COMP270: Mathematics for 3D Worlds and Simulations

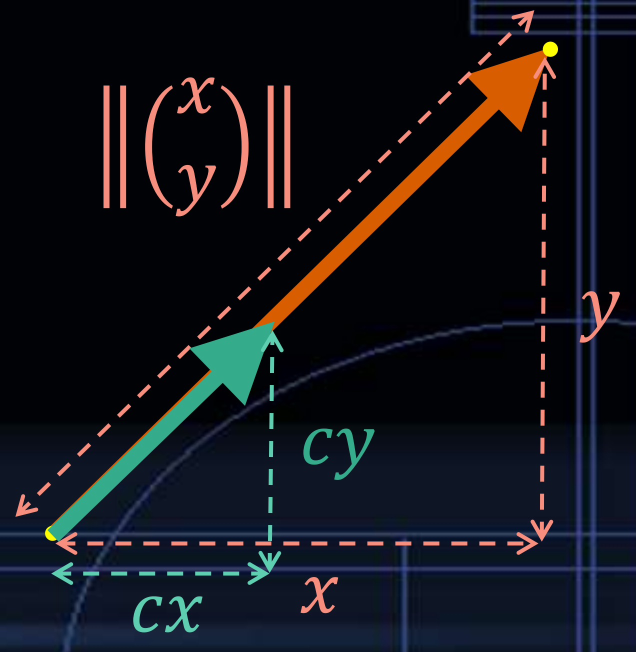
WEEK 3: GEOMETRY II
PART 1: MORE ON VECTORS

Objectives

- **Define** the dot product vector operator
- **Understand** its potential uses in graphics coding applications

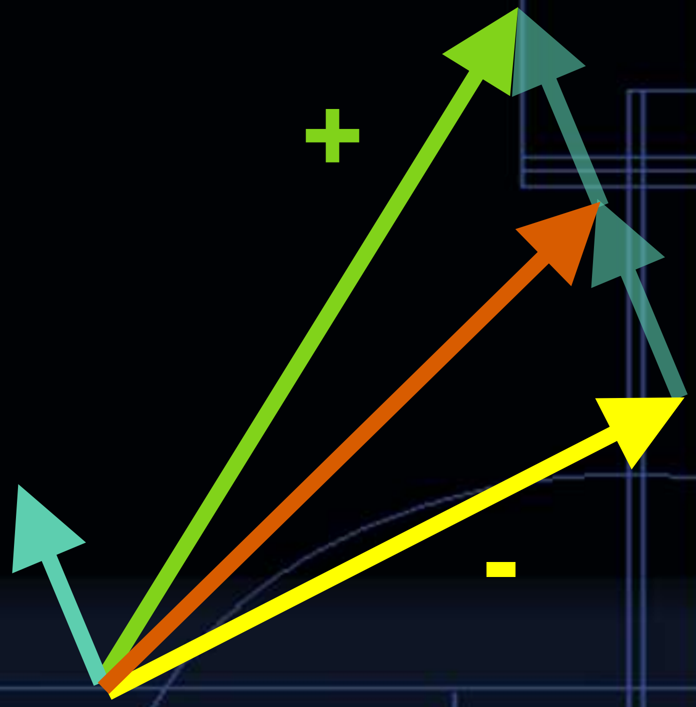
Recap: vector definition

- A **vector** is a directed line segment between 2 points
- Written in column form as $\mathbf{v} = \binom{x}{y}$
- Magnitude $\left\|\binom{x}{y}\right\| = \sqrt{x^2 + y^2}$
- Scalar multiplication $c\binom{x}{y} = \binom{cx}{cy}$



Recap: vector arithmetic

- For two vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$:
  - $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$
  - $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$



Dot product: algebraic definition

- **Definition**: For two vectors $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, the dot product is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 x_2 + y_1 y_2$$

