COMP270 Week 5 Exercise Sheet: Newtonian Mechanics – **Answers**

Here are the equations we need to use; the equations of motion (aka 'suvat equations') describe the relationship between the displacement s from the starting point, the initial velocity $\boldsymbol{u}$, the final velocity $\boldsymbol{v}$, the acceleration $\boldsymbol{a}$ and the time t, while the quadratic formula allows us to find solutions when the required variable appears as both 'itself' and 'itself squared' in an equation.

Equations of motion (for constant acceleration)

1. $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$
2. $\boldsymbol{s} = \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}t^2$
3. $\boldsymbol{s} = \frac{1}{2}(\boldsymbol{u} + \boldsymbol{v})t$
4. $\|\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2\boldsymbol{a} \cdot \boldsymbol{s}$
5. $\boldsymbol{s} = \boldsymbol{v}t - \frac{1}{2}\boldsymbol{a}t^2$

Quadratic formula

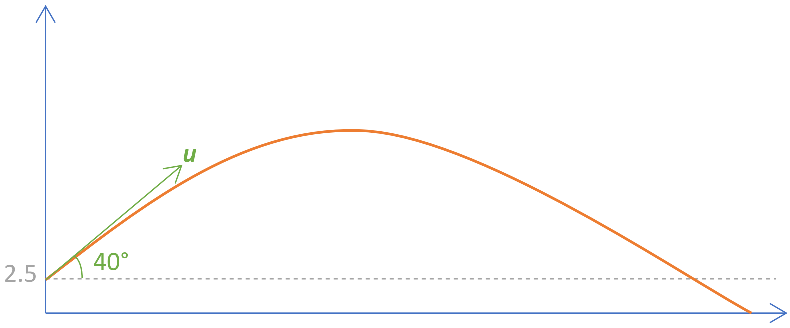
$ax^2 + bx + c = 0$ (where a, b and c are known constants)

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

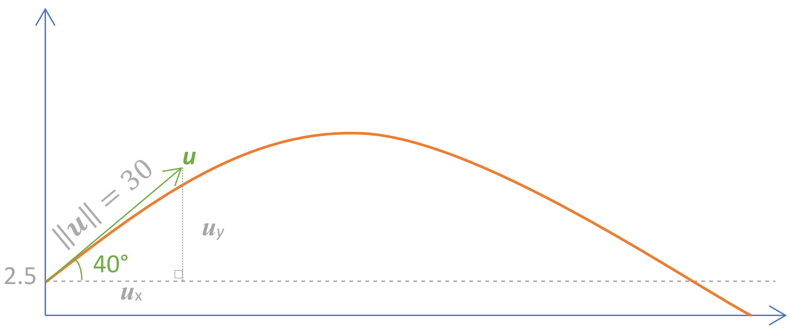
Note: for the following exercises, assume that the acceleration due to gravity, g, is 9.81m/s^2 acting straight downwards, and there is no air resistance or other force acting upon the objects.

Translation: we know that, in the equations above, the value of $\boldsymbol{a}$ will always be $\binom{0}{-9.81}$.

1. A projectile is launched with an initial speed of 30m/s, with an angle of inclination of θ = 40° from the initial position $\boldsymbol{p}_0 = (0\text{m}, 2.5\text{m})$.
   First, let's draw a diagram showing the values we know – the orange arc represents the projectile's motion:



   a. What is the initial velocity $\boldsymbol{u}$ in vector form?
   We know that $\|\boldsymbol{u}\| = 30$, which means that we have two properties of a right-angled triangle formed by the components of u, which is enough to give us the rest through trigonometry:

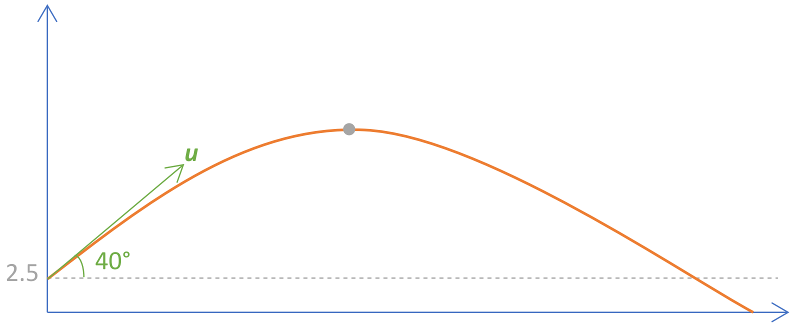


$$\cos 40° = \frac{u_x}{\|\boldsymbol{u}\|} \Rightarrow u_x = 30\cos 40°$$
$$\sin 40° = \frac{u_y}{\|\boldsymbol{u}\|} \Rightarrow u_y = 30\sin 40°$$

