Week 7: 3D Geometry I
Part 2: Lines and planes

COMP270: Mathematics for 3D Worlds and Simulations

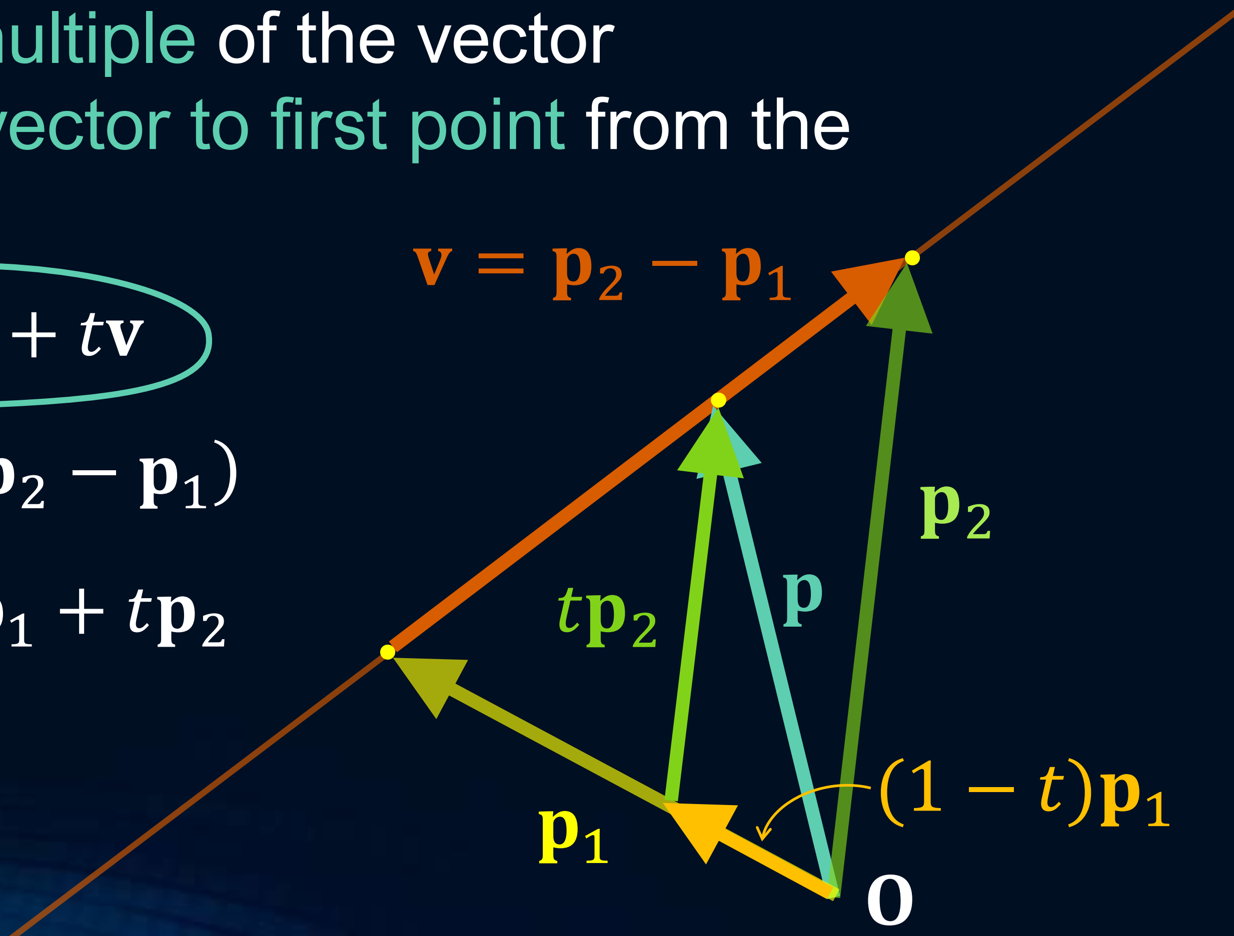
Objectives

- **Define** the equation of a plane
- **Extend** techniques for determining intersections to 3D objects