- The result is a **scalar value**…
- The operation is commutative

Dot product and magnitude

- **Theorem:** for a vector $\mathbf{v}$, $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$
- **Proof:**
  - Let $\mathbf{v} = \binom{x}{y}$
  - Then $||\mathbf{v}||^2 = \sqrt{x^2 + y^2}^2 = x^2 + y^2$
  - Also, $\mathbf{v} \cdot \mathbf{v} = xx + yy = x^2 + y^2$
  - QED

Quod erat demonstradum:
“what was to be demonstrated”

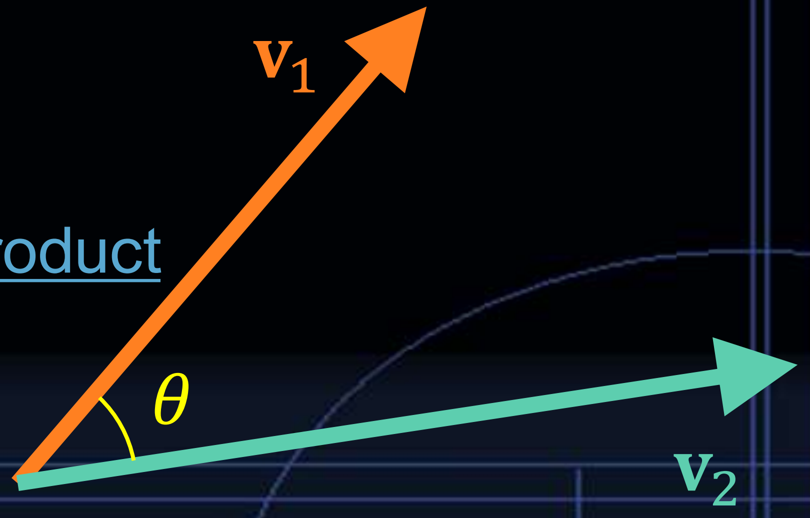
Magnitude and squared magnitude

- Finding the magnitude of a vector involves a **square root**: $\sqrt{x^2 + y^2}$
- Traditionally, calculating square roots (`sqrt`) was expensive
- Common advice: work with squared magnitudes where possible – i.e. calculate $x^2 + y^2$ without the square root
  - e.g. testing for length: don't test if $\|\mathbf{v}\| < r$, test if $\|\mathbf{v}\|^2 < r^2$
- The cost of square roots is negligible on modern hardware – computing `sqrt` is *probably* not the bottleneck in your code!

Dot product: geometric interpretation

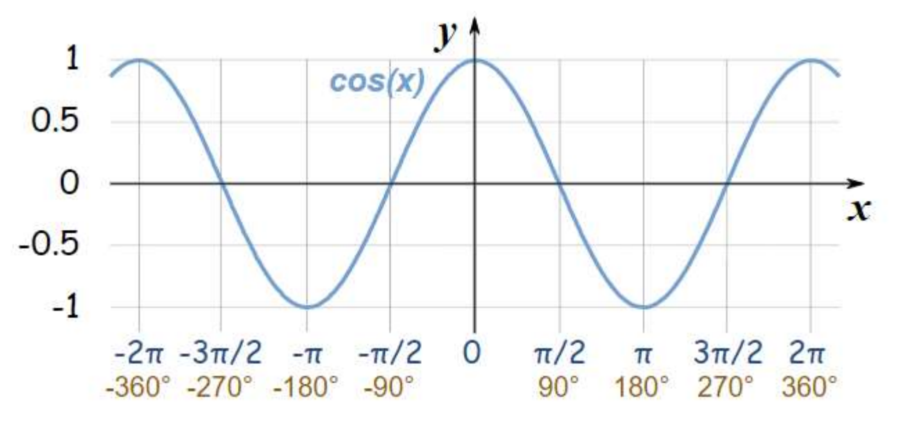
- **Theorem**: for vectors $\mathbf{a}$ and $\mathbf{b}$,
$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta$$
where θ is the angle between the two vectors.

