2: Computational Geometry I

COMP270: MATHEMATICS FOR 3D WORLDS & SIMULATIONS

Number systems

Counting

- Humans first developed numbers as a way of **counting things**
- How many sheep do I have? 1, 2, 3, 4, …
- This gives us the **natural numbers** i.e. the counting numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

We use blackboard bold font for the standard number systems

Curly braces denote a **set**

… means continue this sequence to infinity

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- This gives us the **natural numbers** i.e. the counting numbers
$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$
- What if I have no sheep? This gives us the concept of **zero**
- Some people include 0 in the natural numbers, some don't
- You may also see
$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\} \quad \mathbb{N}_+ = \{1, 2, 3, 4, \dots\}$$

Integers

- We can calculate with natural numbers
- What is the answer to $3 - 5$? We need **negative numbers**
- Adding negative numbers to the natural numbers gives us the **integers** (or **whole numbers**)

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Why Z? From the German *Zahlen*, meaning *numbers*

Infinite sequence to the left as well as to the right

Fractions

- We can do some divisions with integers, e.g. $6 \div 3 = 2$
- But not others, e.g. $7 \div 3 = ?$
- To solve this we need fractions: $7 \div 3 = \frac{7}{3}$
- This gives us the **rational numbers**:

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$$

$\in$ means in, i.e. $x \in S$ means element x is in set S

Why Q? From the German (also English) *Quotient*, meaning *ratio* or *division*

Set builder notation: read as "the set of all $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$"

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- There are multiple ways to write the same fraction, for example $\frac{7}{3} = \frac{14}{6} = \frac{700}{300}$. Mathematically they are all considered to be identical.

Decimals

- In a fraction $\frac{a}{b}$, a is called the **numerator** and b the **denominator**
- Note that decimals are just fractions where the denominator is a power of 10
- E.g.

$$0.7 = \frac{7}{10}$$

$$12.345 = \frac{12345}{1000}$$

- So the decimal numbers are a subset of $\mathbb{Q}$ (or equal to $\mathbb{Q}$ if we allow recurring decimals)

Reals

- Some numbers cannot be written exactly as fractions, e.g. π, $\sqrt{2}$, e
- Such numbers are called **irrational**
- Putting together the rational and irrational numbers gives the **real** numbers

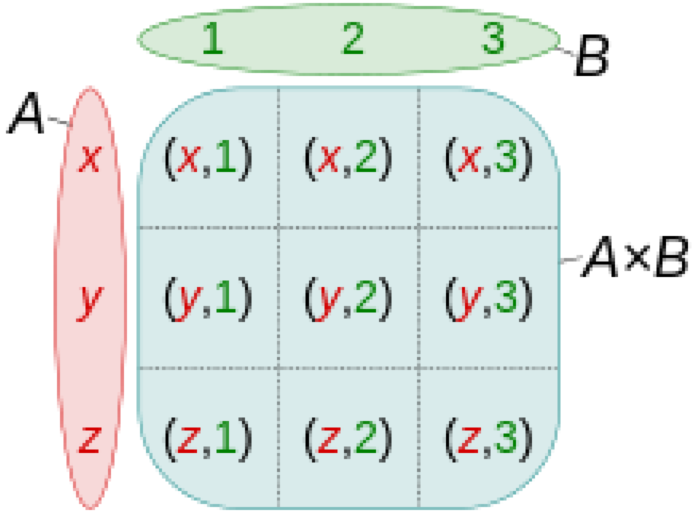
$$\mathbb{R}$$

- (Definition is beyond the scope of this lecture – involves limits of infinite sequences)
- The real numbers can be thought of as the points on an infinite line (from $-\infty$ to $+\infty$)
- Note however that all real numbers are **finite**

