol'n.* Recognize this is the form for the integrating factor with  $p(x) = \tan x$  and  $q(x) = \cos^2 x$ .

$$\begin{aligned} I(x) &= e^{\int \tan x \, dx} = e^{\ln(\sec x)} = \sec x \\ (\sec x)y &= \int \cos(x) \, dx = \sin x + C \\ y &= \sin x \cdot \cos x + C \cdot \cos x \end{aligned}$$

### 1.3.2 Application: Revisiting Newton's Law of Cooling

Newton defined his Law of Cooling as:

$$T' = -k(T - T_m)$$

In its LODE form,  $dT = -k(T - T_m)dt$ . (Now,  $T_m$  could be time-dependent.)

*Example:*

Suppose  $k = \frac{1}{40}$ ,  $T_m(t) = 80e^{-\frac{t}{20}}$ ,  $T(0) = 0$ .

- a) Solve this IVP.
- b) Determine the asymptotic behavior of the solution.
- c) Find the maximum temperature.

- a) Write it out as an ODE:

$$T' = -\frac{1}{40}(T - 80e^{-\frac{t}{20}})$$

Not seperable, 1st order ODE. This is an integrating factor problem. In standard form:

$$T' + \underbrace{\frac{1}{40}}_{p(t)} T = \underbrace{2e^{-\frac{t}{20}}}_{q(t)}$$

$$\implies p(t) = \frac{1}{40} \text{ and } q(t) = 2e^{-\frac{t}{20}}$$

$$I(t) = e^{\int p(t) dt}$$

$$I(t) = e^{\int \frac{1}{40} dx} = e^{t/40}$$

Multiplying the original LODE by  $I(t)$ :

$$\underbrace{I(t)T' + I(t)\frac{1}{40}T}_{(IT)'} = \underbrace{I(t)2e^{-\frac{t}{20}}}_{2e^{-\frac{t}{40}}}$$

$$IT = \int 2e^{-\frac{t}{40}} dt = -80e^{-\frac{t}{40}} + C$$

$$\therefore T = e^{-t/40} \left[ -80e^{-\frac{t}{40}} + C \right]$$

Using IV:  $T(0) = 0 \implies 1[-80 + C] = 0$

$$C = 80$$

$$\therefore T = e^{-t/40} \left[ -80e^{-\frac{t}{40}} + 80 \right]$$

$$\boxed{T = 80 \left( -e^{-\frac{t}{20}} + e^{-\frac{t}{40}} \right)}$$

b)  $\boxed{\lim_{t \rightarrow \infty} T(t) = 0}$

c) Maximum temperature? Occurs when  $\frac{dT}{dt} = 0$  or  $T(t) = T_m(t)$

$$T(t) = T_m(t)$$

$$80(e^{-\frac{t}{40}} - e^{-\frac{t}{20}}) = 80e^{-\frac{t}{20}}$$

$$e^{-\frac{t}{40}} - e^{-\frac{t}{20}} = e^{-\frac{t}{20}}$$

$$e^{-\frac{t}{40}} = 2e^{-\frac{t}{20}}$$

$$-\frac{t}{40} = \ln 2 - \frac{t}{20}$$

$$\frac{t}{40} = \ln 2$$

$$t = 40 \ln 2$$

So  $T_{max} = 80 \left( \frac{1}{2} - \frac{1}{4} \right) = \boxed{20}$

### 1.3.3 Application: RLC Circuits

#### The Physics:

- Voltage is the difference  $\Delta V$  in electric potential  $V$
- Voltage along a wire (usually caused by a battery or EMF) causes charge  $q$  to move, causing a current  $i = \frac{dq}{dt}$
- **Kirchoff's Second Law:**

$$\sum_{\text{closed loop}} \Delta V = 0$$

In a circuit, we have components with opposing voltages.

1. Resistance  $R$  causes  $\Delta V = iR$ . Units are ohms  $\Omega$ .  
(Ohm's Law)
2. Capacitance  $C$  causes  $\Delta V = \frac{1}{C}q$ . Units are farads  $F$ .
3. Inductance  $C$  causes  $\Delta V = L \frac{di}{dt}$ . Units are henrys  $H$ .
4. Driver EMF  $E(t) = \sum_{\text{circuit components}} \Delta V$ .

*Aside.* The work done to move a unit charge from A to B can be found using multivariable calculus!

$$\int_C \vec{E} \cdot \vec{T} ds = \int_C \Delta V \cdot \vec{T} ds \stackrel{FTC}{=} V(b) - V(a)$$

By Kirchoff's Law, we have an LODE:

$$L \frac{di}{dt} + Ri + \frac{1}{C}q = \frac{1}{L}E(t)$$

Since  $i = \frac{dq}{dt}$ ,

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t)}$$

Generally, either  $E(t) = E_0$  (constant as DC) or  $E(t) = E_0 \cos \omega t$  (AC, often 60 Hz). To solve this equation, we must restrict it to either an RL case (first order LODE in  $i$ ) or RC case (first order LODE in  $q$ ).

*Example* (RL Circuit):

Given  $E(t) = E_0 \cos \omega t$ . Find  $i(t)$ , given  $i(0) = 0$ .

*Sol'n.* LODE is  $\frac{di}{dt} + \overbrace{\frac{R}{L}}^{p(t)} i = \overbrace{\frac{E_0}{L}}^{q(t)} \cos \omega t$ . Not seperable, use integrating factor.

$$I(t) = e^{\int \frac{R}{L} dt} = e^{at} \left( a = \frac{R}{L} \right)$$

$$e^{at}i(t) = \int e^{at}E_0 \cos \omega t dt$$

Requires integration by parts

$$i(t) = \underbrace{\frac{E_0 a}{L(a^2 + \omega^2)} (a \cos \omega t + \omega \sin \omega t)}_{\text{steady state}} + \underbrace{\frac{E_0 a}{L(a^2 + \omega^2)} (-ae^{-at})}_{\text{transient}}$$