COMP110: Principles of Computing

10: Algorithm Strategies

Worksheets

- Worksheet 6: due **this Friday**
- Worksheet 7: due **next Friday**

Recursion

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- Example: the **Fibonacci numbers** — each number in the sequence is the sum of the previous two

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- To calculate the *n*th Fibonacci number:

```
int fibonacci(int n)
{
    if (n <= 2)
        return 1;
    else
        return fibonacci(n-1) + fibonacci(n-2);
}
```

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- Recursive functions need a **base case** where they stop recursing, otherwise they will go **forever**

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- I can solve the smallest possible problem
- Therefore I can write a recursive function

Example: file sizes

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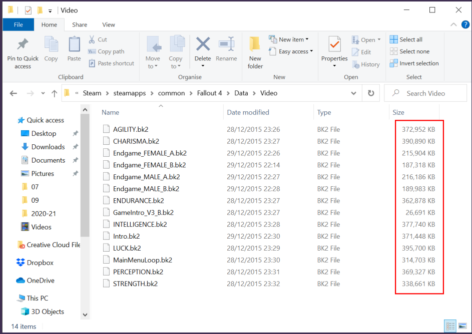
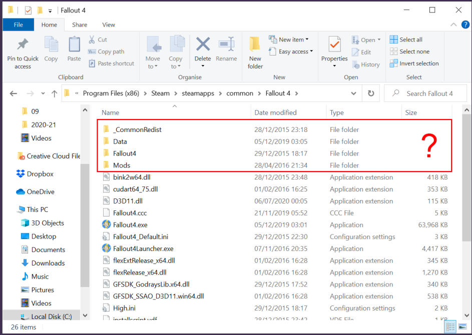


- If the folder contains **only** files, then we can simply add their sizes together

Example: file sizes

- What if the folder contains subfolders?



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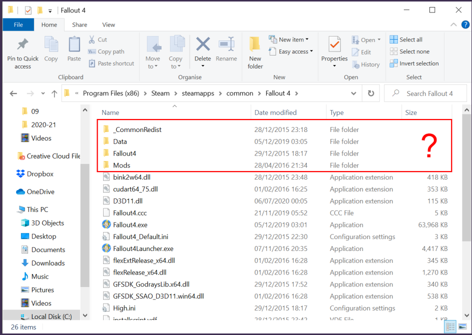

- We need to find the total size of all files in the subfolders and their subsubfolders...

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procedure CALCULATEFOLDERSIZE(*folder*)
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procedure CALCULATEFOLDERSIZE(folder)
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    for each item in folder do
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end procedure

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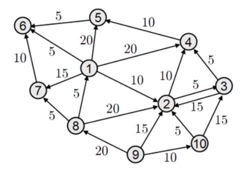
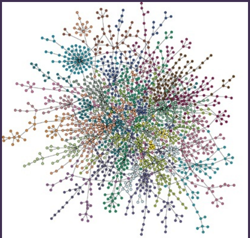
- This means if a recursive function contains **local variables**, they are **independent** between instances of the function
- This is also why careless use of recursion can lead to a **stack overflow**

Graphs and trees

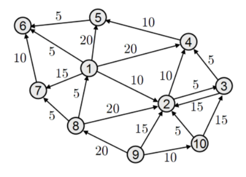
Graphs

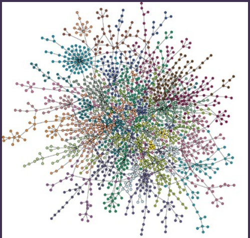
Graphs





Graphs





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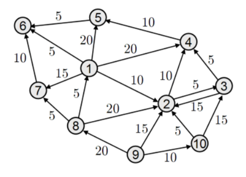
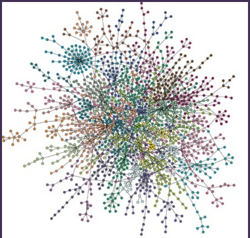
Graphs





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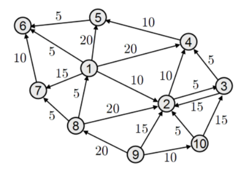
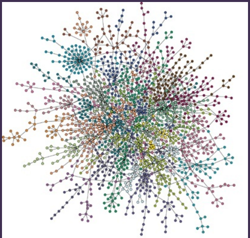
Graphs