$\boldsymbol{u} \approx \binom{22.98}{19.28}$m/s

b. At what time will the projectile reach its apex (highest point)?

The apex is the point at which the projectile stops travelling upwards and starts travelling downwards – in other words, the y component of its velocity changes from positive to negative, and is therefore instantaneously zero:



To find the time, we need to use an equation that includes t, which rules out (4). To narrow the selection further, let's see what values we know:

$$\boldsymbol{s} = \binom{?}{?} \qquad \boldsymbol{u} = \binom{22.98}{19.28} \qquad \boldsymbol{v} = \binom{?}{0} \qquad \boldsymbol{a} = \binom{0}{-9.81} \qquad t = \cdots$$

Since we have no idea what the displacement is, we must use equation (1).

Note: although we don't know the value of v_x, because t is a scalar quantity, and multiplication with a scalar acts component-wise on a vector, we need only solve the equation for the y components (being able to solve components separately in this way is a fundamental concept of mechanics!).

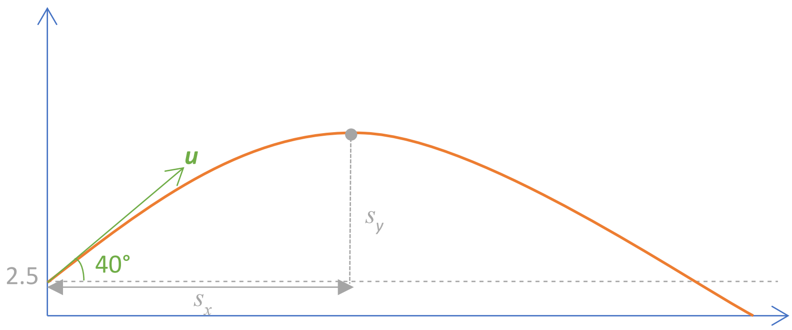
First, we need to rearrange the y-component version of (1) to isolate t:

$$v_y = u_y + a_y t$$
$$\Rightarrow v_y - u_y = a_y t$$
$$\Rightarrow \frac{v_y - u_y}{a_y} = t$$

Now, we can substitute in our known values to get $t = \frac{0-19.28}{-9.81} \approx 1.97$s

c. What are the coordinates of the projectile at the apex?

The value that we need to find here is $\boldsymbol{s}$, the displacement from the projectile's starting point:



Looking at the values we have now:

$$\boldsymbol{s} = \binom{\cdots}{\cdots} \qquad \boldsymbol{u} = \binom{22.98}{19.28} \qquad \boldsymbol{v} = \binom{?}{0} \qquad \boldsymbol{a} = \binom{0}{-9.81} \qquad t = 1.97$$

This time, we need to solve for both components, so we need to exclude $\boldsymbol{v}$, leaving us with equation (2) as the only choice – which helpfully doesn't need rearranging:

$$\boldsymbol{s} = \boldsymbol{u}t + \tfrac{1}{2}\boldsymbol{a}t^2 = \begin{pmatrix}22.98\\19.28\end{pmatrix} \times 1.97 + 0.5 \times \begin{pmatrix}0\\-9.81\end{pmatrix} \times 1.97^2 \approx \begin{pmatrix}45.27\\18.94\end{pmatrix}$$

This isn't our final answer, though, as it only gives us the position relative to where the projectile started, 2.5m above the origin. Thus the actual coordinates are given by $\begin{pmatrix}0\\2.5\end{pmatrix} + \begin{pmatrix}45.27\\18.94\end{pmatrix} = \begin{pmatrix}45.27\\21.44\end{pmatrix}$

d. How long will it take for projectile to come back to an altitude of $y = 2.5\text{m}$?
Since gravity is constant, the projectile will take the same time to reach the starting altitude from its apex as it took to reach the apex from the starting position. We can verify this using the values:

$$\boldsymbol{s} = \begin{pmatrix}?\\0\end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix}22.98\\19.28\end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix}?\\?\end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix}0\\-9.81\end{pmatrix} \qquad t = \cdots$$

