Week 8: 3D Geometry II
Part 1: Matrices in 3D

COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Extend** 2D matrix operations into three dimensions, including:
  - Multiplication
  - Transformations
  - Homogeneous coordinates
  - Inverse

Recap: Matrices as functions

Provided the functions are linear

- The elements of the matrix are the coefficients of the functions they represent
- e.g. $g\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} g_x(x,y) \\ g_y(x,y) \end{pmatrix} = \begin{pmatrix} 0x + 2y \\ x - y \end{pmatrix}$ would be written as

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

x coefficient of g_x

y coefficient of g_x

x coefficient of g_y

y coefficient of g_y

3D functions and matrices

- $m\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} m_1(x,y,z) \\ m_2(x,y,z) \\ m_3(x,y,z) \end{pmatrix} = \begin{pmatrix} m_{11}x + m_{12}y + m_{13}z \\ m_{21}x + m_{22}y + m_{23}z \\ m_{31}x + m_{32}y + m_{33}z \end{pmatrix}$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m_{11}x + m_{12}y + m_{13}z \\ m_{21}x + m_{22}y + m_{23}z \\ m_{31}x + m_{32}y + m_{33}z \end{pmatrix}$$

3×3 matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & \boxed{a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Element at i, j is the dot product of row i with column j

3×3 matrix multiplication – colour-coded

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Recap: 2D affine transformations

- Translation:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation:

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Scale:

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Shear:

$$\mathbf{H}_x = \begin{pmatrix} 1 & \lambda_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{H}_y = \begin{pmatrix} 1 & 0 & 0 \\ \lambda_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Homogeneous coordinates:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

3D homogeneous coordinates

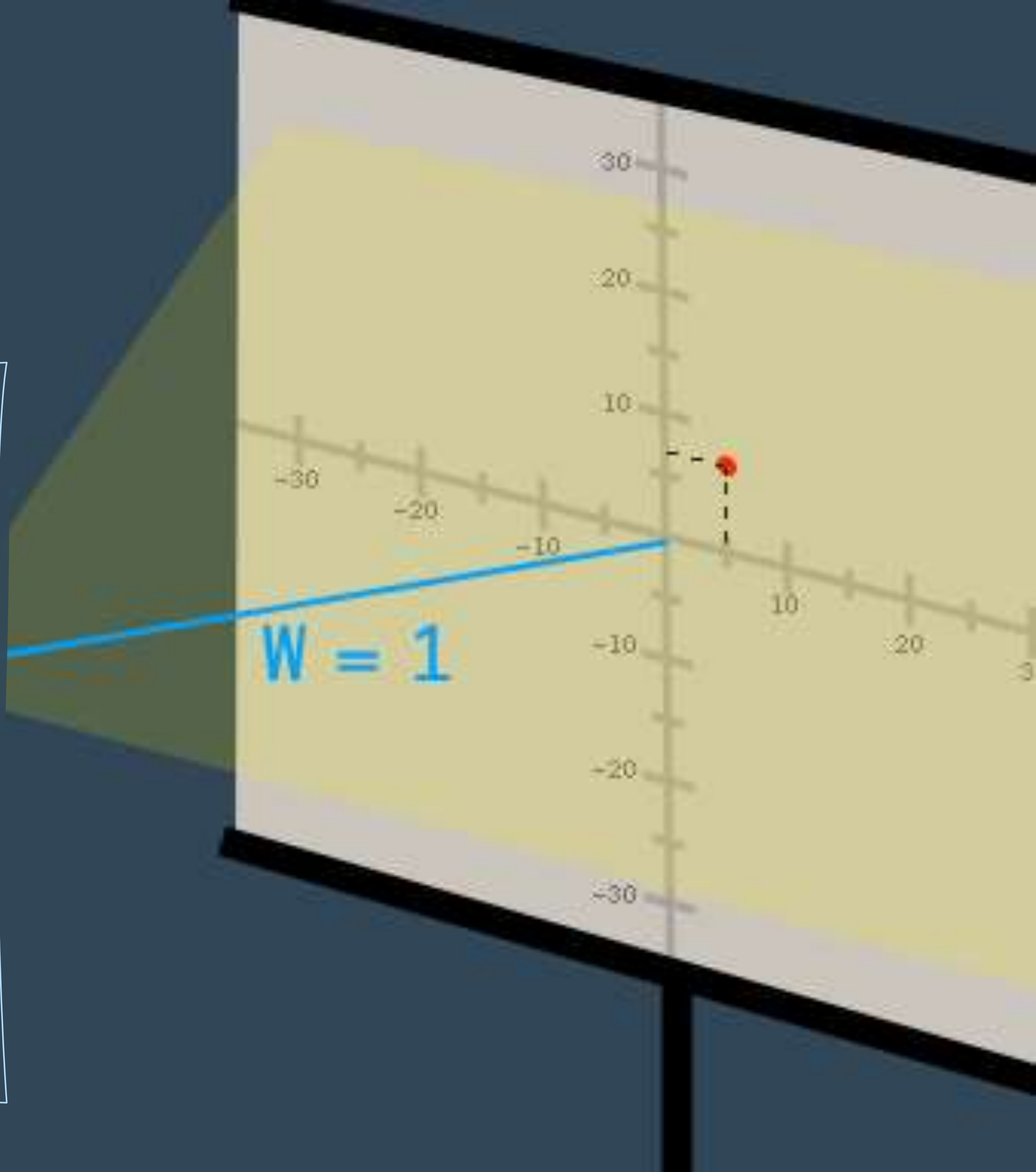
Applying a 4x4 homogeneous matrix to a point/vector:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} r_{11}x + r_{12}y + r_{13}z + \boxed{t_x w} \\ r_{21}x + r_{22}y + r_{23}z + \boxed{t_y w} \\ r_{31}x + r_{32}y + r_{33}z + \boxed{t_z w} \\ w \end{pmatrix}$$

- Note that only the w coordinate is affected by the 4th column (translation values)
  - For points (which have a position), $w = 1$
  - For vectors (which have only direction), $w = 0$

What is w?

