Week 5: Mechanics II
Part 2: Calculating Distances

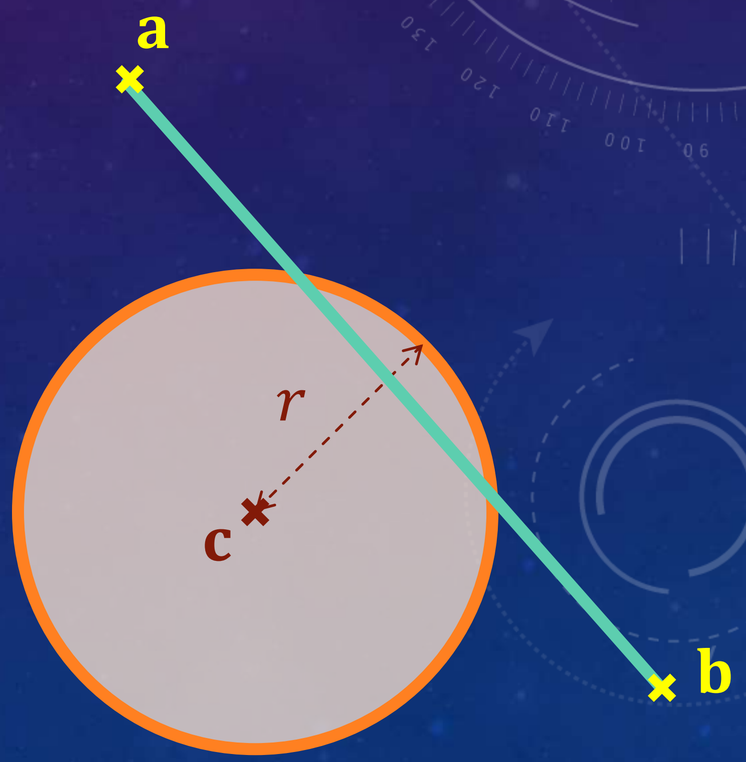
COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Apply** geometric principles and vector arithmetic to **calculate** the distance between a point and a line
- **Recall** a simple method to clamp values in code

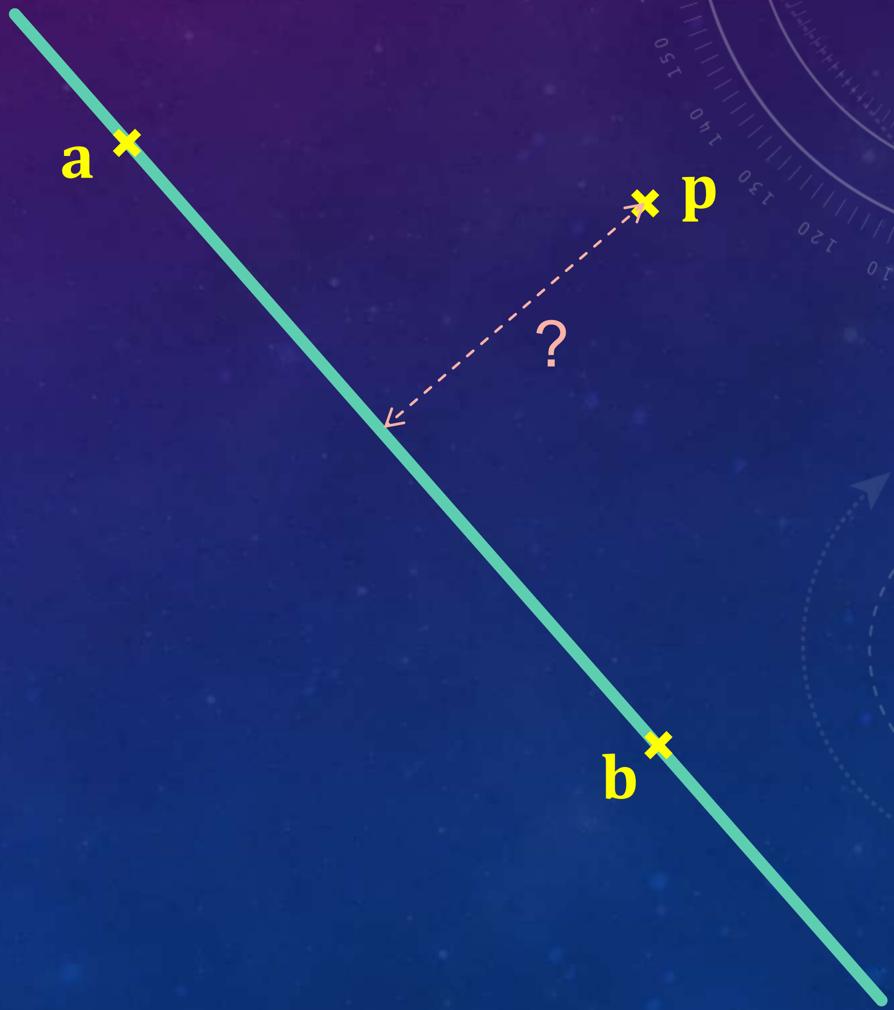
Recap: Circle and line segment collision