Cartesian product

Named after René Descartes, 1596-1650, French mathematician

- For two sets S and T, the Cartesian product $S \times T$ is defined as the set of all **pairs** of elements, the **first** from S and the **second** from T:
$$S \times T = \{(a, b): a \in S, b \in T\}$$
- Generalises to products of 3, 4, … sets
- Cartesian products of a set with itself give **Cartesian powers**:
$$S^2 = S \times S = \{(a, b): a \in S, b \in S\}$$
$$S^3 = S \times S \times S = \{(a, b, c): a \in S, b \in S, c \in S\}$$



Cartesian coordinate systems

- $\mathbb{R}^1 = \mathbb{R}$ is 1-dimensional space, aka the space of **scalars**
- $\mathbb{R}^2$ is 2-dimensional space, aka the space of 2D **vectors,** aka the 2D **plane**
- **Recall:** a 2D vector is a pair of numbers, and $\mathbb{R}^2$ is the set of all pairs of numbers
- $\mathbb{R}^3$ is 3-dimensional space
- $\mathbb{R}^n$ is n-dimensional space

Other number systems

- Complex numbers $\mathbb{C}$
- Quaternions, octonians, …

Functions

Functions

- In mathematics, a **function** is a mapping from one set to another
- If S and T are sets, then a function
$$f : S \to T$$
- Maps each element of S to an element of T
$$s \in S, \qquad f(s) \in T$$
- S is called the **domain** of f, and T is the **codomain**
- **Note:** f maps each element of S to one and only one element of T; however it could map multiple elements of S to the same element of T

Functions vs functions

- In mathematics:

$$f : S \to T$$

- In code:

```
class S {…};
class T {…};
T f(S s) {…}
```

- (Under the assumption that f is implemented to always give the same return value given the same argument – e.g. no internal or external state)

Functions vs functions

- In mathematics:

$$f : \mathbb{R} \to \mathbb{Z}$$

- In code:

```
int f(float x) {…}
```

- OK, so $\mathbb{Z}$ and `int` aren't really the same, nor are $\mathbb{R}$ and `float`, but close enough for computing…

Multiple arguments

- The domain of a function could be a Cartesian product:

$$f : A \times B \to C$$

- In code:

```
C f(A a, B b) {…}
```

Curves

What is a curve?

“The [curved] line is […] the first species of quantity, which has only one dimension, namely length, without any width nor depth, and is nothing else than the flow or run of the point which […] will leave from its imaginary moving some vestige in length, exempt of any width.”

Euclid, *Elements* (English translation from Wikipedia)

