COMP220: Graphics & Simulation

9: Newtonian mechanics

Learning outcomes

- **Recall** the definitions of key concepts such as position, velocity, acceleration, force, friction and restitution
- **Solve** simple mathematical problems involving these key concepts
- **Write** programs which feature realistic physics simulations

Calculus

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- Many other contributions to mathematics and physics

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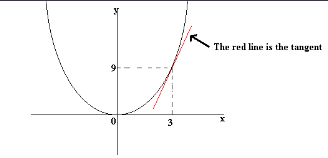

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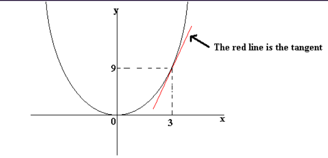

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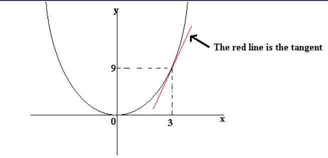

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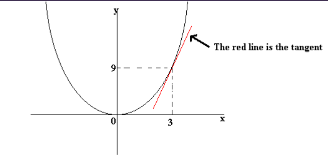

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- The mathematical process of finding $\dfrac{dx}{dt}$ given x is called **differentiation**

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  - If x is distance travelled and t is time, then we have

$$\frac{dx}{dt} = \frac{20}{0.5} = 40$$

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- We are interested in **numerical integration**
  - I.e. integration by computer calculation, not by mathematician with pen and paper...

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- So $h \times \frac{dx}{dt}$ is how much x changes by if t changes by h

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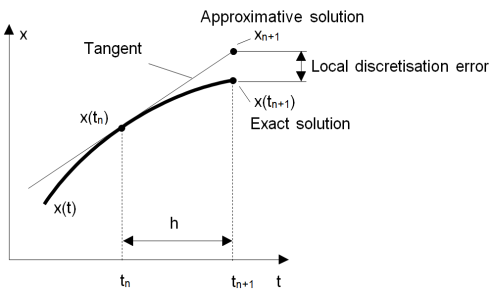

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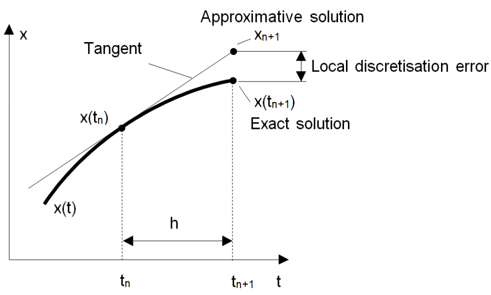

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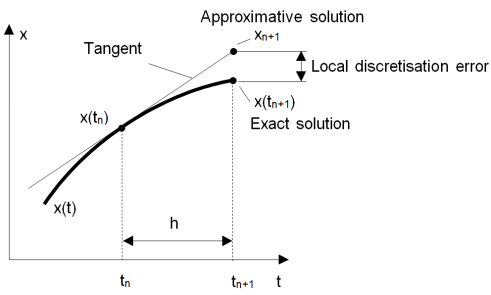

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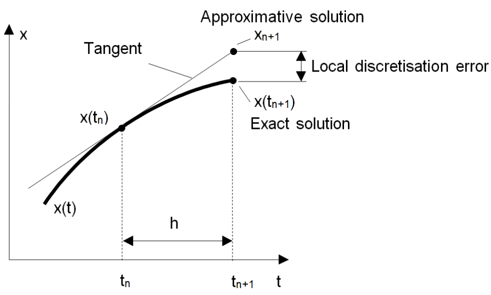

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- There are more advanced forms of numerical integration which give smaller errors

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- If x is an n-vector, then so is $\frac{dx}{dt}$
- Each component of $\frac{dx}{dt}$ is the rate of change of the corresponding component of x

Basic mechanics

Point masses

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- Not to be confused with **weight** (GCSE physics!)

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- E.g. applied forces: car engine, rocket engine, launched projectile, human muscle, ...

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  - Apply **numerical integration** again to determine the new velocity from the current acceleration

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- On Earth, $g \approx 9.81$ (often rounded to $g = 10$)

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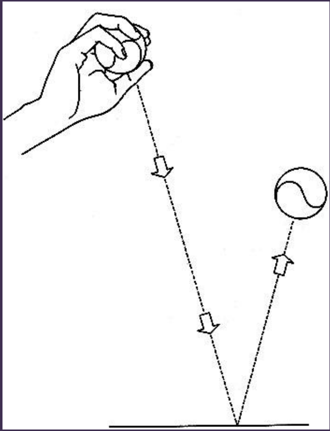
- Famous experiment: in a **vacuum** (no air resistance), a bowling ball falls at the same speed as a feather

Basic collision response



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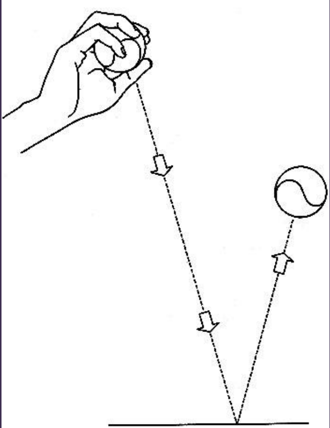

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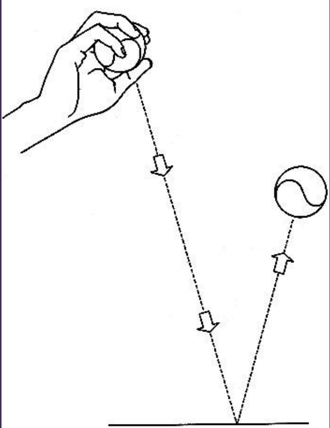

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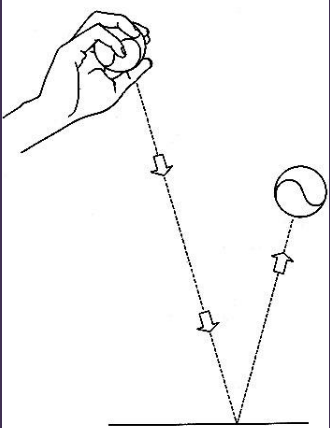

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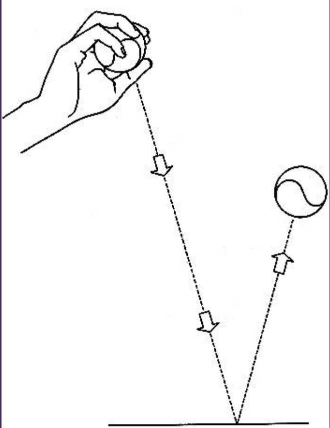

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- For an **inelastic collision**, some velocity is lost
- Flip the *y* component and multiply it by something between 0 and 1

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Sprint review