0.1 Defining Matrices

Let n, m be two integers ≥ 1. A **matrix** is an array of numbers with m rows and n columns (called a $m \times n$ matrix).

We call a_{ij} the **ij-entry** which is the entry in the ith row and the jth column. We write a matrix often as $A = (a_{ij})$ and define a_{ij}.

Each column of an $m \times n$ matrix is a **column vector**. Each row of an $m \times n$ matrix is a **row vector**.

Example (Identity Matrix)*:*
The Kronecker delta is defined as follows:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 2, & i \neq j \end{cases}$$

Then we define the **identity matrix** as:

$$\mathbf{I_{n\times n}} = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example:
If we have a matrix, $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, the second column vector of A is $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and the second row vector of A is $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

We can describe matrices using their column or row vectors. For example:

$$A = \begin{bmatrix} \vec{r} \\ \vec{s} \\ \vec{t} \end{bmatrix}$$

where

$$\vec{r} = \begin{bmatrix} 3 & 4 \end{bmatrix}, \vec{s} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \vec{t} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

Or:

$$A = \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$$

where

$$\vec{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

We treat matricies the same way as numbers. Let A be an $m \times n$ matrix and B be an $p \times q$ matrix.

- We can add A and B. If $m = p$ and $n = q$, then $A + B = (a_{ij} + b_{ij})$
- We can multiply by a scalar c: $c \cdot A = (c \cdot a_{ij})$
- We can multiply A and B. If $n = p$, then $A \cdot B = (c_{ij})$ where $c_{ij} = \vec{r_i}(A) \cdot \vec{c_i}(B)$. <u>Caution:</u> In general, $A \cdot B \neq B \cdot A$.
- We also define the transpose of a matrix. The transpose of A is $A^t = (d_{ij})$ where $d_{ij} = a_{ji}$. When we take a transpose, we switch the columns into rows and vice versa.

Certain special matrices can be described with other terminology. Suppose we have a matrix, $A = (a_{ij}), i = 1, \dots, m$ and $j = 1, \dots, n$.

- If $m = n$, then A is a **square matrix**.
- If $A^t = A$, then A is a **symmetric matrix**. Note: this means A must also be square.
- If $A^t = -A$, then A is said to be **skew-symmetric**.
- If for all i, j such that $i \neq j$, $a_{ij} = 0$, then A is called **diagonal**.

0.2 Matricies as Linear Transformations

Matricies can be used to model transformations of vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. This is accomplished by having an $m \times n$ matrix, A, written as:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

Example ($\mathbb{R}^n$ to $\mathbb{R}^m$)*:*
Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, $A : \mathbb{R}^2$ to $\mathbb{R}^3$ and:

$$A\vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$$

Note. We are getting a linear combination of the column vectors of A. In other words, $A\vec{v} = x\vec{a} + y\vec{b}$.