COMP270: 3D Computational Geometry Worksheet 1 - **Answers**

1. Calculate $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$ for the following vectors:

   a. $\boldsymbol{a} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \times 1 - 0 \times 0 \\ -(0 \times 1 - 0 \times 0) \\ 0 \times 0 - 0 \times (-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \times 0 - 1 \times (-1) \\ -(0 \times 0 - 1 \times 0) \\ 0 \times (-1) - 0 \times 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

   b. $\boldsymbol{a} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \times (-1) - 1 \times (-2) \\ -((-2) \times (-1) - 1 \times 1) \\ (-2) \times (-2) - 4 \times 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2) \times 1 - (-1) \times 4 \\ -(1 \times 1 - (-1) \times (-2)) \\ 1 \times 4 - (-2) \times (-2) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

   c. $\boldsymbol{a} = \begin{pmatrix} 3 \\ 10 \\ 7 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 8 \\ -7 \\ 4 \end{pmatrix}$

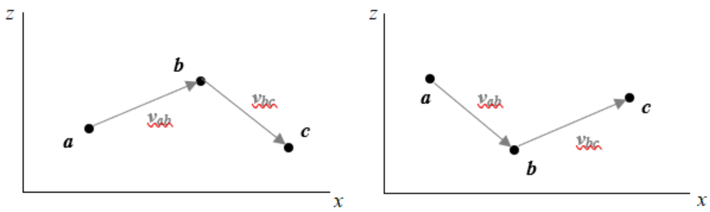
$$\begin{pmatrix} 3 \\ 10 \\ 7 \end{pmatrix} \times \begin{pmatrix} 8 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \times 4 - 7 \times (-7) \\ -(3 \times 4 - 7 \times 8) \\ 3 \times (-7) - 10 \times 8 \end{pmatrix} = \begin{pmatrix} 89 \\ 44 \\ -101 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ -7 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 10 \\ 7 \end{pmatrix} = \begin{pmatrix} (-7) \times 7 - 4 \times 10 \\ -(8 \times 7 - 4 \times 3) \\ 8 \times 10 - (-7) \times 3 \end{pmatrix} = \begin{pmatrix} -89 \\ -44 \\ 101 \end{pmatrix} = -\begin{pmatrix} 89 \\ 44 \\ -101 \end{pmatrix}$$

2. A nonplayer character (NPC) is standing at a location $\boldsymbol{p}$ with a forward direction of $\boldsymbol{v}$. Consider three points $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the xz plane of a left-handed coordinate system, which represent waypoints on the NPC's path.

   a. How can the cross product be used to determine whether, when moving from $\boldsymbol{a}$ to $\boldsymbol{b}$ to $\boldsymbol{c}$, the NPC makes a clockwise or anticlockwise turn at $\boldsymbol{b}$, when viewing the path from above?

   There are essentially two possible configurations in which the three points may be arranged:



The NPC's path consists of two vectors, $\boldsymbol{v_{ab}} = \boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{v_{bc}} = \boldsymbol{c} - \boldsymbol{b}$. In the configuration on the left, taking the cross product $\boldsymbol{v_{ab}}$ x $\boldsymbol{v_{bc}}$ will give a vector pointing the positive y direction ("out of the page", since it's a left-handed coordinate system), while in the second configuration, the direction will be reversed.

We can simplify the calculation by observing that all the points and vectors in question have zero y coordinates, i.e. $\boldsymbol{v_{ab}} = \begin{pmatrix} x_{ab} \\ 0 \\ z_{ab} \end{pmatrix}$, $\boldsymbol{v_{bc}} = \begin{pmatrix} x_{bc} \\ 0 \\ z_{bc} \end{pmatrix}$ and

$$\boldsymbol{v_{ab}} \times \boldsymbol{v_{bc}} = \begin{pmatrix} 0 \\ z_{ab} \times x_{bc} - x_{ab} \times z_{bc} \\ 0 \end{pmatrix}$$

The direction the NPC turns in can therefore be determined by checking the sign of the value $z_{ab} \times x_{bc} - x_{ab} \times z_{bc}$ (positive for clockwise; negative for anticlockwise).

