3: Computational Geometry II

COMP270: MATHEMATICS FOR 3D WORLDS & SIMULATIONS

Dot product

Dot product

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 x_2 + y_1 y_2$$

Dot product and magnitude

Theorem: For a vector $\mathbf{v}$, $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$

Proof:

- Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$
- Then $\|\mathbf{v}\|^2 = \sqrt{x^2 + y^2}^2 = x^2 + y^2$
- And $\mathbf{v} \cdot \mathbf{v} = xx + yy = x^2 + y^2$
- QED

Magnitude and squared magnitude

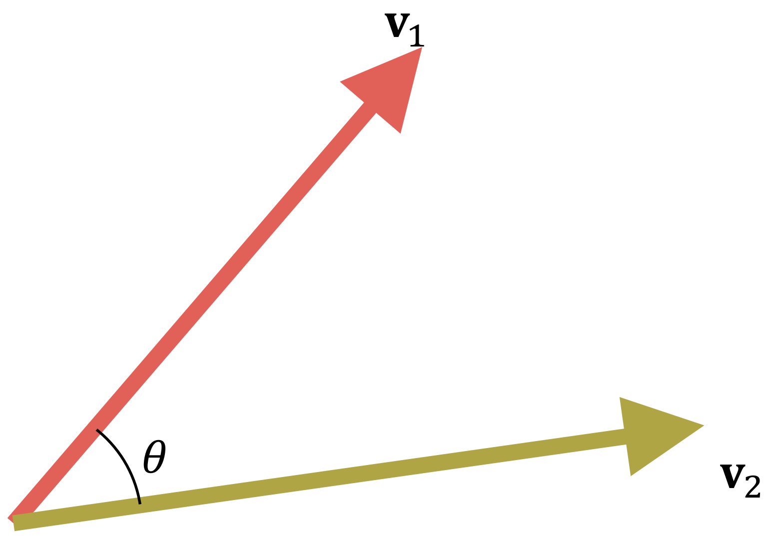
- Finding the magnitude of a vector involves a **square root** $\sqrt{x^2 + y^2}$
- Traditionally, calculating square roots (`sqrt`) was expensive
- So common advice is to work with squared magnitudes where possible – i.e. calculate $x^2 + y^2$ without the square root
- E.g. testing for length: don't test if $||\mathbf{v}|| < r$, test if $||\mathbf{v}||^2 < r^2$
- The cost of square roots is negligible on modern hardware – computing `sqrt` is *probably* not the bottleneck in your code!

Geometric interpretation of dot product

Theorem: For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta$$

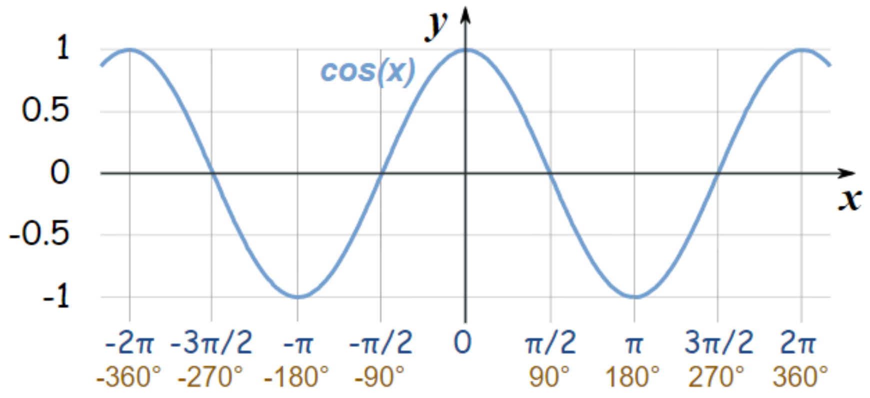
where θ is the angle between the two vectors



