

COMP220: Graphics & Simulation

3: Mathematics for graphics

Learning outcomes

By the end of this session, you should be able to:

- ▶ **Explain** the role of vectors and matrices in computer graphics
- ▶ **Calculate** basic transformation matrices using the GLM library
- ▶ **Explain** the constituents of the model-view-projection matrix

Vectors



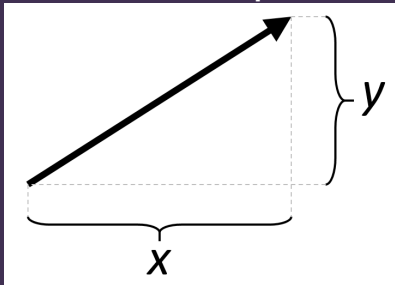
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A vector has **components**

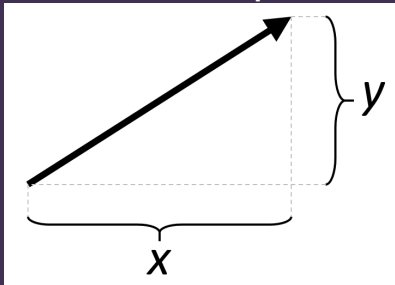
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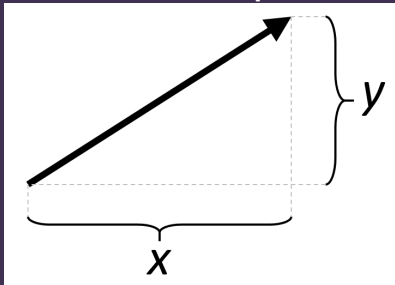
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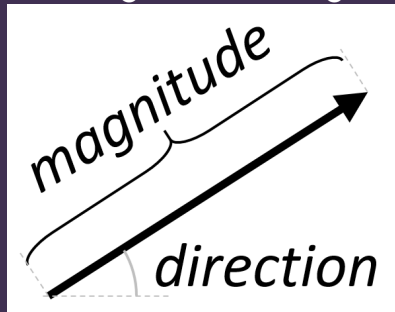
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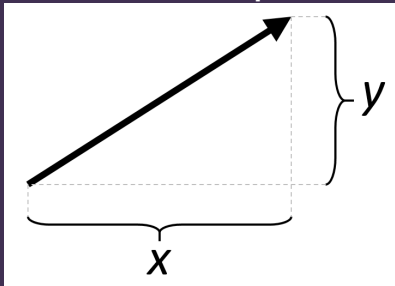


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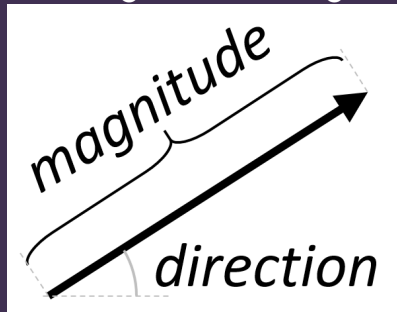


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The **origin** is the point represented by the vector $(0, 0, \dots)$

Radians

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- ▶ Careful! Some things in OpenGL work in **degrees**, others in **radians** (just to confuse you...)

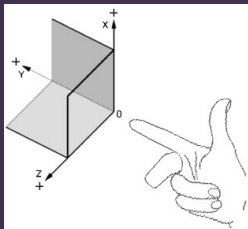
Right hand rule

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OpenGL uses a **right-handed coordinate system**

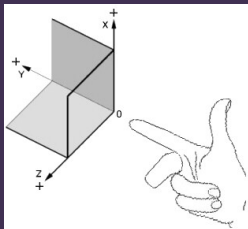
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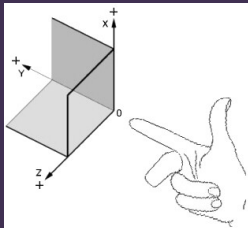
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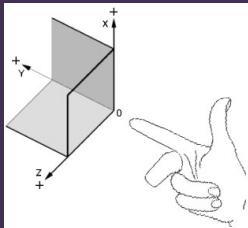
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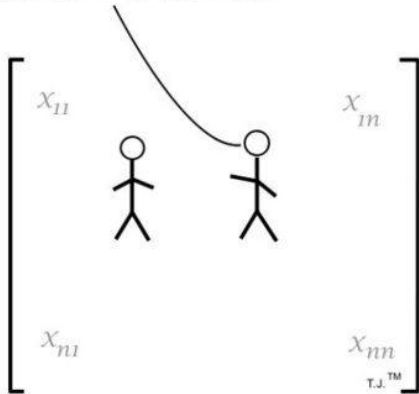
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- ▶ In homogeneous coordinates, the origin is $(0, 0, 0, 1)$ not $(0, 0, 0, 0)$!

