COMP270: Mathematics for 3D Worlds and Simulations

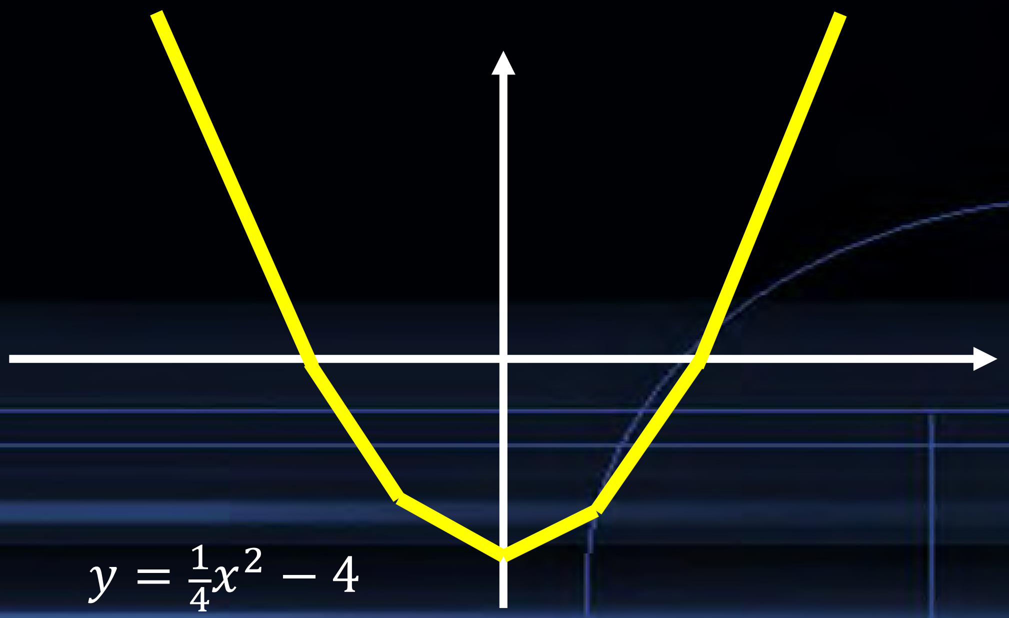
WEEK 2: GEOMETRY I
PART 4: CURVES AND PARAMETERS

Objectives

- **Express** a variety of shapes using parametric equations
- **Compute** the vector equation of a straight line

Recap: drawing functions

- Define a function $f : S \to T$ as $f(s) \in T$ for $s \in S$
- Represent the function as a graph by plotting the points $\left(x, f(x)\right)$ against 2D axes

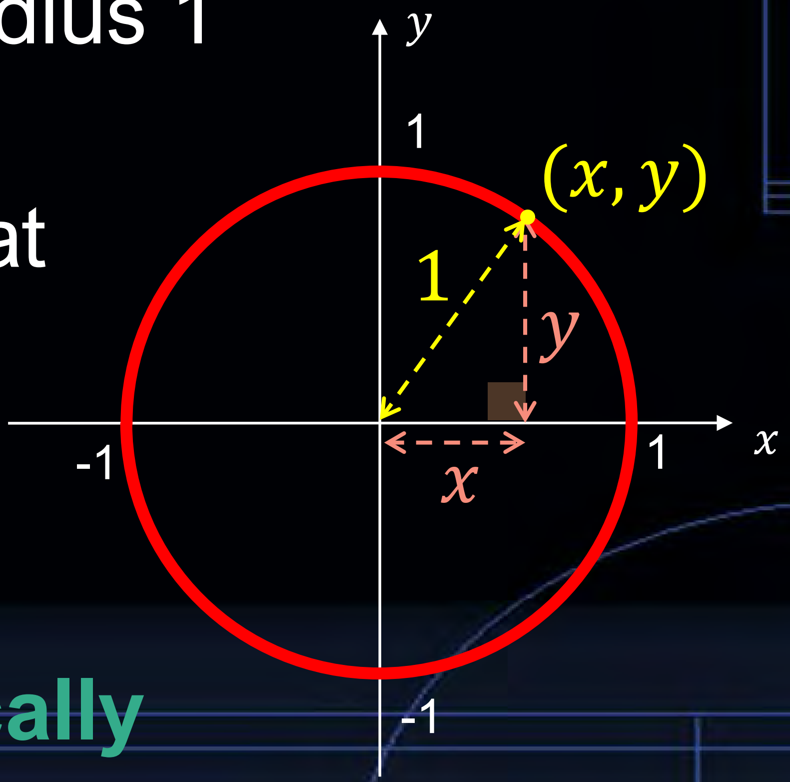


What is a curve?

