Week 7:
3D Computational Geometry I

COMP20: Mathematics for 3D Worlds & Simulations

BSc(Hons) Computing for Games

Vectors in 3D

Addition and subtraction in 3D

Addition:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Subtraction:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

3D dot product and magnitude

Dot product:

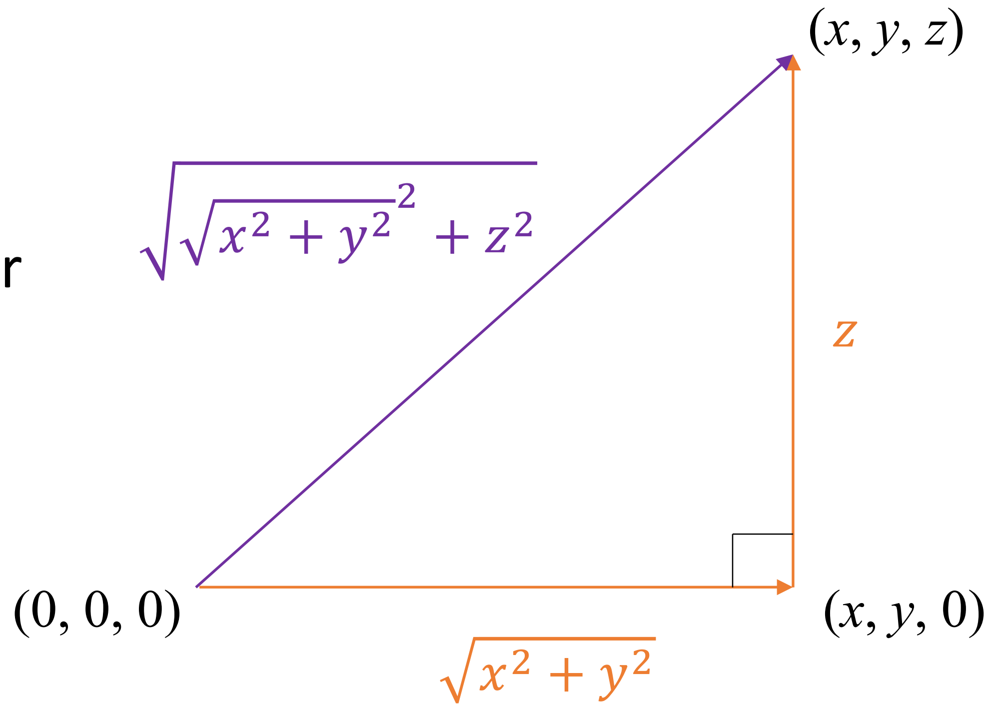
$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Magnitude:

$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

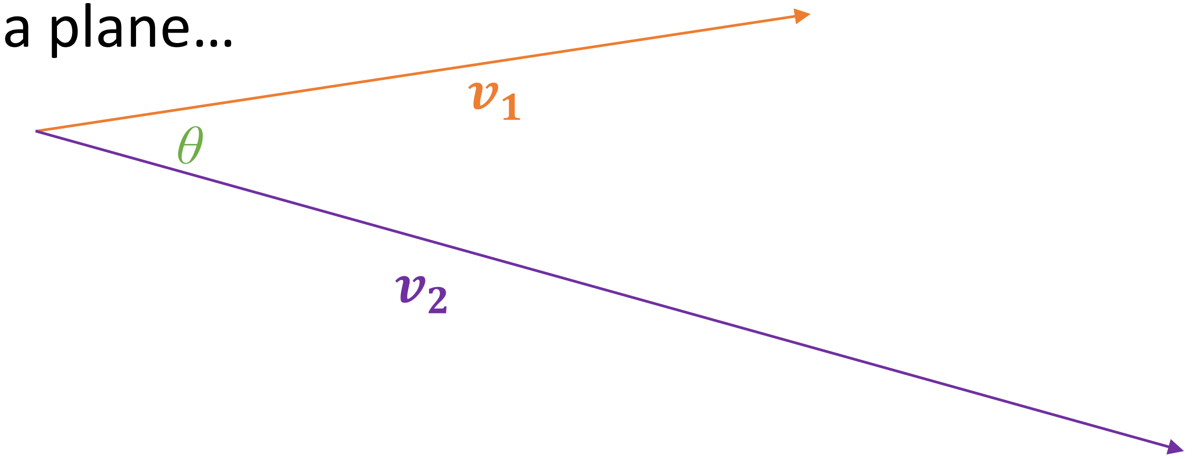
3D vector magnitude: proof

- Consider that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$
- We know that the magnitude of the 2D vector $\binom{x}{y}$ is $\sqrt{x^2 + y^2}$
- Consider the triangle orthogonal to the xy plane, formed by this vector and the z component…



