COMP220: Graphics & Simulation

3: Mathematics for graphics

Learning outcomes

By the end of this session, you should be able to:

- **Explain** the role of vectors and matrices in computer graphics
- **Calculate** basic transformation matrices using the GLM library
- **Explain** the constituents of the model-view-projection matrix

Reminders

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- Next week's **live coding** activity will get you started on implementation

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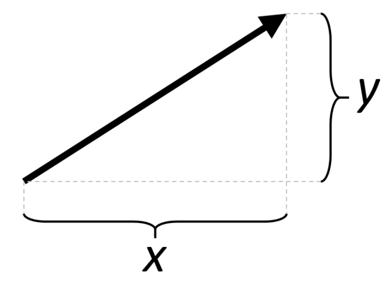

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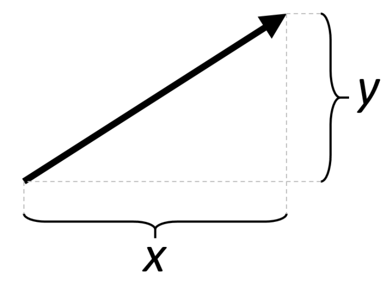

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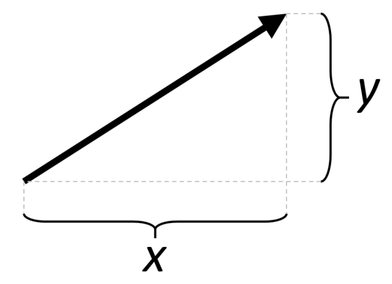

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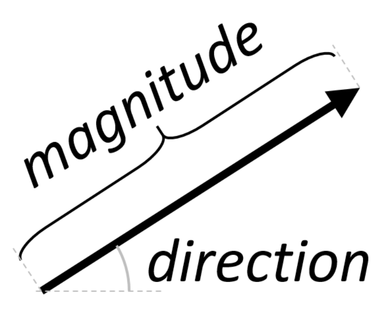

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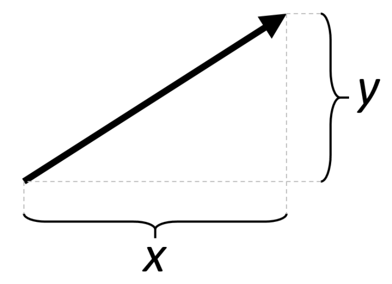

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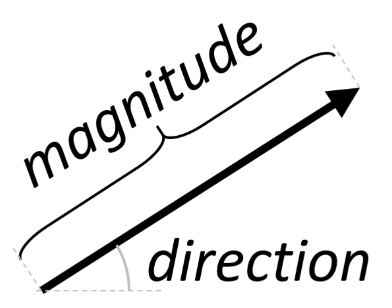

The **origin** is the point represented by the vector $(0, 0, \dots)$

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- Careful! Some things in OpenGL work in **degrees**, others in **radians** (just to confuse you...)

Right hand rule

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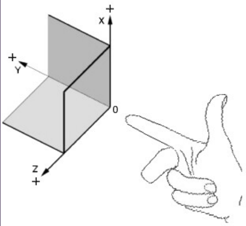

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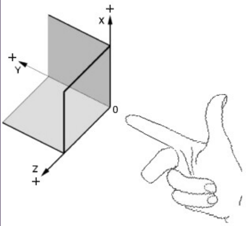

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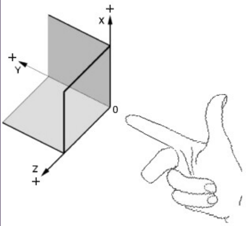

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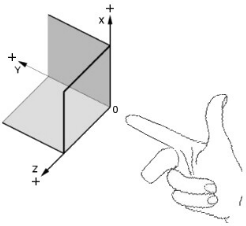

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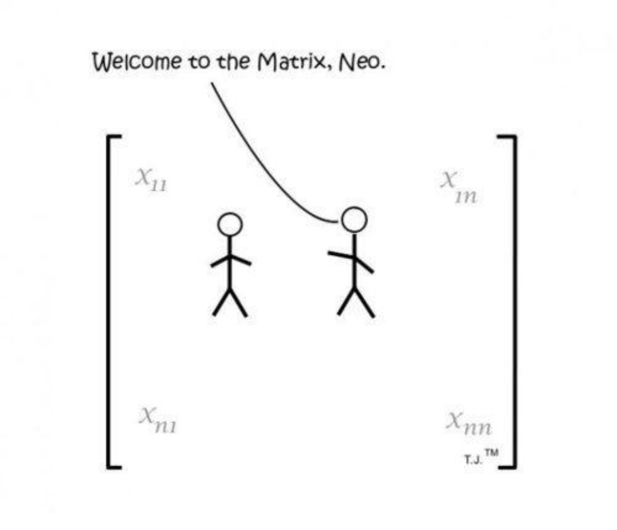
- In homogeneous coordinates, the origin is $(0, 0, 0, 1)$ not $(0, 0, 0, 0)$!