Recap: vector equation of a line

- For a line defined by two points, we can represent any point on the line as a scalar multiple of the vector between the points, plus the vector to first point from the origin

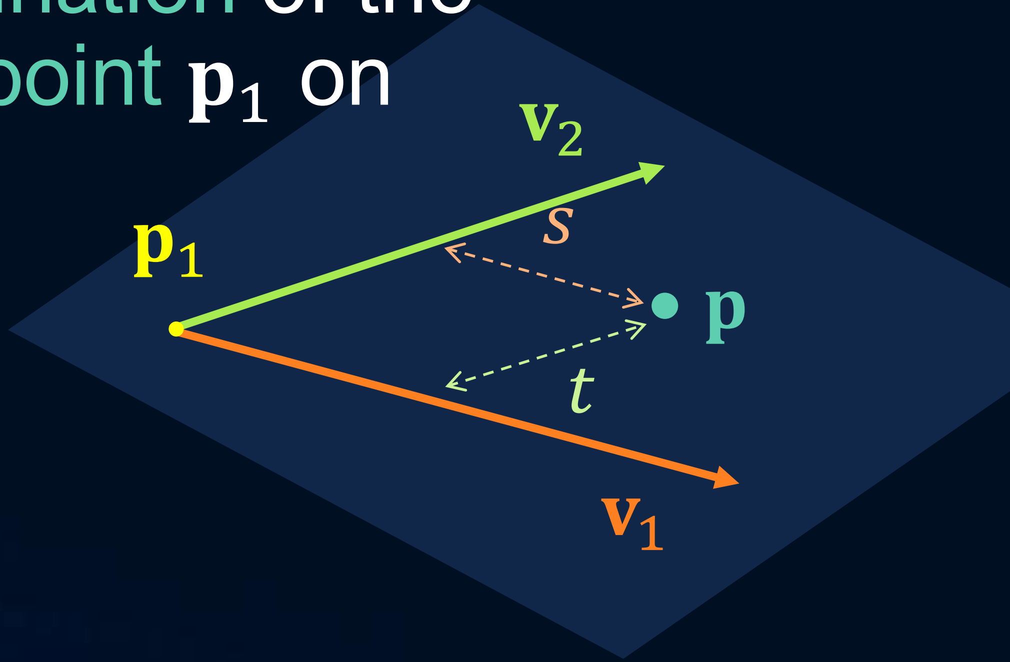
$$\mathbf{p} = \mathbf{p}_1 + t\mathbf{v}$$
$$= \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1)$$
$$= (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$



Vector equation of a plane

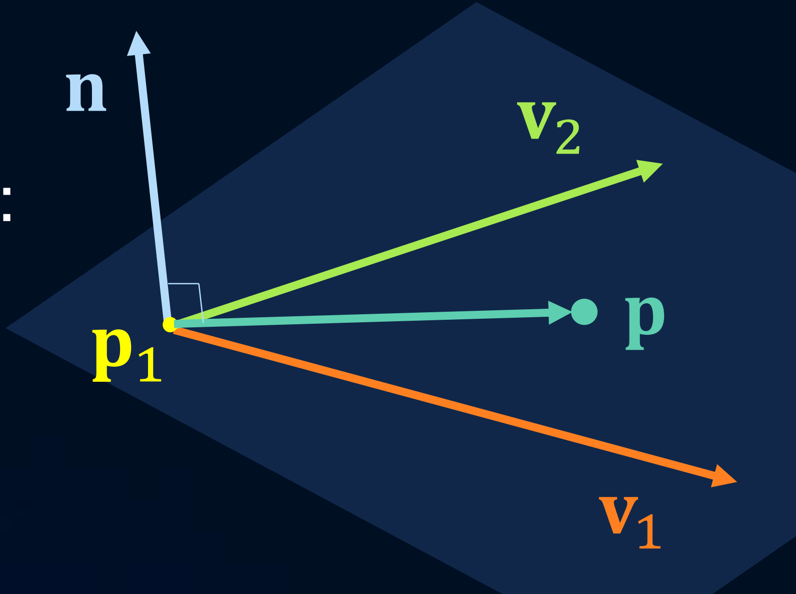
- Any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ define a plane
- Therefore, any point $\mathbf{p}$ lying on the plane can be expressed as a linear combination of the two vectors, starting from any point $\mathbf{p}_1$ on the plane:

$$\mathbf{p} = \mathbf{p}_1 + s\mathbf{v}_1 + t\mathbf{v}_2$$



Implicit equation of a plane

- Any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ define a plane
- The vector perpendicular to both is the plane normal,
$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$
  - The normal completely defines the orientation of the plane
- Choose a point $\mathbf{p}_1$ that lies on the plane
  - For any other point $\mathbf{p}$ that also lies on the plane:
$$(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{n} = 0$$



Geometric equation of a plane

If $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\mathbf{p}_1 = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{n} = 0$ gives:

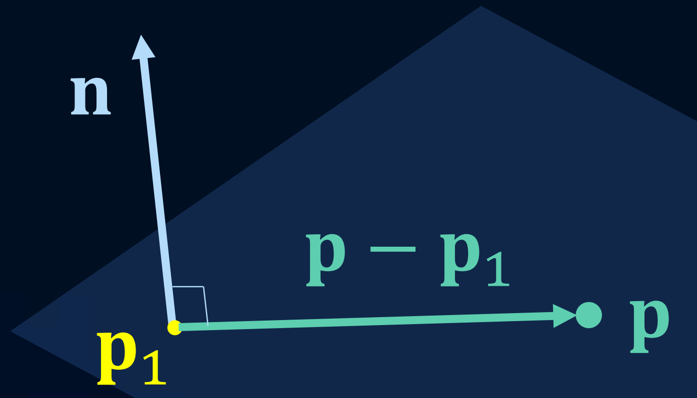
$$\begin{pmatrix} x - p_x \\ y - p_y \\ z - p_z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a(x - p_x) + b\left(y - p_y\right) + c(z - p_z) = 0$$

