Week 7: 3D Geometry I
Part 4: Coordinate spaces

COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Define** the characteristics of a coordinate space
- **Introduce** some common coordinate spaces

What is a coordinate space?

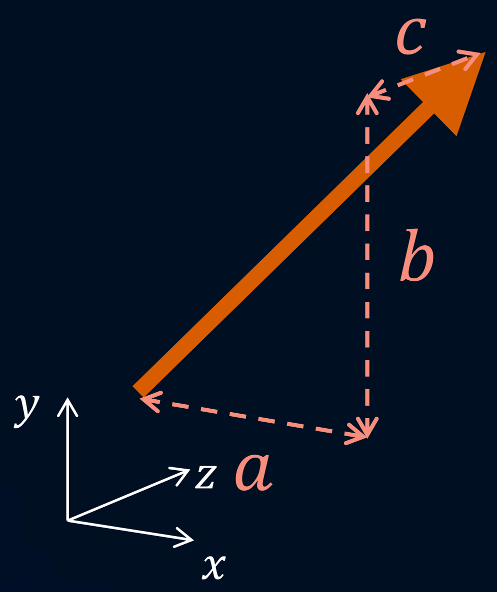
- Definition: a **coordinate space** is a space with a coordinate system defined by an origin and a number of axes equal to the dimension of the space, allowing any point in the space to be uniquely identified as a linear combination of distances along the axes.

- In 3D coordinate space, define the unit vectors along the x, y and z axes to be $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively, i.e.

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in the space can be written as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

A linear combination



Basis vectors

$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{0}$ if and only if $a_1 = a_2 = \ldots = a_n = 0$

- Definition: a set of linearly independent vectors $\mathbf{v}_1 \ldots \mathbf{v}_n$ in n-dimensional space form a **basis** for that space if any vector in that space can be expressed uniquely as a linear combination of the vectors $\mathbf{v}_i$:
$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$
where the coefficients a_i are the coordinates of $\mathbf{x}$
- Any set of n linearly independent vectors form a basis…
  - Therefore, the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ on the previous slide form a basis for 3D space.
  - So do the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ (proof here).

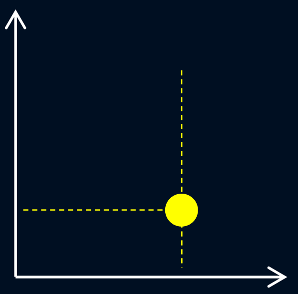
Basis example

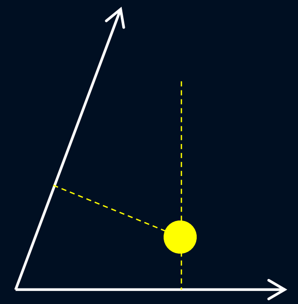
$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 5\begin{pmatrix}1\\0\\-1\end{pmatrix} + 0\begin{pmatrix}2\\1\\-1\end{pmatrix} + 2\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

What do coefficients (5, 0, 2) mean?!

Orthonormal basis vectors





- Definition: an **orthonormal basis** is a set of basis vectors that are orthogonal and unit length.
- This means that:
  - The coordinates are uncoupled, so that any given coordinate of a vector $\mathbf{x}$ can be determined solely from the coefficient and the corresponding basis vector.
  - Displacement along one basis vector does not cause any displacement along any of the others.
  - Each coordinate of $\mathbf{x}$ is the signed displacement in the direction of the corresponding basis vector, which can be computed using the dot product.

Properties of a coordinate space

Has a coordinate system:

- A set of axes (orthonormal basis) → directions
- An origin → position

… relative to what?!

Some common coordinate spaces

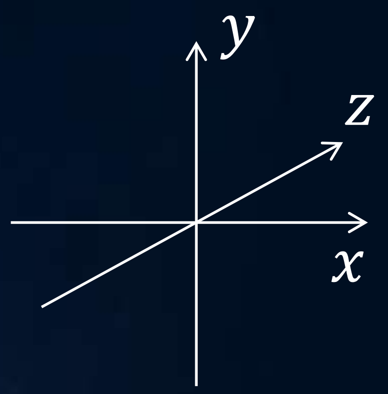
- **World space**: establishes a global reference frame for all other coordinate reference frames.
  - Covers the whole area/volume in which the action is currently taking place
  - Directions are fixed for all objects: e.g. north, south, east, west
- **Object space**: the local coordinate space associated with a particular object.
  - Origin is the object's centre of mass, root joint etc.
    - May have several nested/hierarchical spaces for different components of the model
  - Origin and axes specified in parent/world space
  - Directions are relative to each object: e.g. left, right, up, down
- **Camera space**: the object space associated with the viewpoint used for rendering.
  - Convention: left-handed with viewing direction along the positive z axis from the origin (camera position).
- **Screen space**: the 2D space onto which the camera space view is projected.

Relative spaces

Left or right handed?

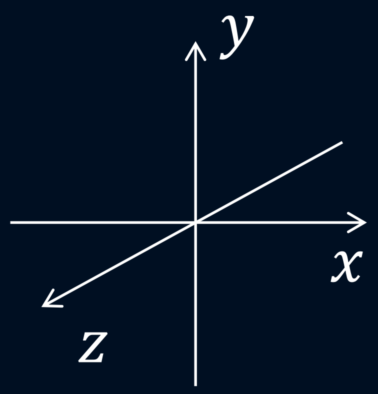
Cross product direction follows the same 'handedness' as the coordinate system

- Left handed:



- Unity (y up)
- Unreal (z up)
- (DirectX)

- Right handed:



- Maths/physics
- (Maya)
- (OpenGL)