FALMOUTH UNIVERSITY

Workshop: 2D Matrices

COMP270: Mathematics for 3D Worlds and Simulation
BSc (Hons) Computing for Games

A few formulae…

Matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Rotation by θ : $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$

Scale by s_x, s_y: $\begin{pmatrix} s_x & 0 \\ 0 & s_x \end{pmatrix}$

Horizontal/vertical reflection: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

FALMOUTH UNIVERSITY

Answers: Question 1

a. $\begin{pmatrix}1 & -2\\ 5 & 0\end{pmatrix}\begin{pmatrix}-3 & 7\\ 4 & \frac{1}{3}\end{pmatrix} = \begin{pmatrix}1\times(-3)+(-2)\times 4 & 1\times 7+(-2)\times\frac{1}{3}\\ 5\times(-3)+0\times 4 & 5\times 7+0\times\frac{1}{3}\end{pmatrix} = \begin{pmatrix}-11 & 6\frac{1}{3}\\ -15 & 35\end{pmatrix}$

b. $\begin{pmatrix}6 & -7\\ -4 & 5\end{pmatrix}\begin{pmatrix}3\\ 3\end{pmatrix} = \begin{pmatrix}6\times 3+(-7)\times 3\\ -4\times 3+5\times 3\end{pmatrix} = \begin{pmatrix}-3\\ 3\end{pmatrix}$

c. $\begin{pmatrix}-3 & -2\\ 5 & 4\end{pmatrix}\begin{pmatrix}-2 & -1\\ 2\frac{1}{2} & 1\frac{1}{2}\end{pmatrix} = \begin{pmatrix}(-3)\times(-2)+(-2)\times(\frac{5}{2}) & (-3)\times(-1)+(-2)\times\frac{3}{2}\\ 5\times(-2)+4\times(\frac{5}{2}) & 5\times(-1)+4\times\frac{3}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

d. $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1\times a+0\times c & 1\times b+0\times d\\ 0\times a+1\times c & 0\times b+1\times d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

e. $\begin{pmatrix}3 & 3\end{pmatrix}\begin{pmatrix}6 & -7\\ -4 & 5\end{pmatrix} = \begin{pmatrix}3\times 6+3\times(-4) & 3\times(-7)+3\times 5\end{pmatrix} = \begin{pmatrix}6 & -6\end{pmatrix}$

(b) vs. (e):

- Values are the same, answers are different: order matters!
- Vector dimensions determine multiplication order

FALMOUTH UNIVERSITY

Answers: Question 2

a. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\-1\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$
The matrix describes a 90° clockwise rotation.

b. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}$
The matrix describes a 45° *anti*clockwise rotation.

c. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}2\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\2\end{pmatrix}$
The matrix describes a uniform scale of 2 units in both directions.

d. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}4\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\7\end{pmatrix}$
The matrix describes a *non*uniform scale of 4 units in the x direction and 7 in y.

e. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-1\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$
The matrix describes a reflection across the y-axis.

f. $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\-2\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}2\\0\end{pmatrix}$
The matrix describes a combination of the transformations from (a) and (c):
$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix}$

Answers: Question 3

a. 3: a reflection across the x-axis
(Compare to the matrix in 2a)

b. 1: a uniform scale by 2.5
(Compare to 2c)

c. 4: a combination of a 45° anticlockwise rotation *and* a reflection across the y-axis.
(Combine 2b with 2e… Question: which transformation is being applied first?)

d. 2: a non-uniform scale of 1.5 in the x direction and 2.0 in y.

FALMOUTH UNIVERSITY

Answers: Question 4

a. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \frac{1}{0\times0-1\times(-1)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^{-1} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

c. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} = \frac{1}{2\times2-0\times0}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

d. $\begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}^{-1} = \frac{1}{4\times7-0\times0}\begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$

e. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{(-1)\times1-0\times0}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

f. $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}^{-1} = \frac{1}{0\times0-(-2)\times2}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

Answers: Question 4 cont.

a. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a 90° anticlockwise rotation.

b. $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ is a 45° clockwise rotation.

c. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is a uniform scaling by 0.5 units in each direction.

d. $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$ is a nonuniform scale of $^1/_4$ units in the x direction and $^1/_7$ in y.

e. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a a reflection across the y-axis.

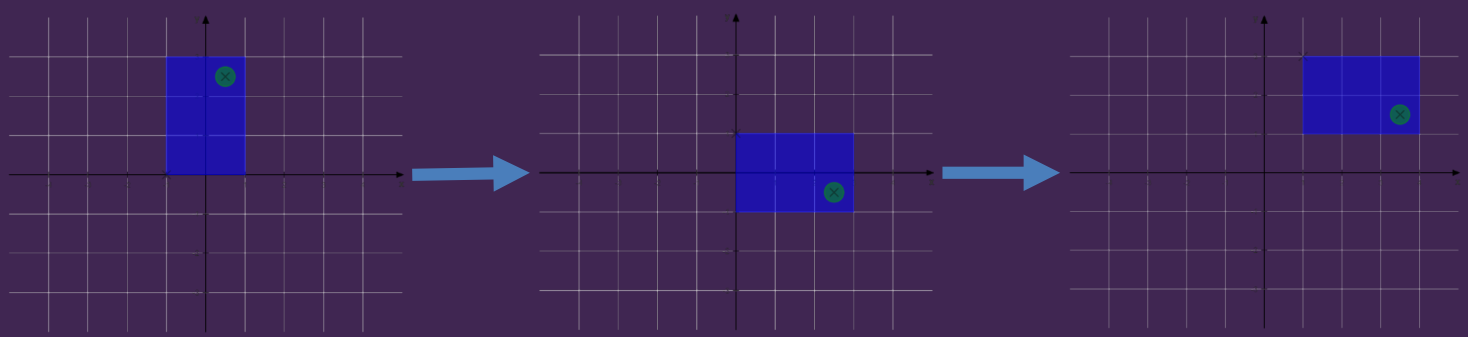
f. $\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ is a combination of (a) and (c) – again!

