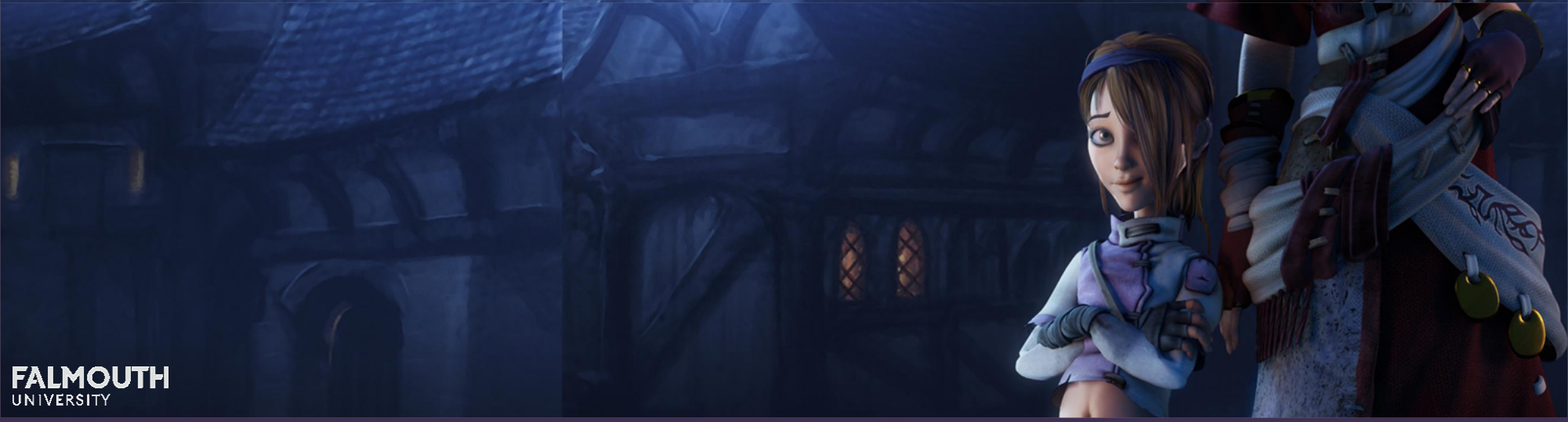

COMP110: Principles of Computing

11: Numerical Representations

Worksheets

- Worksheet 7: due **this Wednesday**
- Worksheet 8: due **next Wednesday**

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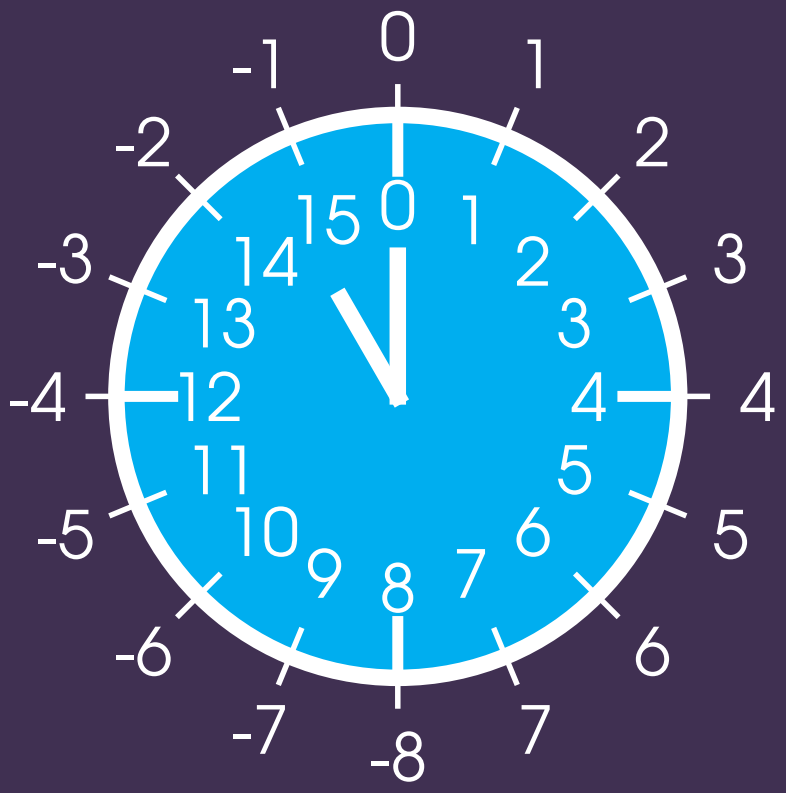
2’s Complement