- An “extra dimension” (not time!) added to allow translations…
- Extends 3D space to projective space
- A scaling factor/“distance to the projector”:
  - $(x, y, z, w) \rightarrow \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$
  - $w = 1$: direct mapping of a point to 3D space
  - $w = 0$: a point that is infinitely far away/a vector with infinite length
- More info:
  - https://www.tomdalling.com/blog/modern-opengl/explaining-homogenous-coordinates-and-projective-geometry/
  - https://hackernoon.com/programmers-guide-to-homogeneous-coordinates-73cbfd2bcc65



3D affine transformations

- Translation:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Scale:

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Shear:

$$\mathbf{H}_{xy} = \begin{pmatrix} 1 & 0 & \lambda_x & 0 \\ 0 & 1 & \lambda_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{H}_{xz} = \begin{pmatrix} 1 & \lambda_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{H}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_y & 1 & 0 & 0 \\ \lambda_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{H}_{ij}$ ‘shifts’ the coordinates along axes i and j by multiples of the other coordinate

3D rotation matrices

Anticlockwise rotation in a right-handed coordinate system about:

- The x-axis:

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The y-axis:

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The z-axis:

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying homogeneous matrices

“Rotation (and scale) part”

“Translation part”

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} & s_{13} & u_x \\ s_{21} & s_{22} & s_{23} & u_y \\ s_{31} & s_{32} & s_{33} & u_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}s_{11} + r_{12}\,s_{21} + r_{13}\,s_{31} & r_{11}s_{12} + r_{12}\,s_{22} + r_{13}\,s_{32} & r_{11}s_{13} + r_{12}s_{23} + r_{13}\,s_{33} & r_{11}u_x + r_{12}u_y + r_{13}u_z + t_x \\ r_{21}s_{11} + r_{22}\,s_{21} + r_{23}\,s_{31} & r_{21}s_{12} + r_{22}\,s_{22} + r_{23}\,s_{32} & r_{21}s_{13} + r_{22}\,s_{23} + r_{23}\,s_{33} & r_{21}u_x + r_{22}u_y + r_{23}u_z + t_y \\ r_{31}s_{11} + r_{32}\,s_{21} + r_{33}\,s_{31} & r_{31}s_{12} + r_{32}\,s_{22} + r_{33}\,s_{32} & r_{31}s_{13} + r_{32}\,s_{23} + r_{33}\,s_{33} & r_{31}u_x + r_{32}u_y + r_{33}u_z + t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3D transformation order

Rotation then translation:

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3D transformation order: reversed

Translation then rotation:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{11}t_x + r_{12}t_y + r_{13}t_z \\ r_{21} & r_{22} & r_{23} & r_{21}t_x + r_{22}t_y + r_{23}t_z \\ r_{31} & r_{32} & r_{33} & r_{31}t_x + r_{32}t_y + r_{33}t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Recap: matrix inverse

- **Definition**: for a square matrix $\mathbf{A}$, the **inverse** of $\mathbf{A}$ is a matrix $\mathbf{A}^{-1}$ such that
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$
- For 2×2 matrices, the inverse is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
- For larger matrices…
  - http://wwwf.imperial.ac.uk/metric/metric_public/matrices/inverses/inverses2.html
  - https://www.khanacademy.org/math/algebra-home/alg-matrices#alg-determinants-and-inverses-of-large-matrices
  - *3D Math Primer for Graphics and Game Development*, Chapter 6

Transformation matrix inverse: rotation

- The inverse of a rotation matrix is its transpose
  - Because: the opposite of rotating by θ is rotating by $-\theta$, e.g.

'Flipped' along the diagonal (convert columns to rows)

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_x{}^{-1}(\theta) = \mathbf{R}_x(-\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\theta) & -\sin(-\theta) & 0 \\ 0 & \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{R}_x{}^{T}(\theta)$$

Transformation matrix inverse: translation

- The inverse of a translation matrix is the same matrix with the signs on the translation components reversed
  - Because: the opposite of travelling t units in one direction is travelling t units in the opposite direction

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformation matrix inverse: scale

Reciprocal of x is $\frac{1}{x}$

- The inverse of a scale matrix is a scale matrix with the reciprocal scale factors
  - Because: the opposite of making something s times bigger is making is s times smaller

$$\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformation matrix inverse: combined

- The inverse of a matrix product is the product of the inverse matrices, ordered in reverse

$$(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$$

$$(\mathbf{AB})^{-1}\mathbf{A}\cancel{\mathbf{B}\mathbf{B}^{-1}} = \mathbf{I}\mathbf{B}^{-1}$$

$$(\mathbf{AB})^{-1}\mathbf{A} = \mathbf{B}^{-1}$$

$$(\mathbf{AB})^{-1}\cancel{\mathbf{A}\mathbf{A}^{-1}} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Transformation matrix inverse: example

Invert a combined rotation and translation:

$$\left(\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)^{-1} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{21} & r_{31} & r_{11}(-t_x) + r_{21}(-t_y) + r_{31}(-t_z) \\ r_{12} & r_{22} & r_{32} & r_{12}(-t_x) + r_{22}(-t_y) + r_{32}(-t_z) \\ r_{13} & r_{23} & r_{33} & r_{13}(-t_x) + r_{23}(-t_y) + r_{33}(-t_z) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$