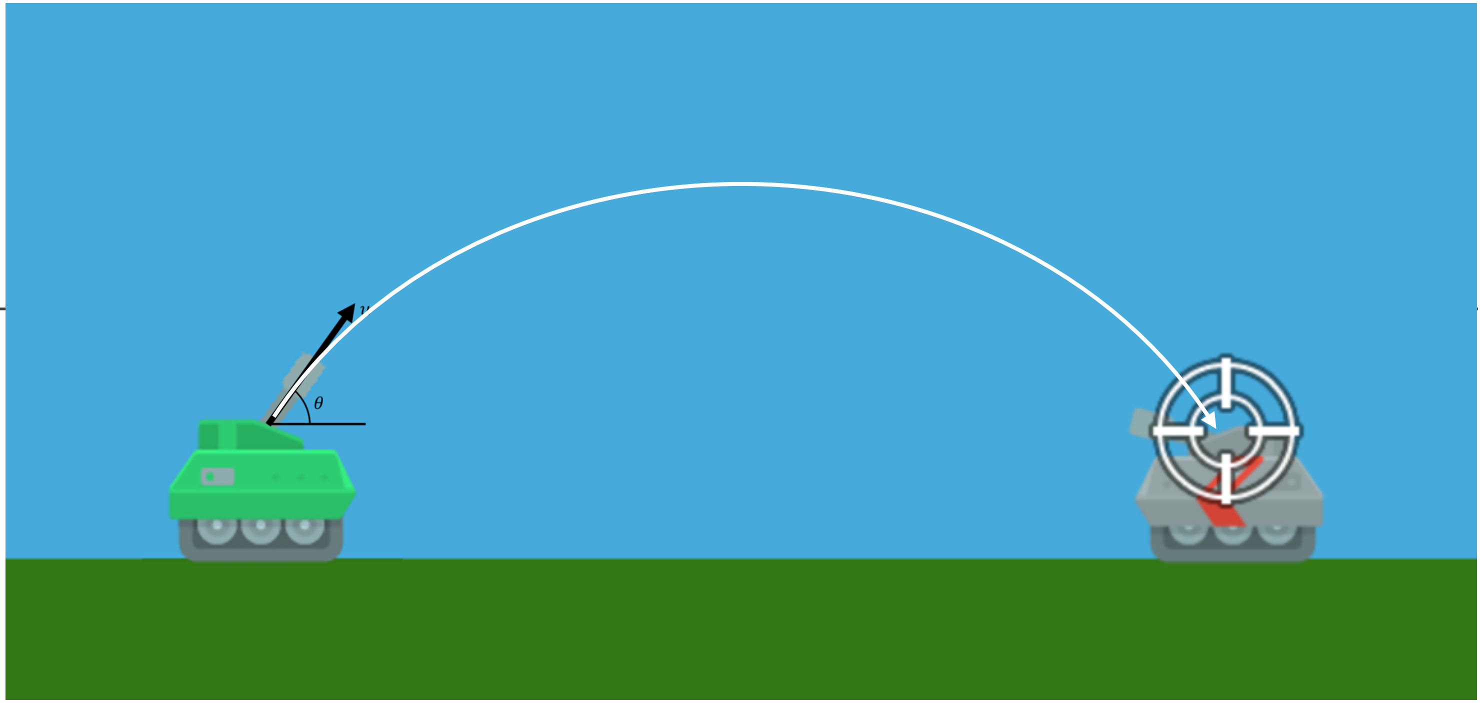
5: Newtonian Mechanics II

COMP270: MATHEMATICS FOR 3D WORLDS & SIMULATIONS

Worksheet B

Projectile motion

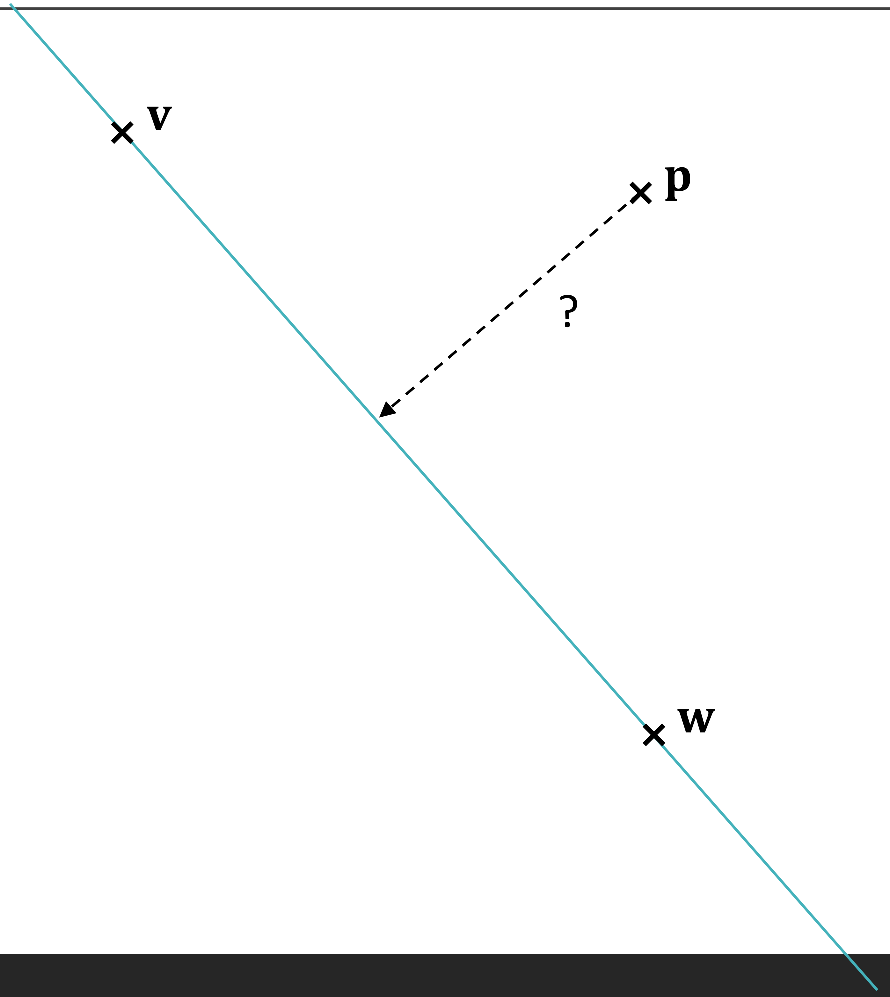


- The enemy tank is a distance of x units away, at the same elevation
- Given angle θ, what shot speed u is needed to hit the enemy tank?
- Answer: $u = \sqrt{\dfrac{xg}{\sin 2\theta}}$

Distance between a point and a line

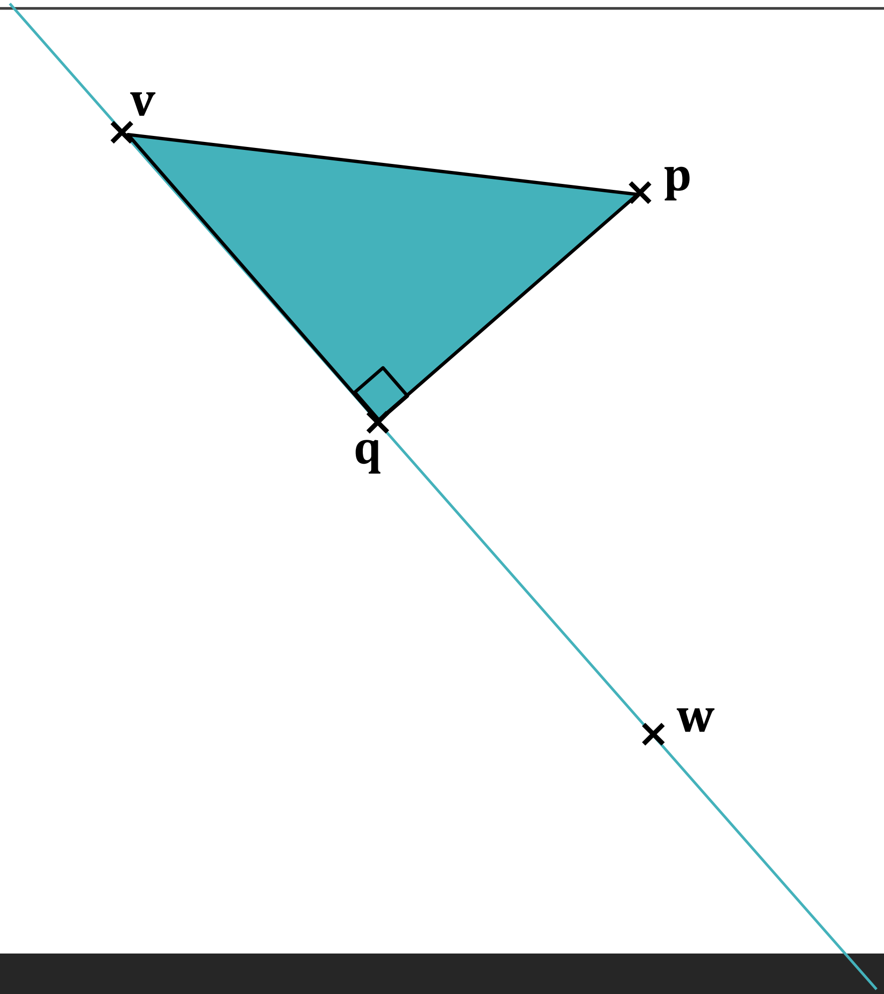
Distance between a point and a line

- Given a point $\mathbf{p}$ and an infinite line through $\mathbf{v}$ and $\mathbf{w}$
- What is the (shortest) distance between the point and the line?



Distance between a point and a line

- Let $\mathbf{q}$ be the point on the line that is closest to $\mathbf{p}$
- Then the line from $\mathbf{q}$ to $\mathbf{p}$ must be perpendicular to the line through $\mathbf{v}$ and $\mathbf{w}$
- Thus we have a right-angled triangle as shown



Distance between a point and a line

- Let θ be the angle shown, then by SOH CAH TOA:

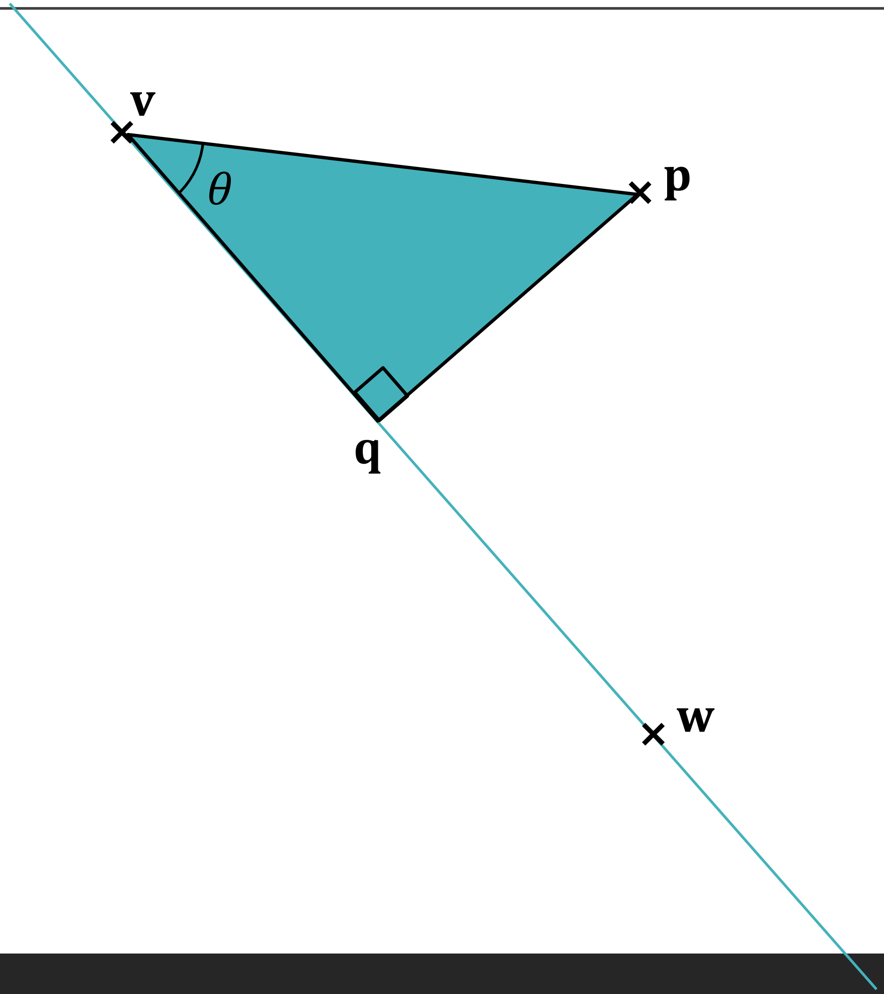
$$\cos\theta = \frac{\|\mathbf{q}-\mathbf{v}\|}{\|\mathbf{p}-\mathbf{v}\|}$$

- But also by dot product:

$$(\mathbf{p}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v}) = \|\mathbf{p}-\mathbf{v}\|\|\mathbf{w}-\mathbf{v}\|\cos\theta$$

- Substituting and rearranging gives

$$\|\mathbf{q}-\mathbf{v}\| = \frac{(\mathbf{p}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v})}{\|\mathbf{w}-\mathbf{v}\|}$$



Distance between a point and a line

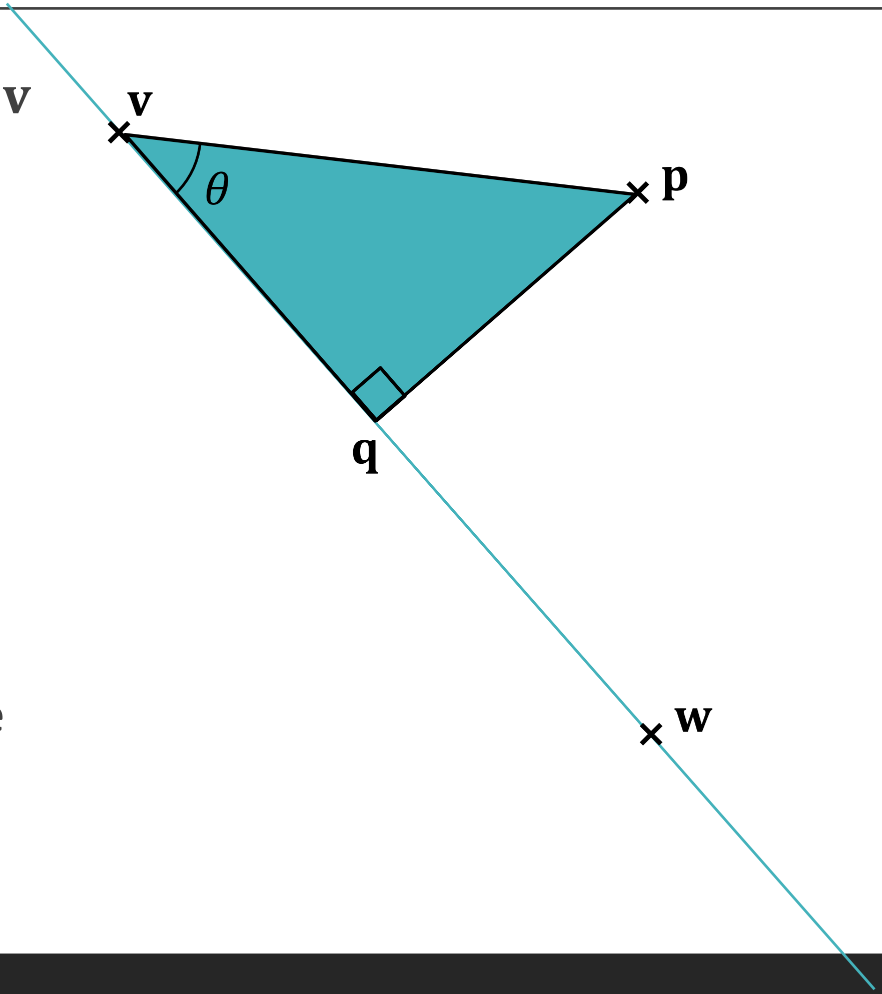
- Since $\mathbf{q}$ is on the line, we know that the vector $\mathbf{q}-\mathbf{v}$ is parallel to $\mathbf{w}-\mathbf{v}$
- In fact,

