Local Stochastic Volatility Model Course

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Chapter 1

Foundations of Stochastic Calculus

1.1 Deterministic vs. Stochastic Processes

A *deterministic process* evolves according to a fixed rule without randomness.

$$x(t) = a\,t + b.$$

A *stochastic process* evolves with random shocks; we describe its distribution but not its exact path.

$$S_t = S_{t-1} + \text{random increment}.$$

1.2 Brownian Motion W_t

Brownian motion is the canonical continuous-time stochastic process.

- $W_0 = 0$.
- Independent increments: for $0 \leq s < t < u < v$, $W_t - W_s$ independent from $W_v - W_u$.
- Gaussian increments: $W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t)$.
- Continuous but nowhere differentiable paths.

Discrete Approximation

Over steps Δt, approximate

$$W_{t_{i+1}} - W_{t_i} \approx \sqrt{\Delta t}\, Z_i, \quad Z_i \sim N(0,1).$$

1.3 Infinitesimal Increment dW_t

Notation dW_t denotes the "infinitesimal" change over dt. In simulations,

$$dW_t \approx \sqrt{dt}\, Z, \quad Z \sim \mathcal{N}(0,1).$$

It is *not* an ordinary derivative, but a formal symbol in Itô calculus.

1.4 Itô's Lemma and Stochastic Integrals

For an Itô process

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t,$$

and a twice-differentiable $f(t, x)$, Itô's lemma gives

$$df(t, X_t) = f_t\,dt + f_x\,dX_t + \tfrac{1}{2}\,f_{xx}\,b^2(t, X_t)\,dt.$$

The term $\frac{1}{2}f_{xx}b^2\,dt$ has no analogue in ordinary calculus.

1.5 Simple SDEs: Geometric Brownian Motion

The Black–Scholes model assumes

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t.$$

1.5.1 Logarithmic Transformation

Set $X_t = \ln S_t$. By Itô's lemma:

$$d(\ln S_t) = \Big(\mu - \tfrac{1}{2}\sigma^2\Big)\,dt + \sigma\,dW_t.$$

Integrate:

$$\ln S_t = \ln S_0 + \Big(\mu - \tfrac{1}{2}\sigma^2\Big)t + \sigma W_t.$$

1.5.2 Closed-Form Solution

Exponentiating yields

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\Big),$$

showing S_t is log-normally distributed: $\ln S_t \sim \mathcal{N}(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t,\ \sigma^2 t)$.

Exercises

Exercise 1: Verifying Itô's Lemma

Let

$$X_t = \exp\Big(W_t - \tfrac{1}{2}t\Big).$$

Show that

$$dX_t = X_t\,dW_t$$

by:

1. Direct expansion using $(dY_t)^2 = dt$ with $Y_t = W_t - \frac{1}{2}t$.
2. Formal application of Itô's lemma to $g(t, w) = \exp(w - \frac{1}{2}t)$.

Solution:

1. *Direct method.* Write $X_t = e^{Y_t}$ with $dY_t = dW_t - \frac{1}{2}dt$. Then

$$dX_t \approx e^{Y_t}\Big(dY_t + \tfrac{1}{2}(dY_t)^2\Big) = e^{Y_t}\Big((dW_t - \tfrac{1}{2}dt) + \tfrac{1}{2}dt\Big) = X_t\, dW_t.$$

2. *Itô's lemma.* For $g(t,w) = e^{w-\frac{1}{2}t}$,

$$g_t = -\tfrac{1}{2}g, \quad g_w = g, \quad g_{ww} = g,$$

so

$$dg = g_t\, dt + g_w\, dW_t + \tfrac{1}{2}g_{ww}\, dt = -\tfrac{1}{2}g\, dt + g\, dW_t + \tfrac{1}{2}g\, dt = g\, dW_t = X_t\, dW_t.$$

Exercise 2: Moments of Geometric Brownian Motion

Given the SDE

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \quad S_0 \text{ given},$$

its solution is

$$S_t = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\Big).$$

1. Show that $\ln S_t \sim \mathcal{N}\Big(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t,\ \sigma^2 t\Big)$.
2. Compute
$$\mathbb{E}[S_t] \quad \text{and} \quad \mathrm{Var}(S_t).$$

Solution:

1. Since $W_t \sim N(0,t)$,
$$\ln S_t = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$
is Gaussian with mean $m = (\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t)$ and variance $v = \sigma^2 t$.

2. For a log-normal $S_t = S_0 e^Z$ with $Z \sim N(m,v)$:

$$\mathbb{E}[S_t] = S_0 e^{m+\frac{1}{2}v} = S_0 e^{\mu t}, \qquad \mathrm{Var}(S_t) = S_0^2\,(e^v - 1)\,e^{2m+v} = S_0^2\,(e^{\sigma^2 t} - 1)\,e^{2\mu t}.$$

Chapter 2

The Black–Scholes Framework

2.1 Risk–Neutral Valuation

In a no-arbitrage market, under the risk-neutral measure $\mathbb{Q}$, every asset drifts at the risk-free rate r:

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dW_t^{\mathbb{Q}}.$$

The price of a derivative with payoff $H(S_T)$ is

$$V(S,t) = e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\Big[H(S_T) \mid S_t = S\Big].$$

2.2 Itô's Lemma on $V(S_t, t)$

For $V = V(S,t)$, Itô's lemma gives

$$dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS_t + \tfrac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\,(dS_t)^2.$$

Under $dS_t = rS_t\,dt + \sigma S_t\,dW_t$ and $(dS_t)^2 = \sigma^2 S_t^2\,dt$:

$$dV = \Big(V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big)\,dt + \sigma S\,V_S\,dW_t.$$

2.3 Delta-Hedging and the Black–Scholes PDE

Define the hedged portfolio $\Pi = V - V_S\,S$. Its differential is

$$d\Pi = dV - V_S\,dS + O(dt) = \Big[V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rV\Big]\,dt.$$

No-arbitrage $\Rightarrow$ Π earns the risk-free rate:

$$d\Pi = r\,(V - V_S\,S)\,dt.$$

Equate and rearrange to obtain the Black–Scholes PDE:

$$V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2\,V_{SS} - rV = 0.$$

2.4 Terminal and Boundary Conditions

- Terminal condition at $t = T$: $V(S,T) = H(S)$.
  For a European call, $H(S) = \max(S - K, 0)$.
- Boundary conditions:
  $V(0,t) = 0, \quad V(S,t) \sim S - Ke^{-r\,(T-t)}$ as $S \to \infty$.

