Linear Algebra[1]

Christopher K. Walsh

April 14, 2025

[1]Partially based on courses taught by Dr. Eric Brussel and Dr. Morgan Sherman at California Polytechnic State University San Luis Obispo

Contents

Preface

This document compiles all my notes on linear algebra including any self-study. This also serves as a foundation for the other analysis topics covered in my notes series.

This topic is what really propelled me to study more advanced math. Linear algebra is a lot more than the study of vectors and systems of equations. It is a methodology for vector spaces including standard Euclidean vectors and functions. For the first time in my life, I started dealing with math beyond calculations and computations. I had to prove things about these systems and I found that incredibly interesting and rewarding.

Hope that these notes make you feel the same way.

- Christopher

Chapter 1

Matricies

1.1 Defining Matrices

Definition (Matrix)*:*
Let n, m be two integers ≥ 1. A **matrix** is an array of numbers with m rows and n columns (called a $m \times n$ matrix).

We call a_{ij} the **ij-entry** which is the entry in the ith row and the jth column. We write a matrix often as $A = (a_{ij})$ and define a_{ij}.

Each column of an $m \times n$ matrix is a **column vector**. Each row of an $m \times n$ matrix is a **row vector**.

Example (Identity Matrix)*:*
The Kronecker delta is defined as follows:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then we define the **identity matrix** as:

$$I_n = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example:
If we have a matrix, $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, the second column vector of A is $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and the second row vector of A is $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

We can describe matrices using their column or row vectors. For example:

$$A = \begin{bmatrix} \vec{r} \\ \vec{s} \\ \vec{t} \end{bmatrix}$$

where

$$\vec{r} = \begin{bmatrix} 3 & 4 \end{bmatrix}, \vec{s} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \vec{t} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

Or:

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

where

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

1.2 Matrix Operations

We treat matricies the same way as numbers. Let A be an $m \times n$ matrix and B be an $p \times q$ matrix.

- We can add A and B. If $m = p$ and $n = q$, then $A + B = (a_{ij} + b_{ij})$
- We can multiply by a scalar c: $c \cdot A = (c \cdot a_{ij})$
- We can multiply A and B. If $n = p$, then $A \cdot B = (c_{ij})$ where $c_{ij} = \vec{r_i}(A) \cdot \vec{c_i}(B)$. Caution: In general, $A \cdot B \neq B \cdot A$.
- We also define the transpose of a matrix. The transpose of A is $A^t = (d_{ij})$ where $d_{ij} = a_{ji}$. When we take a transpose, we switch the columns into rows and vice versa.

Certain special matrices can be described with other terminology. Suppose we have a matrix, $A = (a_{ij}), i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- If $m = n$, then A is a **square matrix**.
- If $A^t = A$, then A is a **symmetric matrix**. Note: this means A must also be square.
- If $A^t = -A$, then A is said to be **skew-symmetric**.
- If for all i, j such that $i \neq j$, $a_{ij} = 0$, then A is called **diagonal**.

1.2.1 Determinants

The determinant is a property of a square matrix, A.

Geomtric Def'n.

Let $A = \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (vectors in $\mathbb{R}^n$). Let Π be the parallelotope defined by basing them all at the same point. Then $V^\sigma(\Pi) = \det \begin{bmatrix} \vec{c_1}, & \vec{c_2}, & \ldots, & \vec{c_n} \end{bmatrix}$ (signed n volume).

Example:

$$A = \begin{bmatrix} \vec{c_1} & \vec{c_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\det A = -1$$

Algebraic Def'n.

Let $A = (a_{ij})$, $n \times n$. Let A_{ij} = submatrix obtained from A by eliminating row i and column j, $(n-1) \times (n-1)$ matrix.

The minor of A is $M_{ij} = \det A_{ij}$. The cofactor of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Then $\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ (for any i, $1 \le i \le n$). We call this the cofactor expansion along the ith row.

Example:

$$A = \begin{bmatrix} 7 & 3 & 12 \\ 2 & 5 & 8 \\ 1 & 5 & 2 \end{bmatrix}$$

Find the determinant.

Use row 3.

$$\det A = \sum_{j=1}^{3} a_{3j} C_{3j} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$\left.\begin{aligned} C_{31} &= (-1)^{3+1} \begin{vmatrix} 3 & 12 \\ 5 & 8 \end{vmatrix} = -36 \\ C_{32} &= (-1)^{3+2} \begin{vmatrix} 7 & 12 \\ 2 & 8 \end{vmatrix} = -32 \\ C_{33} &= (-1)^{3+3} \begin{vmatrix} 7 & 3 \\ 2 & 5 \end{vmatrix} = 29 \end{aligned}\right\} \det A = 1(-36) + 5(-32) + 2(29) = \boxed{-138}$$

Properties of Determinants

1. $\det A = \det A^t$

   (Therefore, we can also use the cofactor expansion along any column as well.)