Dot products and angles

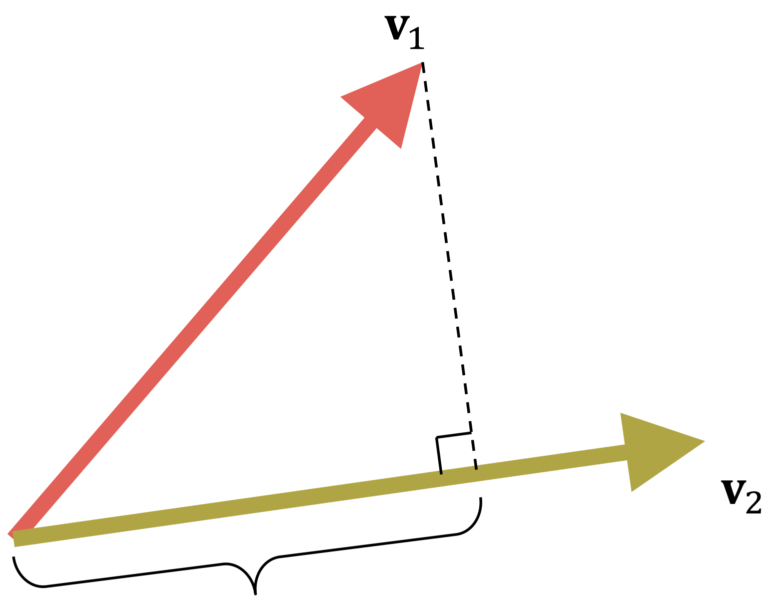
$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$$

Can often test angles without doing the acos, for example $\theta < \phi$ is equivalent to $\cos\theta > \cos\phi$ if θ and ϕ are between 0 and π radians



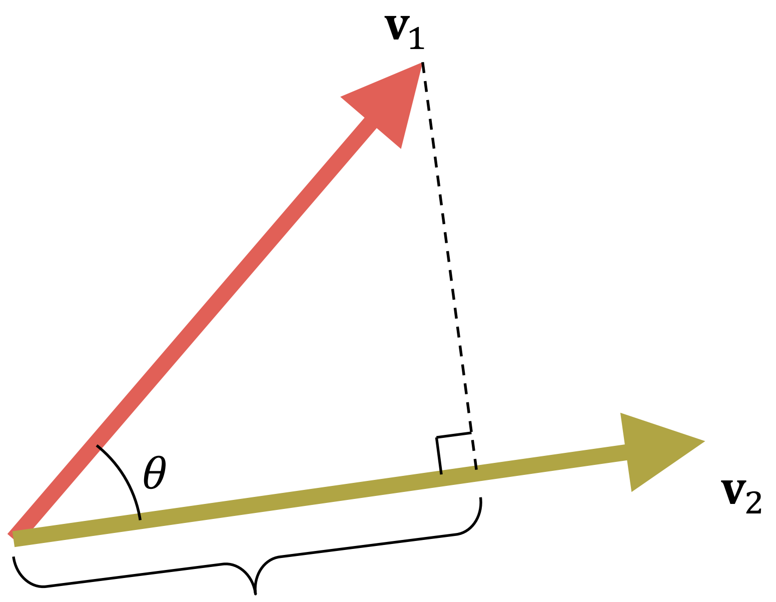
Vector projection

- Take two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ representing points on the plane
- Project a line from point $\mathbf{v}_1$ onto vector $\mathbf{v}_2$, such that it meets $\mathbf{v}_2$ at a right angle
- The **projection** of $\mathbf{v}_1$ onto $\mathbf{v}_2$ is the distance from the origin to the point where the line meets $\mathbf{v}_2$
- A measure of “how much” of $\mathbf{v}_1$ is pointing in the same direction as $\mathbf{v}_2$



Projection and dot product

- From basic trigonometry, the projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$ is
$$\|\mathbf{v}_1\| \cos\theta$$
- But $\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\| \cos\theta$, so this is the same as
$$\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|}$$
- If $\mathbf{v}_2$ is a unit vector (so $\|\mathbf{v}_2\| = 1$) then the projection is just $\mathbf{v}_1 \cdot \mathbf{v}_2$



Matrices

Matrices

The plural of matrix is **matrices**, not “matrixes”!

