COMP110: Principles of Computing

5: Computational Complexity

Learning outcomes

- **Explain** the notion of computability
- **Use** "big O" notation to express computational complexity
- **Apply** appropriate algorithms to achieve efficiency

Computability

Computability theory

- Let A and B be **sets** of elements
  - NB: A may be **infinite**
- A function $f : A \to B$ is **computable** if there exists a Turing machine which computes f
  - I.e. given an encoding of $a \in A$ as input, the Turing machine outputs an encoding of $f(a)$

An uncomputable function

The **halting problem**

- A = the set of all Turing machines (encoded as transition tables)
- $B = \{\text{true}, \text{false}\}$
- $f(a) = \begin{cases} \text{true} & \text{if } a \text{ halts in finite time on all inputs} \\ \text{false} & \text{otherwise} \end{cases}$
- There is **no** Turing machine that computes f
- f is **uncomputable**

Church-Turing Thesis

- A system (e.g. a computer or programming language) is **Turing complete** if it can implement any given Turing machine
- If a function is **effectively calculable**, then it is **computable** by a Turing machine
- Effectively calculable = there is a method or algorithm for computing it
- So in terms of computability, Turing machines are as powerful as computers can be

Halting revisited

- Write a software tool that, given a Python program, predicts whether that program can go into an infinite loop
- Your tool must work for **all** Python programs
- Is this possible?

Computation time

Resources

- All programs use **resources**
  - Time
  - Memory
  - Network bandwidth
  - Power
  - ...
- Often **time** is the resource we care about the most
  - Particularly in games: want to maintain a good **frame rate** free of **lag** or **stuttering**

Basic time measurement in Python

```
import time

start_time = time.clock()

... do something here ...

end_time = time.clock()
print("Computation took", end_time - start_time, " ↩
    seconds")
```

Repeating for better accuracy

```
import time

start_time = time.clock()

repetition_count = 1000

for repetition in range(repetition_count):
    ... do something here ...

end_time = time.clock()
time_per = (end_time - start_time) / repetition_count
print("Computation took", time_per, "seconds")
```

Scaling

- Timing is dependent on hardware and software issues
- We are often less interested in how many milliseconds a particular computation takes on today's hardware, and more interested in how the execution time **scales** with the problem size

Search

Search

- We have a list of names, each with some data associated
- We want to find one of them

Linear search

```
procedure FIND(name, list)
    for each item in list do
        if item.name = name then
            return item
        end if
    end for
    throw "Not found"
end procedure
```

How long does it take?

Socrative room code: `FALCOMPED`

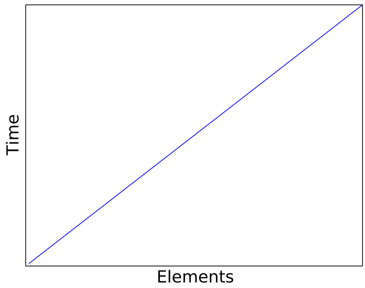
- Suppose there are 25 items in the list
- In the **best case**, how many items do we need to visit before finding the one we want?
- How about in the **worst case**?

How long does it take?

Socrative room code: `FALCOMPED`

- If there are 25 items in the list, the **worst case** number of items visited is 25
- How about if there are 50 items?
- How about 100 items?
- If the number of items **doubles**, what happens to the amount of time the search takes?

Linear time



- The running time of linear search is **proportional** to the size n of the list
- Linear search is said to have **linear time complexity**
- Also written as $O(n)$ **time complexity**

Searching a sorted list

- If the list is **sorted** in alphabetical order, we can do better than linear...

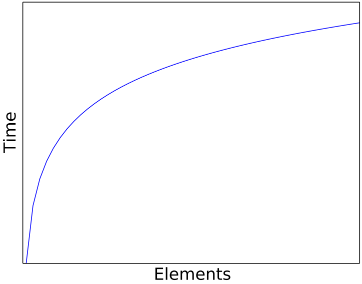
Binary search

```
procedure FIND(name, list)
    if list is empty then
        throw "Not found"
    end if
    mid ← the "middle" item of the list
    if name = mid.name then
        return mid
    else if name < mid.name then
        return FIND(name, first half of list)
    else if name > mid.name then
        return FIND(name, second half of list)
    end if
end procedure
```

How long does it take?