Again, we need to use equation (2), but we can leave out the x components this time:

$$s_y = u_y t + \tfrac{1}{2}a_y t^2$$
$$0 = 19.28t + 0.5 \times (-9.81) \times t^2$$

Rearranging this gives $-4.905t^2 + 19.28t + 0 = 0$, a quadratic equation with coefficients a = -4.905, b = 19.28 and c = 0. Using the formula on the first page, we get:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-19.28 \pm \sqrt{19.28^2 - 4 \times (-4.905) \times 0}}{2 \times (-4.905)} = \frac{-19.28 \pm 19.28}{-9.81} = 0 \text{ or } 3.93$$

t = 0 is the start time, so our answer is t = 3.93s, which is twice the time to reach the apex (with rounding errors taken into consideration!).

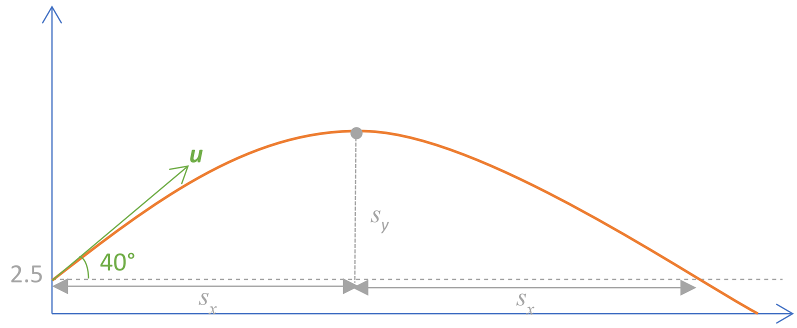
e. What will the horizontal displacement be at this time?
The values we now have are:

$$\boldsymbol{s} = \begin{pmatrix}\cdots\\0\end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix}22.98\\19.28\end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix}?\\?\end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix}0\\-9.81\end{pmatrix} \qquad t = 3.93$$

So, we need to use the same equation, (2), as in part (d), but this time using the new value of t to solve for s_x:

$$s_x = u_x t + \tfrac{1}{2}a_x t^2 = 22.98 \times 3.93 + 0.5 \times 0 \times 3.93^2 \approx 90.31\text{m}$$

Notice that, as the horizontal acceleration is 0, this is just the basic formula "$distance = speed \times time$"... It's also twice the horizontal distance to the apex (rounding errors aside):



f. For how long is the projectile in the air before it hits the ground?

Remember that the projectile was launched from a height of 2.5m, so it will travel a little beyond the point we were considering in parts (d) and (3) before reaching a height of 0m. We solve it in the same way as we did (d), knowing that the vertical displacement from the starting point is now -2.5m:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ -2.5 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 22.98 \\ 19.28 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ ? \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = \cdots$$

$$s_y = u_y t + \tfrac{1}{2} a_y t^2$$
$$-2.5 = 19.28t + 0.5 \times (-9.81) \times t^2$$

This time, rearranging gives $-4.905t^2 + 19.28t + 2.5 = 0$, so the coefficients are now a = -4.905, b = 19.28 and c = 2.5. Using the formula on the first page, we get:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-19.28 \pm \sqrt{19.28^2 - 4 \times (-4.905) \times 2.5}}{2 \times (-4.905)} = \frac{-19.28 \pm 20.51}{-9.81}$$
$$= -0.13 \; or \; 4.06$$

We're not dealing with time travel, so our answer is t=4.06s; it lands very soon after reaching the level of the starting point.

g. Find the values for the projectile's

   i. final velocity, $\boldsymbol{v}$, and

   ii. final horizontal displacement, s_x'

when it hits the ground.