$$ax + by + cz + \left(-ap_x - bp_y - cp_z\right) = 0$$

$$ax + by + cz + d = 0$$

where $d = -ap_x - bp_y - cp_z$

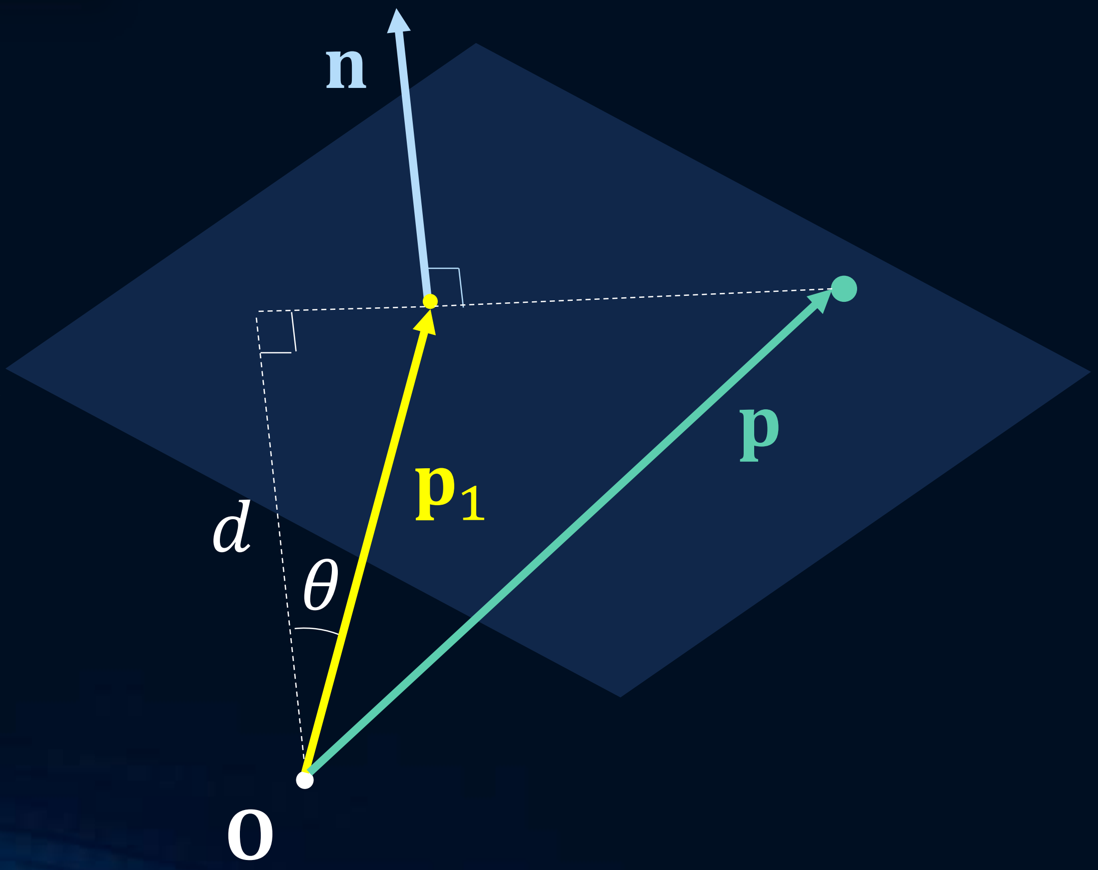


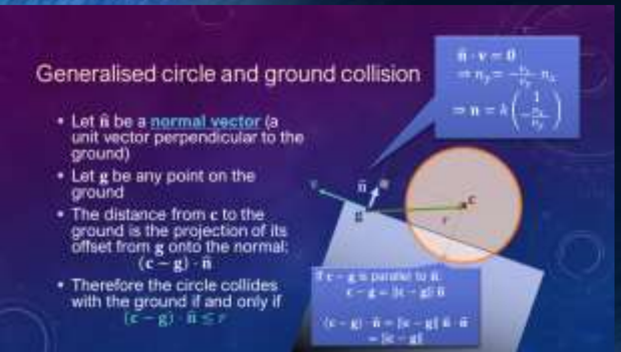
Distance of a plane from the origin

- $d = \|\mathbf{p}_1\| \cos\theta$
  $= \|\mathbf{p}_1\| \|\hat{\mathbf{n}}\| \cos\theta$
  $= \mathbf{p}_1 \cdot \hat{\mathbf{n}}$

  $\|\hat{\mathbf{n}}\| = 1$

- Since $(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{n} = 0 \Rightarrow$ $\mathbf{p} \cdot \mathbf{n} = \mathbf{p}_1 \cdot \mathbf{n}$, then $d = \mathbf{p} \cdot \hat{\mathbf{n}}$