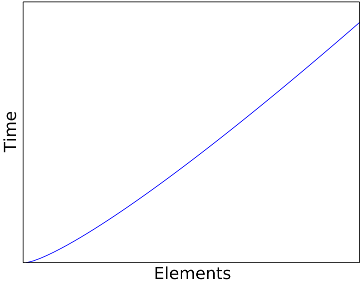
Socrative room code: `FALCOMPED`

- Each iteration cuts the list in **half**
- Worst case: we have to keep halving until we get down to a single element
- If the size of the list is **doubled**, what happens to the worst-case **number of iterations** required?
- The running time is **logarithmic** or $O(\log n)$



Hidden complexity

if name $<$ mid.name **then**
    return FIND(name, first half of list)
else if name $>$ mid.name **then**
    return FIND(name, second half of list)
end if



- Careful how you implement this!
- **Copying** (half of) a list is **linear** $O(n)$
- The actual running time would be $O(n \log n)$
- Use **pointers** into the list instead of copying

Binary search done wrong

```
def binary_search(name, mylist):
    if mylist == []:
        raise ValueError("Not found")

    mid = len(mylist) / 2
    mid_name = mylist[mid_index].name

    if name == mid_name:
        return mid
    elif name < mid_name:
        return binary_search(name, mylist[:mid])
    else:
        return binary_search(name, mylist[mid+1:])
```

Binary search done right

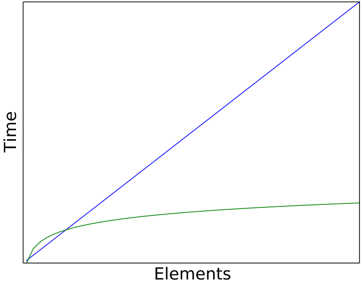
```
def binary_search(name, mylist, start, end):
    if end <= start:
        raise ValueError("Not found")

    mid = (start + end) / 2
    mid_name = mylist[mid].name

    if name == mid_name:
        return mylist[mid]
    elif name < mid_name:
        return binary_search(name, mylist, start, mid)
    else:
        return binary_search(name, mylist, mid+1, end)
```

Binary search vs linear search

Socrative room code: `FALCOMPED`



- So binary search is better than linear search... right?
- Discuss in **pairs**
- On Socrative, post **one reason** why, or **one situation** where, linear search may be a better choice than binary search

Hashing

- Come up with a **hashing function** which maps elements to numbers
- Example: assign $A = 1, B = 2, C = 3$ etc, and add them together
- Use these numbers to assign each element to a "bin" where it can be found

⋮	⋮
112	Ward, Jessica
113	Baker, Theresa
114	Collins, Jane
115	—
116	—
117	Hughes, Aaron
118	—
119	—
120	—
121	—
122	Brown, Janet
123	—
124	—
125	Gonzalez, Adam Lewis, Rose
126	—
127	—
128	—
129	—
130	—
131	—
132	Young, Frank
⋮	⋮

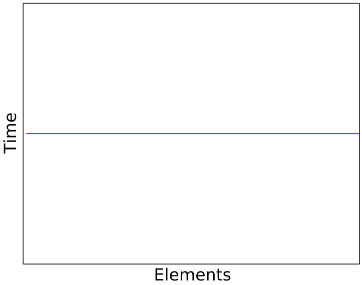
Hash look-up

98	Diaz, Harold
99	Parker, Debra Perez, Diana White, Amanda
112	Ward, Jessica
113	Baker, Theresa
114	Collins, Jane
117	Hughes, Aaron
122	Brown, Janet
125	Gonzalez, Adam Lewis, Rose
132	Young, Frank
135	Kelly, Philip
138	Cox, Shirley
142	Clark, Stephanie
144	Scott, Michelle
145	Miller, Jeremy
147	Davis, Marilyn
149	Lopez, Jeffrey
151	Anderson, Martha
158	Williams, Billy
162	Sanders, Phillip
171	Russell, Mildred
175	Stewart, Howard
183	Henderson, Lawrence

"Lopez, Jeffrey"

$12 + 15 + 16 + 5 + 26 + 10 + 5 + 6 + 6 + 18 + 5 + 25 = 149$

How long does it take?



- If there are no "collisions", look-up time is **constant** or $O(1)$
  - (NB: constant **with respect to** n)
- I.e. doubling the size of the list **does not change** the look-up time
- When there are collisions, need to fall back on something like linear or binary search within each bin

Don't reinvent the wheel!