Modular arithmetic



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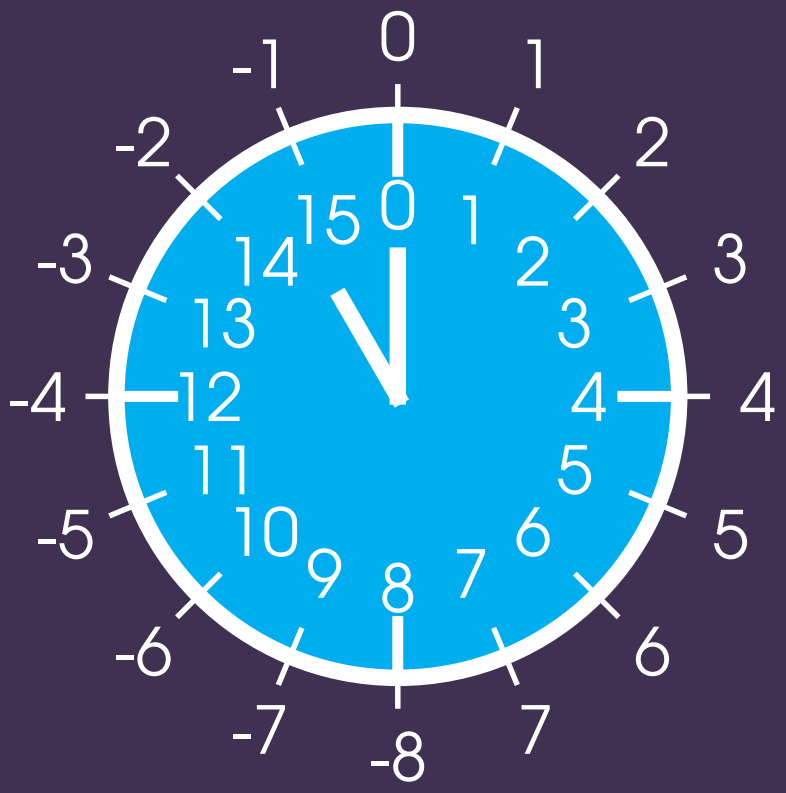

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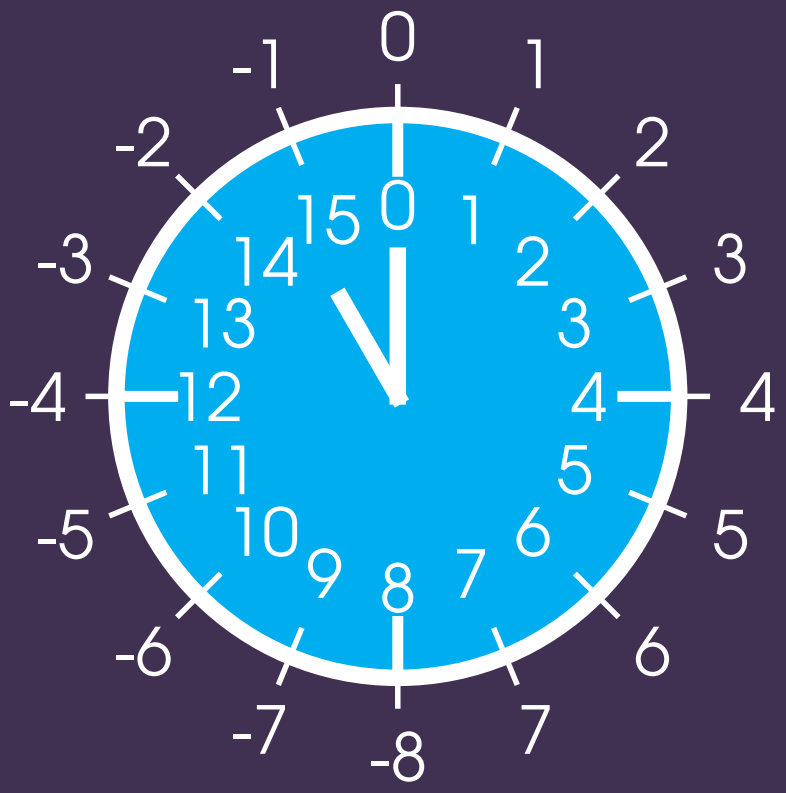

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- Numbers "wrap around" between 0 and $N - 1$

