Week 1: Revision
Numbers and Spaces

COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Recall** the most common *number systems*
- **Understand** the relationships between the different number systems
- **Describe** collections of numbers using *set notation*
- **Introduce** *coordinate systems* as a mathematical concept

Counting

- Humans first developed numbers as a way of **counting things**
- How many sheep do I have? 1, 2, 3, 4, …
- This gives us the **natural numbers** i.e. the counting numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

We use blackboard bold font for the standard number systems

Curly braces denote a **set**

"…" means "continue this sequence to infinity"

Counting

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$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$
- What if I have no sheep? This gives us the concept of **zero**
- Some people include 0 in the natural numbers, some don't
- You may also see
$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\} \quad \mathbb{N}_+ = \{1, 2, 3, 4, \dots\}$$

Integers

- We can calculate with natural numbers: $5 + 3,\ 5\ - 3\ldots$
- What is the answer to $3 - 5$? We need **negative numbers**
- Adding negative numbers to the natural numbers gives us the **integers** (or **whole numbers**)

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Why Z? From the German *Zahlen*, meaning *numbers*

Infinite sequence to the left as well as to the right

Negative Numbers and Inverses

- **Definition**: the **additive inverse** of a number is another number which, when added to the original number, gives zero.
  i.e. if m is the inverse of n, then $n + m = 0$

Zero is called the **additive identity**

- For integers, the inverse of n is $-n$, e.g. $3 + (-3) = 0$
- Clearly, the inverse of $-n$ is n, i.e. $-(-n) = n$
- The inverse is **unique**, i.e. if $n + m = 0$ then $m = -n$

Multiplying Negative Integers

- Inverse: $n + (-n) = 0$

“Doing the same thing to both sides of the equation”

- Using the **substitution property** of equations, we can multiply both sides by another number, k:

$$k \times (n + (-n)) = k \times 0$$

$a \times (b + c) = (a \times b) + (a \times c)$

- Now using the **distributive property** of multiplication:

$$(k \times n) + (k \times (-n)) = 0$$

- i.e. the inverse of $(k \times n)$ is $(k \times (-n))$…

Write ab as shorthand for $a \times b$

- But the inverse of a number is its negative, so the inverse of kn is $-(kn)$ and

$$-(kn) = (k \times (-n))$$

  - i.e. multiplying a positive number by a negative number gives a negative number

Multiplying Negative Integers

- What if we multiply by a negative number, $-k$?
$$(-k) \times (n + (-n)) = 0$$
- Again, using distributivity:
$$((-k) \times n) + ((-k) \times (-n)) = 0$$
- From the previous slide, we know that $((-k) \times n) = -(kn)$, so:
$$-(kn) + ((-k) \times (-n)) = 0$$
- i.e. $((-k) \times (-n))$ is the inverse of $-kn$
- We know that the inverse of $-(kn)$ is kn, so that
$$kn = k \times n = ((-k) \times (-n))$$
  - i.e. multiplying two negative numbers gives a positive number

Division and Fractions

- We can do some divisions with integers, e.g. $6 \div 3 = 2$
- But not others, e.g. $7 \div 3 = ?$
- To solve this we need fractions: $7 \div 3 = \frac{7}{3}$
- This gives us the **rational numbers**:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}_+ \right\}$$

Set builder notation: read as "the set of all $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}_+$"

Note the restriction on b…

Why Q? From *Quotient*, meaning *ratio* or *division*

":" means "where"

"∈" means "in" or "belonging to", i.e. $x \in S$ means element x is in set S

Fractions

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- There are multiple ways to write the same fraction, e.g.

$$\frac{7}{3} = \frac{14}{6} = \frac{700}{300}$$

Mathematically, they are all considered to be identical.