Answers: Question 5

a. $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times (-1) + 0 \times 3 + 1 \times 1 \\ 0 \times (-1) + 1 \times 3 + 2 \times 1 \\ 0 \times (-1) + 0 \times 3 + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$

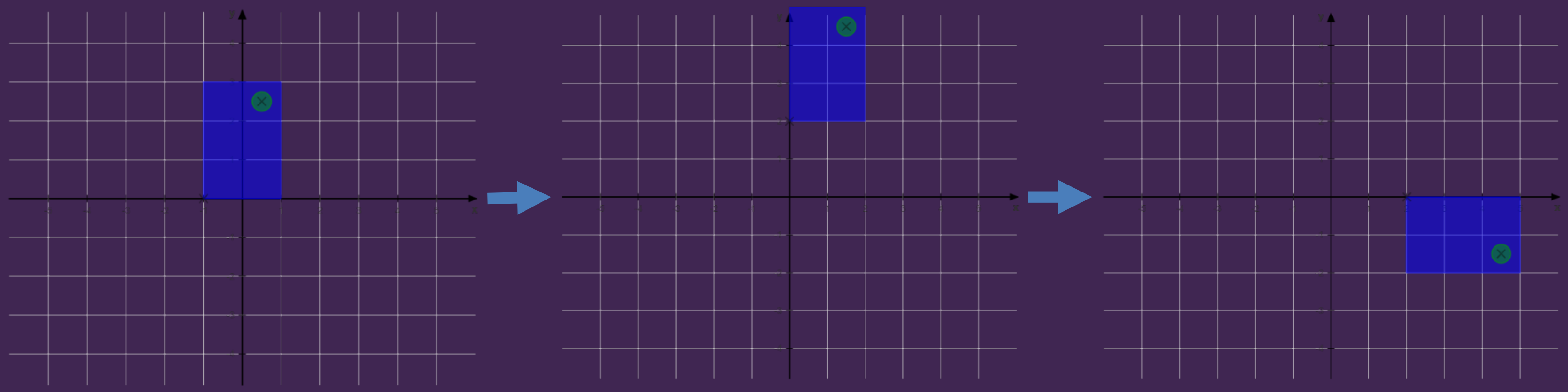
c. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$



Answers: Question 5 cont.

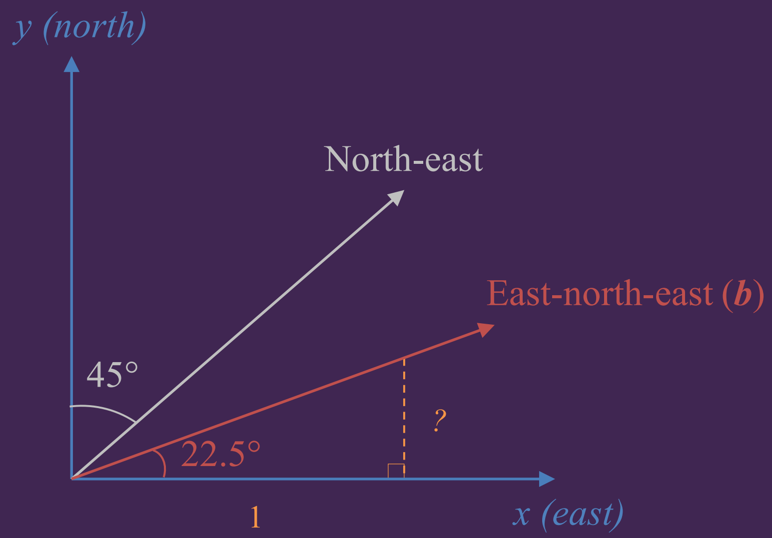
Applied in the opposite order,

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$



Answers: Question 6

a.



$$tan(22.5°) = \sqrt{2} - 1 = \frac{?}{1}$$

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}$$

$$\|\boldsymbol{b}\| = \sqrt{1^2 + \left(\sqrt{2} - 1\right)^2}$$

$$= \sqrt{1 + 2 - 2\sqrt{2} + 1}$$

$$= \sqrt{4 - 2\sqrt{2}} = 2\sqrt{1 - \frac{\sqrt{2}}{2}}$$

$$\hat{\boldsymbol{b}} = \frac{1}{2\sqrt{1 - \frac{\sqrt{2}}{2}}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}$$

Answers: Question 6 cont.

a. $\hat{\boldsymbol{b}} = \frac{1}{2\sqrt{1-\frac{\sqrt{2}}{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$

b. $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

c. $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \frac{1}{2\sqrt{1-\frac{\sqrt{2}}{2}}}\left(2 \times 1 + (-1) \times \left(\sqrt{2} - 1\right)\right) = \frac{1}{2\sqrt{1-\frac{\sqrt{2}}{2}}}\left(3 - \sqrt{2}\right) \approx 1.47$