2. $\det A \neq 0 \iff A$ is invertible

3. $\det AB = \det A \det B$

4. $\det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i} + \vec{r_i}' \\ \vdots \\ \vec{r_n} \end{bmatrix} = \det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i} \\ \vdots \\ \vec{r_n} \end{bmatrix} + \det \begin{bmatrix} \vec{r_1} \\ \vdots \\ \vec{r_i}' \\ \vdots \\ \vec{r_n} \end{bmatrix}$

Note. By (3), $\det(A^{-1}) = (\det A)^{-1}$

(Additionally, elementary row operations affect the determinant in specific ways and are discussed in 2.2.)

1.3 Inverse Matricies

An $n \times n$ matrix A may or may not have an **inverse**: A matrix B such that

$$AB = BA = \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We write $B = A^{-1}$.

For a linear system, $A\vec{x} = \vec{b}$ with A, $n \times n$, if A is invertible:

$$\underbrace{A^{-1}A}_{I_n}\vec{x} = A^{-1}\vec{b}$$

$$\therefore \text{ Sol'n. is } \vec{x} = A^{-1}\vec{b}$$

We can vary $\vec{b}$ and the solutions are immediate.

For the 2×2 case, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A has an inverse $\iff |A| = ad - bc \neq 0$.

Then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$$

Invertible?

$18 - 35 = -17 \neq 0$ ✓

$$A^{-1} = \frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}$$

<u>Check</u>

$$\frac{-1}{17}\begin{bmatrix} 9 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 & 0 \\ 0 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 1. *A $n \times n$ matrix A has an inverse $\iff$ $rkA = n$. If A has an inverse, then A^{-1} is given by $rref[A\, I_n] = [I_n\, A^-1]$*

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Is it invertible?

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[2.\,A_{13}(-1)]{1.\,A_{12}(-2)} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[4.\,A_{31}(1)]{3.\,A_{32}(-1)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{5.\,A_{21}(-2)} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of A^{-1}

1. If A is invertible, so is A^{-1}, $\left(A^{-1}\right)^{-1} = A$.
2. If A, B are invertible, $n \times n$, then so is $A \cdot B$ and $(BA)^{-1} = B^{-1}A^{-1}$.

   Witness $A\,(B \cdot B^{-}1)\,A^{-1} = AI_nA^{-}1 = I_n$.
3. If A is invertible, so is A^t, $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$.

Theorem 2. *Suppose A is $n \times n$. Then the following are equivalent*

1. *A is invertible*
2. *$A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$*
3. *$rk(A) = n$*

4. *rref*$(A) = I_n$

5. $\det(A) \neq 0$

Computing Inverses

Recall if $A = a_{ij}$ is a 2×2 matrix, then if A is invertible,

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 3. *If A is invertible then*

$$A^{-1} = \frac{1}{\det A}(b_{ij})$$

where $b_{ij} =$ the ji-th cofactor of $A = (-1)^{i+j} \det A_{ij}$.

Example:
Is A invertible? If so, find A^{-1}.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 5 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 2 - 1 + 1$$

Therefore, A is invertible.

Then $\frac{1}{\det A} \left.\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\right\} (C_{ij})^t$

$C_{11} = (-1)^2 \cdot 2 = 2$ $\quad C_{21} = -1$ $\quad C_{31} = 0$
$C_{12} = (-1)^3 \cdot 2 = -2$ $\quad C_{22} = 1$ $\quad C_{32} = -1$
$C_{13} = 3$ $\quad C_{23} = 1$ $\quad C_{33} = 1$

Then $A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

1.4 Matricies as Linear Transformations

Matricies can be used to model transformations of vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. This is accomplished by having an $m \times n$ matrix, A, written as:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

Example ($\mathbb{R}^n$ to $\mathbb{R}^m$)*:*

Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, $A : \mathbb{R}^2$ to $\mathbb{R}^3$ and:

$$A\vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$$

Note. We are getting a linear combination of the column vectors of A. In other words, $A\vec{v} = x\vec{a} + y\vec{b}$.

Chapter 2

Application: Systems of Linear Equations

2.1 Systems of Linear Equations

Another application of matricies of great interest is representing systems of linear equations.

Definition (Linear)*:*
An equation is **linear** in n variables x_1, x_2, ... , x_n if each term contains at most one x_i and that x_i appears to the first power.

