COMP270: Mathematics for 3D Worlds and Simulations

# *WEEK 3: GEOMETRY II*
# PART 4: COMBINING TRANSFORMATIONS

# Objectives

- **Calculate** the result of applying successive transformation matrices using matrix multiplication
- **Recall** the concept of an inverse and **understand** how it relates to matrices

# Recap: affine transformations

- Translation:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation:

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Scale:

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Shear:

$$\mathbf{H}_x = \begin{pmatrix} 1 & \lambda_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{H}_y = \begin{pmatrix} 1 & 0 & 0 \\ \lambda_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Combining transformations

- Brute force: apply matrices to a vector in succession

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(b_{11}x + b_{12}y) + a_{12}(b_{21}x + b_{22}y) \\ a_{21}(b_{11}x + b_{12}y) + a_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

# Matrix multiplication

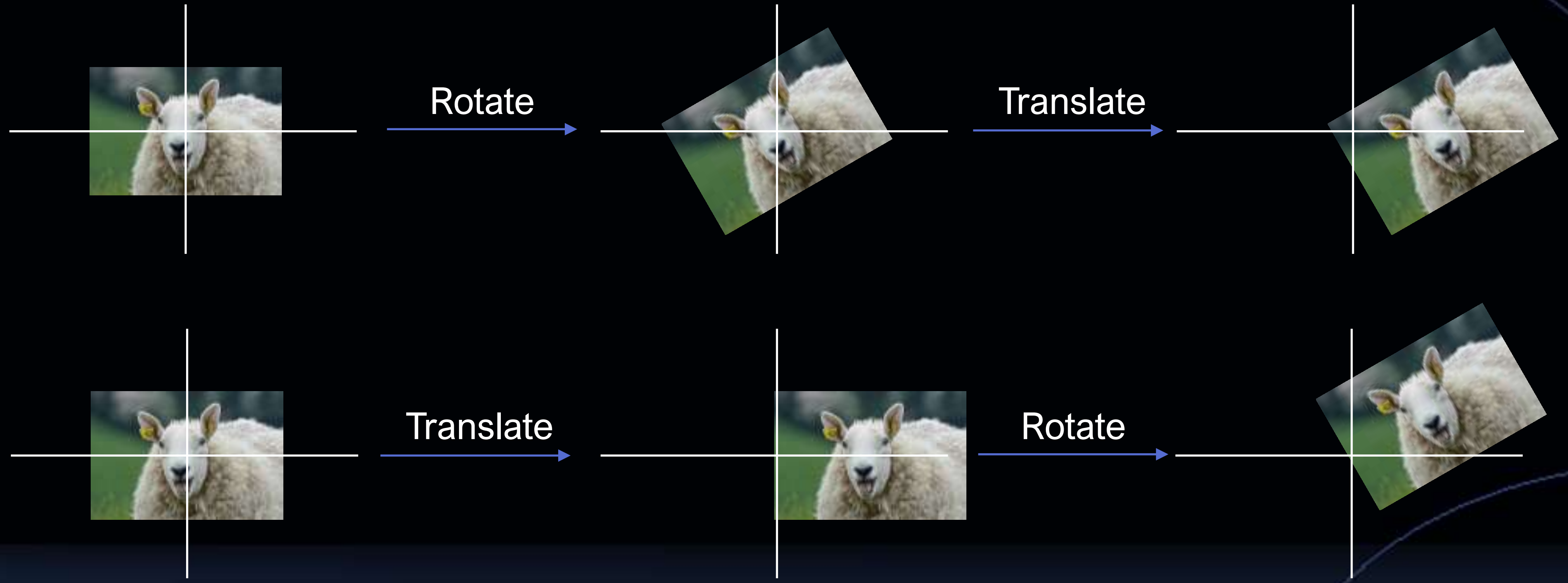
- 2×2 matrices:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