- "The [curved] line is […] the first species of quantity, which has only one dimension, namely length, without any width nor depth, and is nothing else than the flow or run of the point which […] will leave from its imaginary moving some vestige in length, exempt of any width."
  - Euclid, *Elements* (English translation from Wikipedia)

Defining a circle

- How do we define a circle of radius 1 (aka a **unit circle**)?
- The set of points (x, y) such that
$$x^2 + y^2 = 1$$
- The pair of curves
$$y = \pm\sqrt{1 - x^2}$$
- Or we can define it **parametrically**



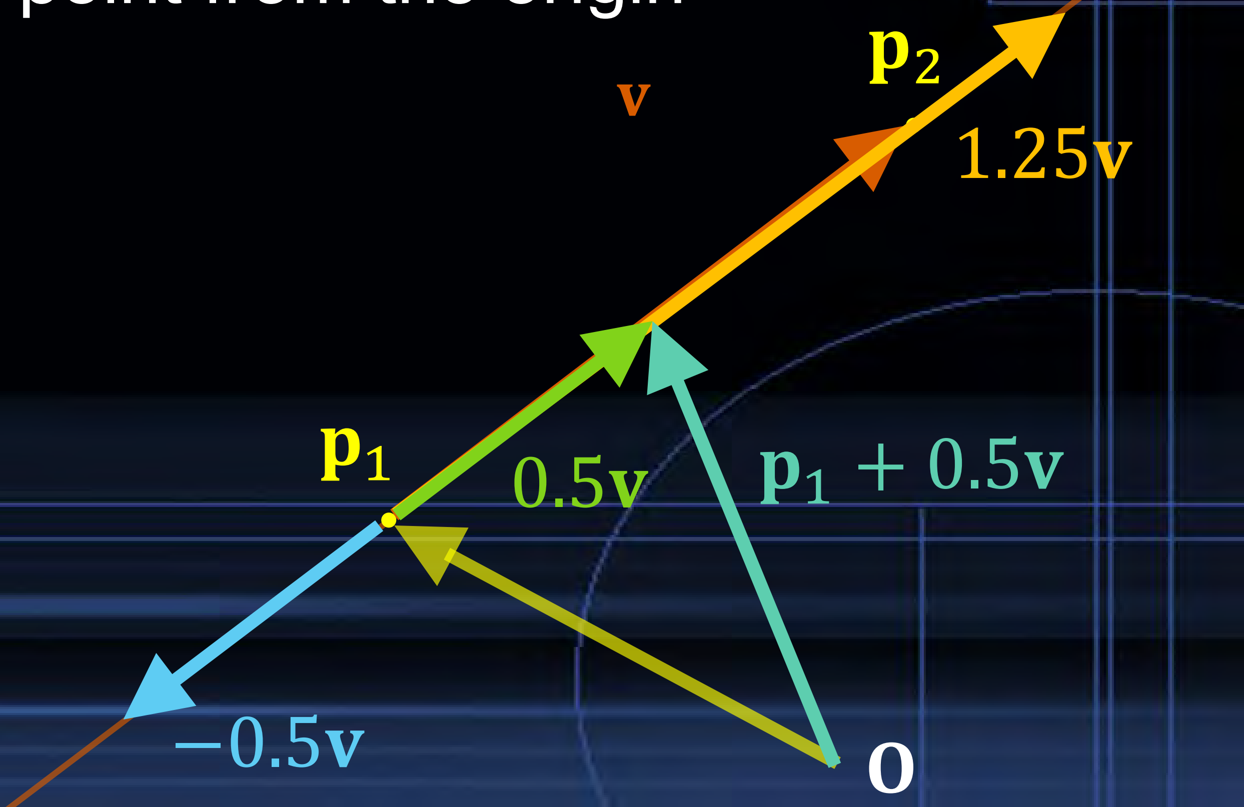
Parametric equations

- Definition: **parametric equations** express a set of quantities as explicit functions of a number of independent variables, known as "parameters"
- e.g. a curve defined by two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, with points (x, y) with $x = f(t)$ and $y = g(t)$ for t in some range
  - t is the parameter
- Equivalently, defined by a single function $h : \mathbb{R} \to \mathbb{R}^2$ which takes a scalar parameter and returns a vector