A "system of m linear equations in n unknowns."

Example (Cutting n by $n-1$ dimensional objects)*:*
Suppose we have two planes which will separate regions in a three dimensional space.

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x + 2y + z &= 0 \end{aligned}$$

Sol'n. $\{(x, y, z) : \text{satisfy all equations}\}$

The solution would be the intersection of these planes. Since they extend out infinitely, we would expect them to meet long a line contained in both planes.

Note. Multiplying by a constant, adding a multiple of one to another, and permuting the rows all do not change the solution.

Possible Outcomes: A system of m linear equations in n unknowns has either $0, 1, \text{or} \inf$ solutions.

The technique to solving systems of linear equations is to develop a faster, simple method using matrices. We represent the system as a matrix, preform a series of operations to obtain a simpler matrix, and get our solution from the simpler solution.

Example:

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x + 2y + z &= 0 \end{aligned}$$

We take the coefficients and constants and put them into a matrix:

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$
$$\Big\downarrow \vec{r_1} \to \vec{r_1} - \vec{r_2}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$
$$\Big\downarrow \vec{r_2} \to \vec{r_2} - 2\vec{r_1}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 5 & 2 \end{bmatrix}$$
$$\Big\downarrow \vec{r_2} \to \frac{1}{2}\vec{r_2}$$
$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{5}{2} & 1 \end{bmatrix}$$

Thus, we find:

$$\begin{aligned} x - 2z &= 1 \\ y + \frac{5}{2}z &= -1 \end{aligned}$$

Sol'n. $z = t$ as a free variable

$$\begin{aligned} x &= 1 + 2t \\ y &= -1 - \frac{5}{2}t \end{aligned}$$

It's a line as expected!

$$L(t) = \left(1 + 2t, -1 - \frac{5}{2}t, t\right)$$

2.2 Vector Form

Given:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

The **vector form** of the system is: $A\vec{x} = b$ The **coefficient matrix** is $A = (a_{ij})$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

The **augmented matrix** is $A^{\#} = \begin{bmatrix} A \; \vec{b} \end{bmatrix}$.

Definition (Elementary Row Operations)*:*
Elementary row operations (EROs) are defined as follows:

- $P_{ij} : \vec{r_i} \leftrightarrow \vec{r_j}$
- $M_i(c) : \vec{r_i} \leftarrow c\vec{r_i}$
- $A_{ij}(c) : \vec{r_j} \leftarrow \vec{r_j} + c\vec{r_i}$

These elementary row operations let us solve systems of linear equations in the same way we learned in early math classes. Adding equations together is the same as the row sum operation etc. Using EROs, the process to solve these systems is to write the augmented matrix of the system, perform EROs to get a reduced matrix and obtain the simpler linear system.

Moreover, suppose $A \xrightarrow{1.} B$ is an ERO. Then,

1. If $1.P_{ij}$, then $\det B = -\det A$.
2. If $1.M_i(k)$, then $\det B = k \cdot$ A.
3. If $1.A_{ij}(k)$, then $\det B = \det A$.

(Other properties of the determinant are addressed in 1.2.1.)

Example:
Solve if possible.

$$\begin{aligned}
x + y + z &= 3 \\
2x + 3y + z &= 5 \\
x - y - 2z &= -5
\end{aligned}$$

We write the augmented matrix of the system and preform EROs:

$$A^{\#} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix} \xrightarrow[\text{2. } A_{13}(-1)]{\text{1. } A_{12}(-2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

$$\xrightarrow{\text{3. } A_{23}(2)} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \operatorname{ref} A^{\#}$$

$$\xrightarrow[\text{6. } A_{31}(-1)]{\text{5. } A_{32}(1)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{7. } A_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \operatorname{rref} A^{\#}$$

We obtain two matrices, the row echelon form and the reduced row echlon form. From ref $A^{\#}$, we find:

$$\begin{aligned} x + y + z &= 3 \\ y - z &= -1 \\ z &= 2 \end{aligned}$$

From rref $A^{\#}$, we find:

$$\begin{aligned} x &= 0 \\ y &= 1 \\ z &= 2 \end{aligned}$$

$$\boxed{(0,\ 1,\ 2)}$$

Thus, (0, 1, 2) solves the system of equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -5 \end{bmatrix}$$

Note. If the number of leading 1's equals the number of unknowns, we obtain one solution.

Definition (Echelon Forms)*:*
The **row echelon form** (ref) of any matrix is one such that:

(a) The first nonzero entry of each nonzero row is 1. ("Leading one")

(b) If a row has a leading one, each row above it has a leading 1 further to the left.