3D dot product: geometric interpretation

- 2D theorem: $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \|\boldsymbol{v}_1\| \|\boldsymbol{v}_2\| cos\theta$
- Still applies in 3D because proof is based only on the two vectors, which will always lie on a plane…



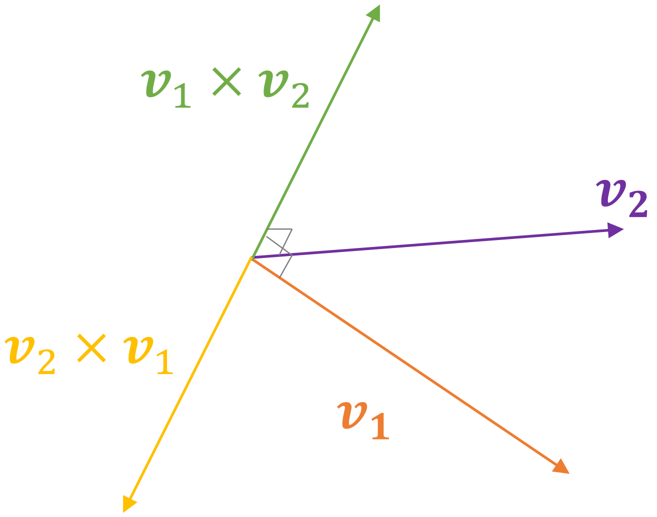
- For proof/derivation of the formula, see proofwiki.org/wiki/Cosine_Formula_for_Dot_Product

Vector cross product

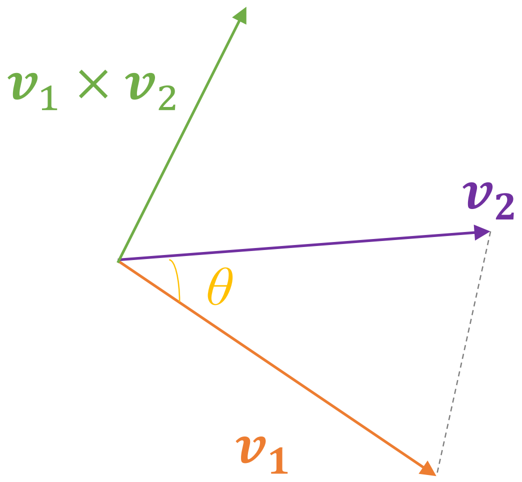
$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Properties:

- $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ is orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and forms a right-handed system
- $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = -(\boldsymbol{v}_2 \times \boldsymbol{v}_1) = -(\boldsymbol{v}_1) \times -(\boldsymbol{v}_2)$
- $\boldsymbol{v}_1 \times \boldsymbol{v}_1 = \boldsymbol{0}$
  - True for any parallel vectors



Vector cross product: geometric interpretation



$\boldsymbol{v}_1 \times \boldsymbol{v}_2 \ = \|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\| sin\theta\ \widehat{\boldsymbol{n}}$, or

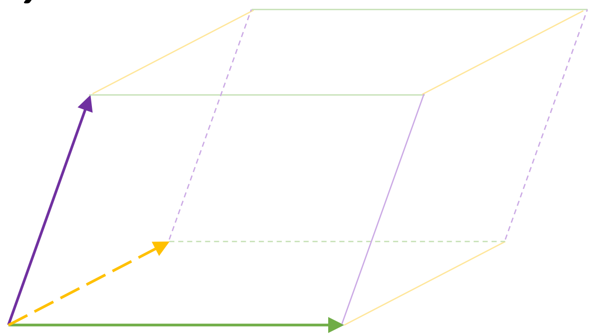
$\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\| = \|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\| sin\theta$

Area of the triangle = $\frac{1}{2}\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|$

Scalar triple product

$$\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3)$$

= the (signed) volume of the parallelepiped defined by the three vectors



Properties:

- $\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3) = \boldsymbol{v}_2 \cdot (\boldsymbol{v}_3 \times \boldsymbol{v}_1) = \boldsymbol{v}_3 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2)$
- $\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3) = (\boldsymbol{v}_1 \times \boldsymbol{v}_2) \cdot \boldsymbol{v}_3$
- $\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3) = -\boldsymbol{v}_1 \cdot (\boldsymbol{v}_3 \times \boldsymbol{v}_2) = -\boldsymbol{v}_2 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_3) = -\boldsymbol{v}_3 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_1)$
- $\boldsymbol{v}_1 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = \boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_1) = \boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_2) = \boldsymbol{v}_2 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_1) = 0$
- $\big(\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \boldsymbol{v}_3)\big)\boldsymbol{v}_1 = (\boldsymbol{v}_1 \times \boldsymbol{v}_2) \times (\boldsymbol{v}_1 \times \boldsymbol{v}_3)$

Vector triple product

$$\boldsymbol{v}_1 \times (\boldsymbol{v}_2 \times \boldsymbol{v}_3) = (\boldsymbol{v}_1 \cdot \boldsymbol{v}_3)\boldsymbol{v}_2 - (\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)\boldsymbol{v}_3$$

Properties:

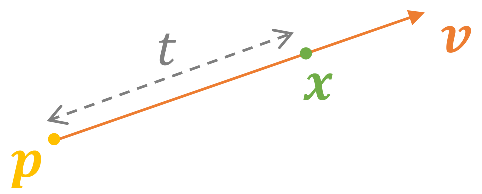
- $(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \times \boldsymbol{v}_3 = -\boldsymbol{v}_3 \times (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = -(\boldsymbol{v}_3 \cdot \boldsymbol{v}_2)\boldsymbol{v}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{v}_1)\boldsymbol{v}_2$
- $\boldsymbol{v}_1 \times (\boldsymbol{v}_2 \times \boldsymbol{v}_3) + \boldsymbol{v}_2 \times (\boldsymbol{v}_3 \times \boldsymbol{v}_1) + \boldsymbol{v}_3 \times (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = 0$
- $(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \times \boldsymbol{v}_3 = \boldsymbol{v}_1 \times (\boldsymbol{v}_2 \times \boldsymbol{v}_3) - \boldsymbol{v}_2 \times (\boldsymbol{v}_1 \times \boldsymbol{v}_3)$

Lines and planes

Vector equation of a line

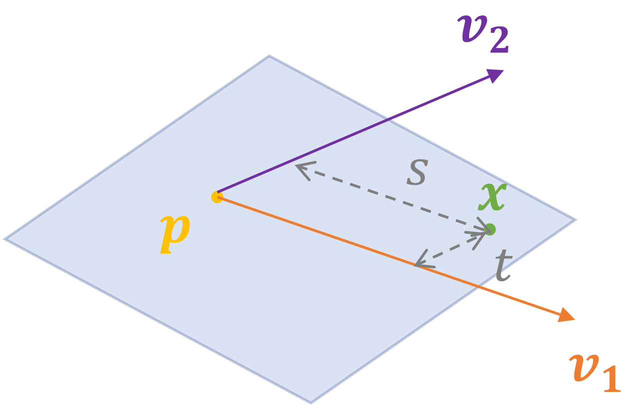
- Remember: a vector only describes a direction, which could be anywhere in space!
- We also need to know where to position it, by specifying a point $\boldsymbol{p}$ on the line.
- Any point $\boldsymbol{x}$ on the line can be expressed parametrically as a vector displacement, first to point $\boldsymbol{p}$ and then some distance along the line's direction $\boldsymbol{v}$:

$$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$$



Vector equation of a plane

- Any two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ define a plane
- Therefore, any point $\boldsymbol{x}$ lying on the plane can be expressed as a linear combination of the two vectors, starting from any point $\boldsymbol{p}$ on the plane:



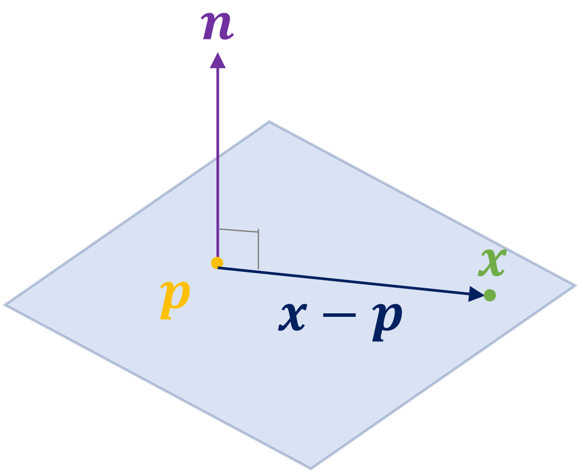
$$\boldsymbol{x} = \boldsymbol{p} + s\boldsymbol{v}_1 + t\boldsymbol{v}_2$$