Matrices



Welcome to the Matrix, Neo.



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- ▶ In computer graphics we mostly work with **square** matrices (number of rows = number of columns)

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- ▶ (But you don't really need to know how to calculate these manually...)

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- ▶ Multiplication of matrices is **not commutative**
 - ▶ In general, $A \times B \neq B \times A$
 - ▶ There may be some matrices where $A \times B = B \times A$, but they are the exception

Transformations



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- ▶ Multiplying a vector by the matrix **applies** the transformation

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- ▶ Lets us perform calculations with vectors and matrices in C++
- ▶ GLM types can be passed into shaders as uniforms, e.g.

```
// transformLocation points to a uniform of type ←  
mat4  
glm::mat4 transform = ...;  
glUniformMatrix4fv(transformLocation, 1, GL_FALSE ←  
    , glm::value_ptr(transform));
```

Identity

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```
// Default constructor for glm::mat4 creates an identity matrix ←  
glm::mat4 transform;
```

Translation

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```
transform = glm::translate(transform, glm::vec3(0.3f, 0.5f, 0.0f));
```

Scaling

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```
transform = glm::scale(transform, glm::vec3(1.2f, 0.5f ↵  
      , 1.0f));
```

Rotation

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```
float angle = glm::pi<float>() * 0.5f;  
glm::vec3 axis(0, 0, 1);  
transform = glm::rotate(transform, angle, axis);
```

Combining transformations

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- ▶ Transformations **do not commute** in general — changing the order will change the result
- ▶ The order they are applied is the **reverse** of what you might think — i.e. the above rotates **then** translates

Euler angles

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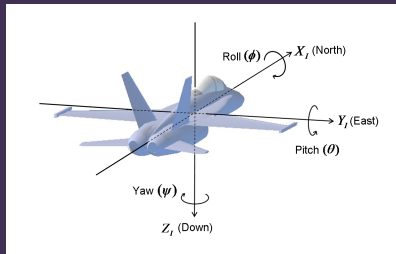
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 - ▶ The x-axis (1, 0, 0)
 - ▶ The y-axis (0, 1, 0)
 - ▶ The z-axis (0, 0, 1)
- ▶ These angles are sometimes called **roll**, **pitch** and **yaw**



Gimbal lock

<https://youtu.be/rrUCBOlJdt4?t=1m55s>

Model, View, Projection



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The **model-view-projection (MVP) matrix**:

$$M_{MVP} = M_{\text{projection}} \times M_{\text{view}} \times M_{\text{model}}$$

(remember, multiplication goes in reverse order)

The model matrix

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Exactly what we've been doing so far today...

The view matrix

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glm::mat4 view = glm::lookAt(  
    glm::vec3(2, 0, 2),    // eye  
    glm::vec3(0, 0, 0),    // centre  
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);
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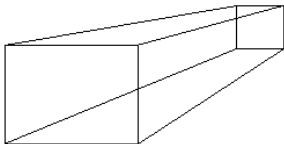
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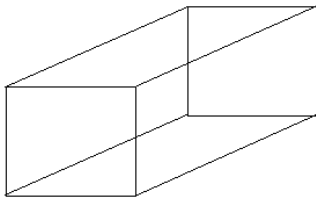
- ▶ `eye` is the position of the camera
- ▶ `centre` is a point for the camera to look at
- ▶ `up` is which direction is “up” for the camera (usually the positive y-axis)

Types of projection

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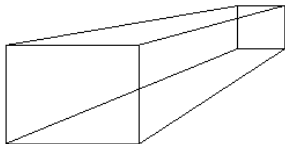