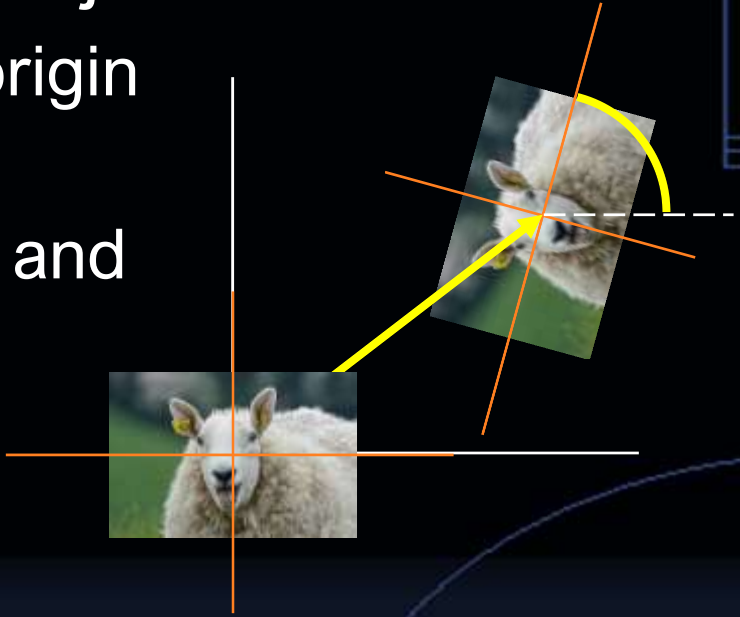
- Note that matrix multiplication is not commutative: in general $\mathbf{AB} \neq \mathbf{BA}$
- Transform order goes from right to left: $\mathbf{AB}$ represents doing $\mathbf{B}$ followed by $\mathbf{A}$

# Transformation order



# Local coordinates

- Define a set of axes relative to the object
  - Object position = translation of local origin relative to scene origin
  - Object rotation = angle between local and scene $x$-axes
- **Question:** How can we rotate an object about its own centre if it's not located at the origin?



# Matrix multiplication: general case

- In order to multiply two matrices, $\mathbf{A}$ and $\mathbf{B}$, the number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$
- If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, $\mathbf{AB}$ is an $m \times p$ matrix
- Square matrices are **closed** under multiplication: if we multiply an $n \times n$ matrix by an $n \times n$ matrix, we get an $n \times n$ matrix

Get the same type of object out as we put in.

- Multiplying an $m \times n$ matrix by a vector in $\mathbb{R}^n$ is equivalent to multiplying an $m \times n$ matrix with an $n \times 1$ 'matrix'
  - The result is a $m \times 1$ 'matrix', i.e. a vector in $\mathbb{R}^m$

# Matrix multiplication and the dot product

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices, then the element at row $i$ column $j$ of $\mathbf{AB}$ is the dot product of row $i$ of $\mathbf{A}$ with column $j$ of $\mathbf{B}$

$$\begin{pmatrix} [a_{11} & a_{12}] \\ [a_{21} & a_{22}] \end{pmatrix} \left( \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \quad \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \right)$$
$$= \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 \\ \mathbf{a}_2 \cdot \mathbf{b}_2 \end{pmatrix}$$

# Identity matrix

- **Definition**: an **identity matrix** $\mathbf{I}$ is a square matrix with 1 on the main diagonal and 0 everywhere else:
$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
- For any square matrix $\mathbf{A}$, $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$
- Equivalent to 1 for multiplication in $\mathbb{R}$
- Equal to applying a uniform scale of 1, or rotating through 0°
  - Uniform scale matrix $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} = s\mathbf{I}$