If the value happens to be zero, then the points are colinear and the NPC either continues straight or doubles back on itself: $\boldsymbol{v_{ab}} = k\boldsymbol{v_{bc}}$

b. For each of the following sets of three points, determine whether the NPC is turning clockwise or anticlockwise when moving from $\boldsymbol{a}$ to $\boldsymbol{b}$ to $\boldsymbol{c}$:

i. $\boldsymbol{a}$ = (2, 0, 3), $\boldsymbol{b}$ = (-1, 0, 5), $\boldsymbol{c}$ = (-4, 0, 1)
$\boldsymbol{v_{ab}} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$, $\boldsymbol{v_{bc}} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$
$z_{ab} \times x_{bc} - x_{ab} \times z_{bc} = 2 \times (-3) - (-3) \times (-4) = -18 < 0$; the NPC turns anticlockwise.

ii. $\boldsymbol{a}$ = (-3, 0, -5), $\boldsymbol{b}$ = (4, 0, 0), $\boldsymbol{c}$ = (3, 0, 3)
$\boldsymbol{v_{ab}} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$, $\boldsymbol{v_{bc}} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$
$z_{ab} \times x_{bc} - x_{ab} \times z_{bc} = 5 \times (-1) - 7 \times 3 = -26 < 0$; the NPC turns anticlockwise.

iii. $\boldsymbol{a}$ = (1, 0, 4), $\boldsymbol{b}$ = (7, 0, -1), $\boldsymbol{c}$ = (-5, 0, -6)
$\boldsymbol{v_{ab}} = \begin{pmatrix} 6 \\ 0 \\ -5 \end{pmatrix}$, $\boldsymbol{v_{bc}} = \begin{pmatrix} -12 \\ 0 \\ -5 \end{pmatrix}$
$z_{ab} \times x_{bc} - x_{ab} \times z_{bc} = (-5) \times (-12) - 6 \times (-5) = 90 > 0$; the NPC turns clockwise.

iv. $\boldsymbol{a}$ = (-2, 0, 1), $\boldsymbol{b}$ = (1, 0, 2), $\boldsymbol{c}$ = (4, 0, 4)
$\boldsymbol{v_{ab}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{v_{bc}} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$
$z_{ab} \times x_{bc} - x_{ab} \times z_{bc} = 1 \times 3 - 3 \times 2 = -3 < 0$; the NPC turns anticlockwise.

3. Consider a triangle defined by the vertices (6, 10, -2), (3, -1, 17) and (-9, 8, 0).

a. What is the equation of the plane containing this triangle?
The standard equation of a plane is the one that expresses the idea that any vector lying along the plane is perpendicular to the plane's normal, i.e. if $\boldsymbol{a}$ is known point

on the plane, then for any point x on the plane, $\boldsymbol{x}\cdot\boldsymbol{n} = \boldsymbol{p}\cdot\boldsymbol{n}$, or $\boldsymbol{x}\cdot\boldsymbol{n} = \boldsymbol{d}$.

Since all the vertices of the triangle lie on the plane, its edges do too, and the cross product of any two of the edges will give a vector perpendicular to the plane (in either possible direction, depending on the order; in some situations, for instance rendering, it's important to note which direction should be the "front" of the triangle, though for this question it doesn't really matter).

Choosing our two edges to follow the order the vertices were given in:

$$\begin{pmatrix}3\\-1\\17\end{pmatrix} - \begin{pmatrix}6\\10\\-2\end{pmatrix} = \begin{pmatrix}-3\\-11\\19\end{pmatrix}$$
$$\begin{pmatrix}-9\\8\\0\end{pmatrix} - \begin{pmatrix}3\\-1\\17\end{pmatrix} = \begin{pmatrix}-12\\9\\-17\end{pmatrix}$$

we get the following cross product:

$$\boldsymbol{n} = \begin{pmatrix}-3\\-11\\19\end{pmatrix} \times \begin{pmatrix}-12\\9\\-17\end{pmatrix} \begin{pmatrix}(-11)\times(-17) - 19\times 9\\ -((-3)\times(-17) - 19\times(-12))\\ (-3)\times 9 - (-11)\times(-12)\end{pmatrix} = \begin{pmatrix}16\\-279\\-159\end{pmatrix}$$