$$\mathbf{q}-\mathbf{v} = \|\mathbf{q}-\mathbf{v}\| \frac{\mathbf{w}-\mathbf{v}}{\|\mathbf{w}-\mathbf{v}\|}$$

- Therefore

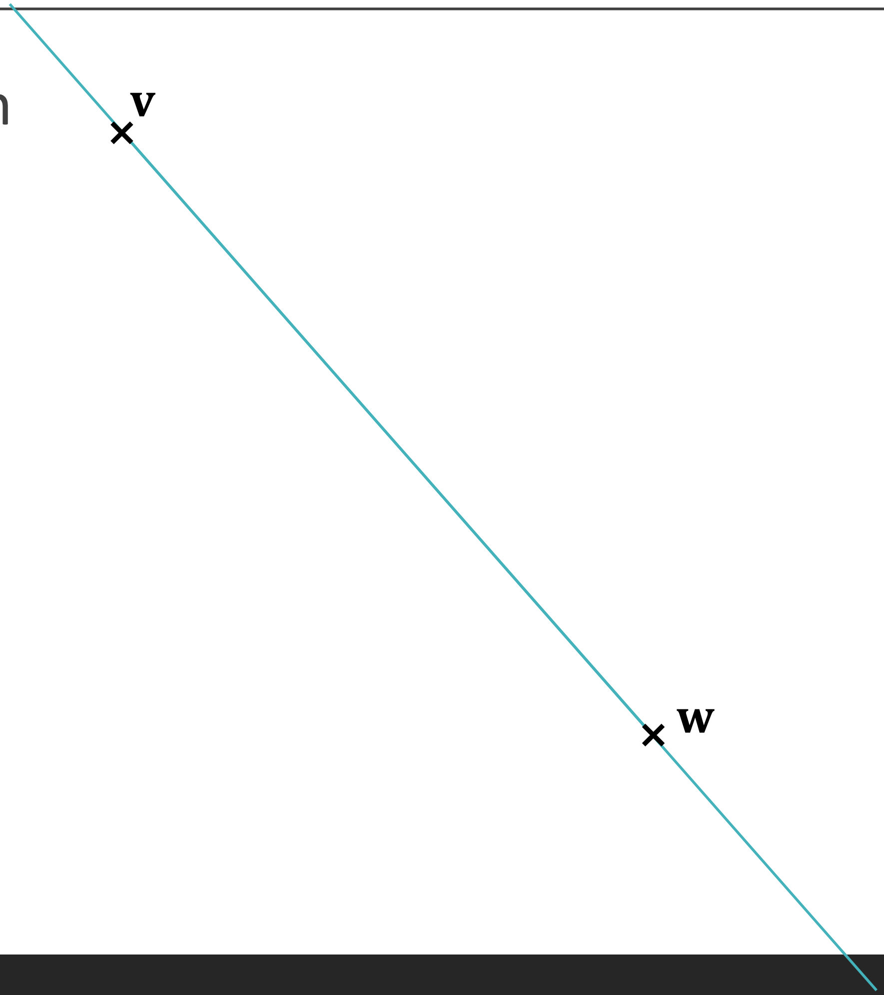
$$\mathbf{q} = \mathbf{v} + \frac{(\mathbf{p}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v})}{\|\mathbf{w}-\mathbf{v}\|^2}(\mathbf{w}-\mathbf{v})$$

- Knowing $\mathbf{q}$, the distance between the point and the line is simply $\|\mathbf{q}-\mathbf{p}\|$



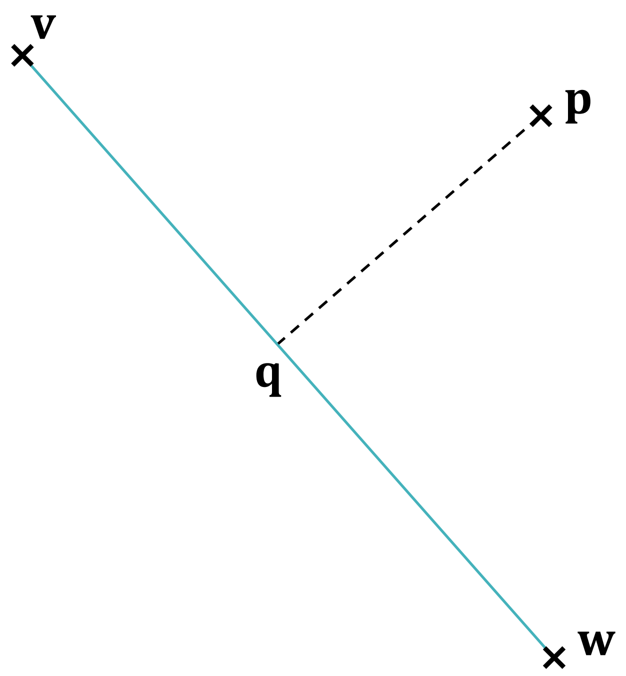
Parametric form of a line

- Any point on the line between $\mathbf{v}$ and $\mathbf{w}$ can be written as
$$\mathbf{v} + t(\mathbf{w} - \mathbf{v}) \ \text{ for some } t$$
- $0 \leq t \leq 1$ for points between $\mathbf{v}$ and $\mathbf{w}$
- Restricting $0 \leq t \leq 1$ gives a **line segment**