- **Proof**: available at
proofwiki.org/wiki/Cosine_Formula_for_Dot_Product



Dot product and angles

- $\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$
- Can often test angles without doing the acos, e.g.
  - $\theta < \phi$ is equivalent to $\cos\theta > \cos\phi$ if θ and ϕ are between 0 and π radians
  - $\mathbf{v}_1 \cdot \mathbf{v}_2 > 0$ for $-90° < \theta < 90°$



- **Useful result**: $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if and only if

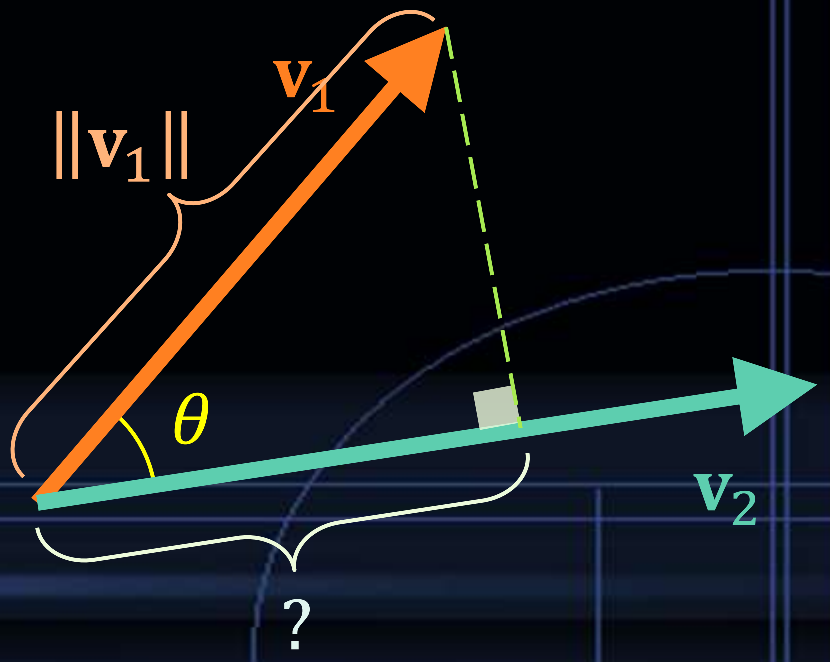
$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \; (= \cos 90°)$$

Vector projection

- Take two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ representing points on the plane
- Project a line from point $\mathbf{v}_1$ onto vector $\mathbf{v}_2$, such that it meets $\mathbf{v}_2$ at a right angle
- The **projection** of $\mathbf{v}_1$ onto $\mathbf{v}_2$ is the distance from the origin to the point where the line meets $\mathbf{v}_2$
- A measure of "how much" of $\mathbf{v}_1$ is pointing in the same direction as $\mathbf{v}_2$

Vector projection and the dot product

- The dot product definition gives $\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$
- From basic trigonometry, the projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$ is
$$\|\mathbf{v}_1\| \cos\theta$$
- Combining the formulae, this is
$$\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|}$$
- If $\mathbf{v}_2$ is a unit vector (so $\|\mathbf{v}_2\| = 1$) then the projection is just $\mathbf{v}_1 \cdot \mathbf{v}_2$



Unit vectors and normalisation

i.e. $||\hat{\mathbf{v}}|| = 1$

- **Theorem**: if $\mathbf{v}$ is a vector of any length, then $\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$ is a **unit vector**
- **Proof**:
  - Let $\mathbf{v} = \binom{x}{y}$
  - Then $\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{x^2+y^2}}\binom{x}{y} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$
  - $||\hat{\mathbf{v}}|| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = 1$
  - QED

Finding $\hat{\mathbf{v}}$ is known as **normalisation**; often performed by functions `normalize()` (in-place) and `normalized()` (returns $\hat{\mathbf{v}}$ keeping $\mathbf{v}$ intact).