2.5 Closed-Form Solution for European Options

After the change of variables $\tau = T - t$, $x = \ln S$, and reduction to the heat equation, the solution is

$$C(S,t) = S\,\Phi(d_1) - K\,e^{-r\,(T-t)}\,\Phi(d_2),$$

$$d_{1,2} = \frac{\ln\!\left(\frac{S}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$P(S,t) = K\,e^{-r\,(T-t)}\,\Phi(-d_2) - S\,\Phi(-d_1),$$

where Φ is the standard normal CDF.

2.6 Implied Volatility and Its Limitations

- **Definition:** The implied volatility $\sigma_{\text{imp}}(K,T)$ is the value of σ that, when input into the Black–Scholes formula, reproduces the market price of an option with strike K and maturity T:
  $$C_{\text{market}} = C_{\text{BS}}(S, K, r, T - t, \sigma_{\text{imp}}).$$
- **Smile and Skew:** Empirically, σ_{imp} varies with K and T, producing the volatility surface.
- **Limitations of Constant-σ:**
  1. Cannot capture the volatility smile/skew across strikes.
  2. No term-structure dynamics: assumes static σ through time.
  3. Misses stylized facts: volatility clustering, mean-reversion, leverage effect.

Exercises

Exercise 1: Derivation of the Black–Scholes PDE

Under the risk-neutral measure $\mathbb{Q}$, the stock price S_t follows

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dW_t^{\mathbb{Q}}.$$

Let $V = V(S,t)$ be the price of a derivative with payoff $H(S_T)$. Derive the Black–Scholes PDE

$$V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0.$$

Solution:

1. **Apply Itô's lemma to $V(S_t, t)$.**
   Itô's lemma gives
   $$dV = V_t\,dt + V_S\,dS_t + \tfrac{1}{2}\,V_{SS}\,(dS_t)^2.$$
   Substitute $dS_t = rS\,dt + \sigma S\,dW$ and $(dS_t)^2 = \sigma^2 S^2\,dt$:
   $$\begin{aligned} dV &= V_t\,dt + V_S\Big(rS\,dt + \sigma S\,dW\Big) + \tfrac{1}{2}V_{SS}\,\sigma^2 S^2\,dt \\ &= \Big(V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big)\,dt + \sigma S\,V_S\,dW. \end{aligned}$$

2. **Construct the delta-hedged portfolio.**
   Define $\Pi = V - V_S\,S$. Then
   $$d\Pi = dV - V_S\,dS - S\,d(V_S).$$
   Continuous rebalancing makes $S\,d(V_S) = O(dt)$. Substitute dV and dS:
   $$\begin{aligned} d\Pi &= \Big(V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big)\,dt + \sigma S\,V_S\,dW \\ &\quad - V_S\Big(rS\,dt + \sigma S\,dW\Big) + O(dt) \\ &= \Big(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rV\Big)\,dt. \end{aligned}$$

3. **Enforce no-arbitrage.**
   The risk-free portfolio Π must earn the rate r:
   $$d\Pi = r\,\Pi\,dt = r\,(V - V_S\,S)\,dt.$$
   Equate with the expression above and drop $O(dt)$:
   $$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rV = r(V - V_S\,S) \implies V_t + rS\,V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0.$$

Exercise 2: Pricing and Implied Volatility

The Black–Scholes formula for a European call with time to maturity $\tau = T - t$ is
$$C(S,t) = S\,\Phi(d_1) - K\,e^{-r\tau}\,\Phi(d_2), \quad d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

1. Compute C for $S = 50$, $K = 55$, $r = 2\%$, $\tau = 0.5$, $\sigma = 30\%$.
2. Compute the call's Delta $\Delta = \partial C/\partial S$.

Solution:

1. **Call price at $\sigma = 0.30$.**
   $$\ln\tfrac{S}{K} = -0.09531, \quad (r \pm \tfrac{1}{2}\sigma^2)\tau = \{0.0325,\ -0.0125\}, \quad \sigma\sqrt{\tau} = 0.21213.$$
   $$d_1 = \frac{-0.09531 + 0.0325}{0.21213} \approx -0.296, \quad d_2 = -0.296 - 0.21213 \approx -0.508.$$
   Using $\Phi(-0.296) \approx 0.383$, $\Phi(-0.508) \approx 0.305$, and $e^{-r\tau} \approx 0.99005$:
   $$C \approx 50 \cdot 0.383 - 55 \cdot 0.99005 \cdot 0.305 \approx 19.15 - 16.60 = 2.55.$$

2. **Delta of the call.**
   $\Delta = \Phi(d_1)$. At $\sigma_{\text{imp}} = 0.35$, $d_1 \approx -0.22$, so
   $$\Delta \approx \Phi(-0.22) \approx 0.412.$$

Chapter 3

Local Volatility Modeling

3.1 Motivation and the Volatility Surface

In practice, when we invert market option prices into Black–Scholes implied volatilities, we observe that the implied volatility depends on strike and maturity. This dependence takes the shape of a "smile" or "skew" in strike for each fixed maturity, and this smile itself changes with maturity. A single constant volatility cannot reproduce all these market quotes simultaneously. The goal of local volatility modeling is to introduce a deterministic function $\sigma_{\text{loc}}(S,t)$ so that our model matches every observed European option price exactly.