We've pretty much ignored $\boldsymbol{v}$ up until now, except for when we knew the vertical component at the apex was zero. Given that we now have the following values:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ -2.5 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 22.98 \\ 19.28 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = 4.06$$

the simplest equation to use is (1):

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t = \begin{pmatrix} 22.98 \\ 19.28 \end{pmatrix} + 4.06 \times \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \approx \begin{pmatrix} 22.98 \\ -20.55 \end{pmatrix}$$

Notice that the horizontal velocity is unchanged, since there is no horizontal acceleration; we might have been tempted to use this fact in earlier parts of the question, however, to be mathematically rigorous we would have had to solve for it first rather than just assume. The negative vertical velocity means that the projectile is travelling downwards, as we'd expect.

The final horizontal displacement is now the only value that we don't have:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ -2.5 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 22.98 \\ 19.28 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} 22.98 \\ -20.55 \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = 4.06$$

It might appear that we could use any of the equations to find it (except for (1), which doesn't involve $\boldsymbol{s}$), but we need to be a little careful – for example, if we were to try using (4) for the x components:

$${v_x}^2 = {u_x}^2 + 2a_x s'_x$$
$$22.98^2 = 22.98^2 + 2 \times 0 \times s'_x$$

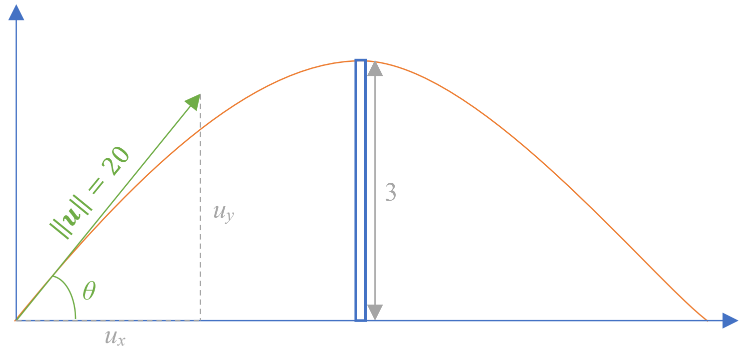
… While true, this isn't very helpful, as our s term has disappeared! (If we had tried to rearrange it before substituting values, to $s'_x = \frac{v_x{}^2 - u_x{}^2}{2a_x}$, we'd have a divide-by-zero error).

Any of the remaining three equations would work, though; for the sake of variety, let's use (3):

$$\boldsymbol{s} = \frac{1}{2}(\boldsymbol{u} + \boldsymbol{v})t = 0.5 \times \left(\begin{pmatrix} 22.98 \\ 19.28 \end{pmatrix} + \begin{pmatrix} 22.98 \\ -20.55 \end{pmatrix}\right) \times 4.06 \ \approx \begin{pmatrix} 93.29 \\ -2.58 \end{pmatrix}$$

We can see that this is roughly right, as the horizontal displacement is just a little greater than it was in part (e), and the vertical displacement is close enough to what we'd expect.
So, our answer is s'_x = 93.29m.

2. A ball is thrown from ground level so that it just clears a wall that is 3m high. If the initial speed of the ball is 20m/s, find the angle of projection.
First, here's our diagram:



This question is similar to the first parts of question 1: as the ball "just clears" the wall, it must pass over at its apex, when the vertical displacement is at its maximum and the vertical velocity is zero. This time, though, instead of knowing the initial velocity and finding the displacement, we have the displacement and need to find the value of $\boldsymbol{u}$ in order to "work backwards" to find the angle:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ 3 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ 0 \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = ?$$

Based on the values we have/want, the only equation we can get something useful from is (4) for the y components only:

$$v_y{}^2 = u_y{}^2 + 2a_y s_y$$

$$0^2 = u_y{}^2 + 2 \times (-9.81) \times 3$$

$$u_y = \sqrt{58.86} \approx \pm 7.67$$

From trigonometry, we know that $sin\theta = \frac{u_y}{\|\boldsymbol{u}\|}$, i.e. $u_y = 20sin\theta$. From the above, this means that

$$20sin\theta = 7.67$$

$$\theta = sin^{-1}\left(\frac{7.67}{20}\right) \approx 22.55°$$

(Note that we don't consider the negative square root because $sin\theta$ is positive between 0° and 180°).