AKA the multiplicative identity

# Matrix inverse

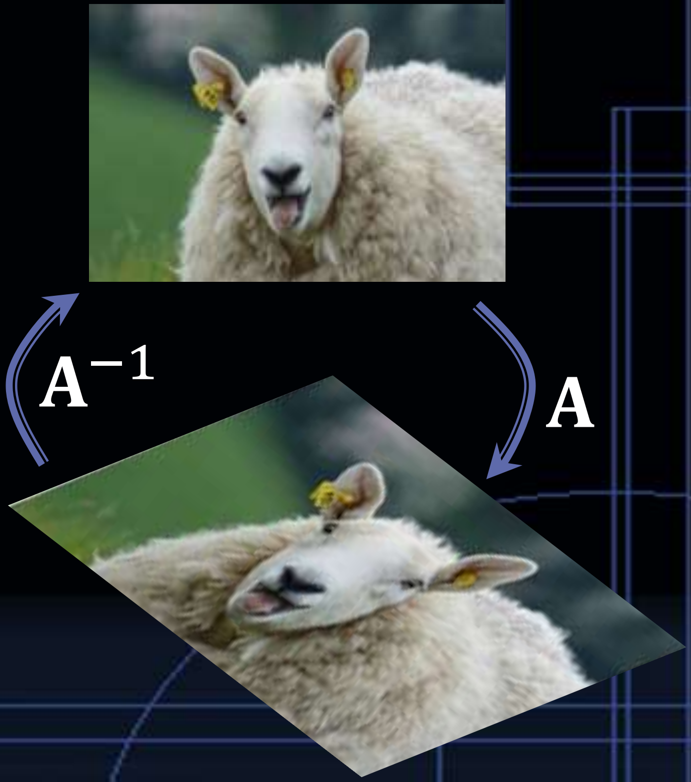
- **Definition**: for a square matrix $\mathbf{A}$, the inverse of $\mathbf{A}$ is a matrix $\mathbf{A}^{-1}$ such that
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$
- For 2×2 matrices, the inverse is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
The determinant
  - If $ad - bc = 0$ then the matrix has no inverse
- For larger matrices the inverse is harder to find, but methods do exist…

# Inverse transformations

- If $\mathbf{A}$ represents a transformation, then $\mathbf{A}^{-1}$ represents the reverse of that transformation
  - If $\mathbf{A}\mathbf{v} = \mathbf{v}'$ then $\mathbf{A}^{-1}\mathbf{v}' = \mathbf{v}$
- Scale: $\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}^{-1} = \begin{pmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{pmatrix}$
- Rotation: $\mathbf{R}_\theta^{-1} = \mathbf{R}_{-\theta}$

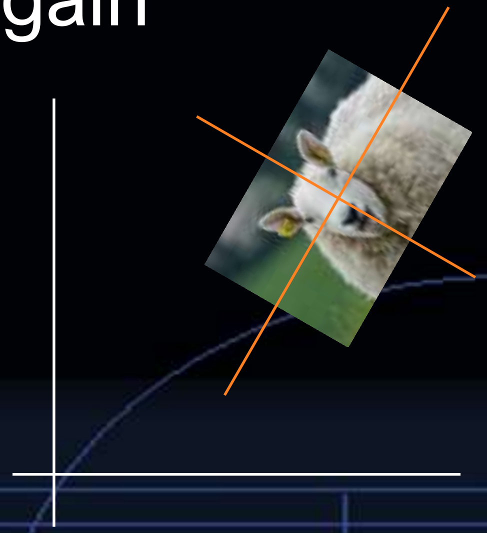
Rotation through $-\theta$

- Translation: $\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$



# Rotating about the origin

- **Question:** How can we rotate an object about its own centre if it's not located at the origin?
- **Answer:** move it to the origin! (Then back again afterwards).
- If the current transformation matrix is $\mathbf{M}$,
  - $\mathbf{M}^{-1}$ realigns the local axes with the scene,
  - $\mathbf{R}$ performs the desired rotation,
  - $\mathbf{M}$ returns the object to its starting position
  - Combined transform: $\mathbf{MRM}^{-1}$



# Basis vectors

**Linearly independent** means that no vector in the set can be written as a linear combination of the others

- **Definition**: a **vector basis** of a space is a subset of linearly independent vectors $\mathbf{v}_1 \dots \mathbf{v}_n$ such that every vector $\mathbf{v}$ in the space can be uniquely written as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

for scalar values $a_1 \dots a_n$

  - e.g. the vectors $\binom{1}{0}$ and $\binom{0}{1}$ form a basis for $\mathbb{R}^2$;

AKA $\mathbf{i}$ and $\mathbf{j}$

$$\binom{x}{y} = x\binom{1}{0} + y\binom{0}{1}$$

i.e. mutually perpendicular

  - Known as the standard basis, these are also orthonormal

# Basis vectors and matrices

- The identity matrix is made of the basis vectors:

$$\mathbf{I} = \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

- Matrix multiplication is distributive, so

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \left[ x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= x \left[ \mathbf{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + y \left[ \mathbf{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$