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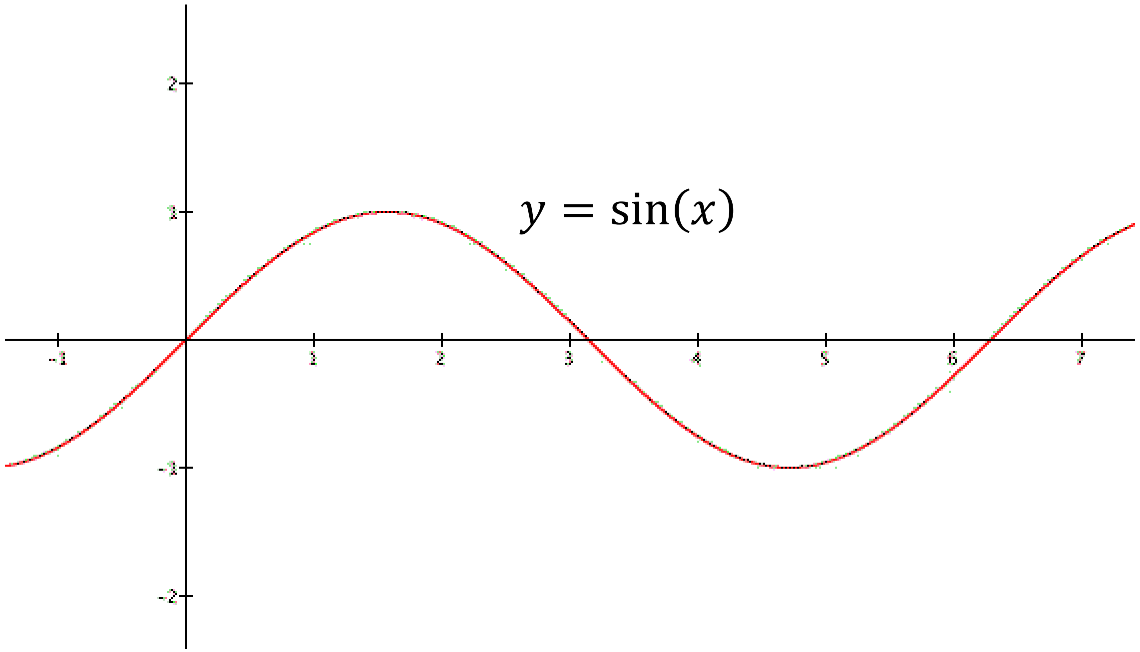
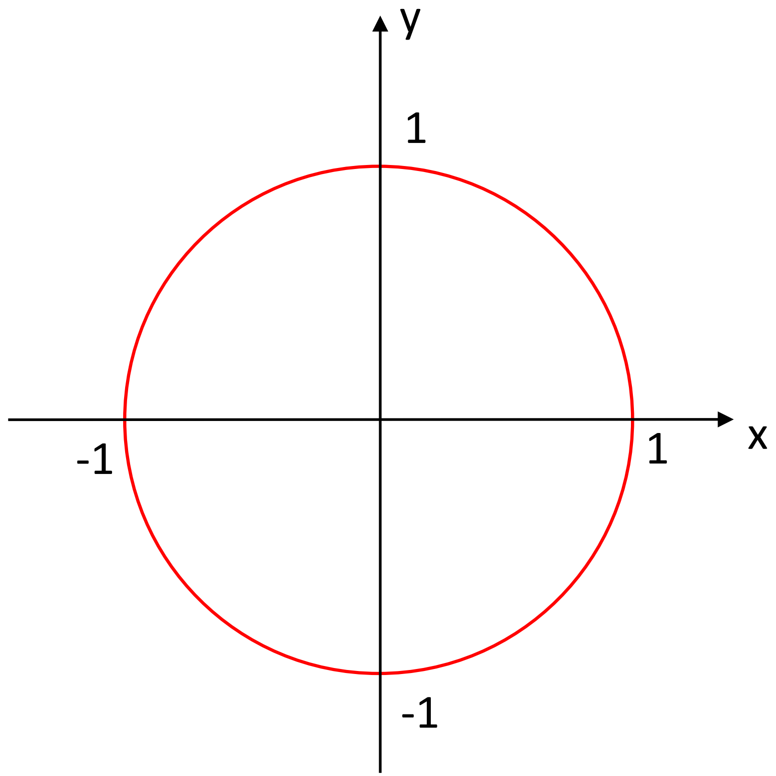
Defining a curve

- For a function $f : \mathbb{R} \to \mathbb{R}$ we can plot the curve $y = f(x)$
- Formed of the points $\begin{pmatrix} x \\ f(x) \end{pmatrix}$ for x in some range
- NB can only have one point per x value, since f maps each x to one **and only one** y value



Defining a curve

- How do we define a circle of radius 1 (aka a **unit circle**)?
- The set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ such that
$$x^2 + y^2 = 1$$
- The pair of curves
$$y = \pm\sqrt{1 - x^2}$$
- Or we can define it **parametrically**



Parametric curve

A curve defined by two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, with points