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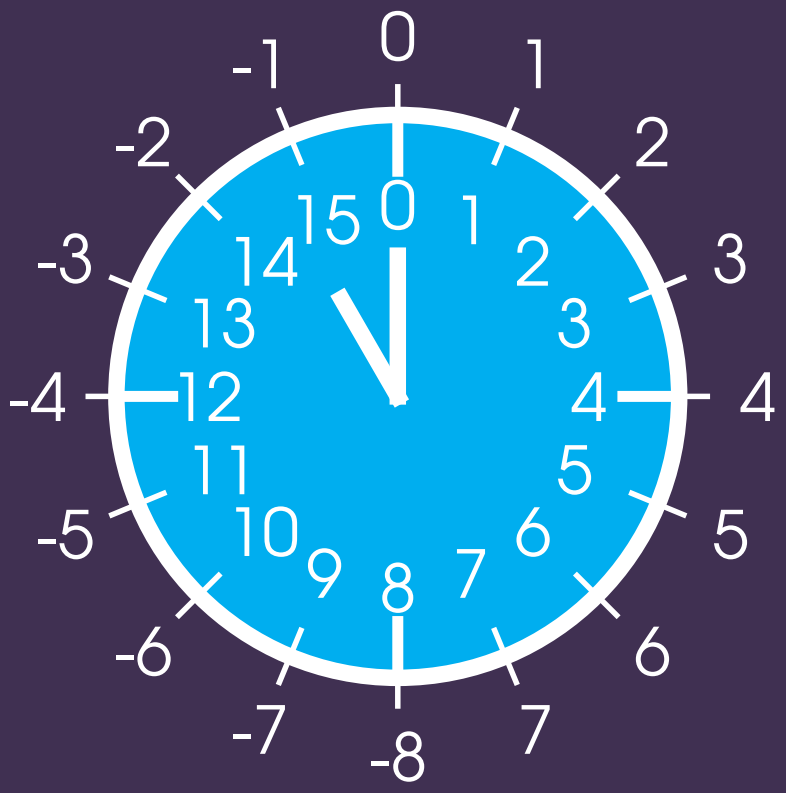

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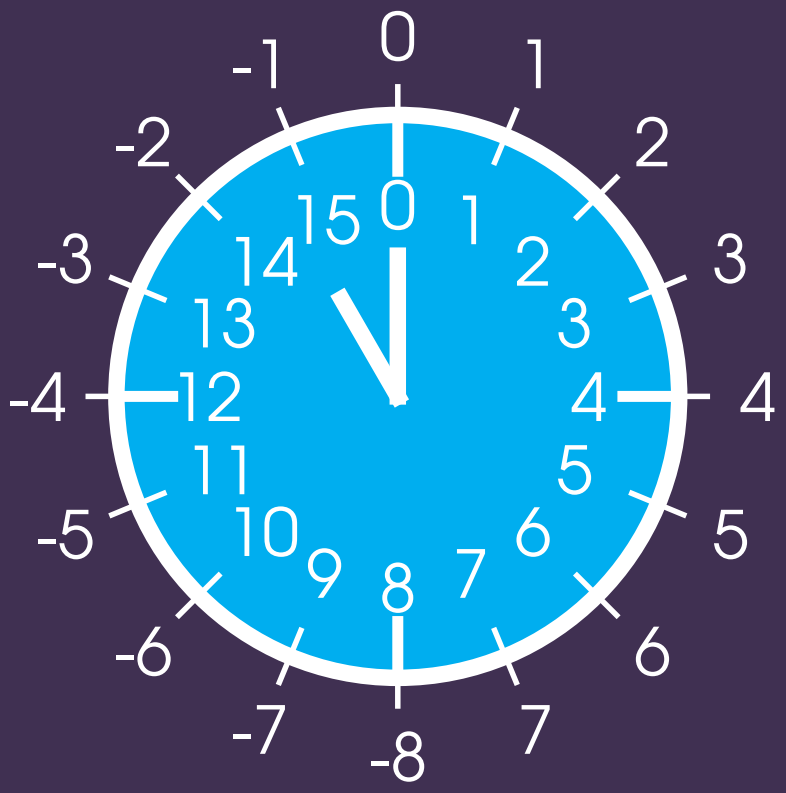

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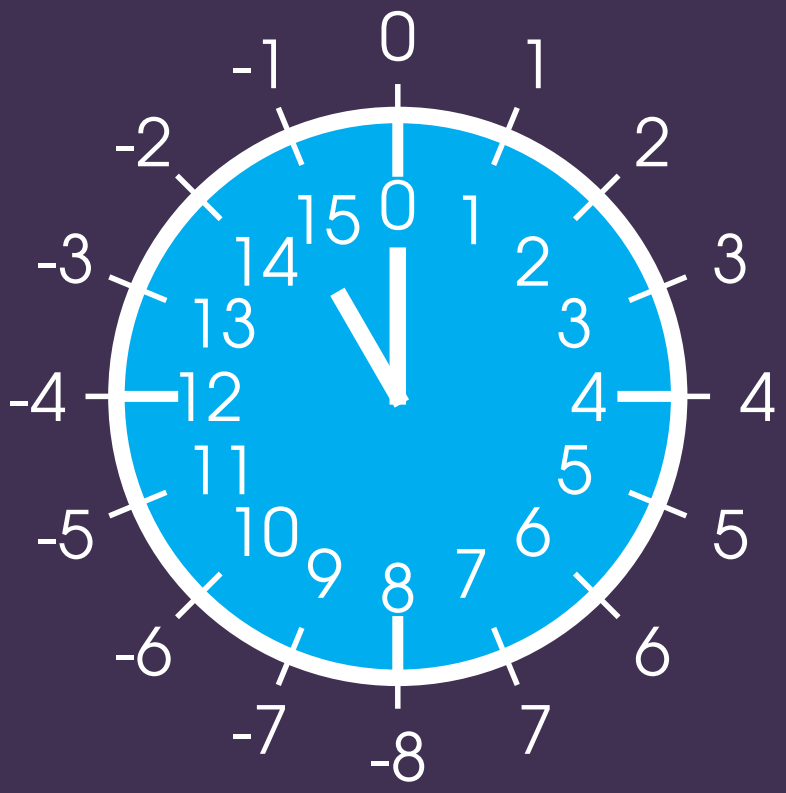

