COMP270: Mathematics for 3D Worlds and Simulations

WEEK 3: GEOMETRY II
PART 3: TYPES OF TRANSFORM

Objectives

- **Identify** the main types of affine transformation
- **Understand** the purpose of homogeneous coordinates and how to **apply** a transform using them

Recap: transformation functions as matrices

- A matrix can represent a transformation applied to a vector:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

  - e.g. $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x - y \end{pmatrix}$

- … Provided that the transformation function is a linear combination of the vector components

Affine transformations

- **Definition**: an affine transformation is any transformation that preserves collinearity and ratios of distances
  - i.e. Straight lines remain straight, and
  - Proportions are preserved
- Does not necessarily preserve angles or lengths
- Includes expansion, dilation, reflection…
- Most are compositions of rotation, translation, scale and shear

Scale matrix

- Multiplying a vector by a scalar s has the effect of scaling about the origin
- Represented by a matrix, this is:

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

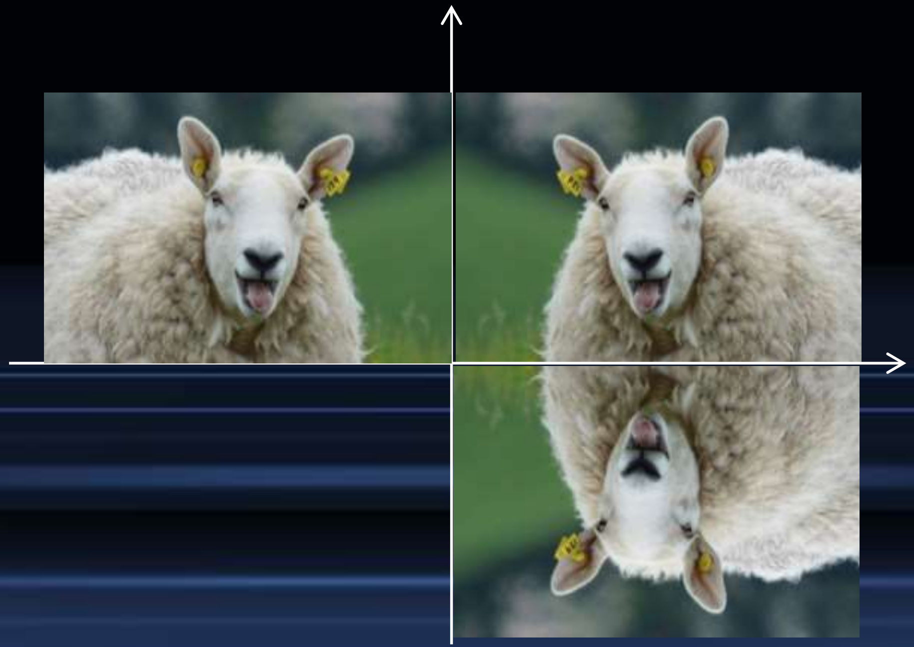
- More generally, can represent a scaling by a factor of s_x horizontally and s_y vertically by the matrix

$$\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

Reflection matrix

- The following matrices represent horizontal and vertical reflections respectively:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Shear matrix

- A shear transformation by a factor of λ parallel to the x-axis is given by the matrix

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \lambda y \\ y \end{pmatrix}$$

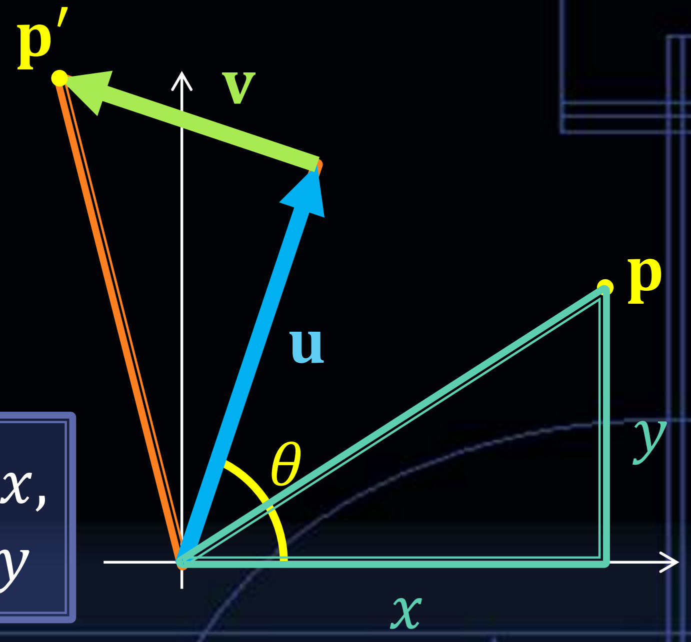
Rotation around the origin

- Consider a point represented by the vector $\mathbf{p} = \binom{x}{y}$
- Rotate p by an angle of θ anticlockwise around the origin
  - by taking this triangle…
  - …and rotating it by θ
- Define vectors along the rotated triangle's sides:

  $\mathbf{u} = \begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -y\sin\theta \\ y\cos\theta \end{pmatrix}$

  $\|\mathbf{u}\| = x,$
  $\|\mathbf{v}\| = y$

- So $\mathbf{p}' = \mathbf{u} + \mathbf{v} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$



Rotation matrix

- $r_\theta\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$
- $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- Some useful rotations:
  - $\mathbf{R}_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\mathbf{R}_{\frac{\pi}{2}} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
  - $\mathbf{R}_\pi = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

The identity matrix: doesn't change the vector.

A uniform scale by -1, or a reflection in both axes.

Translation matrix?

- $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$
- For any matrix $\mathbf{A}$, we have
$$\mathbf{A} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
- i.e. any transformation that can be represented by a matrix must keep the origin fixed
- Therefore translation (i.e. shifting all points by a constant vector) cannot be represented as a matrix
- Neither can rotating / scaling / shearing / reflecting around a point other than the origin…

Homogeneous coordinates

- **Definition:** homogeneous coordinates (x_1, x_2, x_3) of a finite point (x, y) in the plane are three numbers for which $\frac{x_1}{x_3} = x$ and $\frac{x_2}{x_3} = y$
  - i.e. represent points in $\mathbb{R}^2$ by vectors in $\mathbb{R}^3$
- The third component is usually 1 if the vector represents a point – so $\begin{pmatrix} x \\ y \end{pmatrix}$ becomes $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
- Often written (x, y, w)

Homogeneous matrices

- Translation by $\begin{pmatrix} t_x \\ t_y \end{pmatrix}$ is represented by the matrix
$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$
- Multiplying by this matrix is the same as adding $\begin{pmatrix} t_x \\ t_y \\ 0 \end{pmatrix}$:
$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$
- Existing transformation matrices stay similar: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ becomes
$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$