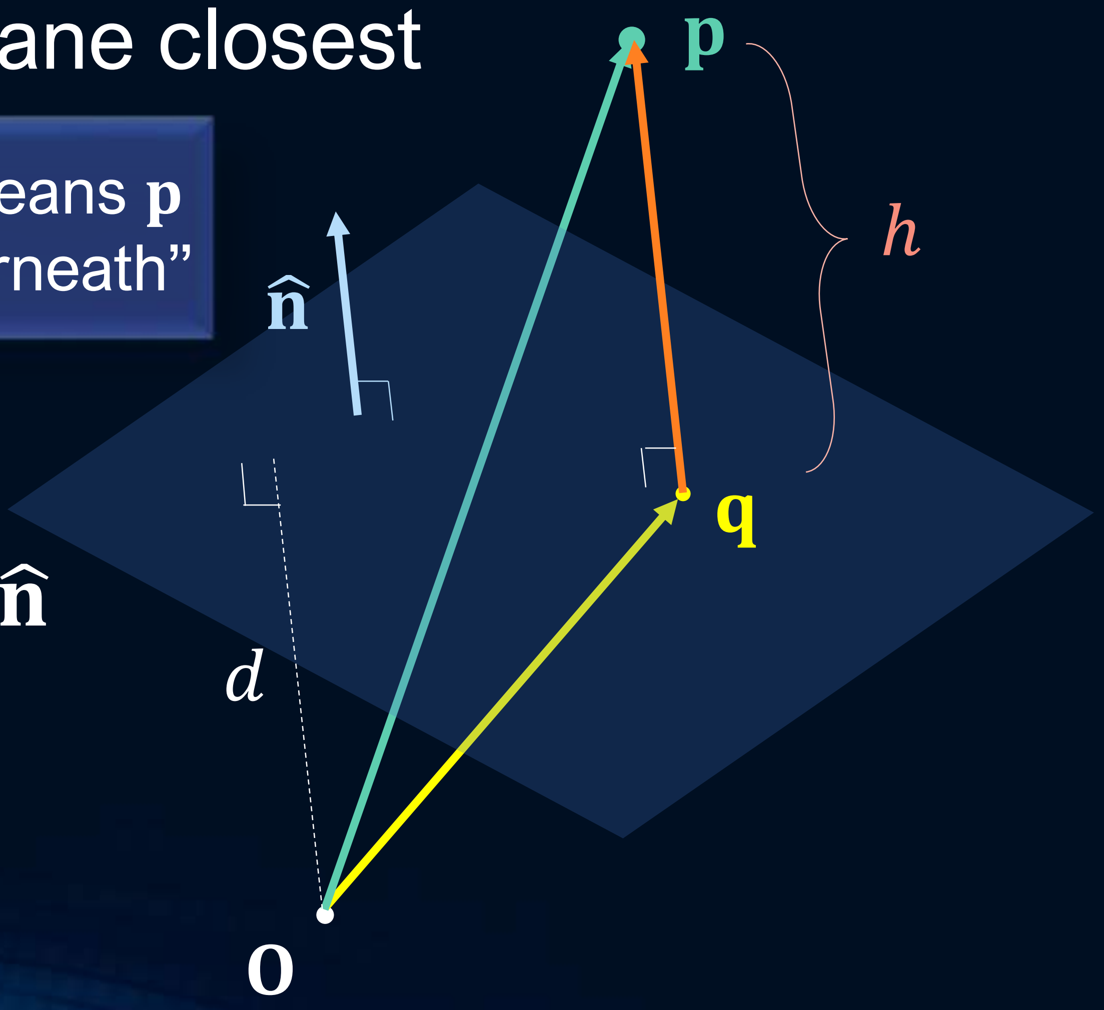
Distance of a point from a plane

- Let $\mathbf{q}$ be the point on the plane closest to $\mathbf{p}$
- Then $\mathbf{p} - \mathbf{q} = h\hat{\mathbf{n}}$
- Also, $\mathbf{p} = \mathbf{q} + (\mathbf{p} - \mathbf{q})$:

$h < 0$ means $\mathbf{p}$ is "underneath"

$$\mathbf{p} = \mathbf{q} + h\hat{\mathbf{n}}$$
$$\mathbf{p} \cdot \hat{\mathbf{n}} = (\mathbf{q} + h\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}$$
$$\mathbf{p} \cdot \hat{\mathbf{n}} = d + h$$
$$h = \mathbf{p} \cdot \hat{\mathbf{n}} - d$$