$$h(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

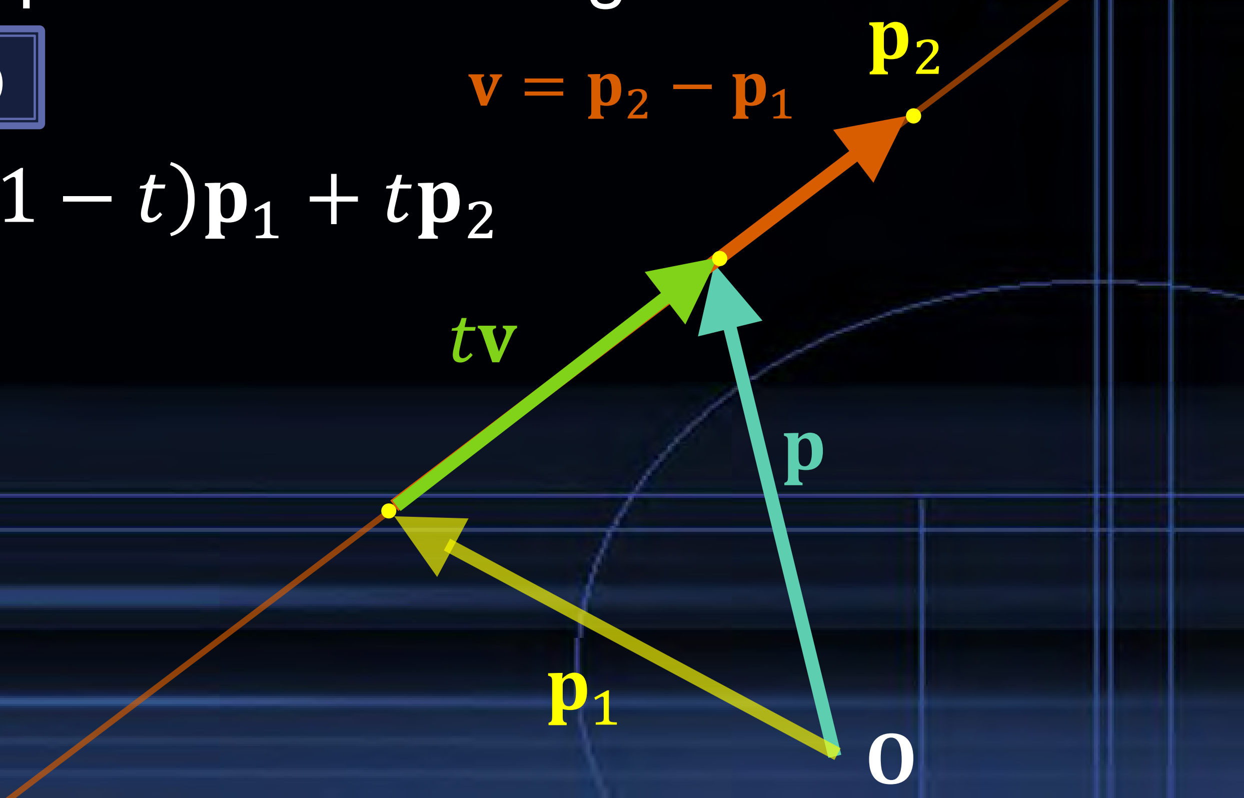
Vector equation of a line

- For a line defined by two points, we can represent any point on the line as a scalar multiple of the vector between the points, plus the vector to first point from the origin



Vector equation of a line

- For a line defined by two points, we can represent any point on the line as a scalar multiple of the vector between the points, plus the vector to first point from the origin
- $\mathbf{p} = \mathbf{p}_1 + t\mathbf{v}$ A function of t, $f(t)$
- $f(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$