An $m \times n$ **matrix** is a rectangular array of numbers, with m **rows** and n **columns**

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{n \text{ columns}} \Bigg\} \, m \text{ rows}$$

For example a 2×2 matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We will mostly work with **square** matrices: matrices where $m = n$

Multiplying matrices

If A and B are matrices, then the element at row i column j of AB
is the dot product of row i of A with column j of B

For 2x2 matrices:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Note that matrix multiplication is **not commutative**:
in general $AB \neq BA$

Multiplying matrices

If we multiply an $m \times n$ matrix by an $n \times p$ matrix, we get an $m \times p$ matrix

If the number of columns in A does not equal the number of rows in B,
then AB cannot be calculated

Square matrices are **closed under multiplication**:
if we multiply an $n \times n$ matrix by an $n \times n$ matrix, we get an $n \times n$ matrix

Multiplying matrices and vectors

We can multiply an $m \times n$ matrix by a vector in $\mathbb{R}^n$, by writing the vector as a column and treating it as an $n \times 1$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

Note that multiplying an $m \times n$ matrix by an $n \times 1$ column vector gives us an $m \times 1$ column vector, i.e. a vector in $\mathbb{R}^m$

Identity matrix

An **identity matrix** is a square matrix with 1 on the main diagonal and 0 everywhere else:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For any square matrix A, $AI = A = IA$

So I works like 1 does for multiplication in $\mathbb{R}$

Inverse matrix

For a square matrix A, the inverse of A is a matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A$$

For 2×2 matrices, there is an easy formula for the inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

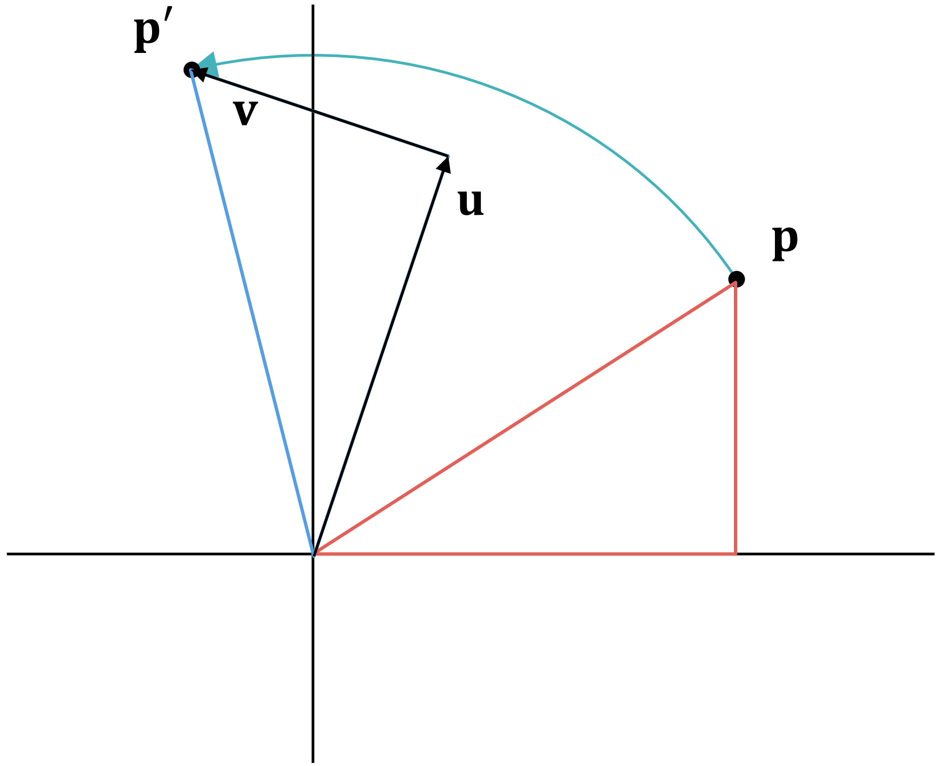
If $ad - bc = 0$ then the matrix has no inverse