- The two collide if and only if the shortest distance between $\mathbf{c}$ and the line is $\leq r$
- The shortest distance may not be the perpendicular distance…
- Need to find the closest point on the line segment to $\mathbf{c}$.



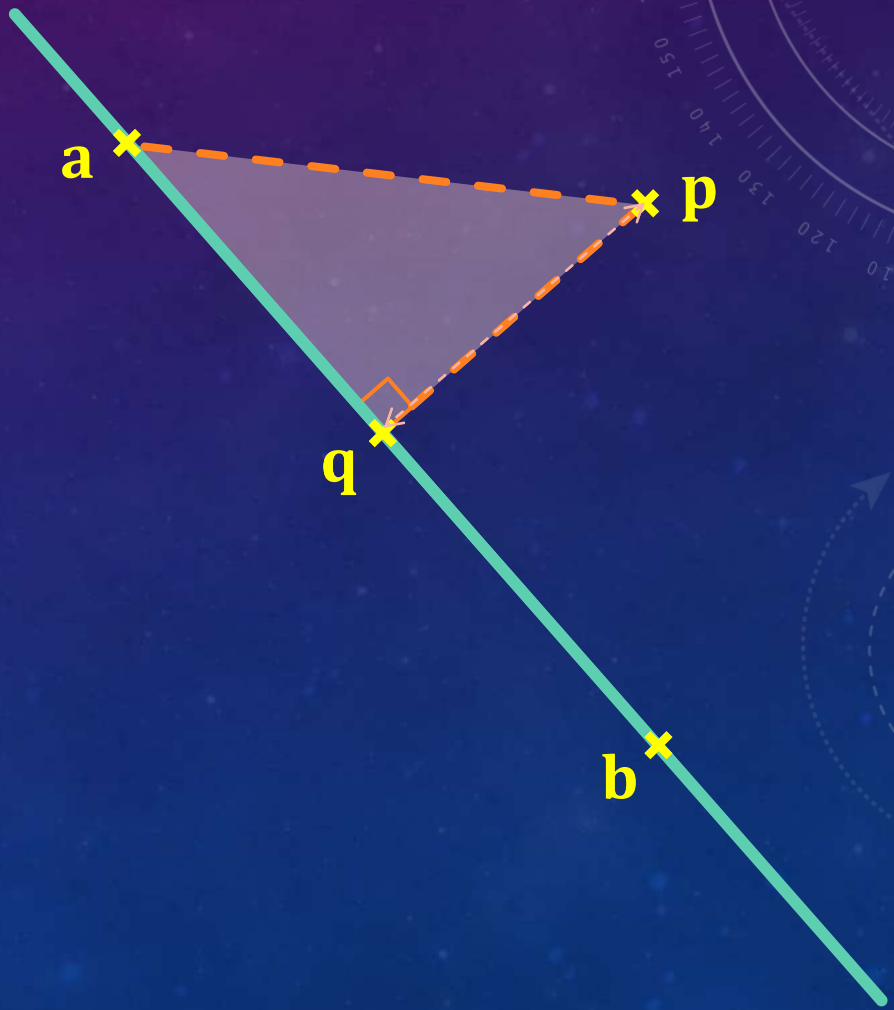
Distance between a point and a line

- Given a point $\mathbf{p}$ and an infinite line through $\mathbf{a}$ and $\mathbf{b}$
- What is the (shortest) distance between the point and the line?



