Vector Analysis (MATH 304)

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Chapter 1

Introduction and Review

1.1 Theory of Curves

Our main tool to understand curves will be vector methods. When we consider $\mathbb{R}^3$, we cannot use equations to describe curves since we obtain surfaces.

How to Describe Curves:

- Solutions to (x, y) to eqn $f(x, y) = 0$.

$$x^2 + y^2 - 1 = 0$$

- Image of vector valued function, a parameterization.

$$\begin{aligned} \gamma : I &\to \mathbb{R}^2 \text{ (or } \mathbb{R}^n) \\ t &\mapsto (x(t), y(t)) \end{aligned}$$

$$\begin{aligned} e.g.\ \gamma(t) &= (\cos t, \sin t),\ 0 \le t \le 2\pi \\ \gamma :\ &[0, 2\pi] \to \mathbb{R}^2 \end{aligned}$$

Example (Parameterization)*:*
Line: Euclid said there exists a unique line between any two points $p, q \in \mathbb{R}^n$ $(p \neq q)$. Parameterize by $\lambda(t)$: $p + t(q - p),\ t \in \mathbb{R}$

Helix: $\gamma(t) = (\cos t, \sin t, -t)$

Basic Fact: Every parameterized curve, $\gamma(t) :\ I \to \mathbb{R}^n$ has a **velocity**, $\gamma'(t)$. $\gamma'(t)$ is the vector tangent to (the trace of) $\gamma(t)$ pointing in the traveling direction.

In $\mathbb{R}^3$, $\gamma'(t) = (x'(t), y'(t), z'(t))$.

$|\gamma'(t)|$ is the speed.

The distance along $\gamma(t)$ is computed using:

$$s = \int_{t_1}^{t_2} |\gamma'(t)|\, dt$$

Another basic fact is that we can apply derivatives to the different (vector) products. We begin by examining how we can use the dot product.

$$(\gamma(t)\cdot\gamma(t))' = \gamma'(t)\cdot\gamma(t) + \gamma(t)\cdot\gamma'(t) = 2\gamma'(t)\cdot\gamma(t)$$

1.2 Determinants and Cross Products

1.2.1 Determinants

Example (Computing areas)*:*
Suppose we have a paralellogram made by the vectors $(1,4)$ and $(8,3)$. Compute the area.

We could compute the area of the paralellogram using geometry, but we can easily see the area as 29 using the determinant.

$$A = \left|\det\begin{bmatrix}1 & 8\\ 4 & 3\end{bmatrix}\right| = |1\cdot 3 - 8\cdot 4| = |-29| = 29$$

We can interpret this matrix as a linear transformation from the standard basis in $\mathbb{R}^2$, a change of basis.

$$e_1 = \begin{bmatrix}0\\1\end{bmatrix} \mapsto f_1 = \begin{bmatrix}1\\4\end{bmatrix}$$

$$e_2 = \begin{bmatrix}1\\0\end{bmatrix} \mapsto f_2 = \begin{bmatrix}8\\3\end{bmatrix}$$

1.2.2 Cross Product

With the standard basis in $\mathbb{R}^3$, let $u = (u_1, u_2, u_3) = u_1e_1 + u_2e_2 + u_3e_3$ and $v = (v_1, v_2, v_3) = v_1e_1 + v_2e_2 + v_3e_3$.

We define the cross product of $u \wedge v$ as:

$$u \wedge v = \left(\begin{bmatrix}u_2 & u_3\\ v_2 & v_3\end{bmatrix}, -\begin{bmatrix}u_1 & u_3\\ v_1 & v_3\end{bmatrix}, \begin{bmatrix}u_1 & u_2\\ v_1 & v_2\end{bmatrix}\right)$$

This also has a geometric interpretation: the three components of the cross product are the areas of the "shadows" of the the paralellogram formed by u and v in $yz,\ xz,\ xy$ -planes respectively.

Properties:

1. The area of the paralellogram u, v is equal to $|u \wedge v|$.
2. The direction of $u \wedge v$ (if non-zero) is perpendicular to the paralellogram u, v using the right hand rule.
3. The volume of the paralellopiped u, v, w is equal to $(u \wedge v) \cdot w = \det \begin{bmatrix} u & v & w \end{bmatrix} = \det \begin{bmatrix} u \\ v \\ w \end{bmatrix}$
4. $u \wedge v = -v \wedge u$
5. $u \wedge (v + w) = u \wedge v + u \wedge w$
6. $(u(t) \wedge v(t))' = u'(t) \wedge v(t) + u(t) \wedge v'(t)$