Matrices


Welcome to the Matrix, Neo.
x_{11}
x_{1n}
x_{n1}
x_{nn}
T.J.™

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- Note: the plural of **matrix** is **matrices**
- In computer graphics we mostly work with **square** matrices (number of rows = number of columns)

Multiplying vectors and matrices

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- (But you don't really need to know how to calculate these manually...)

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- Multiplication of matrices is **not commutative**
  - In general, $A \times B \neq B \times A$
  - There may be some matrices where $A \times B = B \times A$, but they are the exception

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Transformations

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- Multiplying a vector by the matrix **applies** the transformation

GLM

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- Lets us perform calculations with vectors and matrices in C++
- GLM types can be passed into shaders as uniforms, e.g.

```
// transformLocation points to a uniform of type mat4
glm::mat4 transform = ...;
glUniformMatrix4fv(transformLocation, 1, GL_FALSE, glm::value_ptr(transform));
```

Identity

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```
// Default constructor for glm::mat4 creates an identity matrix
glm::mat4 transform;
```

Translation

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```
transform = glm::translate(transform, glm::vec3(0.3f,
    0.5f, 0.0f));
```

Scaling

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```
transform = glm::scale(transform, glm::vec3(1.2f, 0.5f
    , 1.0f));
```

Rotation

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```
float angle = glm::pi<float>() * 0.5f;
glm::vec3 axis(0, 0, 1);
transform = glm::rotate(transform, angle, axis);
```

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- Transformations **do not commute** in general — changing the order will change the result
- The order they are applied is the **reverse** of what you might think — i.e. the above rotates **then** translates

Euler angles

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  - The x-axis $(1, 0, 0)$
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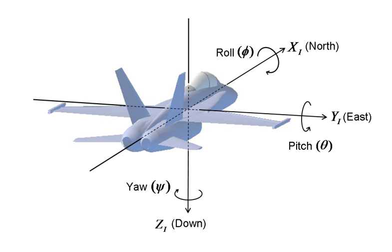
- These angles are sometimes called **roll**, **pitch** and **yaw**



Gimbal lock

https://youtu.be/rrUCBOlJdt4?t=1m55s

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Model, View, Projection

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The **model-view-projection (MVP) matrix**:

$$M_{MVP} = M_{\text{projection}} \times M_{\text{view}} \times M_{\text{model}}$$

(remember, multiplication goes in reverse order)

The model matrix

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Exactly what we've been doing so far today...

The view matrix

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glm::mat4 view = glm::lookAt(
  glm::vec3(2, 0, 2),      // eye
  glm::vec3(0, 0, 0),      // centre
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);
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- `eye` is the position of the camera
- `centre` is a point for the camera to look at
- `up` is which direction is "up" for the camera (usually the positive y-axis)

Types of projection

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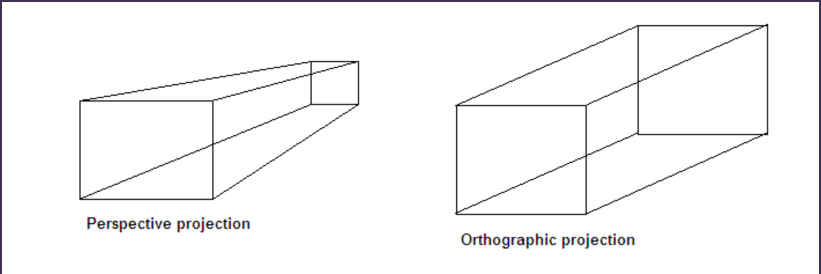