3.2 Formulation of the Local Volatility Model

We work under the risk-neutral measure $\mathbb{Q}$. We assume the stock price S_t follows

$$dS_t = S_t\,\sigma_{\text{loc}}(S_t,t)\,dW_t^{\mathbb{Q}},$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion. The function $\sigma_{\text{loc}}(S,t)$ must be chosen so that for every strike K and maturity T, the model price of the corresponding call

$$C_{\text{model}}(K,T) = \mathbb{E}^{\mathbb{Q}}\Big[(S_T - K)^+\Big]$$

equals the market price $C_{\text{mkt}}(K,T)$.

3.3 Derivation of Dupire's Formula

To find the exact form of $\sigma_{\text{loc}}(K,T)$ in terms of market prices, we proceed as follows.

First, let $p(S,t)$ be the risk-neutral probability density of S_t. From the Itô–Fokker–Planck theorem, p satisfies the forward equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\,\frac{\partial^2}{\partial S^2}\Big(\sigma_{\text{loc}}^2(S,t)\,S^2\,p\Big).$$

Next, express the undiscounted call price as an integral over the density:

$$C(K,T) = \int_K^{\infty} (S-K)\,p(S,T)\,dS.$$

Differentiating once with respect to strike gives

$$\frac{\partial C}{\partial K} = -\int_K^\infty p(S,T)\,dS = -\mathbb{Q}(S_T > K),$$

and differentiating a second time yields

$$\frac{\partial^2 C}{\partial K^2} = p(K,T).$$

Differentiating C with respect to maturity T under the integral sign,

$$\frac{\partial C}{\partial T} = \int_K^\infty (S-K)\,\frac{\partial p}{\partial T}(S,T)\,dS = \frac{1}{2}\int_K^\infty (S-K)\,\frac{\partial^2}{\partial S^2}\Big(\sigma_{\text{loc}}^2\,S^2\,p\Big)\,dS.$$

Integrating by parts twice, the only boundary contribution arises at $S = K$, and one finds

$$\frac{\partial C}{\partial T} = \frac{1}{2}\,\sigma_{\text{loc}}^2(K,T)\,K^2\,p(K,T).$$

Since $p(K,T) = \partial_K^2 C(K,T)$, rearranging gives Dupire's formula:

$$\sigma_{\text{loc}}^2(K,T) = \frac{2\,\partial_T C(K,T)}{K^2\,\partial_K^2 C(K,T)}.$$

3.4 Practical Computation of the Local Volatility Surface

To implement this in practice, one proceeds in these steps:

First, interpolate or fit the market implied volatilities $\sigma_{\text{imp}}(K_i, T_j)$ on a grid of strikes K_i and maturities T_j. Convert each implied volatility into its corresponding call price $C(K_i, T_j)$ via the Black–Scholes formula. Next, approximate the partial derivatives

$$\partial_T C \;\approx\; \frac{C(K,T_{j+1}) - C(K,T_{j-1})}{T_{j+1} - T_{j-1}}, \quad \partial_K^2 C \;\approx\; \frac{C(K+\Delta K,T) - 2C(K,T) + C(K-\Delta K,T)}{(\Delta K)^2}$$

using central finite differences. Applying Dupire's formula at each grid node yields the squared local volatilities $\sigma_{\text{loc}}^2(K_i, T_j)$. Finally, ensure nonnegativity and smoothness by regularizing or smoothing the resulting surface.

3.5 Using the Local Volatility Surface in Simulation

In Monte Carlo simulation of the SDE

$$dS_t = S_t\,\sigma_{\text{loc}}(S_t, t)\,dW_t,$$

we need to evaluate $\sigma_{\text{loc}}(S^*, t^*)$ at off-grid points (S^*, t^*). We locate the surrounding strikes $K_i \le S^* \le K_{i+1}$ and maturities $T_j \le t^* \le T_{j+1}$, then bilinearly interpolate the four values $\sigma_{\text{loc}}(K_i, T_j)$, $\sigma_{\text{loc}}(K_{i+1}, T_j)$, $\sigma_{\text{loc}}(K_i, T_{j+1})$, and $\sigma_{\text{loc}}(K_{i+1}, T_{j+1})$ to obtain $\sigma_{\text{loc}}(S^*, t^*)$. This interpolated local volatility is then used in the discretized SDE step.

3.6 Practical Challenges and Remedies

Noise in market data can produce negative or highly oscillatory estimates of $\partial_K^2 C$, leading to invalid or unstable local variances. To mitigate this, one typically smooths the implied-vol surface before computing derivatives, applies Tikhonov-type regularization to penalize excessive curvature, and imposes reasonable extrapolation rules beyond the observed strike and maturity range.

Exercises

Exercise 1: Estimating Local Volatility via Dupire's Formula

You are given the following undiscounted call-price surface:

K	$C(K, T = 0.5)$	$C(K, T = 1.0)$
90	12.00	13.50
100	6.00	7.00
110	2.00	2.50

1. Using central finite differences in T, compute $\partial_T C(K, 0.5)$ for each strike:

$$\partial_T C(K, 0.5) \;\approx\; \frac{C(K, 1.0) - C(K, 0.5)}{1.0 - 0.5}.$$

2. Using central finite differences in K with $\Delta K = 10$, compute $\partial_{KK}^2 C(100, T)$ at each maturity:

$$\partial_{KK}^2 C(100, T) \;\approx\; \frac{C(110, T) - 2C(100, T) + C(90, T)}{10^2}.$$

3. Apply Dupire's formula

$$\sigma_{\mathrm{loc}}^2(100, 0.5) = \frac{2\,\partial_T C(100, 0.5)}{100^2\,\partial_{KK}^2 C(100, 0.5)}$$

to estimate $\sigma_{\mathrm{loc}}(100, 0.5)$.

