Week 7: 3D Geometry I
Part 1: Vectors in 3D

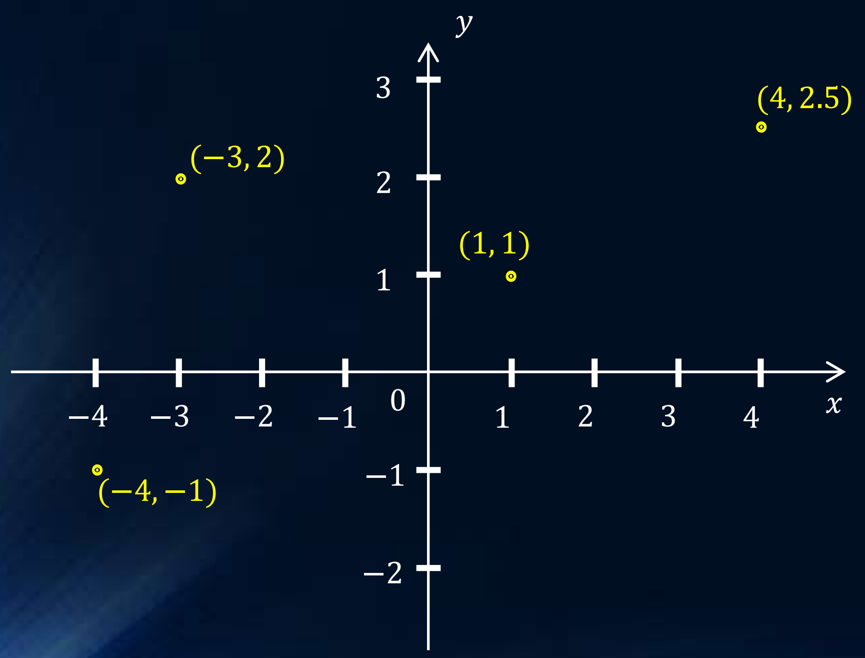
COMP270: Mathematics for 3D Worlds and Simulations

Objectives

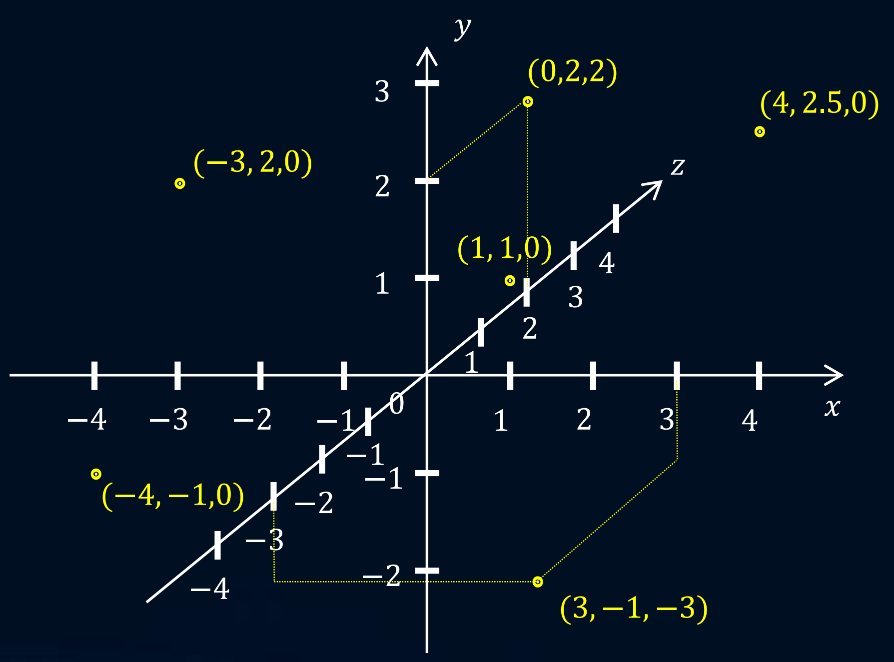
- **Extend** 2D vector arithmetic into three dimensions
- **Define** a new vector operation, the cross product