The **reduced row echlon form** (rref) of any matrix also has the conditon:

(c) Each entry above a leading 1 is 0.

To put a matrix into its ref or rref form:

1. Use P_{ij} so row i has the left most nonzero entry among rows $\geq i$.
2. Use $M_i(k)$ to create a leading 1 in $\vec{r_i}$.
3. Use $A_{ij}(k)$ to zero out column below leading 1 in $\vec{r_i}$. For rref, zero out columns above as well.
4. If there is no nonzero row below i, stop.

   Else:
5. Replace $i = i + 1$, repeat on submatrix with rows $\geq i + 1$.

2.3 Theorem of Rank

Definition (Rank)*:*
The number of leading 1's in ref(A) is the rank of A, usually written as "rk(A)."

Theorem 4 (Theorem of Rank)**.** *Let A, $A^{\#}$ be the coefficient and augmented matrix for a system of linear equations with m equations and n unknowns.*

Then, there are three cases:

1. *No solutions if $\operatorname{rk} A < \operatorname{rk} A^{\#}$.*
2. *Unique solutions if $\operatorname{rk} A = \operatorname{rk} A^{\#} = n$.*
3. *Infinitely many solutions if $\operatorname{rk} A = \operatorname{rk} A^{\#} < n$.*

Note. There are unique solutions if exactly all variables are bound and infinitely many if there are $n - \operatorname{rk}A$ free variables.

Example:
Here is an illustration of this. Suppose we have:

$$\operatorname{rref}A^{\#} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{matrix} \\ \begin{bmatrix} 1 & * & 0 & * & 0 & a \\ 0 & 0 & 1 & * & 0 & b \\ 0 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 0 & d \end{bmatrix} \end{array}$$

If $d \neq 0$, there are no solutions. If $d = 0$, there are infinitely many solutions. There are no unique solutions possible since there are two free variables: $x_2 = s$ and $x_4 = t$. The bound variables are x_1, x_2, and x_5.

From this illustration, we can see two important cases:

1. If we have a homogenous system, $A\vec{x} = 0$, then we always have the trivial solution $\vec{x} = 0$. There exists nontrivial solutions if and only if $\operatorname{rk}(A) < n$.

2. If A is $m \times n$ and $m < n$, then since $\mathrm{rk}(A) \leq m$, $\mathrm{rk}(A) < n$.
   Therefore, there are either 0 or infinitely many solutions.

Example:

Fix $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{w}$ (all) such that $\vec{v} \times \vec{w} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\vec{v} \times \vec{w} = \begin{bmatrix} w_3 - w_2 \\ w_1 - w_3 \\ w_2 - w_1 \end{bmatrix}$$

Now we have a system of linear equations with $\vec{w}$ as the unknown.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^{\#} = \begin{bmatrix} 0 & -1 & 1 & b_1 \\ 1 & 0 & -1 & b_2 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow{1.\ P_{12}} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & -1 & 1 & b_1 \\ -1 & 1 & 0 & b_3 \end{bmatrix} \xrightarrow[3.\ M_2(-1)]{2.\ A_{13}(1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 1 & -1 & b_2 + b_3 \end{bmatrix}$$
$$\xrightarrow{4.\ A_{23}(-1)} \begin{bmatrix} 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & -b_1 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{bmatrix}$$

We have solutions $\iff b_1 + b_2 + b_3 = 0 = \vec{v} \cdot \vec{b}$. Then we have one free variable

$$\left.\begin{aligned} w_3 &= t \\ w_2 &= -b_1 + t \\ w_1 &= b_2 + t \end{aligned}\right\} = \boxed{(b_2 + t,\ -b_1 + t,\ t)}$$

Chapter 3

Vectors and Vector Spaces

3.1 Defining Vector Spaces and Subspaces

We've seen the sets $\mathbb{R}^2$ and $\mathbb{R}^3$ where the elements are vectors as translations in plane and 3-space. These sets have an algebraic structure.

Definition (Vector Space)*:*
Any set following axioms A1-A10 is called a **vector space**.

Given a vector space A, vectors $u, v, w \in A$ and scalars r, s:

A1. We can define addition (and the space is closed under addition).

A2. We can define scalar multiplication (and the space is closed under scalar multiplication).

A3. $u + v = v + u$

A4. $(u + v) + w = u + (v + w)$

A5. There exists an additive identity in A.

A6. There exists an additive inverse for any $u \in A$.

A7. There exists a multiplicative identity in A.

A8. $(rs)v = r(sv)$

A9. $r(u + v) = ru + rv$

A10. $(r + s)v = rv + sv$

By studying vector spaces, we can develop algebraic machinery to apply to all kinds of other objects.