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ with } x = f(t) \text{ and } y = g(t)$$

for t in some range

t is called the **parameter**

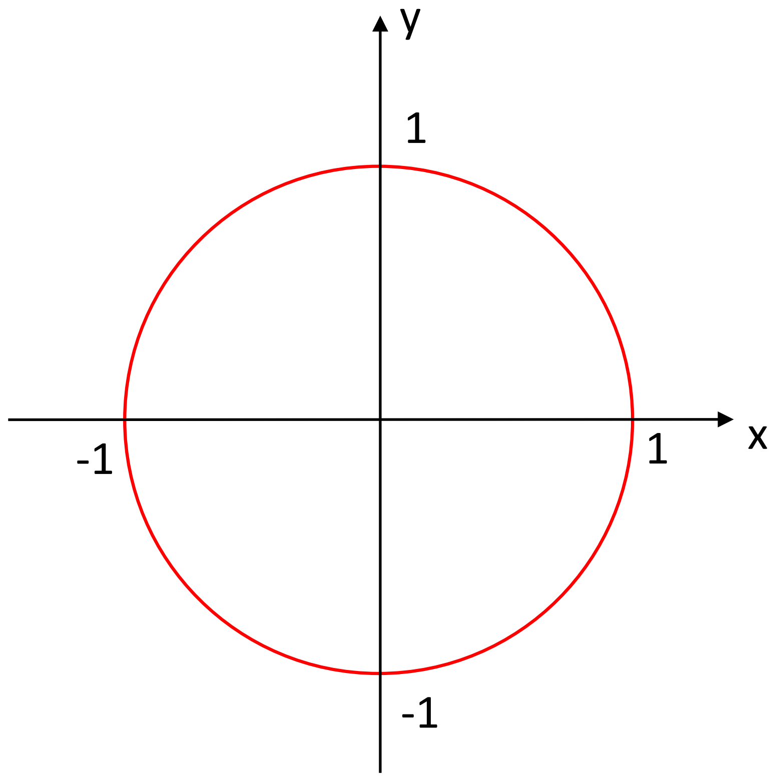
Equivalently, defined by a single function $h : \mathbb{R} \to \mathbb{R}^2$ which takes a scalar parameter and returns a vector

$$h(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

Parametric definition of a unit circle

$$x = \cos(t)$$
$$y = \sin(t)$$

For $0 \leq t < 2\pi$

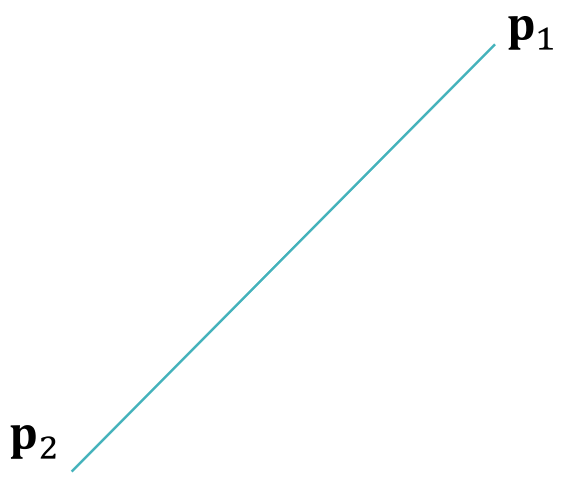


Lines are curves too

- Line between points $\mathbf{p}_1$ and $\mathbf{p}_2$ can be defined parametrically by
$$f(t) = (1 - t)\mathbf{p}_1 + t\mathbf{p}_2$$
for $0 \leq t \leq 1$

- Note:
$$t = 0 \Rightarrow f(0) = 1\mathbf{p}_1 + 0\mathbf{p}_2 = \mathbf{p}_1$$
$$t = 1 \Rightarrow f(1) = 0\mathbf{p}_1 + 1\mathbf{p}_2 = \mathbf{p}_2$$

- This is a **linear interpolation** or **lerp**



Named after Pierre Bézier, 1910-1999, French engineer

Bézier curves

- Defined by a number of **control points**
- Commonly used in computer graphics and game development, as allows artists/designers good control over the precise shape of the curve
- See worksheet A…