4. Interpret your result: does the estimated local volatility at $K = 100, T = 0.5$ seem reasonable given the call prices?

Solution

1.

$$\partial_T C(90, 0.5) = \frac{13.50 - 12.00}{0.5} = 3.00, \quad \partial_T C(100, 0.5) = \frac{7.00 - 6.00}{0.5} = 2.00, \quad \partial_T C(110, 0.5) = \frac{2.}{}$$

2.

$$\partial_{KK}^2 C(100, 0.5) = \frac{2.00 - 2 \cdot 6.00 + 12.00}{10^2} = \frac{2.00 - 12.00 + 12.00}{100} = \frac{2.00}{100} = 0.02.$$

(At $T = 1.0$, the same combination yields $(2.50 - 2 \cdot 7.00 + 13.50)/100 = 0.02$.)

3.

$$\sigma_{\mathrm{loc}}^2(100, 0.5) = \frac{2 \times 2.00}{100^2 \times 0.02} = \frac{4.00}{10000 \times 0.02} = \frac{4.00}{200} = 0.02,$$

hence

$$\sigma_{\mathrm{loc}}(100, 0.5) = \sqrt{0.02} \approx 0.1414 \;=\; 14.14\%.$$

Chapter 4

Stochastic Volatility Modeling

In Modules 1–3 we saw that constant-σ and then local-σ models can match European prices, yet they still fail to reproduce the way real markets behave through time. Module 4 shows why we must make volatility itself random, how the celebrated Heston model does it, and how to fit it to market data.

4.1 Empirical Phenomena: Clustering, Mean-Reversion, Leverage

When you plot daily returns on any equity index, you see "calm" stretches where moves are small followed by "stormy" stretches of large swings. That persistence of volatility—called *volatility clustering*—cannot occur if σ is fixed or merely a deterministic surface. Next, after a burst of volatility, it doesn't stay high forever; it creeps back toward a normal level over weeks or months. That tendency—*mean-reversion*—demands a restoring force in our model. Finally, when the market drops sharply, implied and realized volatility spike upwards: negative return shocks trigger higher volatility. This *leverage effect* reflects market participants' asymmetric reaction to bad news.

4.2 Heston Model Formulation

To capture these effects, one makes variance v_t itself stochastic. Under the risk-neutral measure, Heston proposes:

$$
\begin{aligned}
dS_t &= r\, S_t\, dt \;+\; S_t\, \sqrt{v_t}\, dW_t^S, \\
dv_t &= \kappa\Big(\theta - v_t\Big)\, dt \;+\; \xi\, \sqrt{v_t}\, dW_t^v, \\
d\langle W^S,\, W^v\rangle_t &= \rho\, dt.
\end{aligned}
$$

Here:

1. **Stock equation**: its drift is the risk-free rate r, and its instantaneous volatility is $\sqrt{v_t}$, not a fixed number. 2. **Variance equation**:

- The term $\kappa(\theta - v_t)$ pulls v_t back toward a long-run level θ.
- The term $\xi\sqrt{v_t}\, dW_t^v$ makes v_t itself jump randomly, ensuring $v_t \geq 0$.

3. **Correlation** ρ: if $\rho < 0$, negative shocks to S coincide with positive shocks to v, producing the equity skew.

4.3 Role of Parameters $\kappa, \theta, \xi, \rho$

Each parameter shapes the variance path:
- θ ("long-run variance") is the level toward which variance reverts. If you averaged realized daily variances over many years, you'd approximate θ.
- κ ("speed of mean-reversion") controls how quickly v_t returns to θ. A large κ means shocks to variance fade rapidly; small κ means variance wanders for longer.
- ξ ("vol-of-vol") determines how wildly variance itself moves. If ξ is large, variance paths become jagged, creating fatter tails in stock returns.
- ρ (leverage correlation) ties stock returns and variance shocks together. Empirically, ρ is negative for equities, steepening the implied-vol skew for short maturities.

4.4 Characteristic Function Approach to Pricing

Directly solving the two-dimensional PDE in (S, v) is computationally intensive. Heston's key insight is to work instead with the characteristic function of the log-price $X = \ln S_T$. We proceed in four steps:

1. Risk-Neutral Pricing Formula

Under the risk-neutral measure $\mathbb{Q}$, the undiscounted call price is

$$C = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\Big[(S_T - K)^+\Big] = e^{-rT}\int_{\ln K}^{\infty}\Big(e^x - K\Big)\,f_X(x)\,dx,$$

where $f_X(x)$ is the density of $X = \ln S_T$.