Example:
$Q = \{a_0 + a_1x + a_2x^2 \mid a_i \in \mathbb{R}\}$ (Polynomials of degree 2 or less)

This is a vector space!

A1. Adding doesn't increase the degree. ✓

A2. Scalar multiplication similarly doesn't increase the degree. ✓

A3. Addition in $\mathbb{R}$ is commutative. ✓

A4. By the associative property of $\mathbb{R}$,

$$\begin{aligned}
&\Big((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\Big) + (c_0 + c_1x + c_2x^2) \\
&= \Big((a_0 + b_0) + c_0\Big) + \Big((a_1 + b_1) + c_1\Big)x + \Big((a_2 + b_2) + c_2\Big)x^2 \\
&= \Big(a_0 + (b_0 + c_0)\Big) + \Big(a_1 + (b_1 + c_1)\Big)x + \Big(a_2 + (b_2 + c_2)\Big)x^2 \\
&= a_0 + a_1x + a_2x^2 + (b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 \checkmark
\end{aligned}$$

A5. 0 polynomial ✓

A6. $-(a_0 + a_1x + a_2x^2) = -a_0 - a_1x - a_2x^2$ ✓

A7. $1 \cdot (a_0 + a_1x + a_2x^2) = a_0 + a_1x + a_2x^2$ ✓

A8. ✓

A9. ✓

A10. ✓

Example:
$V = \mathbb{R}^2$
$S = \{(x, y) \mid x^2 - y^2 = 0\}$
Is S a vector space?

No, since addition is not closed.

Example:

$$V = M_{3\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \middle| \; a, b, c, d, e, f \in \mathbb{R} \right\}$$

$$S = \{A \in V \mid \text{columns each sum to } 0\}$$

Is S a vector space?

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -2 \end{bmatrix} \in S$$

. Now,

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

.

Closed under scalar mult? ✓
Consider

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \text{ s.t. } a+b+c=0 \text{ and } d+e+f=0$$

.
Then

$$m \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} ma & md \\ mb & me \\ mc & mf \end{bmatrix}$$

.
So $ma+mb+mc = m(a+b+c) = m(0)$ and $md+me+mf = m(d+e+f) = m(0)$. ✓

Closed under addition? ✓
Consider

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}, \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} \in S$$

.
Now

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} + \begin{bmatrix} a' & d' \\ b' & e' \\ c' & f' \end{bmatrix} = \begin{bmatrix} a+a' & d+d' \\ b+b' & e+e' \\ c+c' & f+f' \end{bmatrix}$$

Check $(a+a')+(b+b')+(c+c') = (a+b+c)+(a'+b'+c') = 0$.

Note. Since the above examples of S are subsets of known vector spaces, we just have to check:

1. Closed under $+, -$
2. Closed under scalar multiplication

These are called **subspaces**.

Vectors spaces do not just include $\mathbb{R}^n$ and $\mathbb{C}^n$. We have:

- $C^k(I)$: function on an internal I with k continuous derivatives
- $P_n(\mathbb{R})$: polynomials with degree up to n
- $M_{m\times n}(\mathbb{R})$

Example:
Consider $y'' + a_1(x)y' + a_2(x) = 0$ on an interval I. Let S be the set of solutions to this LODE. Is S a vector space?

- Addition: If $y_1, y_2 \in S$, is $y_1 + y_2 \in S$?

  $$(y_1+y_2)''+a_1(x)(y_1+y_2)'+a_2(x)(y_1+y_2) = y_1''+y_2''+a_1(x)(y_1'+y_2')+a_2(x)(y_1+y_2) = 0+0$$

  Therefore, S is closed under addition.

- Scalar Multiplication: If $y \in S$, is $cy \in S$? Yes since $(cy)'' = cy''$ and $(cy)' = cy'$.

Therefore S is a vector space, a subspace of $C^2(I)$.

Example:
Let $V = \mathbb{R}^3$ and S be the solutions to the linear system:

$$\begin{aligned} 2x + 3y + z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

Then $S \subset V$ is a subspace. All vectors in the subspace lie on a line through the origin contained on both planes.

Using this concept of subspaces, we can find solutions to certain differential equations or systems of linear equations.

Definition (Null Space)*:*
The solutions of a homogeneous linear system $A\vec{x} = 0$ is called the **null space** of A, any $A_{m\times n}$. It is a subspace of $\mathbb{R}^n$ and is also known as the **kernel** of A.

For $\mathbb{R}^3$, subspaces consist of either a point $\vec{0}$, a line through $\vec{0}$, or a plane through $\vec{0}$.