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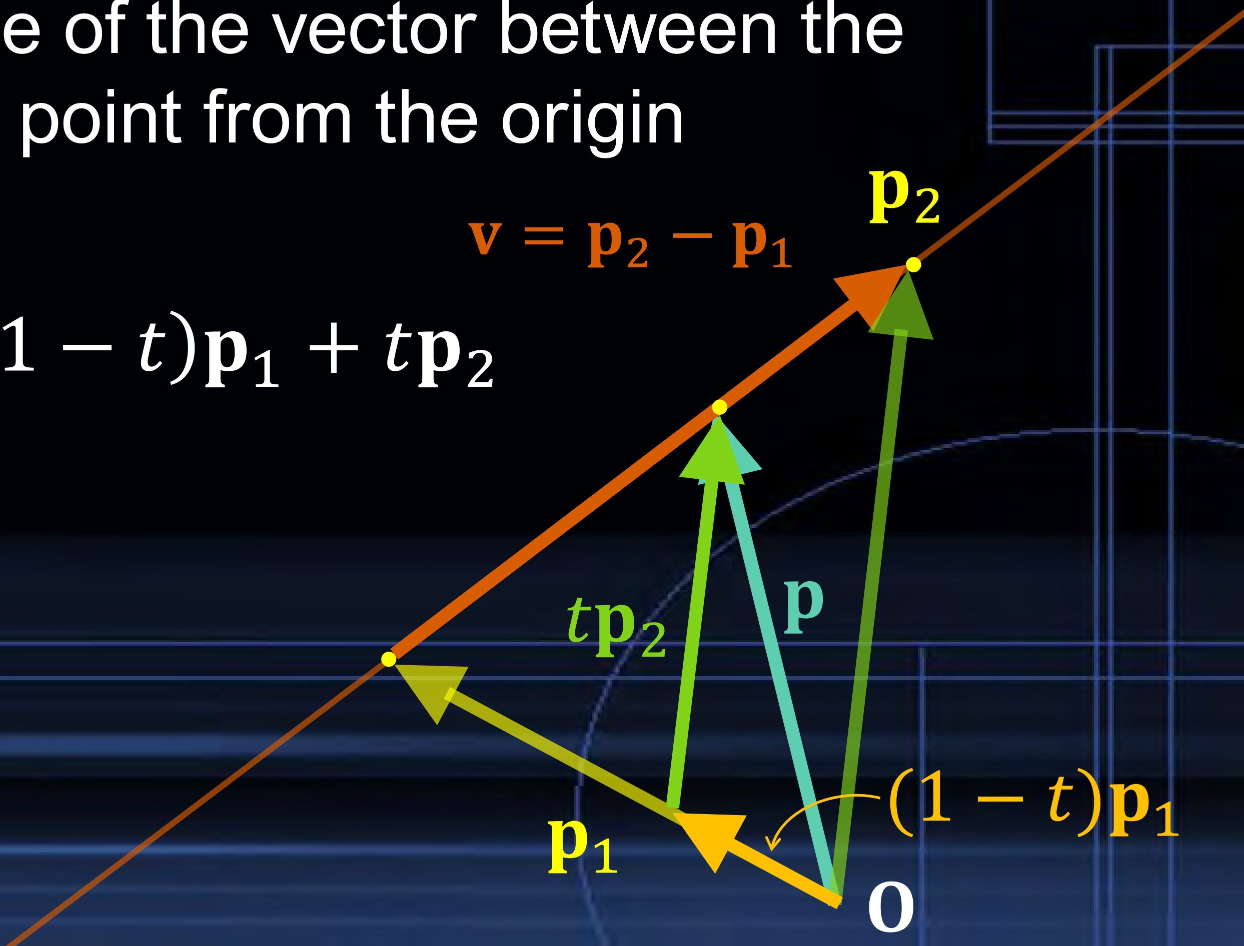