Distance between a point and a line segment

- Consider the point $\mathbf{q}$ we just found
- If $\mathbf{q}$ is between $\mathbf{v}$ and $\mathbf{w}$, then the shortest distance between $\mathbf{p}$ and the line segment is $||\mathbf{q}-\mathbf{p}||$
- If $\mathbf{q}$ is beyond $\mathbf{w}$, then the shortest distance is $||\mathbf{w}-\mathbf{p}||$
- If $\mathbf{q}$ is beyond $\mathbf{v}$, then the shortest distance is $||\mathbf{v}-\mathbf{p}||$



Distance between a point and a line segment

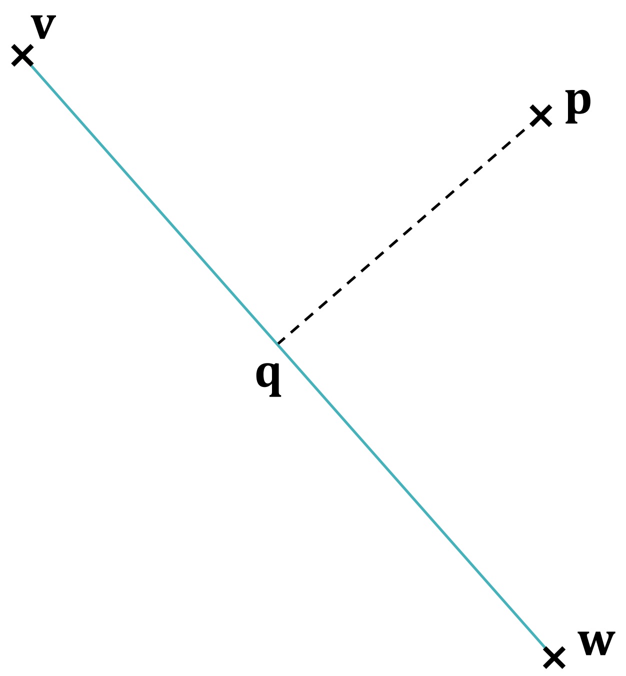
- We have

$$\mathbf{q} = \mathbf{v} + t(\mathbf{w} - \mathbf{v}) \;\; \textbf{where}\; t = \frac{(\mathbf{p} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v})}{\|\mathbf{w} - \mathbf{v}\|^2}$$

- The shortest distance is

$$\begin{cases} \|\mathbf{v} - \mathbf{p}\| & \text{if } t < 0 \\ \|\mathbf{q} - \mathbf{p}\| & \text{if } 0 \le t \le 1 \\ \|\mathbf{w} - \mathbf{p}\| & \text{if } t > 1 \end{cases}$$

- If we clamp $0 \le t \le 1$ then we can just use $\|\mathbf{q} - \mathbf{p}\|$ in all cases (since $t = 0$ gives $\mathbf{q} = \mathbf{v}$ and $t = 1$ gives $\mathbf{q} = \mathbf{w}$)



Clamping in code

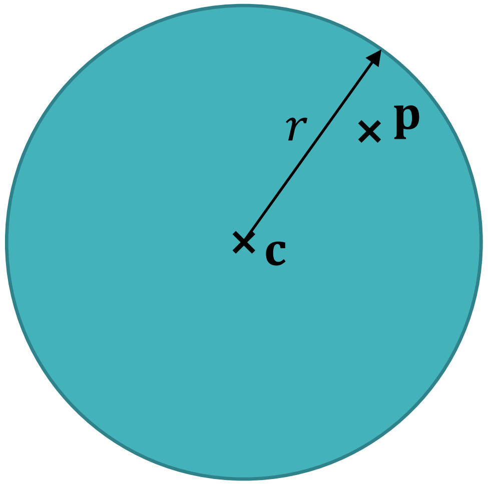
- To clamp a value x between a and b:
- `max(a, min(b, x))`

Collision detection

Point and circle collision

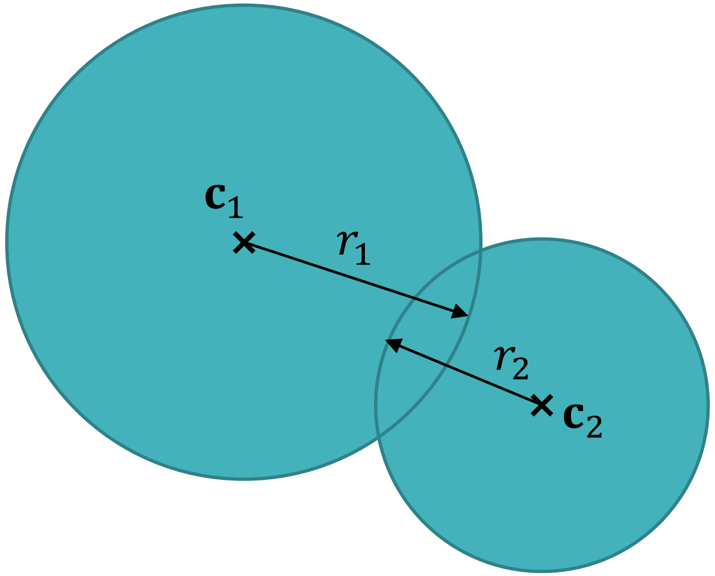
- Consider a circle with centre $\mathbf{c}$ and radius r
- A point $\mathbf{p}$ is inside the circle if and only if the distance between $\mathbf{p}$ and $\mathbf{c}$ is at most r:

$$||\mathbf{p} - \mathbf{c}|| \leq r$$



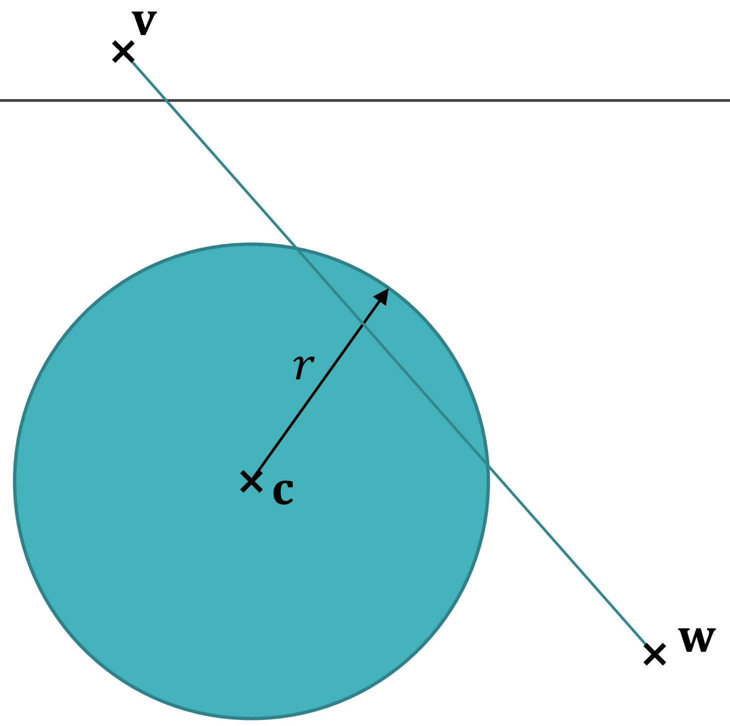
Circle and circle collision

- Consider two circles with centres $\mathbf{c}_1, \mathbf{c}_2$ and radii r_1, r_2
- The circles overlap (collide) if and only if
$$\|\mathbf{c}_1 - \mathbf{c}_2\| \leq r_1 + r_2$$



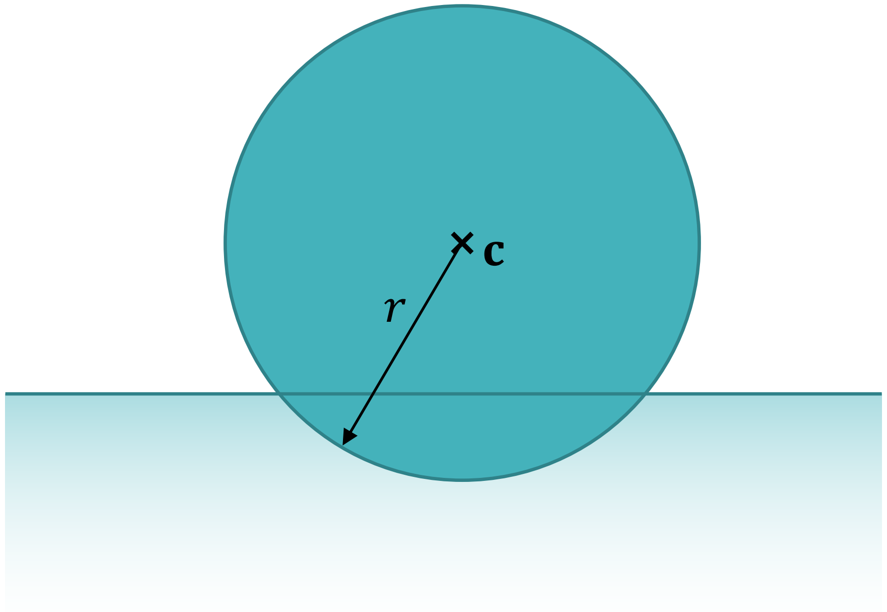
Circle and line collision

- Consider a circle with centre $\mathbf{c}$ and radius r, and a line segment through points $\mathbf{v}$ and $\mathbf{w}$
- The two collide if and only if the distance between $\mathbf{c}$ and the line is $\leq r$
- (Collisions with lines or line segments is the basis of **raycasting**)