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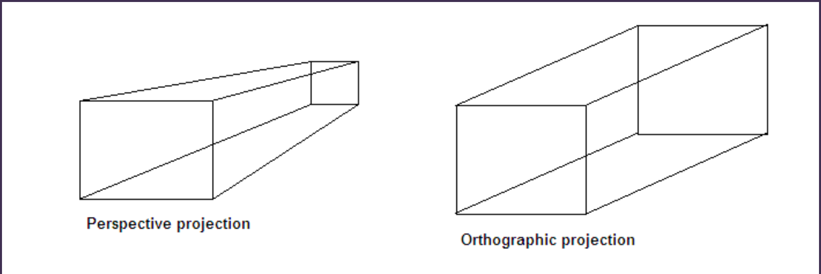

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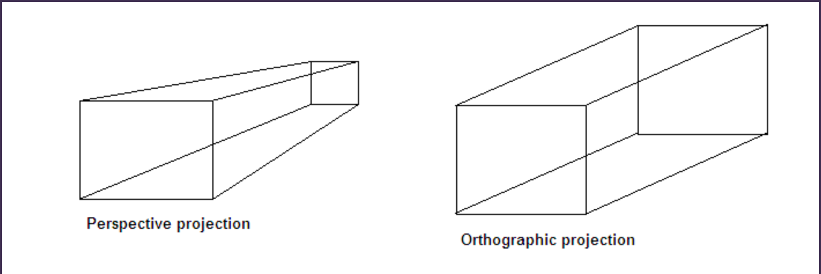

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- **Orthographic** is useful for 2D or pseudo-2D graphics (e.g. isometric perspective)

The projection matrix

The projection matrix

```
glm::mat4 projection = glm::perspective(
    glm::radians(45.0f), // field of view
    4.0f / 3.0f,         // aspect ratio
    0.1f,                // near clip plane
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Also available: `glm::ortho` for orthographic projection

The view frustum

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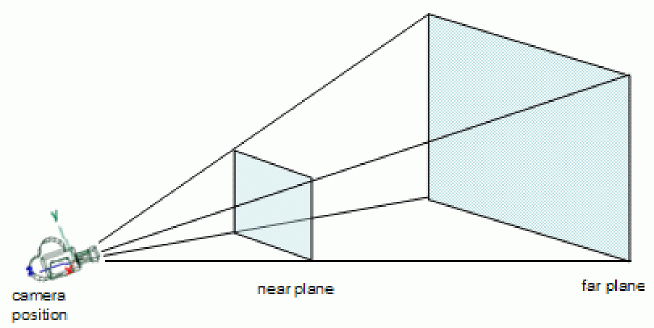

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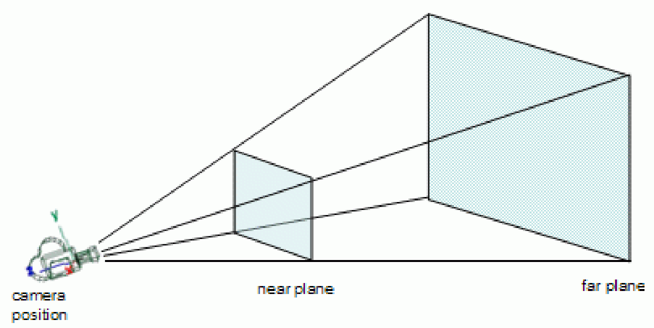

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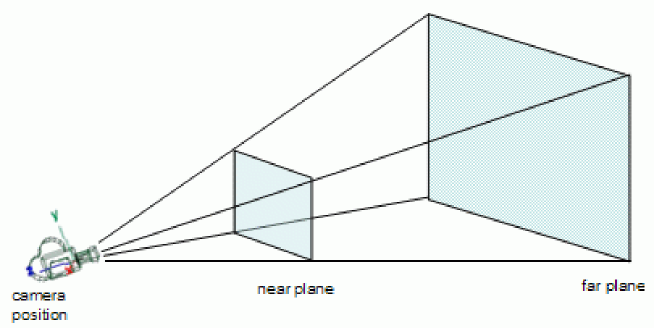

- Defined by the **near and far clipping planes** and the **edges of the screen**
- **Nothing outside** the view frustum is visible

Putting it together

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glm::mat4 mvp = projection * view * modelTransform;
glUniformMatrix4fv(mvpLocation, 1, GL_FALSE, glm::
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And in the vertex shader, simply multiply the vertex position (in homogeneous coordinates) by the MVP matrix:

```
uniform mat4 mvp;

void main()
{
  gl_Position = mvp * vec4(vertexPos, 1.0);
}
```