Perspective projection

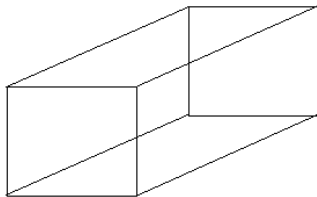


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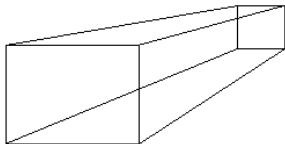
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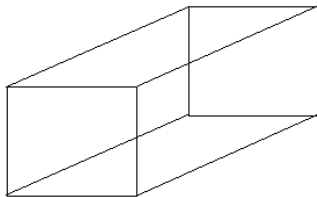
Orthographic projection

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Types of projection



Perspective projection



Orthographic projection

- ▶ Generally use **perspective** for 3D graphics
- ▶ **Orthographic** is useful for 2D or pseudo-2D graphics (e.g. isometric perspective)

The projection matrix

The projection matrix

```
glm::mat4 projection = glm::perspective(  
    glm::radians(45.0f), // field of view  
    4.0f / 3.0f,          // aspect ratio  
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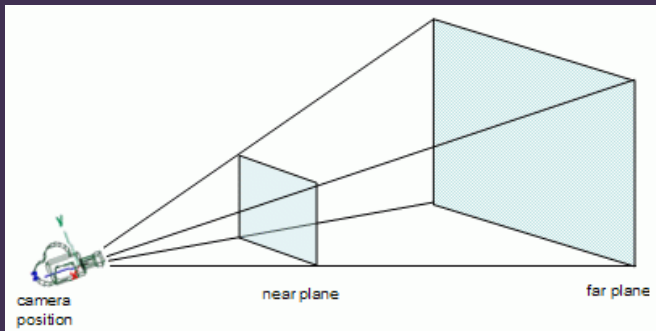
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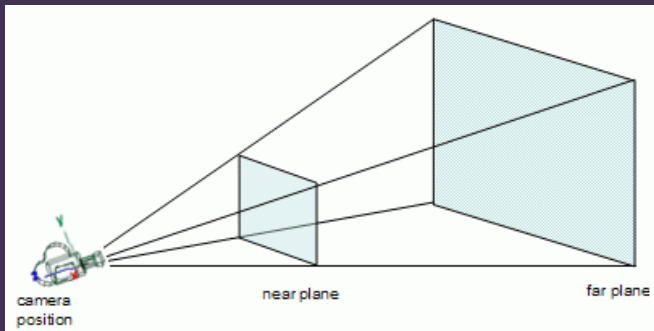
Also available: `glm::ortho` for orthographic projection

The view frustum

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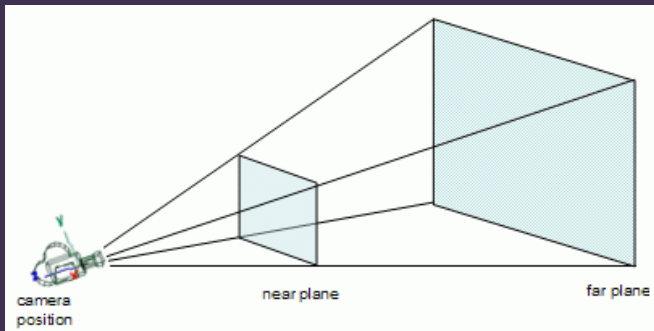


The view frustum



- Defined by the **near and far clipping planes** and the **edges of the screen**

The view frustum



- ▶ Defined by the **near and far clipping planes** and the **edges of the screen**
- ▶ **Nothing outside** the view frustum is visible

Live Coding - Transformations

Exercise

- ▶ Using the following link, first implement model transformations and then view and projection
 - ▶ `http://www.opengl-tutorial.org/beginners-tutorials/tutorial-3-matrices/`
- ▶ Play around with the order of the Model Transformation, what happens when you do this?
- ▶ Use the keyboard to move, rotate and scale the triangle
- ▶ Use the Keyboard to move the camera
- ▶ **Stretch Goal:** Create an FPS camera using the mouse and keyboard