Circle and ground collision

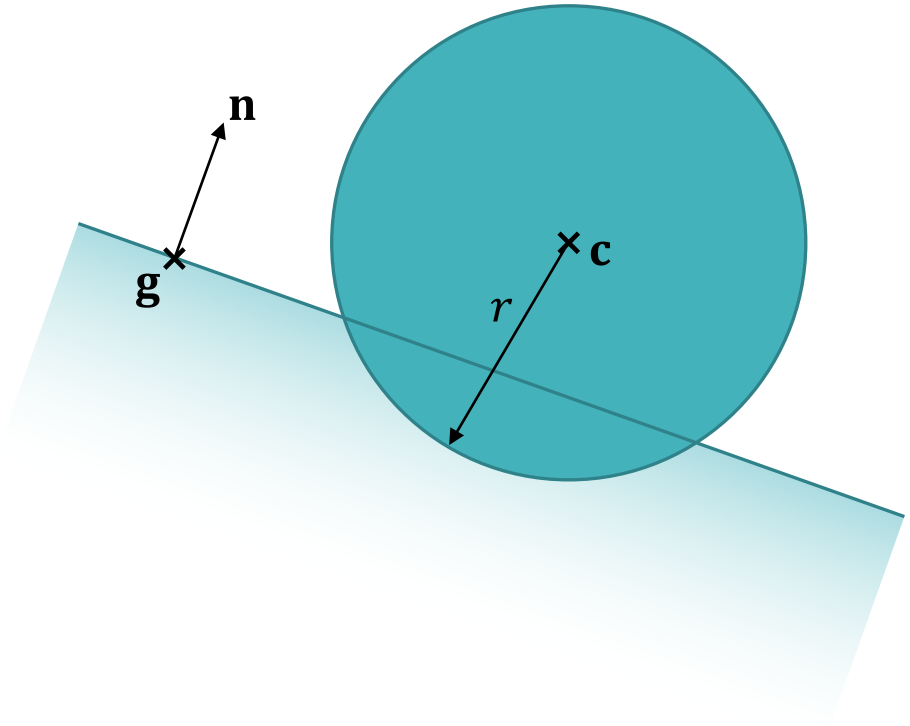
- Consider a circle with centre $\mathbf{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$ and radius r
- Let y_g be the y coordinate of the ground, and let the ground be horizontal
- The circle collides with the ground if and only if
$$c_y - y_g \leq r$$



Generalised circle and ground collision

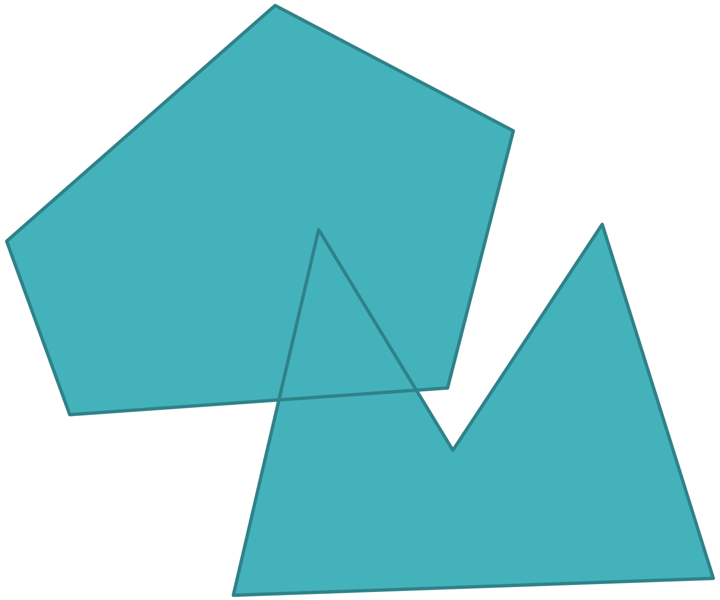
- Let $\mathbf{g}$ be any point on the ground plane, and let $\mathbf{n}$ be a **normal** vector (a unit vector perpendicular to the ground)
- The distance from $\mathbf{c}$ to the ground is $(\mathbf{c} - \mathbf{g}) \cdot \mathbf{n}$ (week 3 – projection)
- Therefore the circle collides with the ground if and only if

$$(\mathbf{c} - \mathbf{g}) \cdot \mathbf{n} \leq r$$



More complex shapes

- General collision detection is beyond the scope of this module
- Algorithms and libraries do exist



Collision response

Energy and momentum

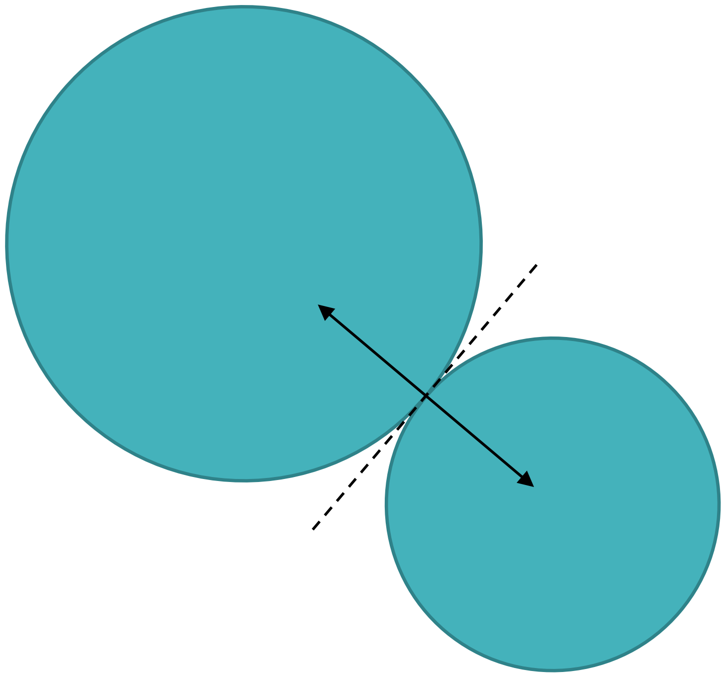
- A moving object has **momentum** proportional to its mass m and velocity $\mathbf{v}$
$$\mathbf{p} = m\mathbf{v}$$
- A moving object also has **kinetic energy** proportional to its mass and the square of its speed
$$E = \frac{1}{2}m\|\mathbf{v}\|^2$$