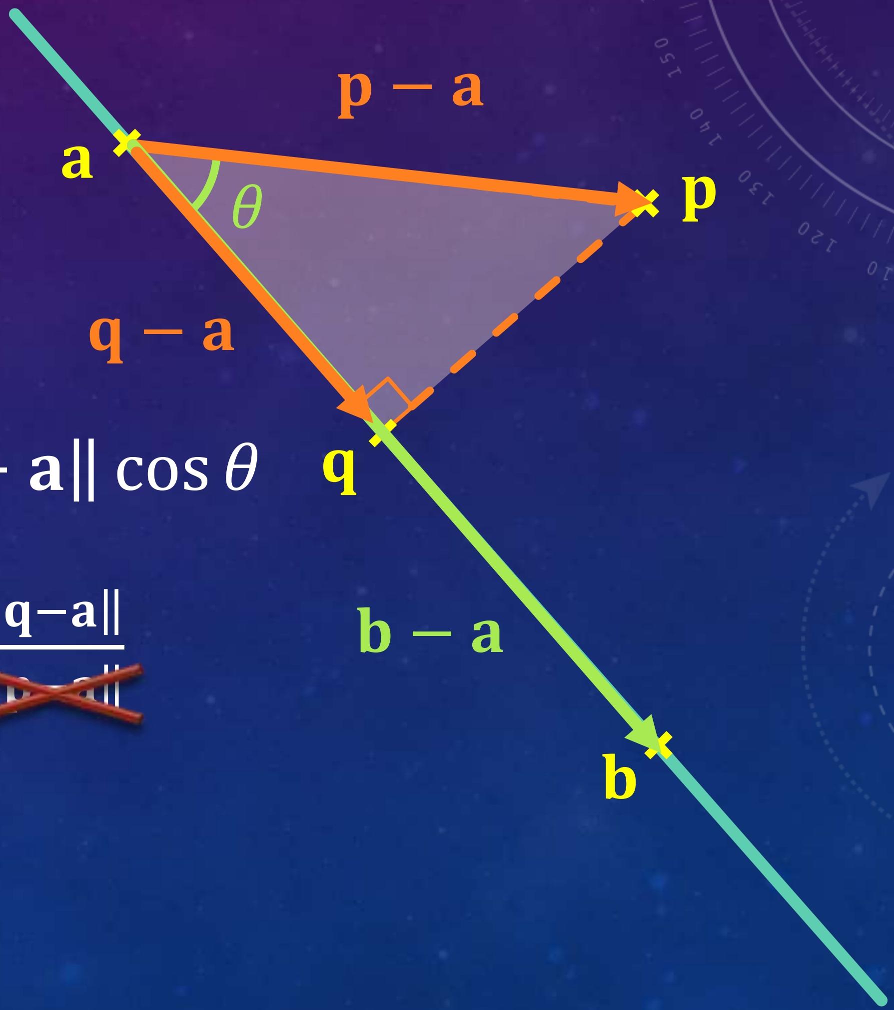
Distance between a point and a line

- Let **q** be the point on the line that is closest to **p**
- Then the line from **q** to **p** must be perpendicular to the line through **a** and **b**
- Thus we have a right-angled triangle as shown



Distance between a point and a line

- Let θ be the angle shown, then by SOH CAH TOA:
$$\cos\theta = \frac{\|\mathbf{q}-\mathbf{a}\|}{\|\mathbf{p}-\mathbf{a}\|}$$
- But also by dot product:
$$(\mathbf{p}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \|\mathbf{p}-\mathbf{a}\|\,\|\mathbf{b}-\mathbf{a}\|\cos\theta$$
- Substituting $\cos\theta$:
$$(\mathbf{p}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \|\cancel{\mathbf{p}-\mathbf{a}}\|\,\|\mathbf{b}-\mathbf{a}\|\,\frac{\|\mathbf{q}-\mathbf{a}\|}{\|\cancel{\mathbf{p}-\mathbf{a}}\|}$$
- Rearranging: $\|\mathbf{q}-\mathbf{a}\| = \frac{(\mathbf{p}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$