Implicit equation of a plane

- Any two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ define a plane
- The vector perpendicular to both is the *plane normal*, $\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$
  - The normal *completely defines* the orientation of the plane; we just need to know its position, too…
- Choose a point with displacement vector $\boldsymbol{p}$ that lies on the plane
  - For any other point $\boldsymbol{x}$ that also lies on the plane, then:

$$(\boldsymbol{x} - \boldsymbol{p}) \cdot \boldsymbol{n} = 0$$

Geometric equation of a plane



If $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\boldsymbol{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then:

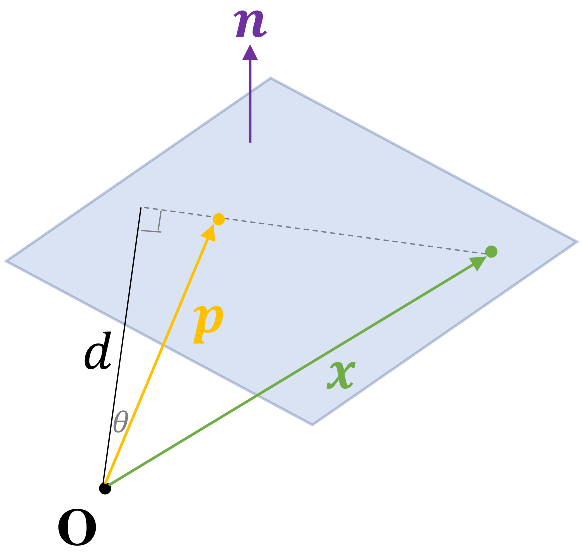
$$\begin{pmatrix} x - p_0 \\ y - p_1 \\ z - p_2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a(x - p_0) + b(y - p_1) + c(z - p_2) = 0$$

$$ax + by + cz + (-ap_0 - bp_1 - cp_2) = 0$$

$ax + by + cz + d = 0$, where $d = -ap_0 - bp_1 - cp_2$

Distance of a plane from the origin



$$d = \|\boldsymbol{p}\| cos\theta$$

$$d = \|\boldsymbol{p}\|\|\boldsymbol{\hat{n}}\| cos\theta$$

$$d = \boldsymbol{p} \cdot \boldsymbol{\hat{n}}$$

Since $(\boldsymbol{x} - \boldsymbol{p}) \cdot \boldsymbol{n} = 0 \Rightarrow \boldsymbol{x} \cdot \boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{n}$, then

$$d = \boldsymbol{p} \cdot \boldsymbol{\hat{n}}$$

Intersection of a line and a plane

A line with direction $\boldsymbol{v}$, passing through the point $\boldsymbol{a}$, has equation $\boldsymbol{x} = \boldsymbol{a} + s\boldsymbol{v}$

A plane passing through the point $\boldsymbol{b}$, with normal $\boldsymbol{n}$, has equation $(\boldsymbol{x} - \boldsymbol{b}) \cdot \boldsymbol{n} = 0$

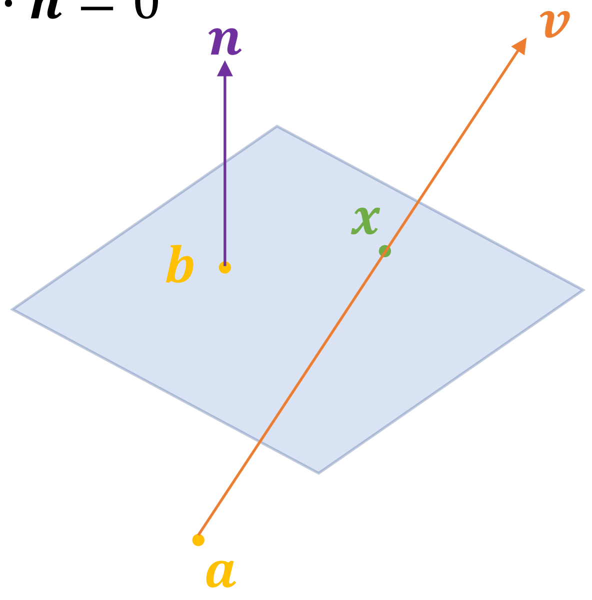
If $\boldsymbol{x}$ is the point of intersection of the line and the plane, then:

$$((\boldsymbol{a} + s\boldsymbol{v}) - \boldsymbol{b}) \cdot \boldsymbol{n} = 0$$