Recap: coordinate systems

- 2D Cartesian coordinates:
  $$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

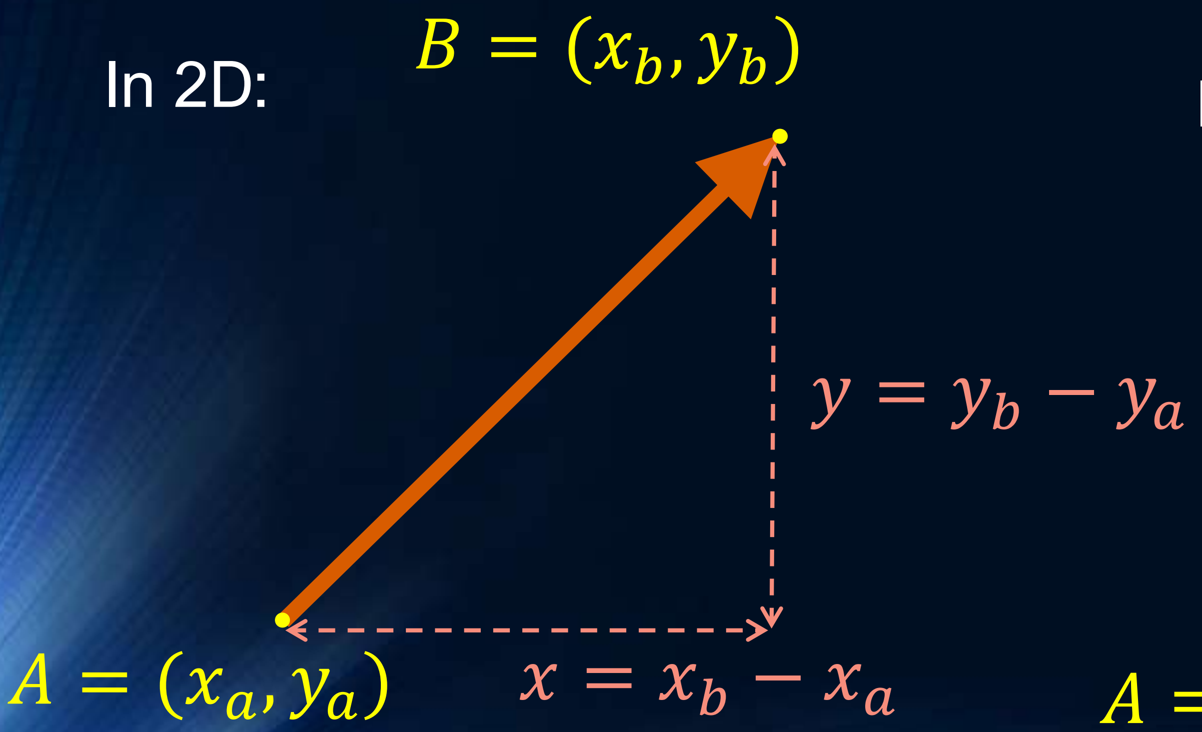


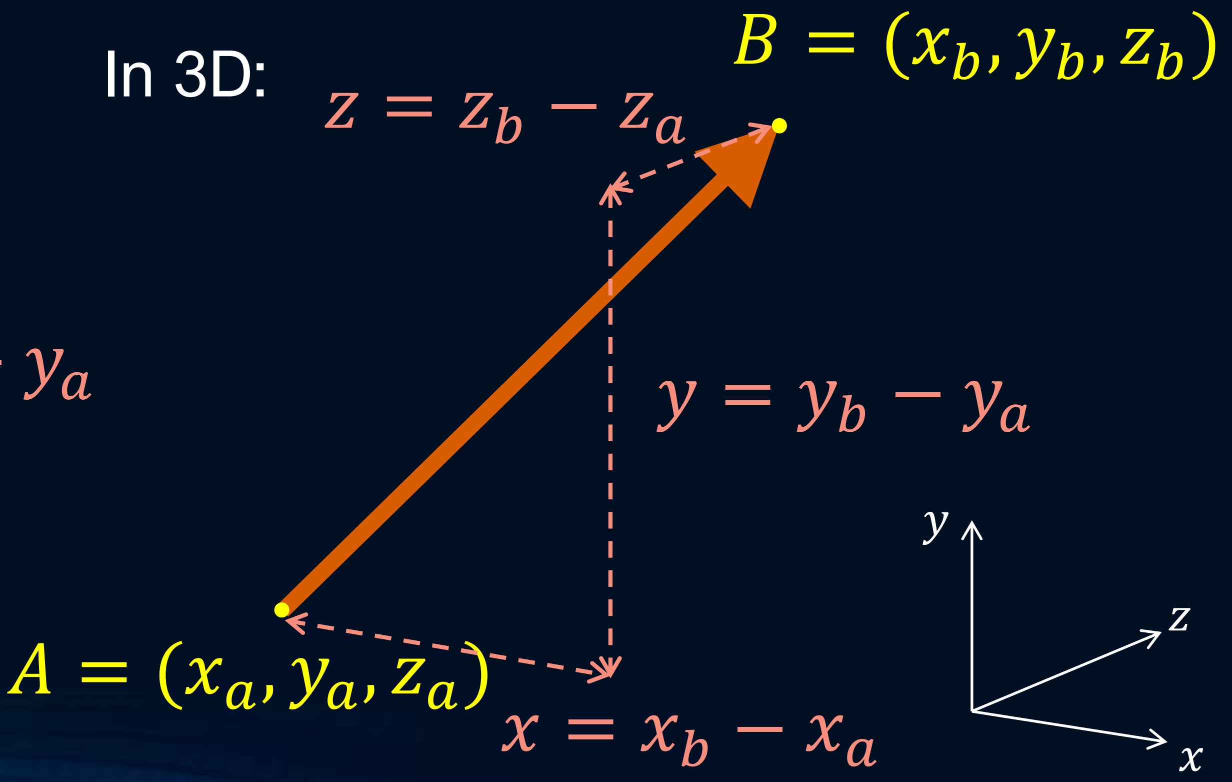
- 3D Cartesian coordinates:
  $$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$