2. Characteristic Function Definition and Inversion

Define the characteristic function

$$\phi(u) = \mathbb{E}^{\mathbb{Q}}\Big[e^{\,i\,u\,X}\Big] = \int_{-\infty}^{\infty} e^{\,i\,u\,x}\,f_X(x)\,dx.$$

By Fourier inversion,

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\,i\,u\,x}\,\phi(u)\,du.$$

Substitute into the call price integral and swap integrals:

$$C = \frac{e^{-rT}}{2\pi}\int_{-\infty}^{\infty}\phi(u)\Big[\int_{\ln K}^{\infty}(e^x - K)\,e^{-iux}\,dx\Big]du.$$

3. Evaluation of the Inner Integral

One shows by direct integration that

$$\int_{\ln K}^{\infty} e^x e^{-iux}\,dx = \frac{K^{1-iu}}{iu-1}, \quad \int_{\ln K}^{\infty} K\,e^{-iux}\,dx = \frac{K^{1-iu}}{iu}.$$

Hence the bracketed term becomes

$$G(u;K) = \frac{K^{1-iu}}{iu-1} - \frac{K^{1-iu}}{iu} = K^{1-iu}\,\frac{1}{iu\,(iu-1)}.$$

4. One-Dimensional Fourier Integral

Putting it all together, and introducing a damping factor $\alpha > 0$ to ensure convergence, one obtains a single real integral of the form

$$C(K,T) = e^{-rT}\,\frac{e^{\alpha \ln K}}{\pi} \int_0^\infty \Re\Big[e^{-i\,u\,\ln K}\,\frac{\phi(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i\,u(2\alpha+1)}\Big]\,du,$$

where:

- $\phi(u)$ is the Heston characteristic function $\exp\Big(A(u;T) + B(u;T)\,v_0 + i\,u\,\ln S_0\Big)$.
- $\Re[\cdot]$ denotes the real part.
- The denominator arises from combining the two terms in $G(u;K)$ and the damping factor.

This one-dimensional integral can be evaluated numerically with high efficiency and accuracy, yielding vanilla option prices across all strikes and maturities without Monte Carlo or PDE solvers.

4.5 Calibration of Heston to Market Data

Calibration means choosing parameters so that model prices match observed market prices. For Heston:

1. **Data collection**

- Pick a set of liquid expiries (e.g. 1 month, 3 months, 6 months, 1 year).
- For each expiry, record implied volatilities across a range of strikes.

2. **Objective function**

$$\text{Error}(\kappa,\theta,\xi,\rho,v_0) = \sum_{i,j}\Big[\sigma_{\text{imp}}^{\text{market}}(K_i,T_j) - \sigma_{\text{imp}}^{\text{model}}(K_i,T_j)\Big]^2,$$

summing over strikes K_i and expiries T_j.

3. **Optimization**

- Constrain $\kappa, \theta, \xi > 0$ and $-1 < \rho < 1$.
- Use initial guesses: $\theta \approx$ average ATM variance squared, $\rho \approx$ slope of the short-dated skew, $\xi \approx$ ATM vol.
- Run a robust solver (Levenberg–Marquardt or hybrid global/local) to minimize the error.

4. **Result** You obtain a parameter set that reproduces the market smile at the chosen expiries. Away from those expiries or for path-dependent payoffs, residual mispricings may remain, motivating the Local Stochastic Volatility model of Module 5.

Exercises

Exercise 1: Mean-Reversion Dynamics

The variance in the Heston model follows the SDE

$$dv_t = \kappa\,(\theta - v_t)\,dt + \xi\sqrt{v_t}\,dW_t^v.$$

1. Show that the conditional expectation $m(t) = \mathbb{E}[v_t]$ satisfies the ODE

$$\frac{dm}{dt} = \kappa\Big(\theta - m(t)\Big).$$

2. Solve this ODE to prove

$$\mathbb{E}[v_t] = \theta + \Big(v_0 - \theta\Big)e^{-\kappa t}.$$

3. Compute $\mathbb{E}[v_{0.5}]$ for $v_0 = 0.09,\ \theta = 0.04,\ \kappa = 2,\ t = 0.5.$

Solution

1. Taking expectation on both sides of the SDE, the stochastic term has zero mean, so
$$\frac{d}{dt}\mathbb{E}[v_t] = \kappa\Big(\theta - \mathbb{E}[v_t]\Big).$$
Denote $m(t) = \mathbb{E}[v_t]$; then $m'(t) = \kappa(\theta - m(t))$.

2. This is a first-order linear ODE. Rewrite as $m' + \kappa\,m = \kappa\,\theta$. The integrating factor is $e^{\kappa t}$. Multiply through:
$$\frac{d}{dt}\Big(e^{\kappa t}m(t)\Big) = \kappa\,\theta\,e^{\kappa t}.$$
Integrate from 0 to t:
$$e^{\kappa t}m(t) - m(0) = \theta\Big(e^{\kappa t} - 1\Big),$$
hence
$$m(t) = \theta + \Big(m(0) - \theta\Big)e^{-\kappa t} = \theta + (v_0 - \theta)e^{-\kappa t}.$$

3. Substitute $v_0 = 0.09$, $\theta = 0.04$, $\kappa = 2$, $t = 0.5$:
$$\mathbb{E}[v_{0.5}] = 0.04 + (0.09 - 0.04)e^{-2\times 0.5} = 0.04 + 0.05\,e^{-1} \approx 0.04 + 0.05\times 0.3679 \approx 0.0584.$$

Exercise 2: Simulating One Heston Step

Starting from (S_t, v_t), an Euler–Maruyama discretization with time-step dt is

$$\begin{aligned} S_{t+dt} &= S_t + r\,S_t\,dt + S_t\sqrt{v_t}\,\Delta W^S, \\ v_{t+dt} &= v_t + \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}\,\Delta W^v, \end{aligned}$$

where $\Delta W^S, \Delta W^v \sim N(0, dt)$ with $\mathrm{corr}(\Delta W^S, \Delta W^v) = \rho$.

1. Show how to generate two correlated normals:

$$\Delta W^S = \sqrt{dt}\, Z_1, \qquad \Delta W^v = \sqrt{dt}\,(\rho\, Z_1 + \sqrt{1-\rho^2}\, Z_2),$$

   with $Z_1, Z_2 \sim N(0,1)$ independent.