Though it's not always necessary to normalise it (if we're only interested in the sign of the direction, say), it's often useful for later calculations – first, we need the magnitude:

$$\|\boldsymbol{n}\| = \sqrt{16^2 + (-279)^2 + (-159)^2} \approx 321.5$$

which we can divide $\boldsymbol{n}$ by to get the normal, $\hat{\boldsymbol{n}} \approx \begin{pmatrix}0.04976\\-0.8677\\-0.4945\end{pmatrix}$

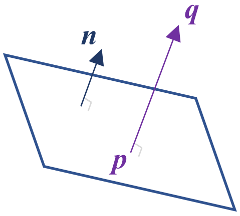
Now we can compute $\boldsymbol{d} = \boldsymbol{p}\cdot\boldsymbol{n}$ using any one of our three triangle vertices; let's use the first one:

$$\begin{pmatrix}6\\10\\-2\end{pmatrix}\cdot\begin{pmatrix}0.04976\\-0.8677\\-0.4945\end{pmatrix} = 6\times 0.04976 + 10\times(-0.8677) + (-2)\times(-0.4945)$$
$$\approx -7.389$$

The equation for the triangle plane is therefore $\boldsymbol{x}\cdot\begin{pmatrix}0.04976\\-0.8677\\-0.4945\end{pmatrix} = -7.389$, or in xyz form, $0.04976x - 0.8677y - 0.4945z = -7.389$

b. Is the point (3, 4, 5) on the front or back side of this plane?
How far is this point from the plane?

The two parts of this question can be answered using the same method, which is to consider the point $\boldsymbol{p}$ on the plane that is closest to the point in question, $\boldsymbol{q}$. The vector from $\boldsymbol{p}$ to $\boldsymbol{q}$ will be perpendicular to the plane, so can be written as $\alpha\boldsymbol{n}$.



If the normal is given as a unit vector, $\hat{\boldsymbol{n}}$, then the distance from $\boldsymbol{p}$ to $\boldsymbol{q}$ is just α (which is a signed distance: it will be negative if $\boldsymbol{q}$ is on the opposite side of the plane to where the normal points).

Interestingly, we don't actually need to know what p is, as we can compute α directly by considering the vector to $\boldsymbol{q}$:

$$
\begin{aligned}
&\boldsymbol{p} + \alpha\hat{\boldsymbol{n}} = \boldsymbol{q}\\
&(\boldsymbol{p} + \alpha\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}} = \boldsymbol{q} \cdot \hat{\boldsymbol{n}}\\
&\boldsymbol{p} \cdot \hat{\boldsymbol{n}} + \alpha\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} = \boldsymbol{q} \cdot \hat{\boldsymbol{n}}\\
&d + \alpha = \boldsymbol{q} \cdot \hat{\boldsymbol{n}}\\
&\alpha = \boldsymbol{q} \cdot \hat{\boldsymbol{n}} - d
\end{aligned}
$$

With $d = \boldsymbol{p} \cdot \hat{\boldsymbol{n}} = -7.389$ from part (a), we get

$$\alpha = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 0.04976 \\ -0.8677 \\ -0.4945 \end{pmatrix} - (-7.398) \approx 1.595$$

which is positive, so the point is on the front side of the plane.

4. Consider the set of five points (7, 11, -5), (2, 3, 8), (-3, 3, 1), (-5, -7, 0) and (6, 3, 4). An *axis aligned bounding box (AABB)* is the smallest box whose edges are aligned with the coordinate axes that contains all the points, defined by its minimum and maximum vertices $\boldsymbol{p}_{min}$ and $\boldsymbol{p}_{max}$.