Recap: what is a vector?

- A **vector** is a directed line segment between 2 points





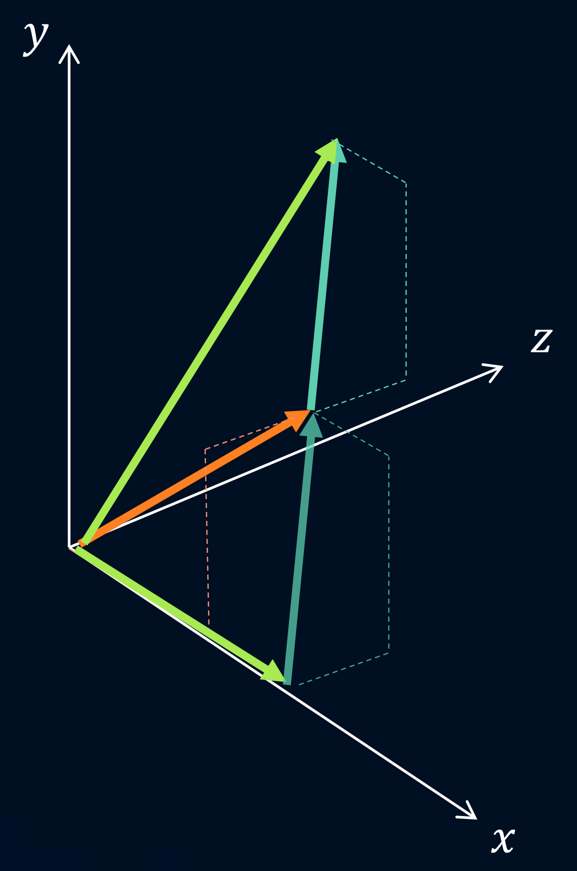
Addition and subtraction in 3D

Addition:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Subtraction:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$