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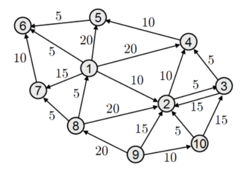

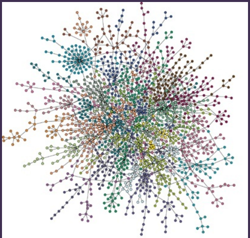

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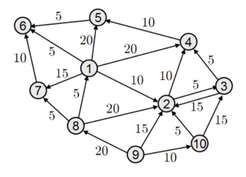

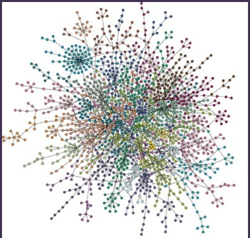

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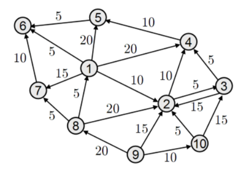

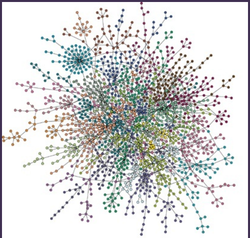

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- **Directed** graph: edges are arrows
- **Undirected** graph: edges are lines

Drawing graphs

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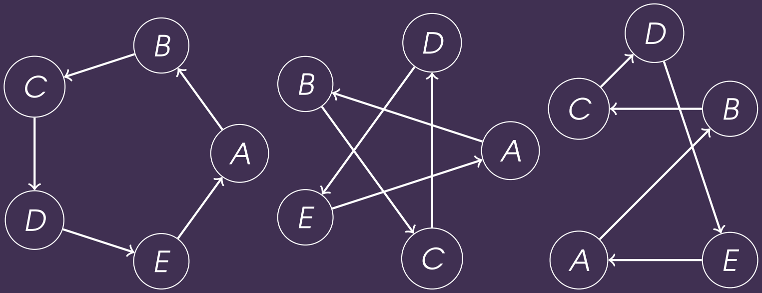
- E.g. these are technically the same graph:



Trees

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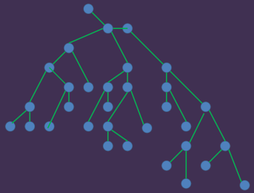

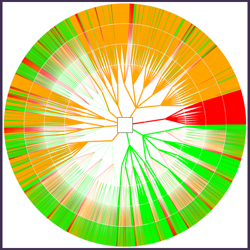

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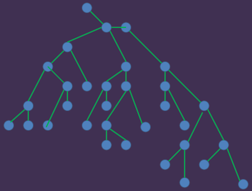

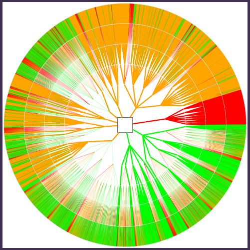

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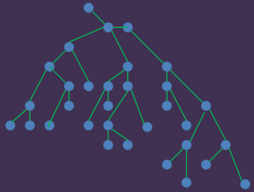

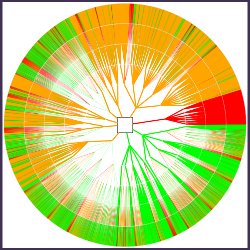

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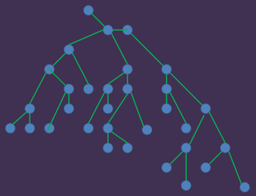

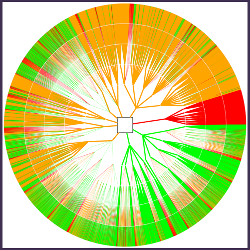

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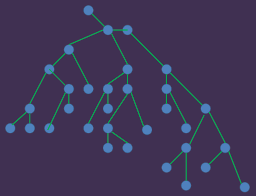

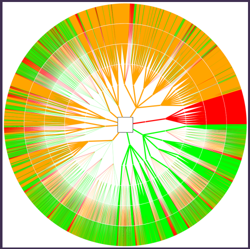

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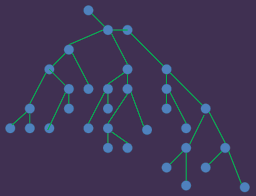