Solving for s, we get:

$$s\boldsymbol{v} \cdot \boldsymbol{n} + (\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{n} = 0$$

$$s = -\frac{(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{n}}{\boldsymbol{v} \cdot \boldsymbol{n}} = \frac{(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{n}}{\boldsymbol{v} \cdot \boldsymbol{n}}$$



Intersection of a line and a sphere

A line with direction $\boldsymbol{v}$, passing through the point $\boldsymbol{a}$, has equation $\boldsymbol{x} = \boldsymbol{a} + s\boldsymbol{v}$

A sphere centred at the origin with radius r has equation $x^2 + y^2 + z^2 = r^2$

If $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the point of intersection of the line and the sphere, then:

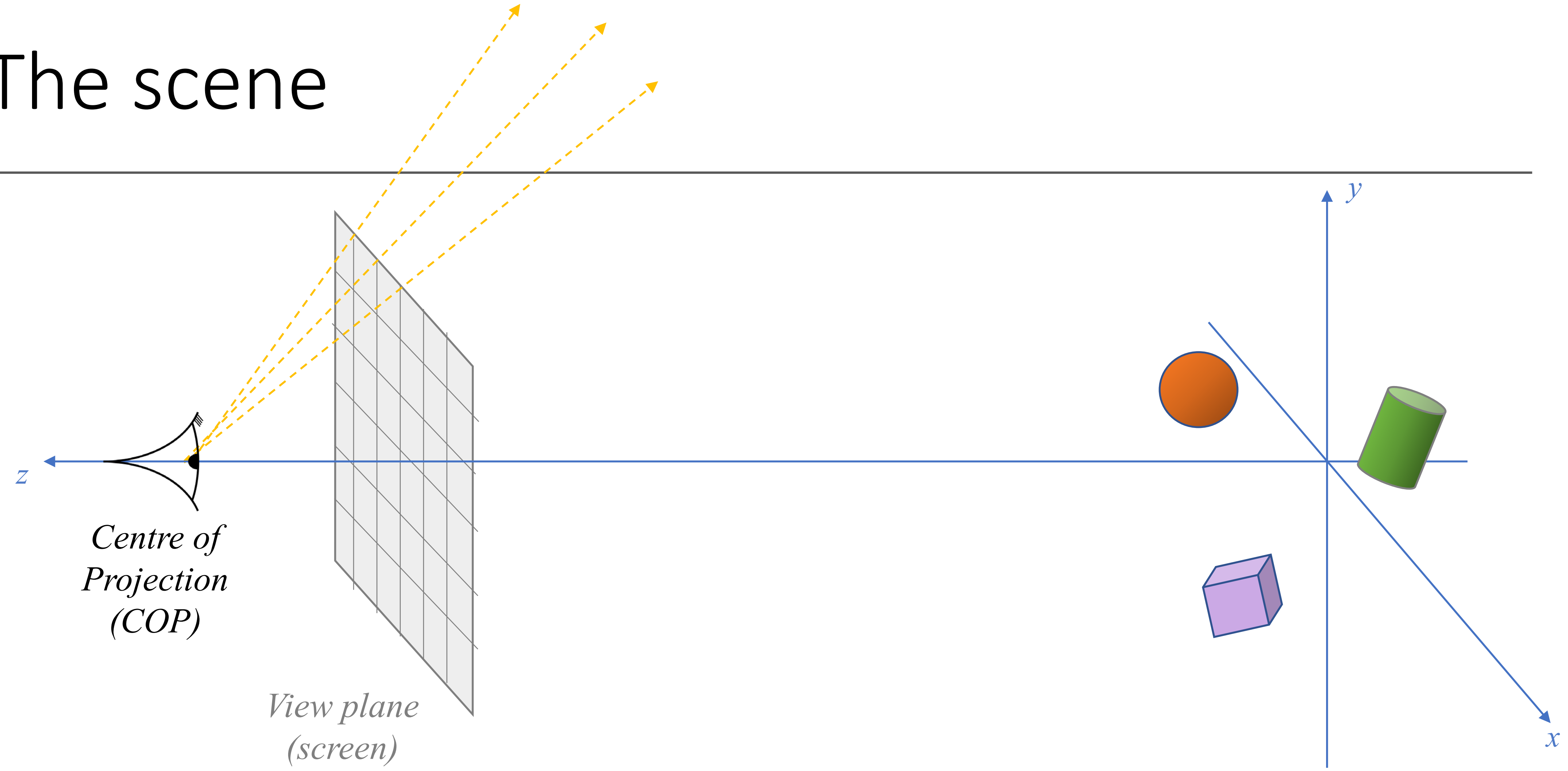
$$(a_x + sv_x)^2 + \left(a_y + sv_y\right)^2 + (a_z + sv_z)^2 = r^2$$