3.2 Spanning Sets

Definition (Span)*:*
A linear combination of vectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n} \in V$ is a vector $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + n\vec{v_n}$ in V.

The **span** of $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ is the set of all linear combinations. The span is a subspace of the vector space V.

Example:
Compute the span of $\{(-4, 1, 3),\ (5, 1, 6),\ (6, 0, 2)\}$ in $\mathbb{R}^3$.

We want the set $\{(a, b, c)\}$ that are linear combinations of the three vectors. In other words, the solutions of the linear system:

$$\begin{bmatrix} -4 & 5 & 6 & a \\ 1 & 1 & 0 & b \\ 3 & 6 & 2 & c \end{bmatrix}$$

$$\xrightarrow{1.\ 2.\ 3.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 9 & 6 & a+4b \\ 0 & 3 & 2 & c-3b \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 1 & 0 & b \\ 0 & 3 & 2 & c-3b \\ 0 & 0 & 0 & a+13b-3c \end{bmatrix}$$

1. P_{12}
2. $A_{12}(4)$
3. $A_{13}(-3)$
4. P_{23}
5. $A_{23}(-3)$

There is a solution if and only if $\text{rk}A = \text{rk}A^{\#}$. So $a + 13b - 3c = 0$. We conclude the span is on a plane $\boxed{x + 13y - 3z = 0}$.

Note. One of the three vectors is redundant.

Example:
Write $(-4, 1, 3)$ as a linear combination of $(5, 1, 6)$ and $(6, 0, 2)$.

We want constants c_1, c_2 such that:

$$c_1 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

We need to solve

$$\begin{bmatrix} 5 & 6 & -4 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow[2.\ 3.]{1.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 6 & -9 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{4.\ 5.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{6.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

.

The moral of this example is that sometimes a spanning set has redundant vectors. We would like a spanning set to have no redundant vectors or a "minimally spanning set."

Definition (Linear Dependence)*:*
A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ in a vector space V is **linearly dependent** if there exists some $c_1, c_2, \ldots, c_n$ not all zero such that: $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$. (This is called the dependence relation, and we would say one of the vectors is redundant.)

Otherwise, the set is **linearly independent** and the relation $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$.

Example:
The vectors

$$\vec{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

are linearly dependent.
Find the dependence relation.

We need to solve $A\vec{x} = \vec{0}$ and $\text{rk}A \leq 2$ for there to be nontrivial solutions.

$$\left[\begin{array}{cccc} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Parameterizing, we find that $c_3 = t, \ c_2 = t, \ c_1 = 2t - t = t$.

Therefore $\boxed{\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}}$.

3.2.1 Method for Testing Spans

Given $\{\vec{v_1}, \ldots, \vec{v_n}\}$, to determine if they are linearly dependent, form a matrix

$$A = \left[\vec{v_1} \ \vec{v_2} \ \ldots \ \vec{v_n}\right]$$

Then the set is linearly dependent if and only if $A\vec{c} = \vec{0}$ has nontrivial solutions. In other words, there exists some non-zero scalars $c_1, \ldots, c_k$ such that $c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}$ if and only if $\text{rk}A < k$.

The technique is to commit ERO's and find rref(A).

For example, suppose we have a span and rref$(A^{\#})$ is:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & \uparrow & & \uparrow & & \end{array} \end{array}$$

We have two free variables.

<u>Claim:</u> Each free variable gives a nontrivial dependence with bound variables.

So for this example, there are two dependence relations where $\vec{v_2}$ and $\vec{v_4}$ are redundant and $\vec{v_1}, \ \vec{v_3}, \ \vec{v_5}$ form a linearly independent span.

In general, we can conclude that $v_1, \ldots, \vec{v_k}$ are linearly independent if and only if there are no free variables in rref$(A^{\#})$ or equivalently $\text{rk}A = k$.

Theorem 5. *In $\mathbb{R}^n$, a set $\{\vec{v_1}, \ldots, \vec{v_n}\}$ is linearly independent if and only if, for the matrix $A = [\vec{v_1}, \ldots, \vec{v_n}]$, $\det A \neq 0$ or rref$A = I_n$.*

Example:
Find all values of k such that: $\{(1,1,0,-1), (1,k,1,1), (2,1,k,1), (-1,1,1,k)\}$ are linearly independent.

Applying the previous theorem, the solution form is:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & k & 1 & 1 \\ 0 & 1 & k & 1 \\ -1 & 1 & 1 & k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then $\det A = k^3 - 2k^2 - 5k + 6 = (k-1)(k+2)(k-3)$. We conclude $\boxed{k \neq -2, 1, -3}$.