Vector equation of a line

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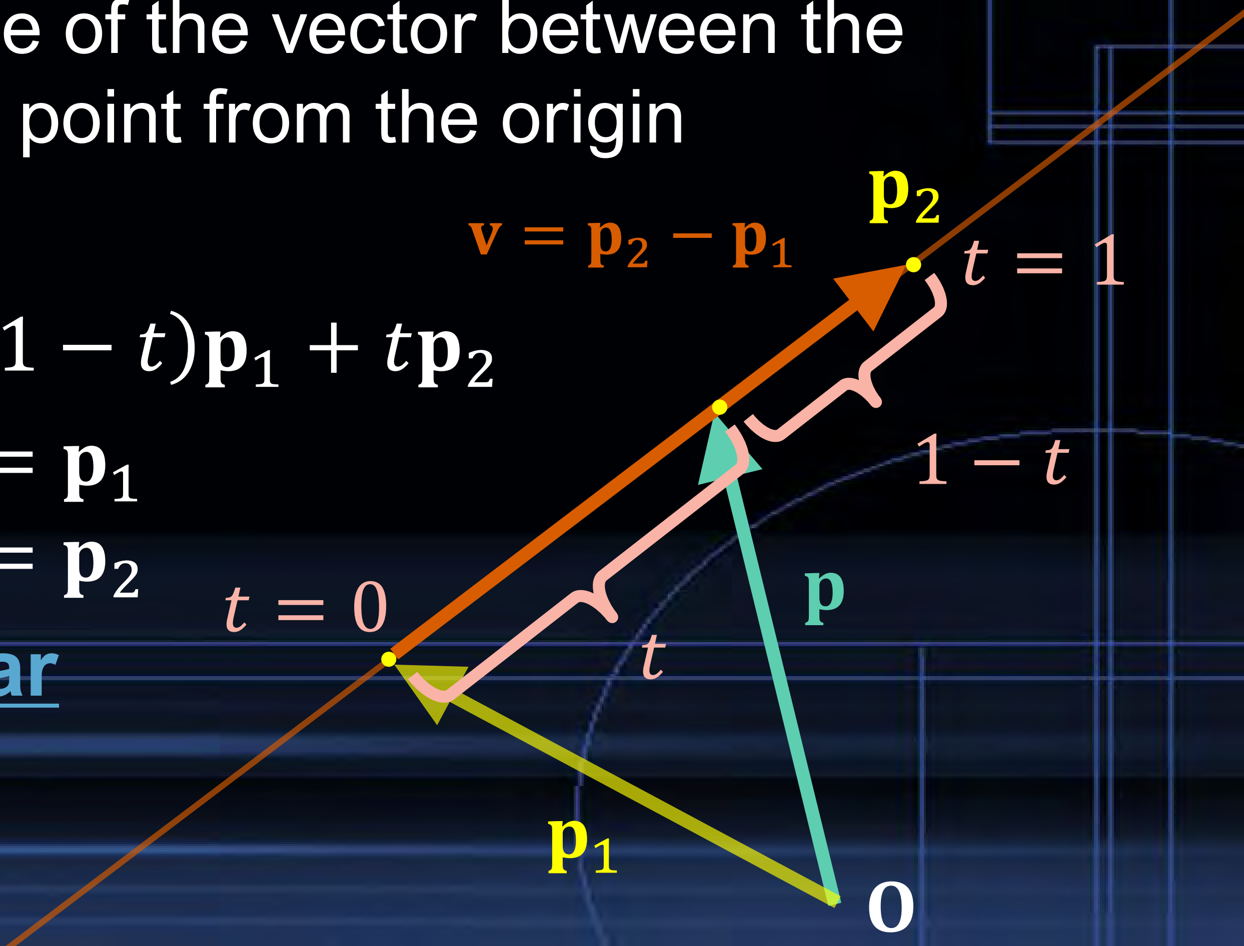
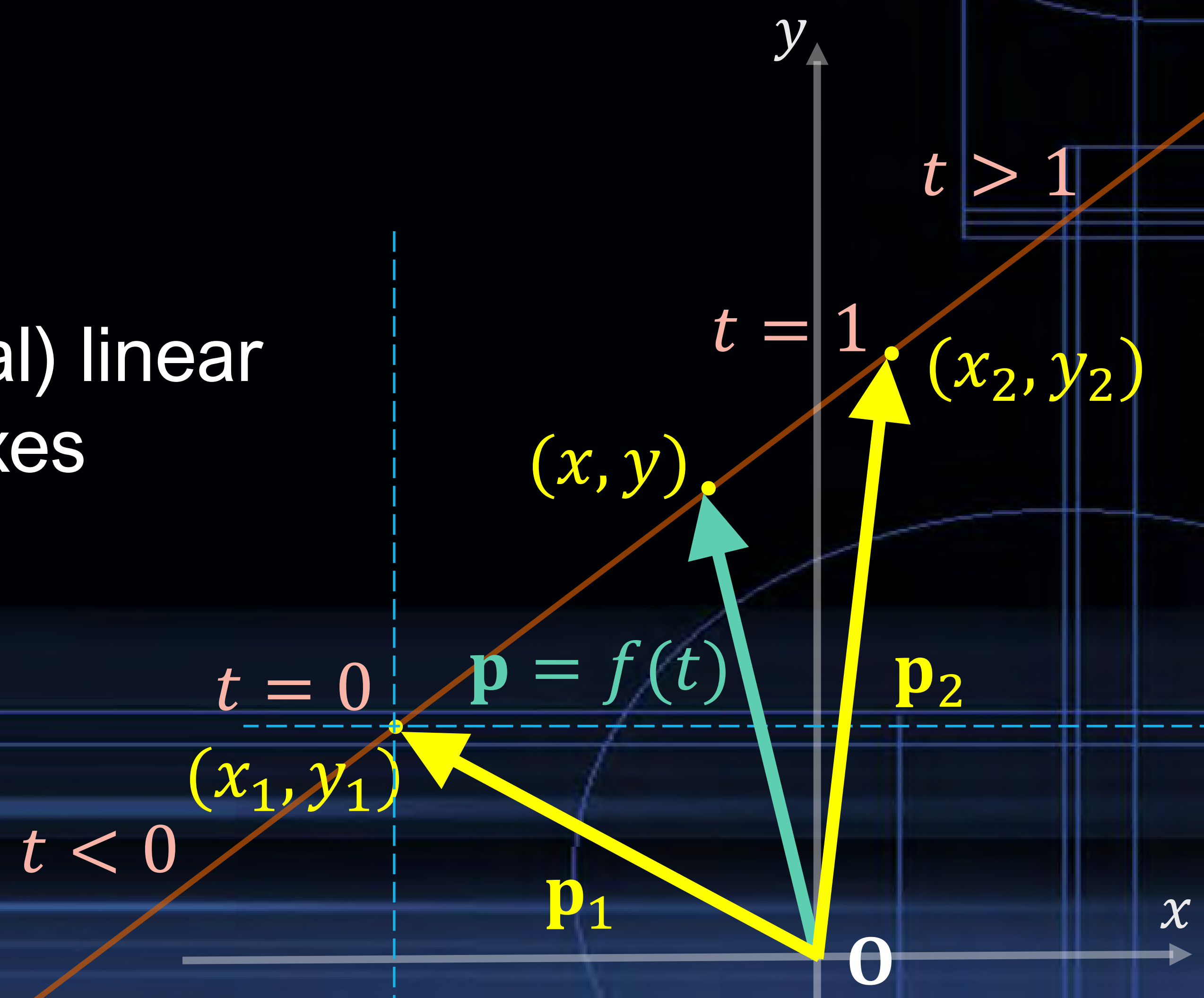
- $f(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$
- $t = 0 \Rightarrow f(0) = 1\mathbf{p}_1 + 0\mathbf{p}_2 = \mathbf{p}_1$
  $t = 1 \Rightarrow f(1) = 0\mathbf{p}_1 + 1\mathbf{p}_2 = \mathbf{p}_2$
- For $0 \leq t \leq 1$, this is a linear interpolation (lerp)