   a. What are $\boldsymbol{p}_{min}$ and $\boldsymbol{p}_{max}$ for the above five points?
   The minimum and maximum points are simply the ones made up of all the smallest/largest x, y and z values out of all the points:
   $\boldsymbol{p}_{min}$ = (-5, -7, -5), $\boldsymbol{p}_{max}$ = (7, 11, 8)

   b. List all eight vertices of the AABB.
   These are just the various combinations of the min and max x, y and z values:
   (-5, -7, -5), (-5, -7, 8), (-5, 11, 8), (-5, 11, -5), (7, -7, -5), (7, -7, 8), (7, 11, 8), (7, 11, -5)

   c. Determine the centre point $\boldsymbol{c}$ of the AABB.
   For a general set of points, the centre point is just their average position. As our AABB is symmetrical, its centre is half-way between $\boldsymbol{p}_{min}$ and $\boldsymbol{p}_{max}$:

$$\boldsymbol{c} = \frac{\boldsymbol{p}_{min} + \boldsymbol{p}_{max}}{2} = (1, 2, 1.5)$$

   d. Multiply the five points by the following matrix (a 45° rotation about the z-axis):

$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 11 \\ -5 \end{pmatrix} = \begin{pmatrix} 0.707 \times 7 + (-0.707) \times 11 + 0 \times (-5) \\ 0.707 \times 7 + 0.707 \times 11 + 0 \times (-5) \\ 0 \times 7 + 0 \times 11 + 1 \times (-5) \end{pmatrix}$$
$$= \begin{pmatrix} -2.828 \\ 12.726 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0.707 \times 2 + (-0.707) \times 3 + 0 \times 8 \\ 0.707 \times 2 + 0.707 \times 3 + 0 \times 8 \\ 0 \times 2 + 0 \times 3 + 1 \times 8 \end{pmatrix} = \begin{pmatrix} -0.707 \\ 3.535 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.707 \times (-3) + (-0.707) \times 3 + 0 \times 1 \\ 0.707 \times (-3) + 0.707 \times 3 + 0 \times 1 \\ 0 \times (-3) + 0 \times 3 + 1 \times 1 \end{pmatrix}$$
$$= \begin{pmatrix} -4.242 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ -7 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \times (-5) + (-0.707) \times (-7) + 0 \times 0 \\ 0.707 \times (-5) + 0.707 \times (-7) + 0 \times 0 \\ 0 \times (-5) + 0 \times (-7) + 1 \times 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1.414 \\ -8.484 \\ 0 \end{pmatrix}$$

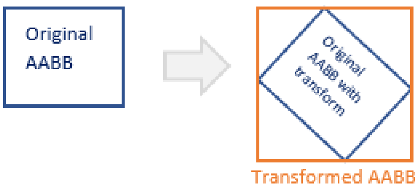
$$\begin{pmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.707 \times 6 + (-0.707) \times 3 + 0 \times 4 \\ 0.707 \times 6 + 0.707 \times 3 + 0 \times 4 \\ 0 \times 6 + 0 \times 3 + 1 \times 4 \end{pmatrix} = \begin{pmatrix} 2.121 \\ 6.363 \\ 4 \end{pmatrix}$$

e. What is the AABB of these transformed points?
$\boldsymbol{p}_{min}$ = (-4.242, -8.484, -5), $\boldsymbol{p}_{max}$ = (2.121, 12.726, 8)

f. What is the AABB we get by transforming the original AABB? (i.e. the bounding box of the transformed corner points).
Before looking at how to find this, note how *not* to:

- if we just transformed the original AABB, we would get a box of the same dimensions but that is *not* axis aligned;
- we can't simply transform the original $\boldsymbol{p}_{min}$ and $\boldsymbol{p}_{max}$, as you can hopefully see from the diagram below

To get a new 'transformed' AABB, we need to find the AABB of the transformed original AABB:



We could do this by transforming all eight points of the original AABB, and then finding *their* $\boldsymbol{p}_{min}$ and $\boldsymbol{p}_{max}$, but a more efficient method is to calculate the new min/max points directly, using the fact that we know what effect the transform will have on each of the input points – i.e. we can find formulae for each transformed point in terms of a linear combination of the original points, as described by the matrix:

$$x' = 0.707x - 0.707y$$
$$y' = 0.707x + 0.707y$$
$$z' = z$$

Since we would find the new min/max values by choosing the components from each transformed point, we just have to decide which combination of the min/max values of x, y and z will give the min/max values of x', y' and z', which in this case gives us:

$$x'_{min} = 0.707 \times (-5) - 0.707 \times 11 = -11.312$$
$$y'_{min} = 0.707 \times (-5) + 0.707 \times (-7) = -8.484$$
$$z'_{min} = -5$$
$$x'_{max} = 0.707 \times (7) - 0.707 \times (-7) = 9.898$$
$$y'_{max} = 0.707 \times (7) + 0.707 \times 11 = 12.726$$
$$z'_{max} = 8$$

Note that this is much larger than the AABB of the transformed points in part (e) … Depending on the increased efficiency of not having to transform all the points may or may not compensate for the less-than-optimal AABB size. (For more details on this approach, please see section 9.4 of "3D Math Primer for Graphics and Game Development", by Fletcher Dunn and Ian Parberry).

5. A robot is at the position $(1, 10, 3)$ and her right, up and forward vectors (expressed in world space) are $\begin{pmatrix} 0.866 \\ 0 \\ -0.5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0.5 \\ 0 \\ 0.866 \end{pmatrix}$ respectively (note that these vectors form an orthonormal basis).

One way to interpret this is to think of the right, up and forward vectors as the axes of the robot's local space, so that a point $\boldsymbol{p}_l = (x_l, y_l, z_l)$ in that space is a linear combination of distances along the local axes, i.e. the world-space vector from the robot's position $\boldsymbol{r}$ to the point in question, $\boldsymbol{p}_w$, is given by:

$$\boldsymbol{p}_w - \boldsymbol{r} = x_l \begin{pmatrix} 0.866 \\ 0 \\ -0.5 \end{pmatrix} + y_l \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z_l \begin{pmatrix} 0.5 \\ 0 \\ 0.866 \end{pmatrix}$$

Expressed in this way, it's easy to see that, to get the full world-space coordinates, we need to add the robot's position to the linear combination:

$$\boldsymbol{p}_w = x_l \begin{pmatrix} 0.866 \\ 0 \\ -0.5 \end{pmatrix} + y_l \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z_l \begin{pmatrix} 0.5 \\ 0 \\ 0.866 \end{pmatrix} + \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix}$$

Notice that this is exactly the result obtained from constructing and applying the homogeneous matrix from the orthonormal basis and the translation components:

$$\begin{pmatrix} 0.866 & 0 & 0.5 & 1 \\ 0 & 1 & 0 & 10 \\ -0.5 & 0 & 0.866 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ z_l \\ 1 \end{pmatrix} = \begin{pmatrix} 0.866x_l + 0y_l + 0.5z_l + 1 \\ 0x_l + 1y_l + 0z_l + 10 \\ -0.5x_l + 0y_l + 0.866z_l + 3 \\ 1 \end{pmatrix}$$

… which shouldn't really be a surprise, as this is exactly what the matrix "is" :D (You may even recognise it as a rotation of 30° about the y-axis, followed by the translation).

From the above, we can obtain the following functions to convert local coordinates to world-space ones:

$$\begin{aligned} x_w &= 0.866x_l + 0.5z_l + 1 \\ y_w &= y_l + 10 \\ z_w &= -0.5x_l + 0.866z_l + 3 \end{aligned}$$

The following points are expressed in object space; calculate their coordinates in world space:

a. (-1, 2, 0)
(0.866 x (-1) + 0.5 x 0 + 1, 2 + 10, -0.5 x (-1) + 0.866 x 0 + 3) = (0.134, 12, 3.5)

b. (1, 2, 0)
(0.866 x 1 + 0.5 x 0 + 1, 2 + 10, -0.5 x 1 + 0.866 x 0 + 3) = (1.866, 12, 2.5)

c. (0, 0, 0)
(0.866 x 0 + 0.5 x 0 + 1, 0 + 10, -0.5 x 0 + 0.866 x 0 + 3) = (1, 10, 3)
… The robot's position is the origin of its local space!

d. (1, 5, 0.5)
(0.866 x 1 + 0.5 x 0.5 + 1, 5 + 10, -0.5 x 1 + 0.866 x 0.5 + 3) = (2.116, 15, 2.933)

e. (0, 5, 10)
(0.866 x 0 + 0.5 x 10 + 1, 5 + 10, -0.5 x 0 + 0.866 x 10 + 3) = (6, 15, 11.66)

The coordinates below are in world space; find their positions relative to the robot:
We now need to convert the coordinates that are a linear combination of our standard basis vectors (the usual x, y and z axes) into a linear combination of the

robot's axes by finding the local coordinates/coefficients.

