Week 8: 3D Geometry II
Part 4: Quaternions

COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Introduce** the key properties of quaternions
- **Identify** their key use, i.e. SLERP

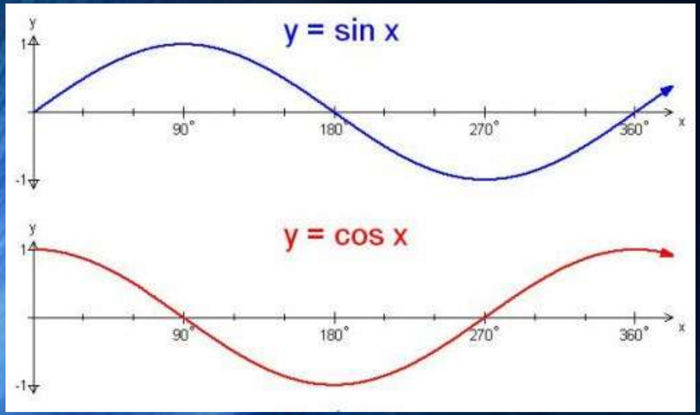
Quaternion representation

- Encode the axis and angle of rotation as a scalar component w and a 3D vector component $\mathbf{v}$:

$$[w \quad \mathbf{v}] = \left[\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\,\hat{\mathbf{n}}\right]$$
$$\mathbf{v} = [x \quad y \quad z]$$

- **Negation:**

$$\begin{aligned}
-\mathbf{q} &= -[w \quad (x \quad y \quad z)] \\
&= [-w \quad (-x \quad -y \quad -z)] \\
&= -[w \quad \mathbf{v}] = [-w \quad -\mathbf{v}] \\
&= \left[-\cos\frac{\theta}{2} \quad -\sin\frac{\theta}{2}\,\hat{\mathbf{n}}\right] \\
&= \left[\cos\left(\frac{\theta}{2}+\pi\right) \quad \sin\left(\frac{\theta}{2}+\pi\right)\hat{\mathbf{n}}\right] \\
&= \left[\cos\left(\frac{\theta+2\pi}{2}\right) \quad \sin\left(\frac{\theta+2\pi}{2}\right)\hat{\mathbf{n}}\right]
\end{aligned}$$



Doesn't really do anything: $\mathbf{q}$ and $-\mathbf{q}$ describe the same angular displacement (though are not mathematically the same)

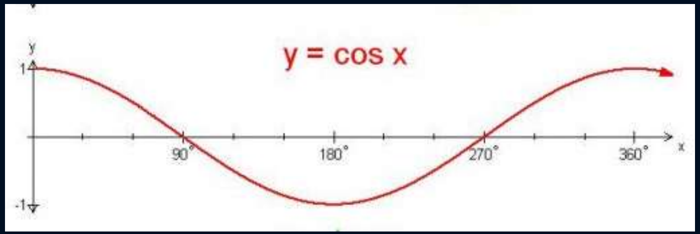
Quaternion magnitude

$$\|\mathbf{q}\| = \|[w \quad (x \quad y \quad z)]\|$$
$$= \sqrt{w^2 + x^2 + y^2 + z^2}$$
$$= \|[\mathrm{w} \quad \mathbf{v}]\|$$
$$= \sqrt{w^2 + \|\mathbf{v}\|^2}$$
$$= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \left(\sin\left(\frac{\theta}{2}\right)\|\hat{\boldsymbol{n}}\|\right)^2}$$
$$= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)} = 1$$

$$\cos^2\alpha + \sin^2\alpha \equiv 1$$

Quaternion identity and inverse

- **Identity**: $[1\ \mathbf{0}]$ and (geometrically) $[-1\ \mathbf{0}] = -[1\ \mathbf{0}]$
  - Complete rotation (360°) about any axis
  - $[-1\ \mathbf{0}]$ is from an odd multiple of 360°



- **Conjugate:** $\mathbf{q}^* = [\mathrm{w} \quad \mathbf{v}]^* = [w \quad -\mathbf{v}]$
- **Inverse**: $\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|} = \mathbf{q}^*$
  - Negating the rotation axis flips the positive rotation direction
  - Negating the angle is geometrically equivalent, but not mathematically

Quaternion multiplication

- Hamilton product:

$$\mathbf{q}_1\mathbf{q}_2 = [w_1 \quad \mathbf{v}_1][w_2 \quad \mathbf{v}_2]$$
$$= [w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \quad w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$$

- Properties:
  - Associative but not commutative
  - $\|\mathbf{q}_1\mathbf{q}_2\| = \|\mathbf{q}_1\|\|\mathbf{q}_2\| = 1$
  - The inverse of a quaternion product is equal to the product of the inverses in reverse order,

$$(\mathbf{q}_1\mathbf{q}_2 \dots \mathbf{q}_n)^{-1} = \mathbf{q}_n{}^{-1}\mathbf{q}_{n-1}{}^{-1} \dots \mathbf{q}_1{}^{-1}$$