3D dot product and magnitude

Dot product:

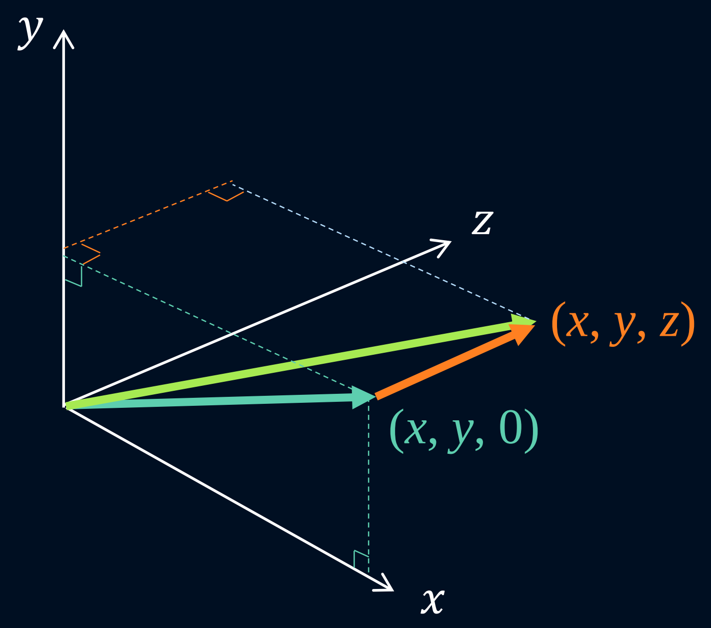
$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

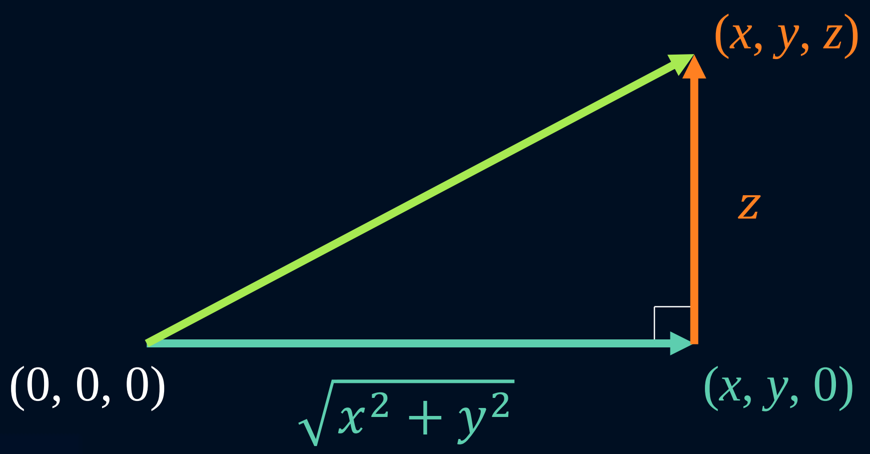
Magnitude:

$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

3D vector magnitude: proof

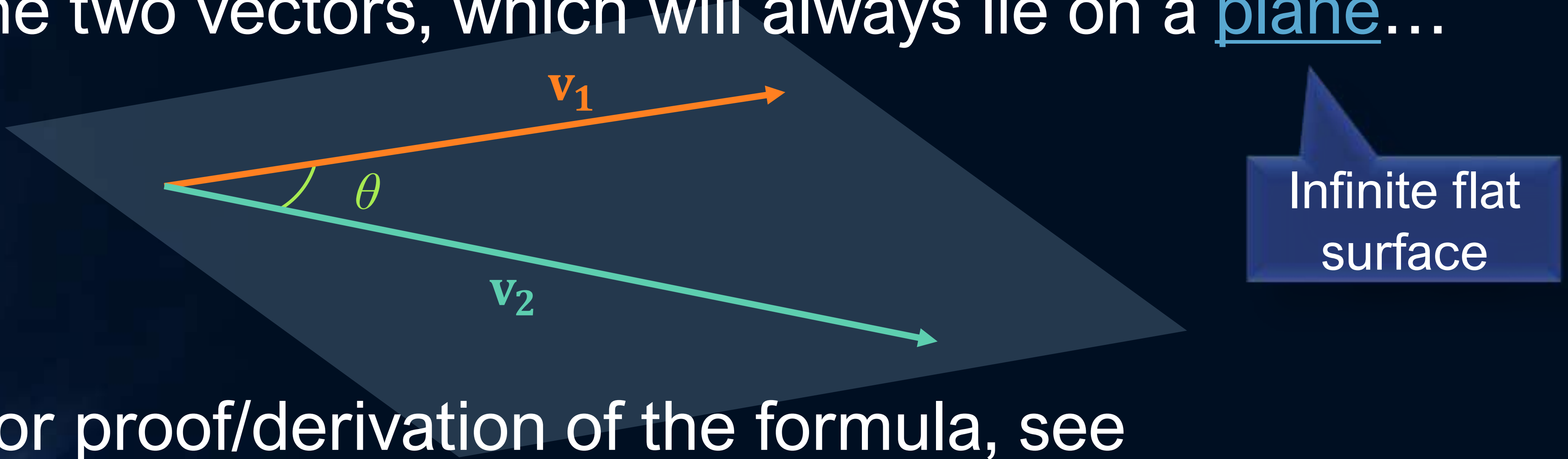
- Consider that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$
- We know that the magnitude of the 2D vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$
- Consider the triangle orthogonal to the xy plane, formed by this vector and the z component…





3D dot product: geometric interpretation

- 2D theorem: $\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\| cos\theta$
- Still applies in 3D because proof is based only on the two vectors, which will always lie on a plane…



- For proof/derivation of the formula, see proofwiki.org/wiki/Cosine_Formula_for_Dot_Product

Vector cross product

The **cross product** of two vectors is given as:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

- Properties

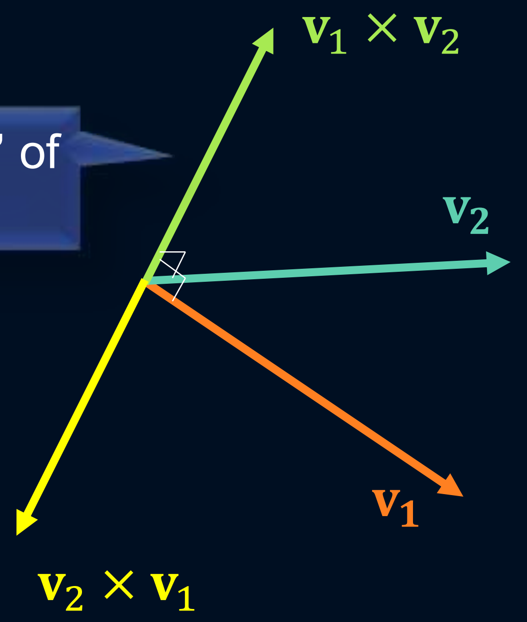
Direction is determined by the 'handedness' of the coordinate system – see part 4!

  - $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2 = 0$$

  - Anticommutative:

$$\mathbf{v}_2 \times \mathbf{v}_1 = -(\mathbf{v}_1 \times \mathbf{v}_2)$$



Cross product: geometric interpretation

- $\mathbf{v}_1 \times \mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\| sin\theta\ \hat{\mathbf{n}}$ (proof here)
  - $\hat{\mathbf{n}}$ is the unit normal to the plane containing $\mathbf{v}_1$ and $\mathbf{v}_2$
- $\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\|\|\mathbf{v}_2\| sin\theta$
- Useful property:
  $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0} \Leftrightarrow \mathbf{v}_1$ and $\mathbf{v}_2$ are parallel
- Area of the triangle between $\mathbf{v}_1$ and $\mathbf{v}_2$
  $= \frac{1}{2}\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|$