COMP110: Principles of Computing

7: Algorithm Strategies

FALMOUTH UNIVERSITY

Recursion and induction

A boolean identity

$$\neg(X_1 \vee X_2 \vee \cdots \vee X_n) = \neg X_1 \wedge \neg X_2 \wedge \cdots \wedge \neg X_n$$

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Case $n = 1$

$$\neg(X_1) = \neg X_1$$

Case $n = 2$

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Exercise Sheet ii, question 3(a)

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$$
\begin{aligned}
\neg(X_1 \vee X_2 \vee \cdots \vee X_k) &= \neg(X_1 \vee (X_2 \vee \cdots \vee X_k)) \\
&= \neg X_1 \wedge \neg(X_2 \vee \cdots \vee X_k) \text{ } (n = 2 \text{ case}) \\
&= \neg X_1 \wedge (\neg X_2 \wedge \cdots \wedge \neg X_k) \text{ } (n = k - 1 \text{ case})
\end{aligned}
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- Therefore the formula works for all positive integers n

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Therefore

- $\sum_{i=1}^{k} i = \left(\sum_{i=1}^{k-1} i\right) + k$

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- $\sum_{i=1}^{k} i = \left(\sum_{i=1}^{k-1} i\right) + k$
- $= \frac{1}{2}(k-1)k + k$ (by inductive assumption)

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Therefore

- $\sum_{i=1}^{k} i = \left(\sum_{i=1}^{k-1} i\right) + k$
- $= \frac{1}{2}(k-1)k + k$ (by inductive assumption)
- $= \frac{1}{2}k^2 - \frac{1}{2}k + k$
- $= \frac{1}{2}k^2 + \frac{1}{2}k$
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So **if** the formula works for $n = k - 1$, **then** it works for $n = k$

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Exercise

Prove

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

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- Given k, if I had already proved $n = k - 1$ then I could prove $n = k$

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- Given k, if I had already proved $n = k - 1$ then I could prove $n = k$
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- Therefore by induction I can prove the result for all n

Recursion

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```
def factorial(n):
    if n <= 1:
        return 1
    else:
        return n * factorial(n-1)
```

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- I can solve the smallest possible problem
- Therefore I can write a recursive function

Exercise

- **Write** a pseudocode function to calculate the total size of all files in a directory and its subdirectories
- You may use the following functions in your pseudocode:
  - LISTDIR(directory): return a list of names of all files and folders in the given directory
  - GETSIZE(filename): return the size, in bytes, of the given file
  - ISDIR(name), ISFILE(name): determine whether the given name refers to a file or a directory

procedure CALCDIRSIZE(directory)
  ... ▷ return total size in bytes
end procedure