COMP270: Mathematics for 3D Worlds and Simulations

WEEK 3: GEOMETRY II
PART 2: MOVING ON TO MATRICES

Objectives

- **Describe** the appearance and purpose of a matrix
- **Calculate** the result of applying a matrix to a vector

Recap: functions and vectors

- A parametric function $f : \mathbb{R} \to \mathbb{R}^2$ can map a scalar to a vector:

$$f(t) = \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix}$$

- Functions can also map vectors to vectors:

$$g : \mathbb{R}^2 \to \mathbb{R}^2$$

$$g\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} g_x(x, y) \\ g_y(x, y) \end{pmatrix}$$

Example

$$g\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 2y \\ x - y \end{pmatrix}$$

$$g\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Example

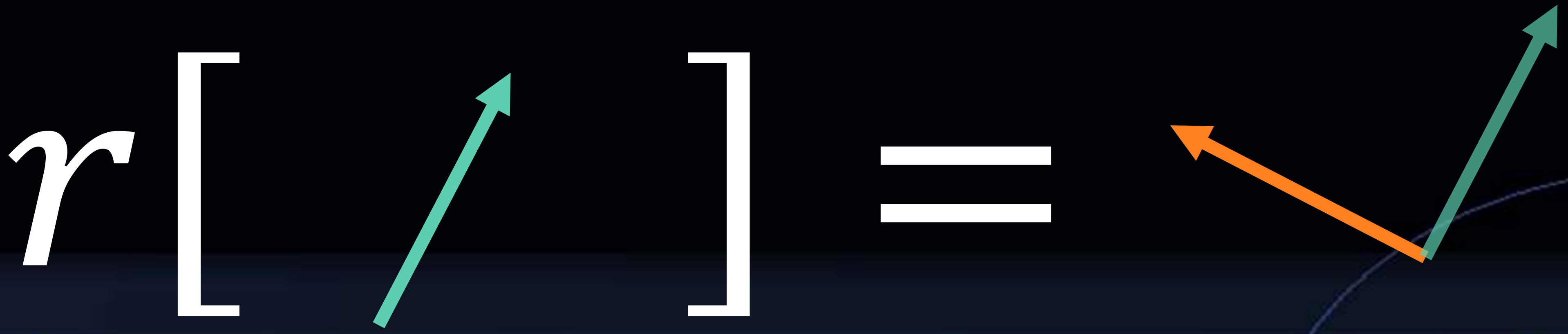
$$s\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$s\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Example

$$r\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$r\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



Example

$$r\left(s\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]\right) = r\left(\begin{pmatrix} 2x \\ 2y \end{pmatrix}\right) = \begin{pmatrix} -2y \\ 2x \end{pmatrix}$$

$$rs\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$rs[\quad\quad] =$$

Matrices

- **Definition**: an $m \times n$ **matrix** is a rectangular array of numbers, with m **rows** and n **columns**

Note indices are row first

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{pmatrix} a \\ a \\ \vdots \end{pmatrix}}\right\} m \text{ rows}$$

- We will mostly work with square matrices: matrices where $m = n$
  - For example a 2×2 matrix: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Matrices as functions

- The elements of the matrix are the coefficients of the functions they represent
- e.g. $g\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} g_x(x,y) \\ g_y(x,y) \end{pmatrix} = \begin{pmatrix} 0x + 2y \\ x - y \end{pmatrix}$ would be written as

x coefficient of g_x

y coefficient of g_x

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

x coefficient of g_y

y coefficient of g_y

Applying matrices to vectors

- To apply a matrix to a vector, use a special kind of multiplication:

$$\begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} a_{11}x \\ a_{21}x \end{matrix}} \begin{matrix} + \\ + \end{matrix} \boxed{\begin{matrix} a_{12}y \\ a_{22}y \end{matrix}} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$$

Applying matrices to vectors – examples

- g: $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0x + 2y \\ 1x + (-1)y \end{pmatrix} = \begin{pmatrix} 2y \\ x - y \end{pmatrix}$
- s: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$
- r: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0x + (-1)y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$
- rs: $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0x + (-2)y \\ 2x + 0y \end{pmatrix} = \begin{pmatrix} -2y \\ 2x \end{pmatrix}$

Multiplying matrices and vectors

- General rule: an $m \times n$ matrix can be multiplied by a vector in $\mathbb{R}^n$ to give a vector in $\mathbb{R}^m$

$$\overbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}^{n \text{ columns}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left.\begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}\right\} n \text{ rows}$$

Mismatched matrix…

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{31} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$g\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} g_x(x,y) \\ g_y(x,y) \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

Non-square matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix}$$

$g : \mathbb{R}^3 \to \mathbb{R}^2$

$$g\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} g_x(x,y,z) \\ g_y(x,y,z) \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \end{pmatrix}$$

Matrix limitations

- **Note**: matrices can only represent **linear combinations** of components, i.e. of the form
$$f(x_1, x_2, \ldots, x_2) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$
where $a_1 \ldots a_n$ are scalars.
- Cannot represent x^a, a^x, $\frac{1}{x}$, xy etc.