Parameterising a line

- $f(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$
- $x = f(t) = (1-t)x_1 + tx_2$
  $y = g(t) = (1-t)y_1 + ty_2$
- i.e. performing a(n identical) linear interpolation along both axes

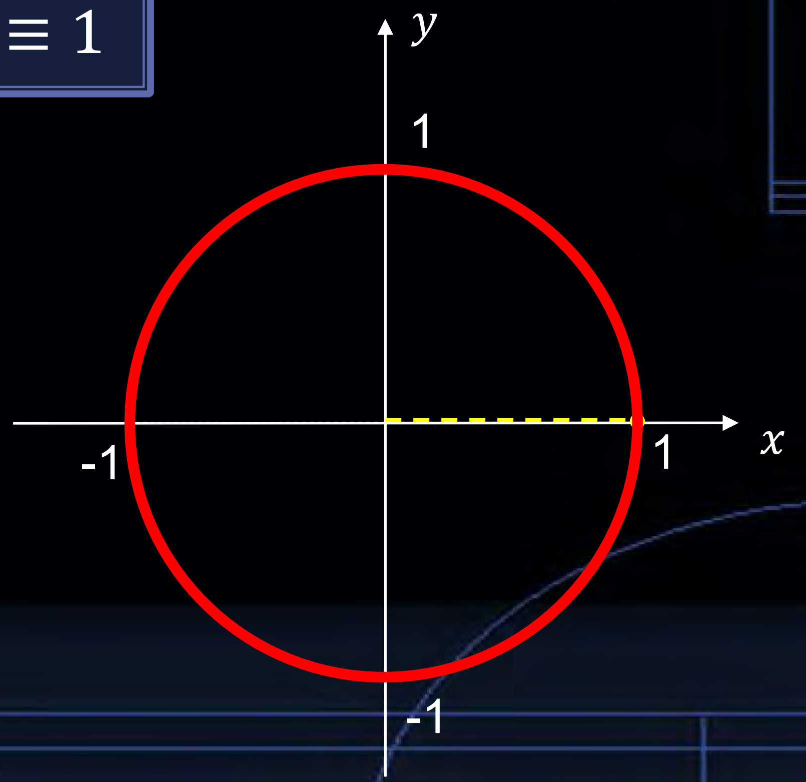


Parametric definition of a unit circle

$\sin^2 t + \cos^2 t \equiv 1$

$$x = \cos t$$
$$y = \sin t$$

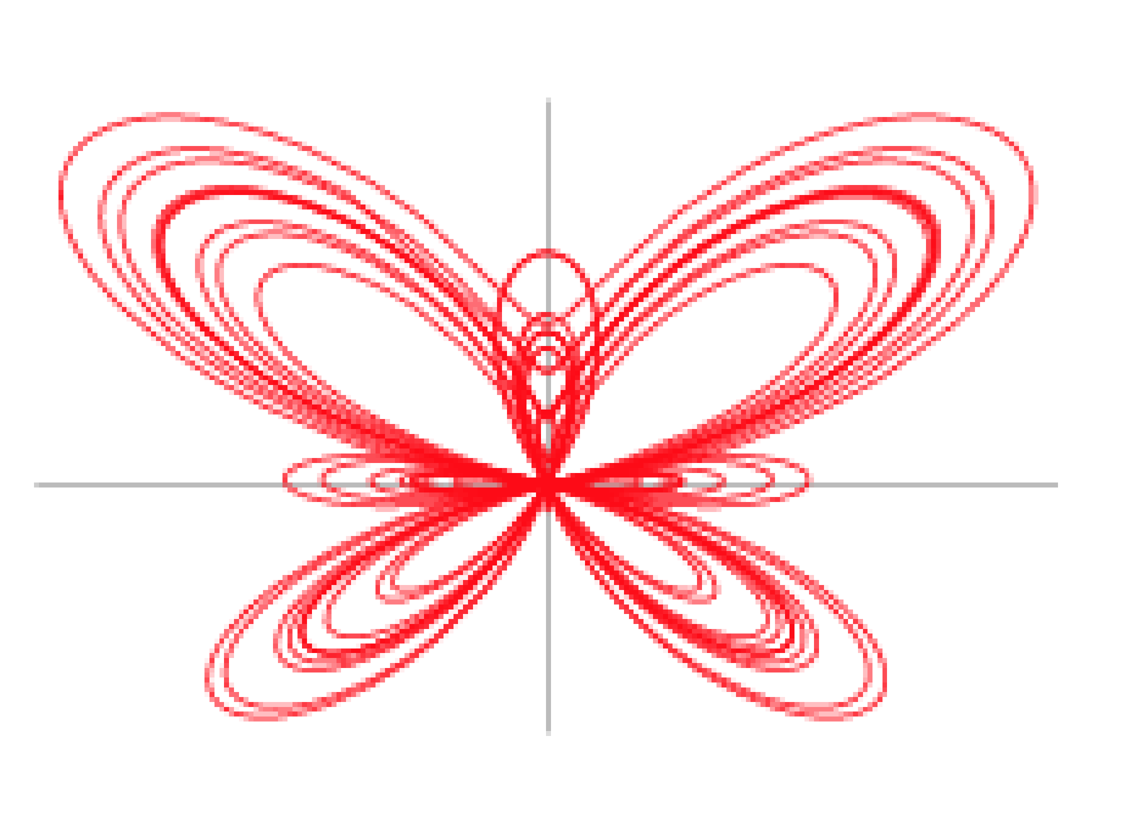
For $0 \leq t < 2\pi$



Parametric definition of a butterfly

$$x = \sin t\left[e^{\cos t} - 2\cos(4t) + \left(\sin\left(\tfrac{1}{12}t\right)\right)^5\right]$$
$$y = \cos t\left[e^{\cos t} - 2\cos(4t) + \left(\sin\left(\tfrac{1}{12}t\right)\right)^5\right]$$

https://mathworld.wolfram.com/ButterflyCurve.html



Bézier curves

Named after Pierre Bézier, 1910-1999, French engineer

- Defined by a weighted blend of a number of **control points**
- Commonly used in computer graphics and game development, as allows artists/designers good control over the precise shape of the curve
- See worksheet A…