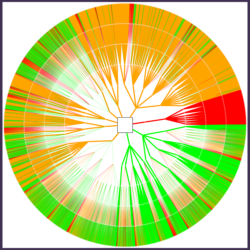

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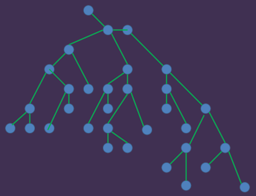

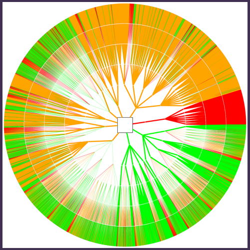

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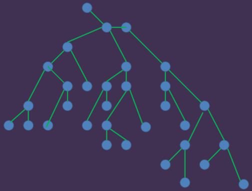

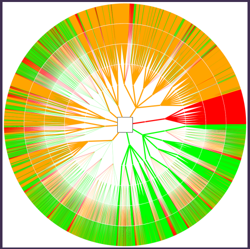

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- Edges go from **parent** to **child**
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  - Nodes can have 0, 1 or many children
- Used to model **hierarchies** (e.g. file systems, object inheritance, scene graphs, state-action trees, behaviour trees, ...)

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Tree traversal

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  - Depth first
  - Breadth first

Tree traversal

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procedure DEPTHFIRSTSEARCH

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Tree traversal

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let S be a stack

Tree traversal

procedure DEPTHFIRSTSEARCH
    let S be a stack
    push root node onto S

Tree traversal

procedure DepthFirstSearch
    let *S* be a stack
    push root node onto *S*
    while *S* is not empty **do**

Tree traversal

procedure DepthFirstSearch
    let S be a stack
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Tree traversal

procedure DEPTHFIRSTSEARCH
    let S be a stack
    push root node onto S
    while S is not empty **do**
        pop n from S
        print n

Tree traversal

procedure DEPTHFIRSTSEARCH
    let S be a stack
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        push children of n onto S

Tree traversal

procedure DepthFirstSearch
    let *S* be a stack
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    end while
end procedure

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procedure BREADTHFIRSTSEARCH

Tree traversal

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    let S be a stack
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    while S is not empty do
        pop n from S
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procedure BreadthFirstSearch
    let Q be a queue
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Tree traversal example

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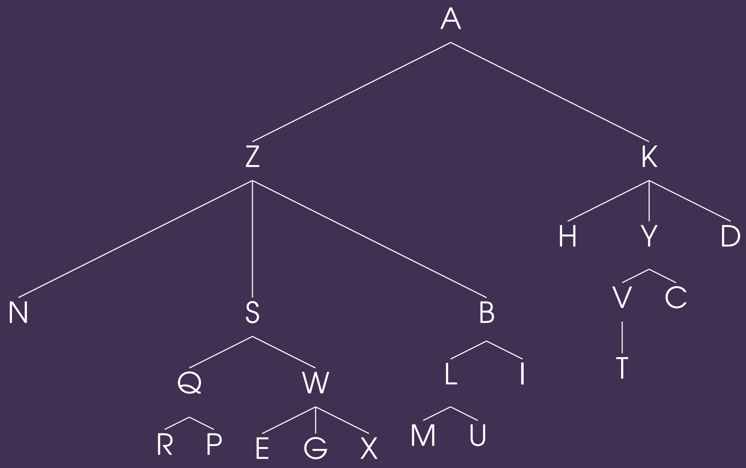

Recursive depth first search

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procedure DepthFirstSearch(n)

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Recursive depth first search

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    for each child c of n **do**

Recursive depth first search

procedure DEPTHFIRSTSEARCH(*n*)
    print *n*
    for each child *c* of *n* **do**
        DEPTHFIRSTSEARCH(*c*)

Recursive depth first search

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  for each child c of n **do**
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  end for
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Recursive depth first search