- We are using search as an **example**, to learn the **principles** — in practice you should hardly ever implement your own search
- Linear search in Python:
  - `list.index()` method
  - **List comprehension**, e.g.

```
[person for person in people if person.name ↩
    == "Lopez, Jeffrey"]
```

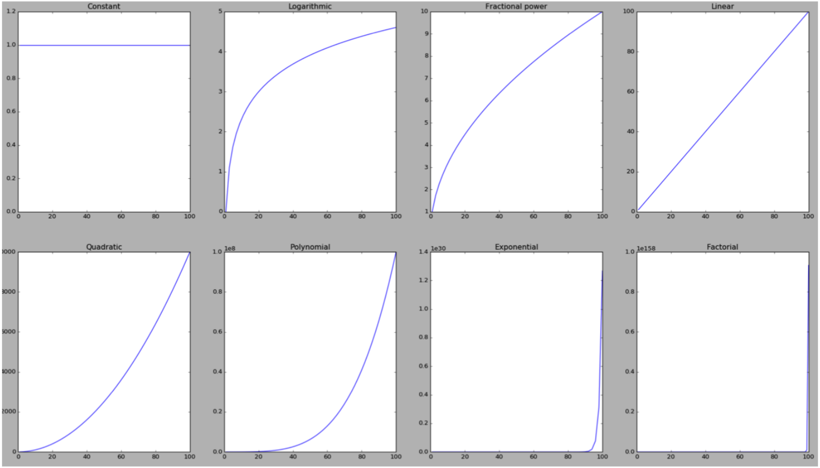
- Binary search in Python:
  - The `bisect` module
- Hash tables in Python:
  - The `dict` (dictionary) data structure

More on complexity

Common complexity classes

"Faster"	Constant	$O(1)$
$\uparrow$	Logarithmic	$O(\log n)$
$\mid$	Fractional power	$O(n^k), k < 1$
$\mid$	Linear	$O(n)$
$\mid$	Quadratic	$O(n^2)$
$\mid$	Polynomial	$O(n^k), k > 1$
$\downarrow$	Exponential	$O(e^n)$
"Slower"	Factorial	$O(n!)$

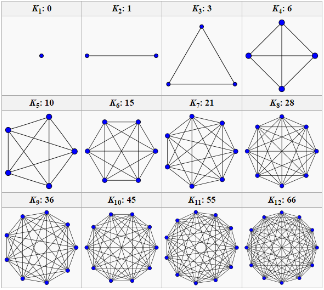
Common complexity classes



Working with big O notation

- Can ignore **leading constants**
  - If one algorithm takes n^2 operations, another takes $500n^2$ and a third takes $0.00000001n^2$, all three are $O(n^2)$
- Take only the **dominant term**
  - The term that is largest when n is large
  - If an algorithm takes $0.1n^3 + 300n^2 + 7000$ operations, it is $O(n^3)$
- Multiply **compound** algorithms
  - If an algorithm does n "things" and each "thing" is $O(n)$, then the overall algorithm is $O(n^2)$

Quadratic complexity



- Collision detection between n objects
- The naïve way: check **each pair** of objects to see whether they have collided
- This is **quadratic** or $O(n^2)$
- Doubling the number of objects would **quadruple** the time required!
- Cleverer methods exist that are more scalable
  - Further reading: spatial hashing, quadtrees, octrees, Verlet lists

Exponential complexity

- A prime number is a number that is divisible only by 1 and itself
- Given an n-bit number $m = pq$ that is a product of two primes p and q, find p and q.

for $p = 2, 3, \ldots, m$ **do**
    $q \leftarrow m/p$
    if q is an integer **then**
        return p, q
    end if
end for

- Since $m \leq 2^n - 1$, in the worst case this is $O(2^n)$
  - Actually even slower because division is not $O(1)$
- Adding 1 to n potentially **doubles** the running time!

Aside: a famous unanswered question in computing

- A problem is "in P" if it can be solved with an algorithm running in $O(n^k)$ time
- A problem is in NP if a potential solution can be checked in $O(n^k)$ time
  - Equivalently, it can be solved with an algorithm running in $O(n^k)$ time on an infinitely parallel machine
- Are there any problems in NP but not in P?

P versus NP

- If you can find a **mathematical proof** that either $P = NP$ or $P \neq NP$, there's a \$1 million prize...
- It is believed that $P \neq NP$, so large instances of *NP*-hard problems are not solvable in a feasible amount of time
  - Many types of cryptography are based on this assumption
  - Quantum computers are "infinitely parallel" in a sense so *can* solve some large *NP*-hard problems

Caveats

- Time complexity only tells us how an algorithm **scales** with the size of the input
  - If we know the input will always be **small**, time complexity is not so important
  - Linear search is quicker than binary search if you only ever have 3 elements
  - Naïve collision detection is fine if your game only ever has 4 objects on screen
  - Sometimes complexity in terms of other resources (e.g. space, bandwidth) are more important than time
- Software development is all about choosing **the right tool for the job**
  - If you need scalability, choose a scalable algorithm
  - Otherwise, choose simplicity

Summary

- Time complexity tells us how the running time of an algorithm **scales** with the size of the data it is given
- Choice of data structures and algorithms can have a large impact on the efficiency of your software
- ... but only if scalability is actually a factor