For larger matrices the inverse is harder to find, but methods do exist

Rotation

Rotation around the origin

- Consider a point $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$
- Suppose we want to rotate $\mathbf{p}$ by an angle of θ anticlockwise around the origin
- By taking this triangle…
- … and rotating it by θ…
- We see that $\mathbf{u} = \begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -y\sin\theta \\ y\cos\theta \end{pmatrix}$
- So $\mathbf{p}' = \mathbf{u} + \mathbf{v} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$



Rotation matrix

- $\mathbf{p}' = \mathbf{u} + \mathbf{v} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$
- We can write this as a matrix multiplication:

$$p' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- We call $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ a **rotation matrix**

Inverting the rotation matrix

Theorem:

$$R_\theta^{-1} = R_{-\theta}$$

The inverse of the rotation matrix for angle θ is the rotation matrix for angle $-\theta$

Useful rotation matrices

- $R_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  so a rotation of 0 is equivalent to the identity
- $R_{\frac{\pi}{2}} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $R_{\frac{\pi}{2}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$
  so $\begin{pmatrix} -y \\ x \end{pmatrix}$ is a vector perpendicular to $\begin{pmatrix} x \\ y \end{pmatrix}$

Other transformation matrices

Scaling

- Multiplying a vector by a scalar s has the effect of scaling about the origin
- Can also represent this by a matrix:

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

- (Note that multiplying a vector by a scalar s is the same as multiplying by the matrix sI)
- More generally, can represent a scaling by a factor of s_x horizontally and s_y vertically by the matrix

$$\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

Shearing

- A shear transformation by a factor of λ parallel to the x-axis is given by the matrix

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

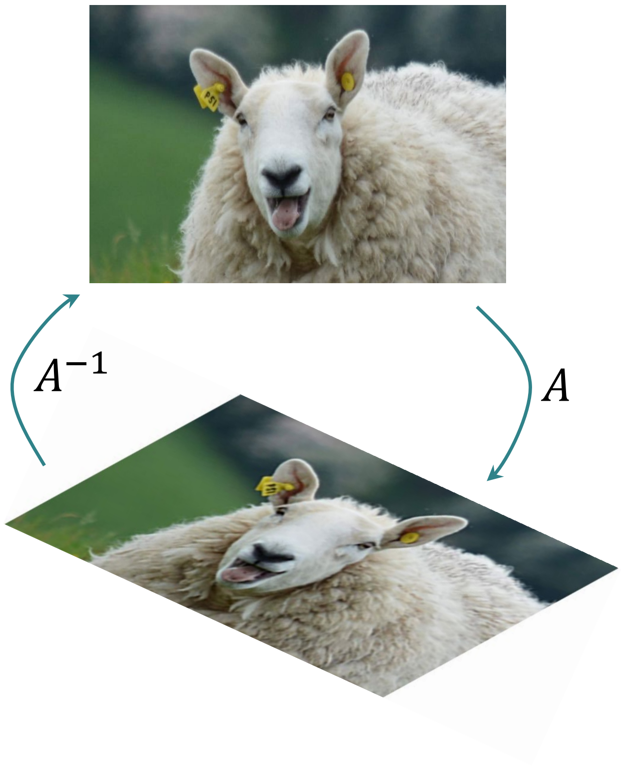
Reflection

- The following matrices represent horizontal and vertical reflections respectively:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Inverses

- If A represents a transformation, then A^{-1} represents the reverse of that transformation
- If $A\mathbf{v} = \mathbf{v}'$ then $A^{-1}\mathbf{v}' = \mathbf{v}$



Translation?