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    for each child c of n do
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    end for
end procedure
```

- Compare to the pseudocode on the previous slide. Where is the stack?

Algorithm strategies

The knapsack problem — informally

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The knapsack problem — informally

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- You have a **maximum carry weight**
- Which items should you pick up to maximise the total value without exceeding your carry weight?

The knapsack problem — formally

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- What subset $S \subseteq X$ maximises the total value, whilst not exceeding the maximum weight?

The knapsack problem — formally

- There is a set X of **items**
- Each item x has a weight $\mathrm{weight}(x)$ and a value $\mathrm{value}(x)$
- There is a maximum weight W
- What subset $S \subseteq X$ maximises the total value, whilst not exceeding the maximum weight?
- In other words: find $S \subseteq X$ to maximise

$$\sum_{x \in S} \mathrm{value}(x)$$

subject to

$$\sum_{x \in S} \mathrm{weight}(x) \leq W$$

Algorithm strategies

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Algorithm strategies

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- Greedy
- Divide-and-conquer
- Dynamic programming

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- Try **every possible** solution and decide which is best

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procedure KNAPSACK(X, W)

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      $S_{\text{best}} \leftarrow S$
      $v_{\text{best}} \leftarrow \text{value}(S)$
    end if
  end for
  return S_{best}
end procedure

Socrative `FALCOMPED`

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  - Hint: think about constructing a subset as a series of "yes or no" questions

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- If X contains n elements, how many subsets of X are there?
  - Hint: think about constructing a subset as a series of "yes or no" questions
- Therefore what is the time complexity of the brute force algorithm?
- If we add one element to X, what happens to the running time of the algorithm?

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- For example (maximum carry weight is 100)
  - Greedy algorithm takes 1 set of horse armour (weight 100, value 500)
  - ... instead of 100 silver coins (each weight 1, value 10)

Divide and conquer strategies

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- Requires that the solution to the original problem is composed of the solutions to the smaller problem
- Example from earlier in the module: **binary search**
  - Problem: find an element in a list
  - Subproblem: find the element in a list of half the size

Divide and conquer for the knapsack problem

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- Consider an element $x \in X$ with $\mathtt{weight}(x) \leq W$

Divide and conquer for the knapsack problem

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- Let X' be X with x removed

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- ... whichever has the greater value

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- ... whichever has the greater value
- Base case: the solution to the knapsack problem on the empty set **is** the empty set

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- For each piece of loot, try:
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  - Leaving it and solving the problem with the original carry capacity
- Whichever of those two gives the best result, go with it

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procedure KNAPSACK(X, W)

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    if $X = \{\}$ or $W \leq 0$ **then**
        return $\{\}$
    end if

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  $X' \leftarrow X$ without x

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  $S \leftarrow \textsc{Knapsack}(X', W)$
  if $\text{weight}(x) \leq W$ **then**
    $S' \leftarrow \textsc{Knapsack}(X', W - \text{weight}(x))$
    add x_k to S'

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- Thus the worst case time complexity is $O(2^n)$ — still exponential!
- However in the **average** case many of the calls have only a single recursive call, so this is still more efficient than brute force

Overlapping subproblems

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- This is called **memoization**
  - **Not** memorization!
- One of several techniques in the category of **dynamic programming**

Dynamic programming for the knapsack problem

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```
procedure KNAPSACK(X, W)
    if KNAPSACK(X, W) has already been computed then
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    end if
    if X = {} or W ≤ 0 then
        return {}
    end if
    x ← last element of X
    X′ ← X without x
    S ← KNAPSACK(X′, W)
    if weight(x) ≤ W then
        S′ ← KNAPSACK(X′, W − weight(x))
        add x_k to S′
        cache and return whichever of S, S′ has the larger value
    else
        cache and return S
    end if
end procedure
```

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- Remaining carry weight is an integer between 0 and W — so there are $W + 1$ possibilities

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- What is the maximum possible number of entries in the table of intermediate results?
- Therefore what is the time complexity of the dynamic programming algorithm?

Another example of dynamic programming

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- From the beginning of the lecture:

```
int fibonacci(int n)
{
    if (n <= 2)
        return 1;
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- `fibonacci(9)` calls `fibonacci(8)` and `fibonacci(7)`
- `fibonacci(8)` calls `fibonacci(7)` and `fibonacci(6)`
- So if we memoize, we can vastly reduce the number of recursive calls

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- Dynamic programming
  - Makes divide-and-conquer more efficient if subproblems often reoccur

Workshop