2. For parameters

$$S_t = 100,\ v_t = 0.04,\ r = 0.03,\ \theta = 0.04,\ \kappa = 1,\ \xi = 0.2,\ \rho = -0.7,\ dt = \tfrac{1}{252},$$

   and a particular draw $Z_1 = 0.1$, $Z_2 = -0.2$, compute $\Delta W^S, \Delta W^v$ and the resulting (S_{t+dt}, v_{t+dt}).

3. Explain why, if you repeat steps (1)–(2) 10,000 times, the sample mean of v_{t+dt} should be close to the theoretical $\mathbb{E}[v_{t+dt}]$ from Exercise 1.

Solution

1. To obtain two correlated normals with correlation ρ, draw independent $Z_1, Z_2 \sim N(0,1)$. Set

$$\Delta W^S = \sqrt{dt}\, Z_1, \quad \Delta W^v = \sqrt{dt}\Big(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2\Big).$$

   Then $\mathbb{E}[\Delta W^S \Delta W^v] = \rho\, dt$.

2. With $dt = 1/252 \approx 0.00397$, $Z_1 = 0.1$, $Z_2 = -0.2$:

$$\Delta W^S = \sqrt{0.00397}\times 0.1 \approx 0.00630, \qquad \Delta W^v = \sqrt{0.00397}\Big(-0.7\times 0.1 + 0.714\times(-0.2)\Big) \approx -0.0131.$$

   Then

$$\begin{aligned} S_{t+dt} &= 100 + 0.03\times 100\times 0.00397 + 100\times 0.2\times 0.00630 \approx 100 + 0.0119 + 0.126 \approx 100.1379,\\ v_{t+dt} &= 0.04 + 1\times(0.04-0.04)\times 0.00397 + 0.2\times 0.2\times(-0.0131) \approx 0.04 - 0.000524 \approx 0.03948. \end{aligned}$$

3. By the law of large numbers, averaging 10,000 independent draws of v_{t+dt} will converge to its expectation. From Exercise 1, $\mathbb{E}[v_{t+dt}] = 0.04 + (v_t - 0.04)e^{-\kappa dt} \approx 0.04$. Thus the sample mean of the simulated v_{t+dt} should be very close to 0.04.

Chapter 5

Local Stochastic Volatility Modeling

5.1 Motivation for LSV

Local-vol models calibrate exactly to every European option but fail to reproduce volatility clustering and realistic dynamics. Stochastic-vol models capture clustering, mean-reversion and leverage but only fit the surface approximately. The Local Stochastic Volatility (LSV) framework merges both: it forces exact calibration of European marginals while retaining stochastic dynamics.

5.2 LSV SDE with Leverage Function

Under the risk-neutral measure,

$$
\begin{aligned}
dS_t &= r\,S_t\,dt \;+\; L(S_t,t)\,\sqrt{v_t}\;S_t\,dW_t^S,\\
dv_t &= \kappa\Big(\theta - v_t\Big)\,dt \;+\; \xi\,\sqrt{v_t}\,dW_t^v,\\
d\langle W^S,\,W^v\rangle_t &= \rho\,dt.
\end{aligned}
$$

Here:

- v_t follows the Heston/CIR variance dynamics.
- The deterministic *leverage function* $L(S,t)$ scales the stochastic volatility so that

$$
\sigma_{\mathrm{loc}}^2(S,t) = L^2(S,t)\,\mathbb{E}\Big[v_t \mid S_t = S\Big].
$$

5.3 Calibrating the Leverage Function

1. **Compute local-vol grid:** use Dupire's formula to obtain $\sigma_{\mathrm{loc}}(K_i,T_j)$.
2. **Estimate conditional variance:** simulate pure-Heston paths ($L \equiv 1$) up to each T_j; bucket all v_{T_j} values by final spot $S_{T_j} \approx K_i$; compute $\widehat{\mathbb{E}}[v_{T_j} \mid S_{T_j} = K_i]$.
3. **Define leverage on grid:**

$$
L(K_i,T_j) = \frac{\sigma_{\mathrm{loc}}(K_i,T_j)}{\sqrt{\widehat{\mathbb{E}}[v_{T_j} \mid S_{T_j} = K_i]}}.
$$

Example: if $\sigma_{\text{loc}}(100, 0.5) = 0.25$ and $\widehat{\mathbb{E}}[v_{0.5} \mid S_{0.5} = 100] = 0.04$, then $L(100, 0.5) = 1.25$.

4. **Interpolate** $L(S,t)$ off-grid via bilinear interpolation over (K_i, T_j).

5.4 Simulation under LSV

To price exotics, simulate the coupled SDEs using a discrete-time scheme:

- **Euler–Maruyama step:**

$$
\begin{aligned}
S_{t+dt} &= S_t + r\, S_t\, dt + L(S_t, t)\, \sqrt{v_t}\; S_t\, \Delta W^S, \\
v_{t+dt} &= v_t + \kappa(\theta - v_t)\, dt + \xi\sqrt{v_t}\; \Delta W^v,
\end{aligned}
$$

  where $\Delta W^S, \Delta W^v \sim N(0, dt)$ with $\operatorname{corr}(\Delta W^S, \Delta W^v) = \rho$.

- **Or QE-scheme for v_t:** to preserve $v \geq 0$ and reduce bias, sample v_{t+dt} from a moment-matching distribution at each step.

- **And Bilinear interpolation of L:** at each step, given S_t and t, locate the four surrounding grid nodes (K_i, T_j) and interpolate $L(S_t, t)$ from $L(K_i, T_j)$.

These simulated paths display both realistic volatility clustering (from v_t) and exact calibration to European options (via L).

5.5 Practical Considerations

- **Nested simulation:** estimating $\mathbb{E}[v_t \mid S_t]$ requires a preliminary Monte Carlo or PDE solve. Variance-reduction (particle-filter, control variates) helps efficiency.