Distance between a point and a line

- Since $\mathbf{q}$ is on the line, we know that the vector $\mathbf{q} - \mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{a}$
- In fact,

Unit vector in the direction from $\mathbf{a}$ to $\mathbf{b}$

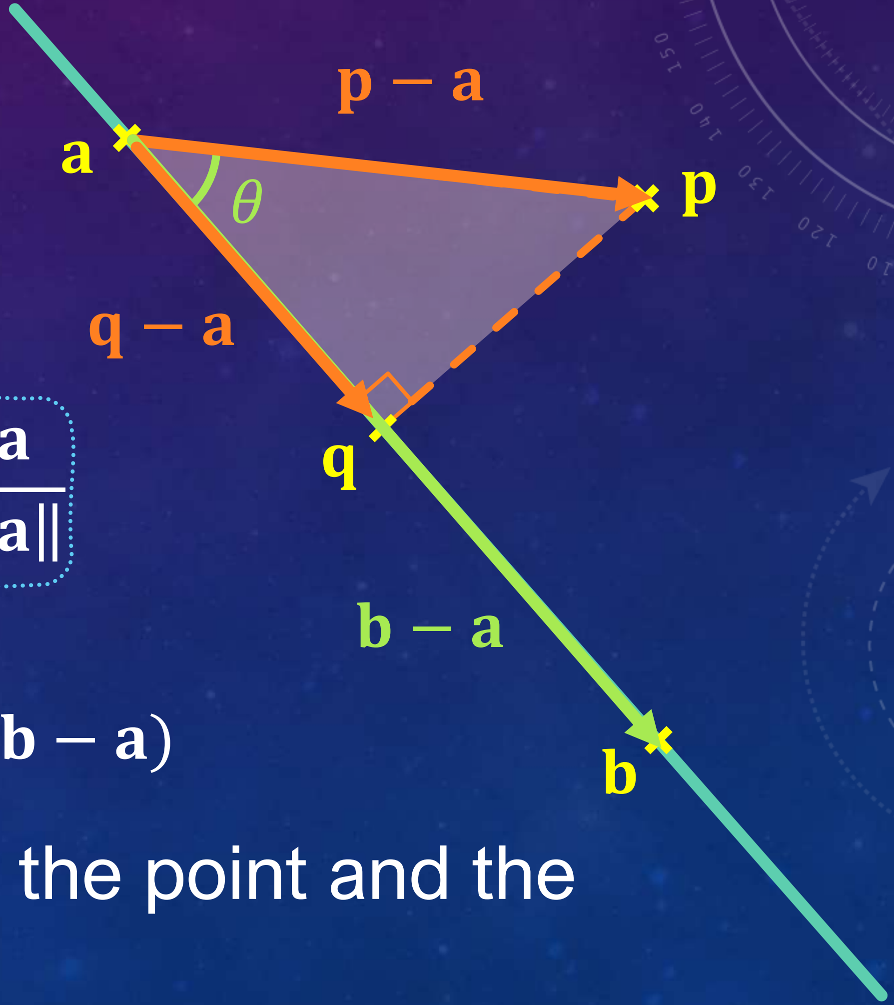
$$\mathbf{q} - \mathbf{a} = \|\mathbf{q} - \mathbf{a}\| \boxed{\frac{\mathbf{b} - \mathbf{a}}{\|\mathbf{b} - \mathbf{a}\|}}$$

- Therefore

"t"

