COMP270
Mathematics for 3D Worlds and Simulations

Week 8 Workshop exercises: 3D Transformations and Rotations

INTRODUCTION

This worksheet is split into two sections; the first deals with matrix transformations between coordinate spaces, while the second contains some exercises involving quaternion calculations.

You may find the Symbolab matrix multiplication calculator useful for finding/checking your results.

MATRIX TRANSFORMATIONS

1. An object initially had its axes and origin coincident with the world axes and origin. It was first rotated 30° about the y-axis, and then -22° about the world x-axis.
    a. What is the matrix that can be used to transform column vectors from object space to world space?
    b. What about the matrix to transform vectors from world space to object space?
    c. Express the object's z-axis using world coordinates.

2. A robot is at the position $(1, 10, 3)$ and her right, up and forward vectors (expressed in world space) are $\begin{pmatrix} 0.866 \\ 0 \\ -0.5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0.5 \\ 0 \\ 0.866 \end{pmatrix}$ respectively (note that these vectors form an orthonormal basis).
    a. The following points are expressed in object space; calculate their coordinates in world space:
        i. (-1, 2, 0)
        ii. (1, 2, 0)
        iii. (0, 0, 0)
        iv. (1, 5, 0.5)
        v. (0, 5, 10)
    b. The coordinates below are in world space; find their positions relative to the robot:
        i. (1, 10, 3)
        ii. (0, 0, 0)
        iii. (2.732, 10, 2)
        iv. (2, 11, 4)
        v. (1, 20, 3)

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QUATERNIONS

3. A quaternion $\mathbf{q}$ to rotate through an angle θ is written as $\mathbf{q} = [w \quad \boldsymbol{v}] = \left[\cos\left(\frac{\theta}{2}\right) \quad \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{n}}\right]$.
   a. Construct a quaternion to rotate 30° about the x-axis.
   b. What is the magnitude of this quaternion?
   c. What is its conjugate, $\mathbf{q}^*$?
   d. Assume the quaternion is used to rotate points from object space to world space. What would the position of the point $\mathbf{p} = (0.5, -0.7, 2.3)$ be under this rotation?

4. Compute a quaternion that performs twice the rotation of the quaternion $[0.965 \quad (0.149 \quad -0.149 \quad 0.149)]$.

5. Consider the quaternions:

$$\mathbf{a} = [0.233 \quad (0.060 \quad -0.257 \quad -0.935)]$$
$$\mathbf{b} = [-0.752 \quad (0.286 \quad 0.374 \quad 0.459)]$$

   a. Compute the dot product $\mathbf{a} \cdot \mathbf{b}$, given by the formula

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = [w_1 \quad \mathbf{v}_1] \cdot [w_2 \quad \mathbf{v}_2] = w_1 w_2 + \mathbf{v}_1 \cdot \mathbf{v}_2$$
$$= [w_1 \quad (x_1 \quad y_1 \quad z_1)][w_2 \quad (x_2 \quad y_2 \quad z_2)] = w_1 w_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

   b. Compute the quaternion product $\mathbf{ab}$, given by the Hamilton product

$$\mathbf{q}_1 \mathbf{q}_2 = [w_1 \quad \mathbf{v}_1][w_2 \quad \mathbf{v}_2]$$
$$= [w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \quad w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$$

   c. Compute the difference from $\mathbf{a}$ to $\mathbf{b}$, given by the quaternion $\mathbf{d} = \mathbf{b}\mathbf{a}^{-1}$ (with $\mathbf{a}^{-1} = \mathbf{a}^*$).