- **Smoothing:** noisy estimates of $\widehat{\mathbb{E}}[v_t \mid S_t]$ can destabilize L; apply smoothing or regularization before interpolation.

- **Applications:** use LSV for pricing barriers, cliquets, forward-start options—products sensitive to both marginals and path-dynamics.

Chapter 6

Implementation and Derivative Pricing with Local Stochastic Volatility

6.1 Market Data Collection and Preprocessing

1. **Underlying price history:** Download daily closing prices S_t for your stock over several years. Compute daily returns $\Delta \ln S_t$ and realized variances to observe clustering and average levels.
2. **Option quotes:** For each maturity T_j (e.g. 1 m, 3 m, 6 m, 1 y) and strike K_i:
   - Retrieve bid and ask prices $(P_{\text{bid}}, P_{\text{ask}})$.
   - Convert each to Black–Scholes implied volatilities $\sigma_{\text{bid}}, \sigma_{\text{ask}}$.
   - Form the mid implied vol: $\sigma_{\text{mid}} = (\sigma_{\text{bid}} + \sigma_{\text{ask}})/2$.
3. **Cleaning stale or illiquid strikes:**
   - Discard strikes with bid–ask spread in vol $> 3\%$ or open interest < 10 contracts.
   - Remove quotes older than a few minutes at snapshot time.

6.2 Constructing the Local-Volatility Surface

1. **Black–Scholes inversion:** Compute call prices $C(K_i, T_j)$ from $\sigma_{\text{mid}}(K_i, T_j)$.
2. **Finite differences:**
$$\partial_T C \approx \frac{C(K, T_{j+1}) - C(K, T_{j-1})}{T_{j+1} - T_{j-1}}, \quad \partial^2_{KK} C \approx \frac{C(K + \Delta K, T) - 2C(K, T) + C(K - \Delta K, T)}{(\Delta K)^2}.$$
3. **Dupire's formula:**
$$\sigma^2_{\text{loc}}(K_i, T_j) = \frac{2\,\partial_T C(K_i, T_j)}{K_i^2\,\partial^2_{KK} C(K_i, T_j)}.$$
4. **Regularization:** Smooth σ^2_{loc} to enforce nonnegativity and remove spurious oscillations.

6.3 Calibrating the Heston Backbone

1. **Select expiries:** match $\{T_j\}$ used above.
2. **Objective function:**

$$\min_{\kappa,\theta,\xi,\rho,v_0} \sum_{i,j} \Big[\sigma_{\text{imp}}^{\text{market}}(K_i, T_j) - \sigma_{\text{imp}}^{\text{Heston}}(K_i, T_j)\Big]^2.$$

3. **Pricing via characteristic function** (Module 4) for speed.
4. **Optimization:** constrain $\kappa, \theta, \xi > 0$, $-1 < \rho < 1$; initialize $\theta \approx$ squared ATM vol, $\rho \approx$ skew slope, $\xi \approx$ ATM vol; solve with Levenberg–Marquardt.
5. **Validation:** compare Heston-implied smiles to market at out-of-sample expiries.

6.4 Computing the Leverage Function

1. **Simulate pure-Heston paths.**
   - Set $L \equiv 1$ in the SDEs of Module 5.
   - Generate N sample paths of (S_t, v_t) from $t = 0$ to each maturity T_j.
2. **Bin end-points by strike.**
   - For each grid node (K_i, T_j), collect all simulated end-of-period variances $v_{T_j}^{(k)}$ from those paths whose final price satisfies

$$S_{T_j}^{(k)} \in [\, K_i - \delta,\ K_i + \delta\,].$$

   - Compute the sample average

$$\widehat{\mathbb{E}}\Big[v_{T_j} \mid S_{T_j} \approx K_i\Big] = \frac{1}{\#\{\text{paths}\}} \sum_{k:\, S_{T_j}^{(k)} \approx K_i} v_{T_j}^{(k)}.$$

3. **Compute leverage at each node.**

$$L(K_i, T_j) = \frac{\sigma_{\text{loc}}(K_i, T_j)}{\sqrt{\widehat{\mathbb{E}}\Big[v_{T_j} \mid S_{T_j} \approx K_i\Big]}}.$$

   Example: if $\sigma_{\text{loc}}(100, 0.5) = 0.25$ and $\widehat{\mathbb{E}}[v_{0.5} \mid S = 100] = 0.04$, then $L(100, 0.5) = 0.25/0.2 = 1.25$.

4. **Construct continuous leverage surface.**
   - Use bilinear interpolation on the rectangle grid $\{(K_i, T_j, L(K_i, T_j))\}$.
   - For any (S, t) during simulation, find the four surrounding (K_i, T_j) and interpolate to get $L(S, t)$.

6.5 Monte Carlo Simulation of the LSV Model

1. **Choose discretization scheme.**

   - *Euler–Maruyama:*

$$\begin{cases} \Delta W^S = \sqrt{dt}\, Z_1, \\ \Delta W^v = \sqrt{dt}\, (\rho\, Z_1 + \sqrt{1-\rho^2}\, Z_2), \end{cases} \qquad Z_1, Z_2 \sim N(0,1).$$

   - *QE-scheme* (if you require nonnegativity of v).

2. **Time-step update for each path.** At each time t, with known (S_t, v_t):

   (a) Compute $L = L(S_t, t)$ by bilinear interpolation.
   $\Delta W^S, \Delta W^v$ as above, with $\operatorname{corr}(\Delta W^S, \Delta W^v) = \rho$.
   $S_{t+dt} = S_t + r\, S_t\, dt + L\, \sqrt{v_t}\, S_t\, \Delta W^S,$
   $v_{t+dt} = v_t + \kappa(\theta - v_t)\, dt + \xi\, \sqrt{v_t}\, \Delta W^v$ (or QE-step).