Solving for s, we get:

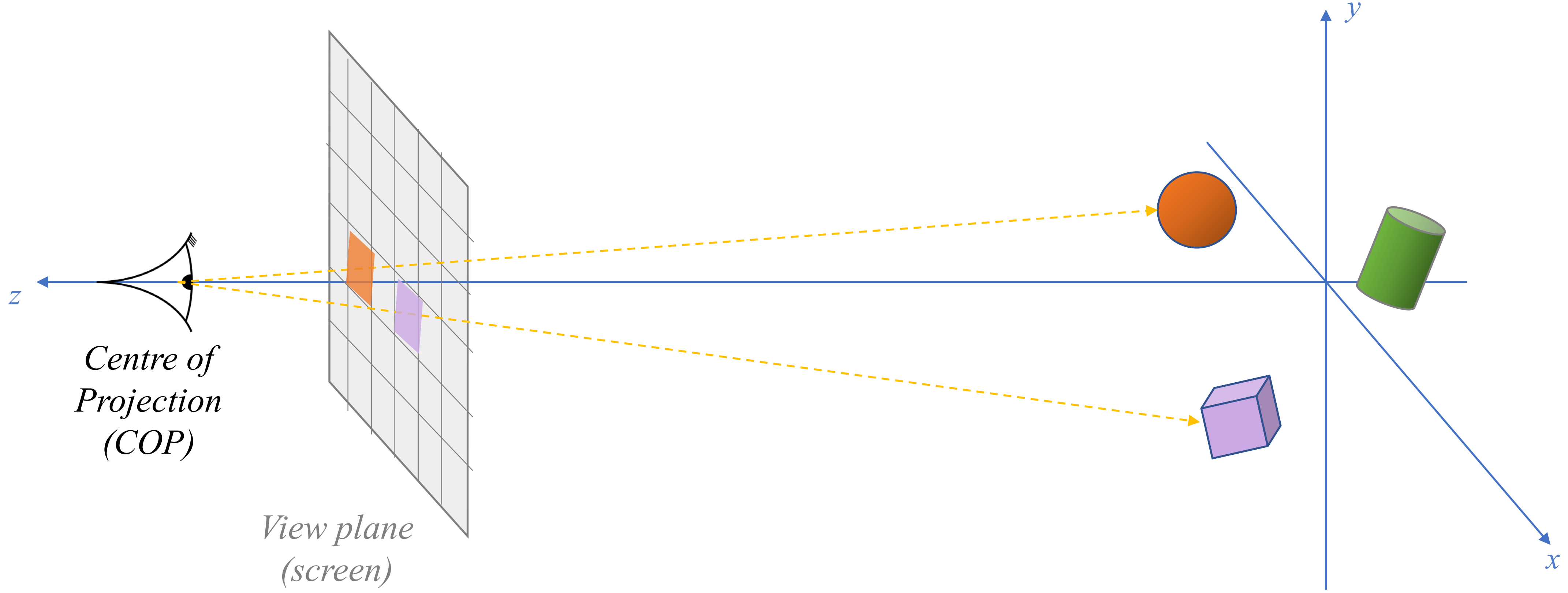
$$\left({v_x}^2 + {v_y}^2 + {v_z}^2\right)s^2 + 2\left(a_x v_x + a_y v_y + a_z v_z\right)s + \left({a_x}^2 + {a_y}^2 + {a_z}^2 - r^2\right) = 0$$

Putting vectors to use:
a simple camera model

The scene



The scene



Worksheet C: due Monday 18th November