3.3 Linear Independence of Functions

Definition (Wronskian)*:*
Let $f_1, f_2, \ldots, f_n$ be in $C^{n=1}(I)$. The **Wronskian** is:

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1 & f_2 & \ldots & f_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1 & f_2 & \ldots & f_n \end{vmatrix}$$

Then if $W[f_1, f_2, \ldots, f_n](x) \neq 0$ for some $x \in I$, then the functions are linearly independent.

Sketch (of Proof)

The dependence relation has the form:

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \text{ for all } x$$

This implies:

$$c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' = 0 \text{ for all } x$$

and so on, ending with

$$c_1 f_1^{n-1} + c_2 f_2^{n-1} + \cdots + c_n f_n^{n-1} = 0 \text{ for all } x$$

.

We conclude that $W[f_1, f_2, \ldots, f_n](x) = 0$ for all x.

Therefore if $W[f_1, f_2, \ldots, f_n](x) \neq 0$, then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \neq 0$ for some x.

Example (Exponentials)*:*
Determine whether e^x and e^{-x} are linearly independent on $(-\infty, \infty)$.

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

We conclude they are linearly independent on $(-\infty, \infty)$.

What about e^x and e^{2x}?

$$W[e^x, e^{2x}] = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

We also conclude they are linearly independent.

Example ($P_2(x)$)*:*
Determine the linear independence of $\{1, x, x^2\}$.

$$W[1, x, x^2] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

Therefore, $\{1, x, x^2\}$ are linearly independent.

What about $\{1, x, x^2\}$?

$$W[1, x, 2x] = \begin{vmatrix} 1 & x & 2x \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

We cannnot draw a conclusion soley on the Wronskian, but look:

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot 2x = 0$$

$$c_1 = 0,\, c_2 = -2,\, c_3 = 1$$

Therefore, $\{1, x, x^2\}$ are linearly dependent.

3.4 Bases and Dimensions

Given any number of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ in a vector space V, we can define a subspace of V as $W = \text{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ and write $W \subset V$. Equivalently we can write W as $W = \{\sum a_i \vec{i} : a_i \in \mathbb{R}\}$.

Definition (Basis)*:*
A **basis** for a vector space V is a set of vectors that:

1. span V.
2. are linearly independent.

Example:
In $\mathbb{R}^3$, the standard basis is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

.

In P_2, the standard basis is: $\{1, x, x^2\}$.

Note. These are not unique. (We can write multiple bases.)

Theorem 6. *Any two bases for V have the same number of vectors (assuming they are finite).*

Sketch of Proof

If we had two bases:

$$\{\vec{v_1}, \ldots, \vec{v_m}\}, \{\vec{w}_1, \ldots \vec{w}_n\}$$

Let say that $m \leq n$ and $A = [\vec{v_1}, \ldots, \vec{v_m}]$. Now $A\vec{c} = \vec{w_j}$ has unique solutions for all j since $\vec{v_i}$ span V. Since A has m columns, By Thm. of Rank, rk$A = rk(A^{\#}) = n$. *Finish*

Thus, from the previous theorem, we can define the following.

Definition (Dimension)*:*
The **dimension** of a vector space V is the number of elements in any basis.

Example:
Find a basis for the subspace of $\mathbb{R}^4$ spanned by:

$$\underline{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \underline{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}, \begin{bmatrix}3\\4\\5\\0\end{bmatrix}, \underline{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}$$

.

We can simply choose the underlined vectors. The dimension of this subspace is 3.

3.4.1 Secret Trick

Given a $m \times n$ matrix A, it is useful to know:

- A basis for im(A).
- A basis for ker$(A) =$ null(A).
- A basis for the solution space for the linear system, $A\mathbf{c} = \mathbf{0}$.

This section will detail a shorthand method to find all these bases quickly.

Let $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$, an $m \times n$ matrix. First, compute rref(A) in the usual way. Suppose:

$$\text{rref}(A) = \begin{bmatrix} 0 & 1 & b_{13} & 0 & b_{15} & 0 \\ 0 & 0 & 0 & 1 & b_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The free-variable columns correspond to redundant vectors. The bound-variable columns correspond to a basis of the span. Here, $\mathbf{v}_1, \mathbf{v}_3,$ and $\mathbf{v}_5$ are redundant vectors, and $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$ are a basis for im(A).

The dependence-relation vector $\mathbf{c}_j$ corresponding to each free-variable column can be constructed by:

1. Put 1 into the j-th spot.
2. If $b_{ij} \neq 0$, and the leading 1 is in the i-th row is in column $k < j$, put $-b_{ij}$ in the k-th spot.
3. Put 0's everywhere else.