3. **Compute path-dependent payoffs.**

   - For a barrier: record whether S_t crosses a barrier level.
   - For an Asian: accumulate the average of S_t.
   - Etc.

4. **Estimate price.**

$$\hat{C} = e^{-rT}\, \frac{1}{N} \sum_{k=1}^{N} \text{payoff}^{(k)}.$$

   Increase N and/or decrease dt until the Monte Carlo error is acceptable.

6.6 Validation and Diagnostics

- **Marginal check:** compare simulated European prices to market/local-vol via histograms.
- **Dynamics check:** verify volatility clustering (autocorrelation of realized var) and leverage effect (correlation of returns and future var).
- **Convergence:** test different dt and interpolation grids to ensure stability.

Chapter 7

Unified Overview of Volatility Modeling

This single module brings together all key ideas from constant-vol models through Local Stochastic Volatility (LSV). Read this and you will grasp the motivation, the core equations, and the practical pipeline without diving into each prior module in full detail.

7.1 From Black–Scholes to Market Realities

In the Black–Scholes world we assume volatility is a fixed constant σ. Under risk-neutral pricing the stock follows

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dW_t,$$

and one obtains the famous closed-form call price

$$C_{\text{BS}}(S_0, K, r, T, \sigma).$$

In practice, however, implied volatilities quoted by markets vary by strike K and maturity T: the so-called volatility smile or skew. A single constant σ cannot reproduce this entire surface.

7.2 Local Volatility: Exact Fit to European Prices

Local volatility replaces constant σ with a deterministic surface $\sigma_{\text{loc}}(S, t)$. Dupire's formula shows how to extract σ_{loc} from market call prices $C(K, T)$:

$$\sigma_{\text{loc}}^2(K, T) = \frac{2\,\partial_T C(K, T)}{K^2\,\partial_{KK}^2 C(K, T)}.$$

One then simulates

$$dS_t = r\,S_t\,dt + \sigma_{\text{loc}}(S_t, t)\,S_t\,dW_t$$

to recover all European prices exactly. But this model lacks realistic temporal dynamics: volatility clustering and the leverage effect are absent.

7.3 Stochastic Volatility: Realistic Dynamics

Empirical studies show (i) volatility clusters in time, (ii) it mean-reverts toward a long-run level, and (iii) negative stock returns tend to spike future volatility (leverage effect). The Heston model captures these by introducing a stochastic variance v_t:

$$\begin{cases} dS_t = r\, S_t\, dt + \sqrt{v_t}\, S_t\, dW_t^S, \\ dv_t = \kappa(\theta - v_t)\, dt + \xi\, \sqrt{v_t}\, dW_t^v, \\ d\langle W^S, W^v\rangle_t = \rho\, dt. \end{cases}$$

Here θ is the long-run variance, κ its mean-reversion speed, ξ the "vol-of-vol," and $\rho < 0$ builds in the leverage effect. Heston admits a semi-analytic characteristic function $\phi(u) = \mathbb{E}[e^{iu \ln S_T}]$, enabling fast Fourier-based European pricing—but it only approximately matches the full implied-vol surface.

7.4 Local Stochastic Volatility: Combining Strengths

LSV merges local and stochastic approaches. We drive the stock with stochastic variance v_t but scale each shock by a deterministic *leverage function* $L(S, t)$:

$$dS_t = r\, S_t\, dt + L(S_t, t)\, \sqrt{v_t}\, S_t\, dW_t^S, \quad dv_t = \kappa(\theta - v_t)\, dt + \xi \sqrt{v_t}\, dW_t^v.$$

L is chosen so that the model's instantaneous variance matches the local-vol surface:

$$\sigma_{\text{loc}}^2(S, t) = L^2(S, t)\, \mathbb{E}[v_t \mid S_t = S].$$

In effect, L "leverages" or "de-leverages" the stochastic backbone to enforce exact calibration of every European option, while preserving realistic clustering and skew dynamics.

7.5 Practical Pipeline for LSV Implementation

1. **Market data:** collect bid/ask quotes, form mid implied vols, remove stale/illiquid strikes.

2. **Local-vol surface:** fit $\sigma_{\text{imp}}(K, T)$, invert Black–Scholes to get $C(K, T)$, compute $\partial_T C$ and $\partial_{KK}^2 C$, apply Dupire.

3. **Heston calibration:** fit $(\kappa, \theta, \xi, \rho, v_0)$ by minimizing squared error between model and market implied vols, using the characteristic-function engine.

4. **Leverage grid:** simulate pure-Heston paths to estimate $\mathbb{E}[v_T \mid S_T = K]$; compute $L(K, T) = \sigma_{\text{loc}}(K, T)/\sqrt{\mathbb{E}[v_T \mid S_T = K]}$; interpolate $L(S, t)$.

5. **Monte Carlo LSV:** choose Euler-Maruyama or QE for (S_t, v_t); at each step bilinearly interpolate L; draw correlated normals $(\Delta W^S, \Delta W^v)$; update

$$S_{t+dt} = S_t + rS_t dt + L\sqrt{v_t} S_t\, \Delta W^S, \quad v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\, \Delta W^v.$$

6. **Pricing:** for European options marginals match by construction; for exotics (barriers, Asians, cliquets), compute path payoffs and average.

7.6 Key Takeaways

- *Black–Scholes* gives closed-form European prices under constant volatility.
- *Local volatility* fits every European quote exactly but misses real-world dynamics.
- *Stochastic volatility* captures clustering and skew but only approx-fits the implied-vol surface.
- *LSV* unites both: exact marginal calibration plus realistic path dynamics, yielding a robust tool for vanilla and exotic pricing.