Conservation

- When two objects collide, the **total momentum** is conserved
- In an **elastic** collision, the **total kinetic energy** is also conserved
- In an **inelastic** collision, some kinetic energy is **lost** (e.g. as sound, heat etc)
- These can be used to calculate the velocities of the objects after collision

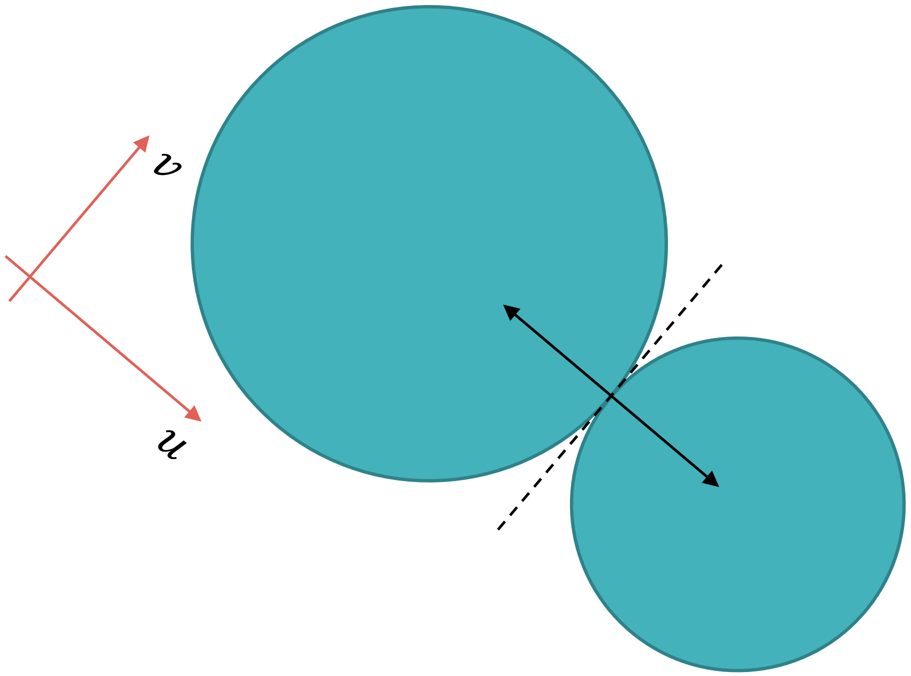
Contact normal

- Reaction force acts along the **contact normal –** perpendicular to both surfaces
- Component of velocity parallel to the normal changes; component perpendicular to the normal does not (disregarding friction)



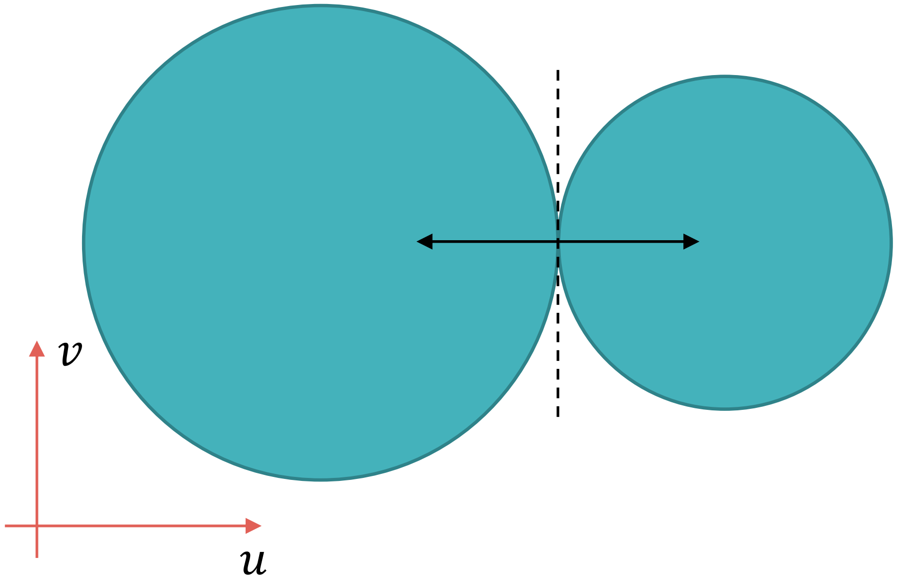
Change in coordinate systems

- Useful to consider the situation in a different **coordinate system**
- Instead of x-y axes, use u-v axes parallel and perpendicular to the normal
- Equivalent to **rotation**
- Can use **dot product** to convert between systems



Change in coordinate systems

- Under the new coordinates, reaction force acts along the u axis
- So can calculate collision response just by using u components of vectors
- This is a common trick – solving problems by **transforming** to a more convenient coordinate system



Worksheet B

Worksheet B

- Due next week!
- Worksheet review tutorials (with Joan) next Monday