One way to do this is to rearrange the functions we were using to do the object-to-world conversion to isolate the world coordinate values:

$$x_w = 0.866x_l + 0.5z_l + 1$$
$$\Longrightarrow z_l = \frac{x_w - 0.866x_l - 1}{0.5} = 2x_w - 1.732x_l - 2$$

Substituting this into the formula for z_w gives x_l in terms of world coordinates:

$$\begin{aligned} z_w &= -0.5x_l + 0.866z_l + 3 \\ &= -0.5x_l + 0.866(2x_w - 1.732x_l - 2) + 3 \\ &= (-0.5 - 1.5)x_l + 1.732x_w - 1.732 + 3 \\ z_w &= -2x_l + 1.732x_w + 1.268 \\ \Rightarrow x_l &= 0.866x_w - 0.5z_w + 0.634 \end{aligned}$$

Now putting this back into the expression for z_l:

$$\begin{aligned} z_l &= 2x_w - 1.732x_l - 2 \\ &= 2x_w - 1.732(0.866x_w - 0.5z_w + 0.634) - 2 \\ &= (2 - 1.5)x_w + 0.866z_w - 1.098 - 2 \\ &= 0.5x_w + 0.866z_w - 3.098 \end{aligned}$$

Finally, the y coordinates aren't affected by the others, so our final functions are:

$$\begin{aligned} x_l &= 0.866x_w - 0.5z_w + 0.634 \\ y_l &= y_w - 10 \\ z_l &= 0.5x_w + 0.866z_w - 3.098 \end{aligned}$$

Of course, we could also achieve the same result by finding the inverse matrix, either using numerical methods or by constructing the matrices for the reverse transformations – rotating by -30° about the y-axis and translating by (-1, 10, 3) – and multiplying them, remembering that the translation must be applied first in this case:

$$\begin{pmatrix} 0.866 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0.866 & 0 & -0.5 & (-1) \times 0.866 + (-3) \times (-0.5) \\ 0 & 1 & 0 & -10 \\ 0.5 & 0 & 0.866 & (-1) \times 0.5 + (-3) \times 0.866 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0.866 & 0 & -0.5 & 0.634 \\ 0 & 1 & 0 & -10 \\ 0.5 & 0 & 0.866 & -3.098 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Whichever method you use should give exactly the same results, but let's use the formulae again this time:

f. (1, 10, 3)
(0.866 x 1 - 0.5 x 3 + 0.634, 10 - 10, 0.5 x 1 + 0.866 x 3 - 3.098) = (0, 0, 0)
… As we would expect, since this is the robot's position in world space.

g. (0, 0, 0)
(0.866 x 0 - 0.5 x 0 + 0.634, 0 - 10, 0.5 x 0 + 0.866 x 0 - 3.098) = (0.634, -10, -3.098)
If we view this transformation as one that moves an object around in world space,

then rotating the origin has no effect (since rotations are about the origin) and the net result is the translation values.

h. (2.732, 10, 2)
(0.866 x 2.732 - 0.5 x 2 + 0.634, 10 - 10, 0.5 x 2.732 + 0.866 x 2 - 3.098) = (2, 0, 0)

i. (2, 11, 4)
(0.866 x 2 - 0.5 x 4 + 0.634, 11 - 10, 0.5 x 2 + 0.866 x 4 - 3.098) = (0.366, 1, 1.366)

j. (1, 20, 3)
(0.866 x 1 - 0.5 x 3 + 0.634, 20 - 10, 0.5 x 1 + 0.866 x 3 - 3.098) = (0, 10, 0)