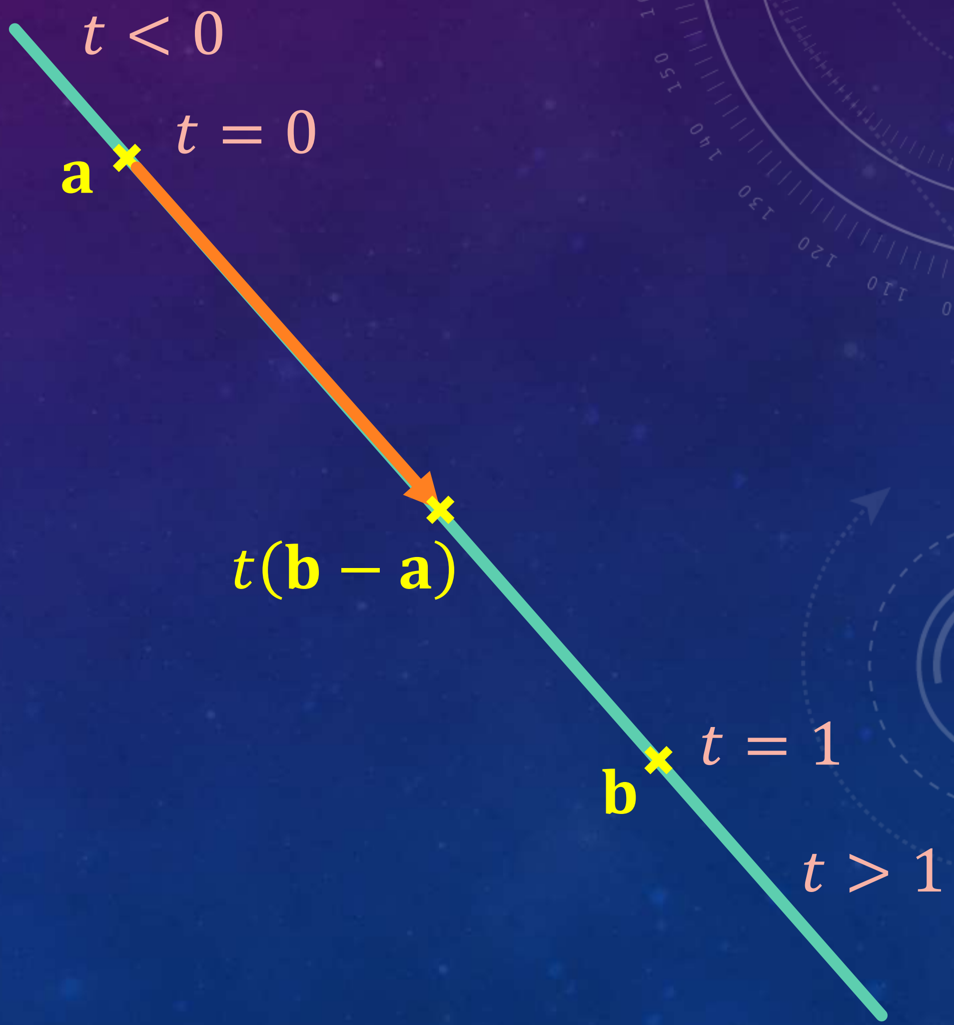
$$\mathbf{q} = \mathbf{a} + \boxed{\frac{(\mathbf{p} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}{\|\mathbf{b} - \mathbf{a}\|^2}} (\mathbf{b} - \mathbf{a})$$

- Knowing $\mathbf{q}$, the distance between the point and the line is simply $\|\mathbf{q} - \mathbf{p}\|$