$\mathbf{q} \cdot \hat{\mathbf{n}} = d$
$\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$



Intersection of a line with a plane

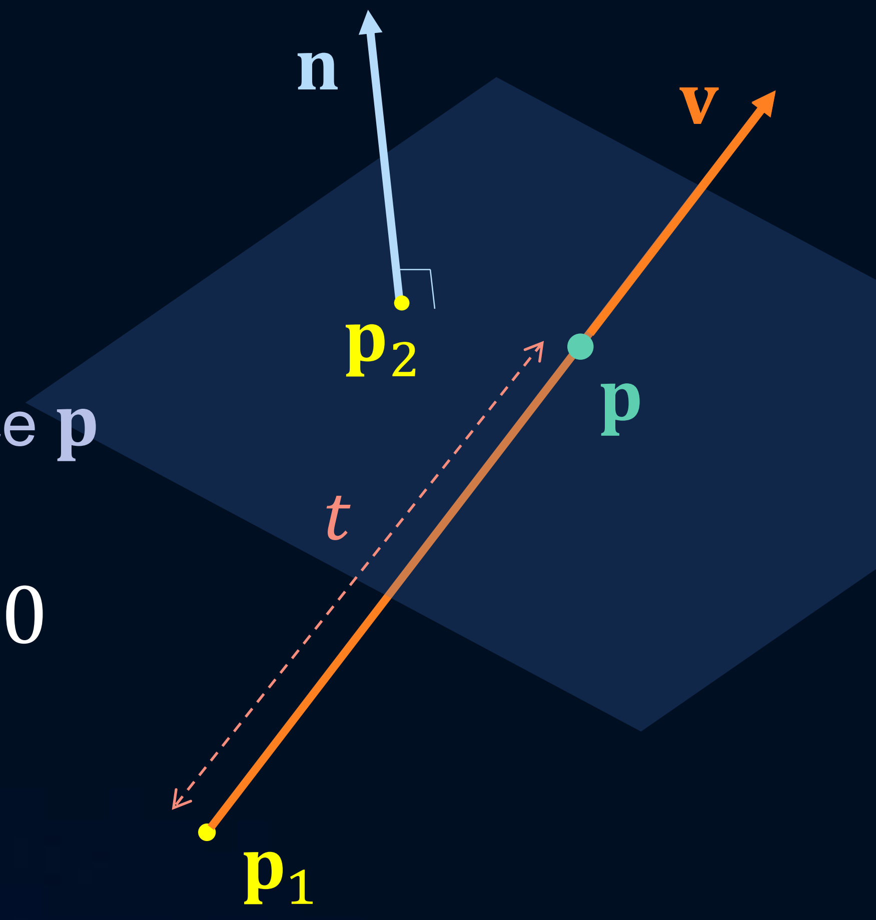
- A line through point $\mathbf{p}_1$ with direction $\mathbf{v}$ has equation $\mathbf{p} = \mathbf{p}_1 + t\mathbf{v}$
- A plane with normal $\mathbf{n}$ passing through the point $\mathbf{p}_2$ has equation $(\mathbf{p} - \mathbf{p}_2) \cdot \mathbf{n} = 0$

Replace $\mathbf{p}$

- If the line intersects the plane at $\mathbf{p}$,

$$\big((\mathbf{p}_1 + t\mathbf{v}\,) - \mathbf{p}_2\big) \cdot \mathbf{n} = 0$$

- Rearranging: $t = \frac{(\mathbf{p}_2 - \mathbf{p}_1)\cdot\mathbf{n}}{\mathbf{v}\cdot\mathbf{n}}$



Intersections with other objects

- Similar to in 2D!
  - Sphere – circle
  - Cuboid – rectangle
- Some examples here: https://www.scratchapixel.com/lessons/3d-basic-rendering/minimal-ray-tracer-rendering-simple-shapes