- Arithmetic **modulo** N
- Numbers "wrap around" between 0 and $N - 1$
- E.g. modulo 16:
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  - $4 - 7 = 13$

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- Useful for wrapping around e.g. loop indexes or screen coordinates

2's complement

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- E.g. 16-bit number ranges from -32768 to $+32767$
- Note that the left-most bit can be interpreted as a **sign** bit: 1 if negative, 0 if positive or zero

Converting to 2's complement

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- (This is equivalent to subtracting the number from 2^n... why?)
- This is also the process for converting back from 2's complement, i.e. doing it twice should give the original number

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- In fact, subtraction can just be done as addition
- I.e. $a - b$ is the same as $a + (-b)$, where a and $-b$ are just n-bit numbers

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Scientific notation

Integer powers

Let $a \neq 0$ be a real number, and let $b > 0$ be an integer

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$$123.45 \times 10^{-2} = 1.2345$$

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- Socrative `FALCOMPED`

Scientific notation in code

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Instead of writing $\times 10$, write e (no spaces)

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Instead of writing $\times 10$, write `e` (no spaces)

```
double lightYear = 9.461e15;
double plancksConstant = 6.626e-34;
```

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Floating point numbers

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- Sign is stored as a single bit: $0 = +$, $1 = -$
- We know there's a 1 before the point so no need to store it — just store the binary digits after the point

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- NB we don't use 2's complement to store it!
- Instead it is stored in binary as a positive integer with a **bias** added on
- (This is so that exponents can be efficiently compared (less/greater than) — 2's complement would be less efficient for this)

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An exponent of...	... is stored as
-126	00000001 (1)
⋮	⋮
-1	01111110 (126)
0	01111111 (127)
1	10000000 (128)
⋮	⋮
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- (Exponents of 00000000 and 11111111 have special meaning — more on this later)

IEEE 754 floating point formats

Type	Sign	Exponent	Mantissa	Total
Single precision	1 bit	8 bits bias 127	23 bits	32 bits
Double precision	1 bit	11 bits bias 1023	52 bits	64 bits
Extended precision	1 bit	15 bits bias 16383	64 bits	80 bits

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- Extended precision is not usually used in programs, but is used internally on Intel CPUs

Example

Example

What is the value stored in the following IEEE single-precision floating point number?

```
01000000110100000000000000000000
```

Example

Example

Example

$$\boxed{0}10000001101000000000000000000000$$

- Sign bit is 0

Example

$$\boxed{0}10000001101000000000000000000000$$

- Sign bit is 0
- Therefore the number is positive

Example

Example

0 10000001 10100000000000000000000

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- Exponent is 10000001
- This is 129 in binary
- Exponent is stored with a bias of 127, therefore the actual exponent is $129 - 127 = 2$

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`0 10000001 10100000000000000000000`

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`010000001` `10100000000000000000000`

- Mantissa is 101000...
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- The exponent is 2, so we move the point 2 places to the right: 110.1000...
- $4 + 2 + \frac{1}{2} = 6.5$

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- Can check for these with `float.IsInfinity`, `double.IsNaN`, etc.
- Infinities and NaNs sometimes arise from calculations (e.g. dividing by zero)

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Numerical precision

Precision of floating point numbers

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  - Around 3.142×10^6: can represent a difference of 1000

Range of floating point numbers

Type	Smallest value (closest to 0)	Largest value (furthest from 0)
Single precision	$\pm 1.175 \times 10^{-38}$	$\pm 3.403 \times 10^{38}$
Double precision	$\pm 2.225 \times 10^{-308}$	$\pm 1.798 \times 10^{308}$

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- Binary: can represent $\frac{a}{b}$ exactly iff $b = 2^n$
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- This can lead to **rounding errors** with some calculations

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- Better to check for **approximate equality**: calculate the difference between the numbers, and check that it's smaller than some threshold
- E.g. Unity has `Mathf.Approximately` which does exactly this

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- ... however not natively supported by the CPU, hence much slower than `float`/`double`