Recap: Parametric form of a line

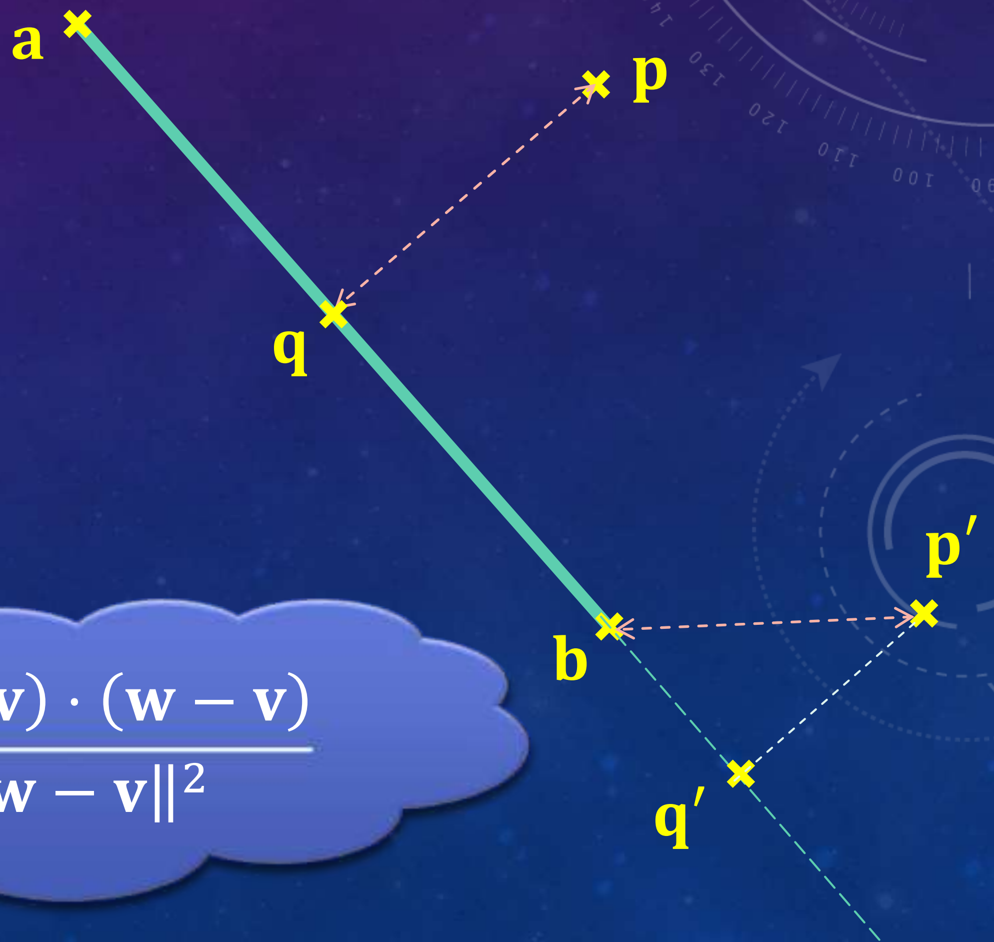
- Any point on the line between $\mathbf{a}$ and $\mathbf{b}$ can be written as $\mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some scalar t
- $0 \le t \le 1$ for points between $\mathbf{a}$ and $\mathbf{b}$
- Restricting $0 \le t \le 1$ gives a line segment



Distance between a point and a line segment

- Consider the point $\mathbf{q}$ we just found
- If $\mathbf{q}$ is between $\mathbf{a}$ and $\mathbf{b}$ $(0 < t < 1)$, then the shortest distance between $\mathbf{p}$ and the line segment is $\|\mathbf{q} - \mathbf{p}\|$
- If $\mathbf{q}$ is beyond $\mathbf{b}$ $(t > 1)$, then the shortest distance is $\|\mathbf{b} - \mathbf{p}\|$
- If $\mathbf{q}$ is beyond $\mathbf{a}$ $(t < 0)$ then the shortest distance is $\|\mathbf{a} - \mathbf{p}\|$

$$t = \frac{(\mathbf{p} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v})}{\|\mathbf{w} - \mathbf{v}\|^2}$$



Computing the distance for all cases

To clamp a value x between m and m: `max(m, min(x, n))`

```
if (t < 0)
    d = (a - p).magnitude();
else if (t > 1)
    d = (b - p).magnitude();
else
    d = (q - p).magnitude();
```

```
t = max(0, min(1, t));
q = a + t(b - a);
d = (q - p).magnitude();
```

If we clamp $0 \le t \le 1$ then we can just use $\|\mathbf{q} - \mathbf{p}\|$ in all cases (since $t = 0$ gives $\mathbf{q} = \mathbf{a}$ and $t = 1$ gives $\mathbf{q} = \mathbf{b}$)