3. A ball is thrown vertically upwards with a speed of 21m/s.

   a. In which direction is the ball travelling after 3 seconds?

   b. What is the total distance it has travelled in this time?

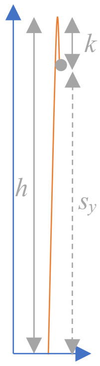
The diagram of the ball's motion here is effectively a straight vertical line – since there is no horizontal component, it travels straight up before falling back down again. Part (a) is simply asking whether, after 3 seconds, it is rising or falling, which we can determine by looking at the sign of the y component of the velocity at this time – which we first need to calculate from the values we're given:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ ? \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 21 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ \ldots \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = 3$$

We don't care about $\boldsymbol{s}$, so using equation (1) for the vertical components:
$v_y = u_y + a_y t = 21 + (-9.81) \times 3 = 21 - 29.43 = -8.43\text{m/s}$

The velocity is negative, so the answer to part (a) is that the ball is travelling downwards.

Note that part (b) asks for the total distance travelled by the ball, which is not quite the same as the displacement $\boldsymbol{s}$ given in the equations, which is measure to its current position from the point of projection – and will decrease as the ball falls back to the ground. At the time we're considering, the ball has travelled up to its apex and some distance back down; if h is the maximum height it reaches, the total distance travelled will be $d = h + k$, where k is the distance fallen from the apex at $t = 3$:



From the diagram, it's clear that $k = h - s_y$, where s_y is the vertical displacement at $t = 3$, while h is simply the vertical displacement at the apex (when $v_y = 0$).

So, to find h, we have:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ h \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 21 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ 0 \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = ?$$

Using equation (4), we have ${v_y}^2 = {u_y}^2 + 2a_y h \Rightarrow 0 = 21^2 + 2 \times (-9.81) \times h$, giving

$$h = \frac{-441}{-19.62} \approx 22.48$$

Next, to find s_y, we have:

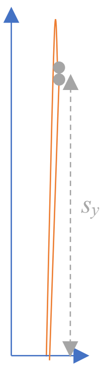
$$\boldsymbol{s} = \begin{pmatrix} ? \\ s_y \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 21 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ ? \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad t = 3$$

This time, we use equation (2) for the vertical components:

$$s_y = u_y t + \tfrac{1}{2} a_y t^2 = 21 \times 3 + 0.5 \times (-9.81) \times 3^2 \approx 18.86$$

This gives our final answer of $d = h + k = h + (h - s_y) = 22.48 + (22.48 - 18.86) = 26.1\text{m}$

4. A stone is thrown vertically upwards with a speed of 7m/s, and one second later, a second stone is thrown vertically upwards from the same point with the same speed.
   Find the height at which the two stones collide.



The stones' trajectories here will be the same as the ball's in question 3 – straight up and down again. By logic/common sense, we know that they will collide when the first one is travelling downwards, though we don't really need to care as the equations will figure it out for us; it just helps us draw the diagram!

When the stones meet, we have the following values for both of them:

$$\boldsymbol{s} = \begin{pmatrix} ? \\ s_y \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} ? \\ ? \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix} \qquad \begin{matrix} t = t \text{ and} \\ t - 1 \end{matrix}$$

Note that, although we don't know the *absolute* time of the collision, we know that the second stone has been in the air for one second less than the first – so if the first has been in the air for t seconds when they collide, the second has been in the air for t-1 seconds.

Using this in equation (2), for the first stone, we get:

$$s_y = u_y t + \frac{1}{2}a_y t^2 = 7t - 4.905t^2$$

while for the second stone, we have:

$$s_y = 7(t-1) - 4.905(t-1)^2 = 7t - 7 - 4.905(t^2 - 2t + 1)$$

Equating the two expressions for s_y, we get:

$$7t - 4.905t^2 = 7t - 7 - 4.905t^2 + 9.81t - 4.905$$

The green and orange terms appear on both sides of the equation, cancelling each other out, leaving us with:

$$0 = -7 + 9.81t - 4.905 = 9.81t - 11.905$$

Which gives us a value of $t = \frac{11.905}{9.81} \approx 1.21$

We can now put this value of t back into our first equation for s_y to give a value of $7 \times 1.21 - 4.905 \times 1.21^2 \approx$ 1.29m