Week 4: Mechanics I

Part 3: Equations of motion

COMP270: Mathematics for 3D Worlds and Simulations

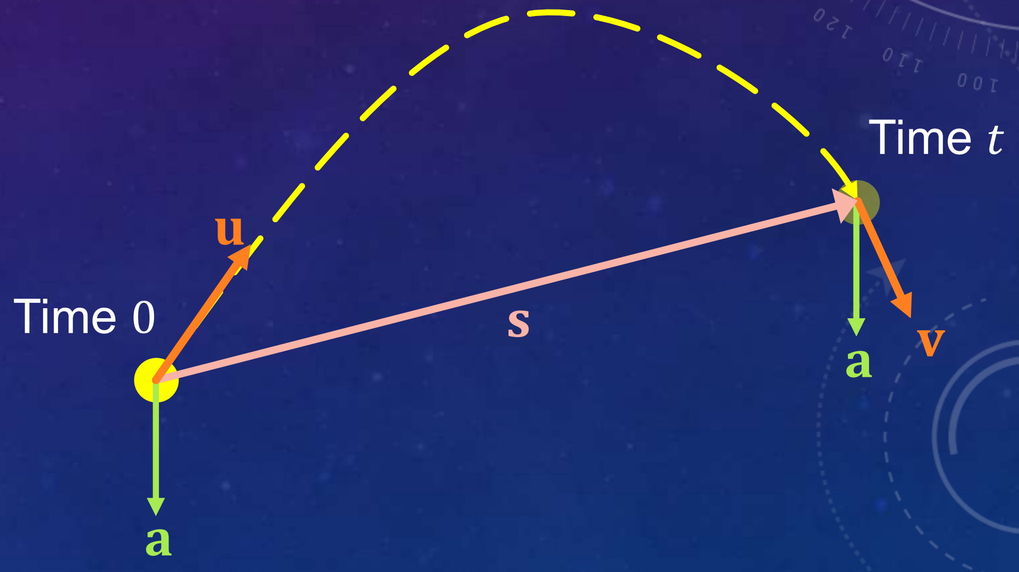
Objectives

- **Apply** techniques from Calculus to **extend** the relationships between physical quantities to a set of equations that **predict** an object's motion under constant acceleration.

Setup

"Point mass" (0 dimensions)

- Consider a particle under constant acceleration
  - e.g. under gravity with no other forces acting
- At all times, the acceleration of the particle is $\mathbf{a}$
- At time 0, assume the particle is at the origin and has velocity $\mathbf{u}$
- At time t, let $\mathbf{s}$ be the particle's position (or displacement) and let $\mathbf{v}$ be its velocity



Recap: Simulating Newtonian physics

- For each object, store its position $\mathbf{x}$ and velocity $\mathbf{v}$
- On each time step:
  - Apply numerical integration to the velocity to determine the new position, $\mathbf{x}' = \mathbf{x} + \mathbf{v}\Delta t$
  - ~~Calculate the forces acting upon the object, and thus the acceleration $\mathbf{a}$ from Newton's 2nd law~~
  - Apply numerical integration to the acceleration to determine the new velocity, $\mathbf{v}' = \mathbf{v} + \mathbf{a}\Delta t$

Recap: velocity and acceleration

- Acceleration is the derivative of velocity:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a}$$

$$= \mathbf{a}t^0$$

- Integrate w.r.t. t:

$$\mathbf{v} = \mathbf{a}t + \mathbf{c}$$

- When $t = 0, \mathbf{v} = \mathbf{u}$:

$$\mathbf{u} = \mathbf{a} \times 0 + \mathbf{c} \Rightarrow \mathbf{c} = \mathbf{u}$$

So

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t \quad (1)$$

'suvat' equations:
s – displacement
$\mathbf{u}$ – initial velocity
$\mathbf{v}$ – general/final velocity
$\mathbf{a}$ – acceleration
t – time

Equations of motion: displacement

- Velocity is the derivative of displacement:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{u} + \mathbf{a}t \text{ (from 1)}$$

- Integrate w.r.t. t:

$$\mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{c}'$$

- When $t = 0$, $\mathbf{s} = \mathbf{0}$:

$$\mathbf{0} = \mathbf{u} \times 0 + \tfrac{1}{2}\mathbf{a} \times 0^2 + \mathbf{c}' \Rightarrow \mathbf{c} = \mathbf{0}$$

So

$$\text{2} \quad \mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$

$\mathbf{a} = \mathbf{0} \Rightarrow \mathbf{s} = \mathbf{u}t$

"distance = speed × time"

Equations of motion: excluding acceleration

- Rearrange 1 :

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t}$$

$= \frac{\text{Change in quantity}}{\text{Change in time}}$

- Substitute in 2 :

$$\mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2 = \mathbf{u}t + \frac{1}{2}\left(\frac{\mathbf{v} - \mathbf{u}}{\cancel{t}}\right)t^{\cancel{2}}$$

So

3 $$\mathbf{s} = \tfrac{1}{2}(\mathbf{u} + \mathbf{v})t$$

Equations of motion: excluding time

- Rearrange 1 :

$$\mathbf{a}t = \mathbf{v} - \mathbf{u}$$

- Modify 3 :

$$\mathbf{s} \cdot \mathbf{a} = \tfrac{1}{2}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{a}t$$

$$\mathbf{s} \cdot \mathbf{a} = \tfrac{1}{2}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} - \mathbf{u})$$

$$\mathbf{s} \cdot \mathbf{a} = \tfrac{1}{2}(\cancel{\mathbf{u} \cdot \mathbf{v}} - \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \cancel{\mathbf{v} \cdot \mathbf{u}})$$

So

4 $$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{a} \cdot \mathbf{s}$$

Or $\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{a} \cdot \mathbf{s}$

Equations of motion: excluding initial velocity

- Rearrange (1):

$$\mathbf{u} = \mathbf{v} - \mathbf{a}t$$

- Substitute in (2):

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 = (\mathbf{v} - \mathbf{a}t)t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \mathbf{v}t - \mathbf{a}t^2 + \frac{1}{2}\mathbf{a}t^2$$

So

(5) $$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

Example

- A particle is dropped and falls under gravity: $\mathbf{u} = 0$, $\mathbf{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$
- At time $t = 5$ seconds:
  - $\mathbf{v} = \mathbf{u} + \mathbf{a}t = \begin{pmatrix} 0 \\ 5 \times -9.81 \end{pmatrix}$ - the particle is falling downwards at 49.05 metres per second
  - $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 = \frac{25}{2}\begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$ - the particle has fallen down a distance of 122.625 metres