Applying a quaternion to a point

- "Extend" a point (x, y, z) into quaternion space by defining the quaternion $\mathbf{p} = [0 \quad (x \quad y \quad z)]$
  - In general, $\mathbf{p}$ is not a rotation as it may have any magnitude
- Rotate $\mathbf{p}$ about the axis $\hat{\mathbf{n}}$ of a rotation quaternion $\mathbf{q} = \left[cos\left(\frac{\theta}{2}\right) \quad sin\left(\frac{\theta}{2}\right)\hat{\mathbf{n}}\right]$ by performing the multiplication

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

  - Can verify by conversion to a matrix
  - Uses about the same number of operations as matrix multiplications

Combining quaternions

- Rotating $\mathbf{p}$ first by a quaternion $\mathbf{a}$ and then by another, $\mathbf{b}$, is equivalent to performing a single rotation by the quaternion product $\mathbf{ba}$:

$$
\begin{aligned}
\mathbf{p}' &= \mathbf{b}(\mathbf{apa}^{-1})\mathbf{b}^{-1} \\
&= (\mathbf{ba})\mathbf{p}(\mathbf{ba})^{-1}
\end{aligned}
$$

Quaternion difference and exponentiation

- **Difference**: given orientations $\mathbf{a}$ and $\mathbf{b}$, the angular displacement $\mathbf{d}$ that rotates from $\mathbf{a}$ to $\mathbf{b}$ is given by

$$\mathbf{da} = \mathbf{b}$$
$$\mathbf{d} = \mathbf{ba}^{-1}$$

- **log**: $\log\mathbf{q} = \log([cos\alpha \quad \hat{\mathbf{n}}sin\alpha]) \equiv [0 \quad \alpha\hat{\mathbf{n}}]$, with $\alpha = \frac{\theta}{2}$
- **Exponentiation**: $\mathbf{q}^t =$ "$\mathbf{q}$ multiplied by itself t times" $= \exp(t\log\mathbf{q})$
  - As t varies from 0 to 1, $\mathbf{q}^t$ varies from $[1 \ \mathbf{0}]$ to $\mathbf{q}$
    - Allows extraction of a "fraction" of an angular displacement
  - $\mathbf{q}^2$ represents twice the angular displacement of $\mathbf{q}$
    - Always uses the shortest arc; cannot represent multiple spins

Quaternion interpolation

- **Spherical linear interpolation (SLERP)**
- Algebraic form:

$$\mathrm{slerp}(\mathbf{q}_0, \mathbf{q}_1, t) = \left(\mathbf{q}_0 \mathbf{q}_1{}^{-1}\right)^t \mathbf{q}_0$$

- Analogous to standard linear interpolation (LERP):

$$\Delta a = a_1 - a_0$$
$$\mathrm{lerp}(a_0, a_1, t) = a_0 + t\Delta a$$

1. Compute the difference between the values
2. Take a fraction of the difference
3. Adjust the original value by the fraction of the difference

SLERP algebraic derivation

1. Compute the difference between the values
   The angular displacement from $\mathbf{q}_0$ to $\mathbf{q}_1$ is given by the quaternion difference,
   $$\Delta\mathbf{q} = \mathbf{q}_1 {\mathbf{q}_0}^{-1}$$
2. Take a fraction of the difference
   Given by quaternion exponentiation, $(\Delta\mathbf{q})^t$
3. Adjust the original value by the fraction of the difference
   Combine the rotations $\mathbf{q}_0$ and $(\Delta\mathbf{q})^t$ via quaternion multiplication,
   $$(\Delta\mathbf{q})^t \mathbf{q}_0$$

SLERP computation: alternate approach

- Interpret quaternions as existing in a 4D Euclidean space
- Since all rotation quaternions are unit length, they "live" on the surface of a 4D hypersphere
  - Interpolate around the arc along the surface of the hypersphere, which connects the quaternions…
- $\mathrm{slerp}(\mathbf{q}_0, \mathbf{q}_1, t) = \dfrac{\sin(1-t)\omega}{\sin\omega}\mathbf{q}_0 + \dfrac{\sin t\omega}{\sin\omega}\mathbf{q}_1$

$$\cos\omega = \mathbf{q}_0 \cdot \mathbf{q}_1$$

SLERPing problems

- $\mathbf{q}$ and $-\mathbf{q}$ represent the same orientation, but may give different results when SLERPed (because a 4D hypersphere has a different topology from Euclidean space)
  - Solution: choose signs of $\mathbf{q}_1$ and $\mathbf{q}_2$ so that the dot product is non-negative, i.e. selecting the shortest rotational arc between them.
- If $\mathbf{q}_1$ and $\mathbf{q}_2$ are very close, then ω is very small and so is $\sin\omega$, which can cause problems with the division.
  - Use simple linear interpolation in these cases.