- For any matrix A, we have

$$A\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

- Therefore any transformation that can be represented by a matrix must keep the origin fixed
- Therefore translation (i.e. shifting all points by a constant vector) cannot be represented as a matrix
- Neither can rotating / scaling / shearing / reflecting around a point other than the origin

Homogeneous coordinates

- A workaround for the inability to represent translation as a matrix
- Add an extra dimension: represent points in $\mathbb{R}^2$ by vectors in $\mathbb{R}^3$
- The third component is usually 1 if the vector represents a point – so $\begin{pmatrix} x \\ y \end{pmatrix}$ becomes $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
- Existing transformation matrices stay similar:
  $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ becomes $\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Translation in homogeneous coordinates

- Translation by $\begin{pmatrix} t_x \\ t_y \end{pmatrix}$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- Multiplying by this matrix is the same as adding $\begin{pmatrix} t_x \\ t_y \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

Sequences of transformations

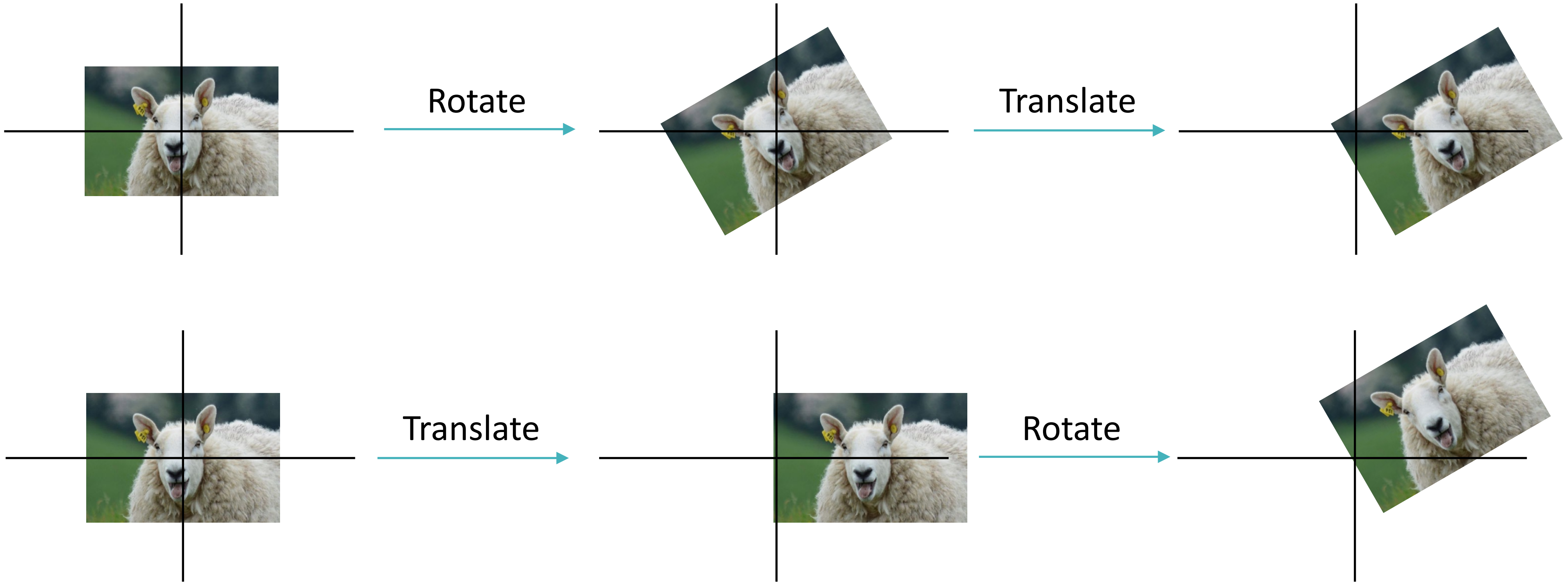
If A and B are matrices representing transformations,
then AB represents doing B followed by A

Note the right to left order!

This is because the vector being transformed appears on the far right of the product

Order of transformation matters!

$$AB \neq BA$$



Reminder: Worksheet A due next Monday!