Here, $\mathbf{c}_1 = \langle 1, 0, 0, 0, 0, 0 \rangle$, $\mathbf{c}_3 = \langle 0, -b_{13}, 1, 0, 0, 0 \rangle$, $\mathbf{c}_5 = \langle 0, -b_{15}, 0, -b_{25}, 1, 0 \rangle$

Recall that the solution space is equivalent to $\ker(A) = \text{null}(A)$.

Example:
Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 3 & 6 & 9 & 6 & 2 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$$

. Find bases for the $\operatorname{Im} A$ and $\ker A$.

Now $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Applying the secret trick ...

A basis for $\operatorname{Im} A$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$$

Recall $\operatorname{Im} A = \{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_5\vec{v_5} : c_i \in \mathbb{R}\}$.

$$\begin{array}{ccccc} \left[\begin{matrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} -2 & 1 & 0 & 0 & 0 \\ -5 & 0 & 1 & 1 & 0 \end{matrix} \end{array}$$

A basis for $\ker A$ is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Recall $\ker A = \{\vec{c} \in \mathbb{R}^5 : A\vec{c} = 0\}$.

An important result regarding the kernel and image of a matrix is the rank-nullity theorem.

Theorem 7 (Rank-Nullity Theorem)**.** *Let A be an $m \times n$ matrix. Then*

$$\dim(\ker A) + \dim(\operatorname{Im} A) = n$$

.

Another important result regards the dimension of a basis.

Theorem 8. *Let V be a vector space. If V has a finite basis, then any basis for V has the same number of vectors.*

Corollary 8.1. If $\dim V = n$, then any set of n linearly independent vectors in V is a basis.

The secret trick gives us a method to find a basis and the dimension of the space. Then, any other vector can be expressed using this basis.

Chapter 4

Linear Transformations

4.1 Introduction to Linear Transformations

Let A be an $m \times n$ matrix. We can interpret it as a function or transformation between vector spaces, where $A : \mathbb{R}^n \to \mathbb{R}^m$.

Note that A is a linear transformation since $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$ and $A(c\vec{v}) = cA\vec{v}$.

Example:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

What does it do?

Let

$$\vec{e_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

.

Example:
Find a matrix for $\frac{\Pi}{2}$ rotation.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4.2 Eigenvalues and Eigenvectors

Most linear transformations can be understood with eigenvalues and eigenvectors.

Definition:
Let A be $n \times n$.

An **eigenvalue** of A is a scalar λ such that $A\vec{v} = \lambda\vec{v}$ has a nonzero solution $\vec{v}$.

An **eigenvector** $\vec{v}$ for λ is a nonzero $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenspace** for λ is the set of all $\vec{v}$: $A\vec{v} = \lambda\vec{v}$.

An **eigenbasis** for λ is a basis λ's eigenspace.

We will look at a simple matrix to give concrete examples for all of these definitions.

Example:
Given A $n \times n$, find its eigenvalues, eigenvectors, eigenspace, and eigenbasis.

There are two eigenvalues, $\lambda = 2, 1$.

For $\lambda = 2$, $\vec{e_1}$ is one possible eigenvector. Another possible eigenvector is $3\vec{e_1}$. The eigenspace for $\lambda = 2$ is the x-axis. The eigenbasis is simply $\{\vec{e_1}\}$.

For $\lambda = 1$, the eigenbasis is simply $\{\vec{e_2}\}$.

We can determine this intutitively by considering some vectors and applying the linear transformation A.

To determine the eigenvalues and eigenvectors analytically, note that $A\vec{v} = \lambda\vec{v}$ for nonzero $\vec{v}$ is the same as $(A - \lambda I)\vec{v} = \vec{0}$. Thus all λ satisfy $\det(A - \lambda I) = 0$. This is a polynomial in λ with degree n.

The eigenvalues are the roots of $P_A(x)$ and the eigenvectors are $\ker A - \lambda I = \{\vec{v} \neq 0 : (A - \lambda I)\vec{v} = \vec{0}\}$

Definition:
The **algebraic multiplicity** of λ is the multiplicity of the factor $(x - \lambda)^m$ in $P_A(x)$.

Example:
For the matrix A, find its eigenvalues and a basis for the corresponding eigenspaces.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{bmatrix}$$

Sol'n.

1. 1.
   $$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 & 0 \\ -1 & 0-\lambda & 1 \\ -2 & -1 & 4-\lambda \end{bmatrix}$$
   $\det A - \lambda I = -(\lambda - 2)^3$ $\lambda = 2$ is an eigenvalue.

2. Want basis for

Chapter 5

Eigenvalue Problems

The main way we can