Decimals

- In a fraction $\frac{a}{b}$, a is called the **numerator** and b the **denominator**
- Note that decimals are just fractions where the denominator is a power of 10
- e.g.

$$0.7 = \frac{7}{10}$$

$$12.345 = \frac{12345}{1000}$$

- So the decimal numbers are a subset of $\mathbb{Q}$ (or equal to $\mathbb{Q}$ if we allow recurring decimals)

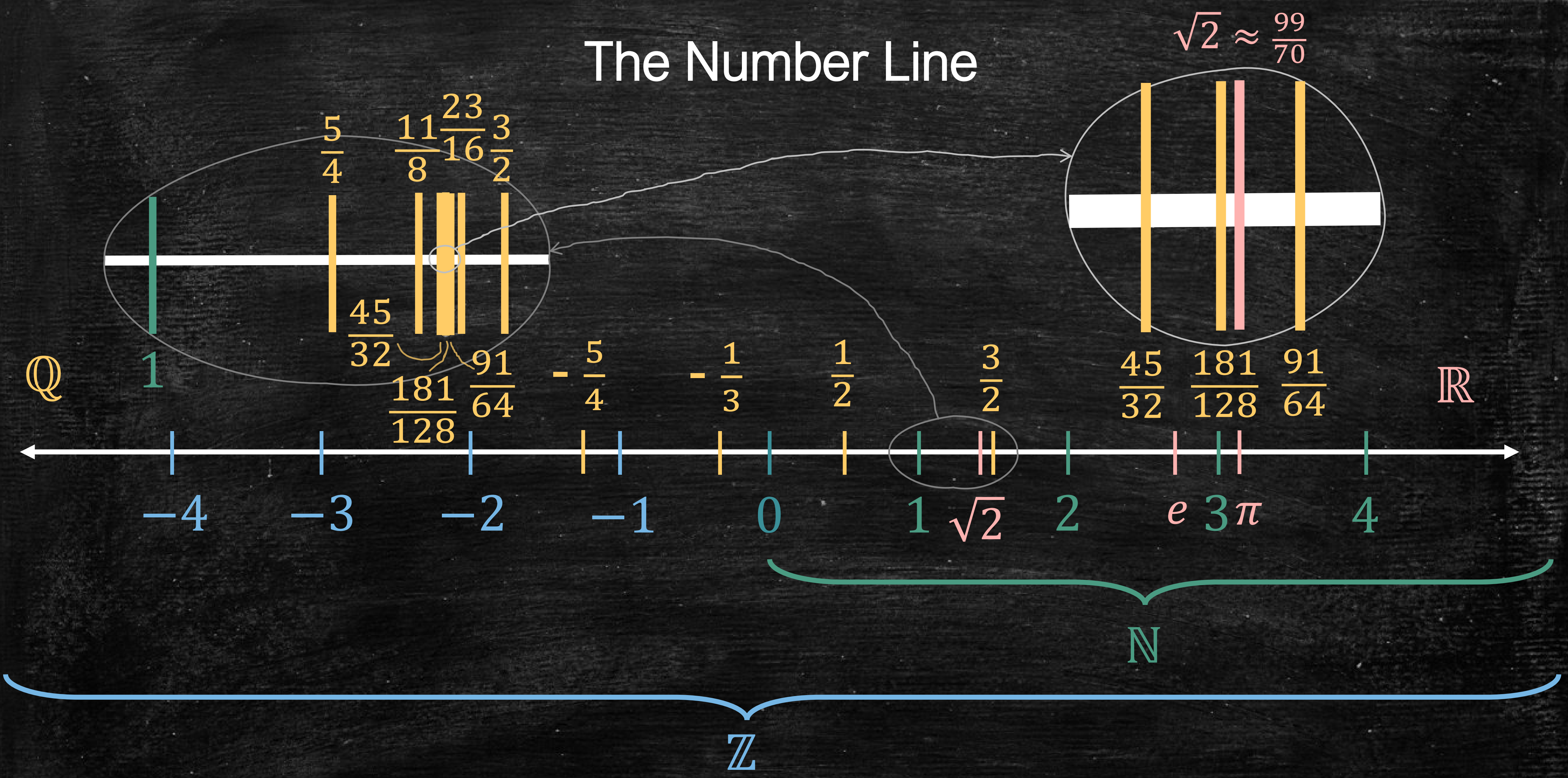
Reals

- Some numbers cannot be written exactly as fractions, e.g. π, $\sqrt{2}$, e
- Such numbers are called **irrational**
- Putting together the rational and irrational numbers gives the **real numbers**

$$\mathbb{R}$$

- The real numbers can be thought of as the points on an infinite line (from $-\infty$ to $+\infty$)
- Note however that all real numbers are **finite**

The Number Line



The Cartesian Product

Named after René Descartes, 1596-1650, French mathematician

- For two sets S and T, the **Cartesian product** $S \times T$ is defined as the set of all **pairs** of elements, the **first** from S and the **second** from T:

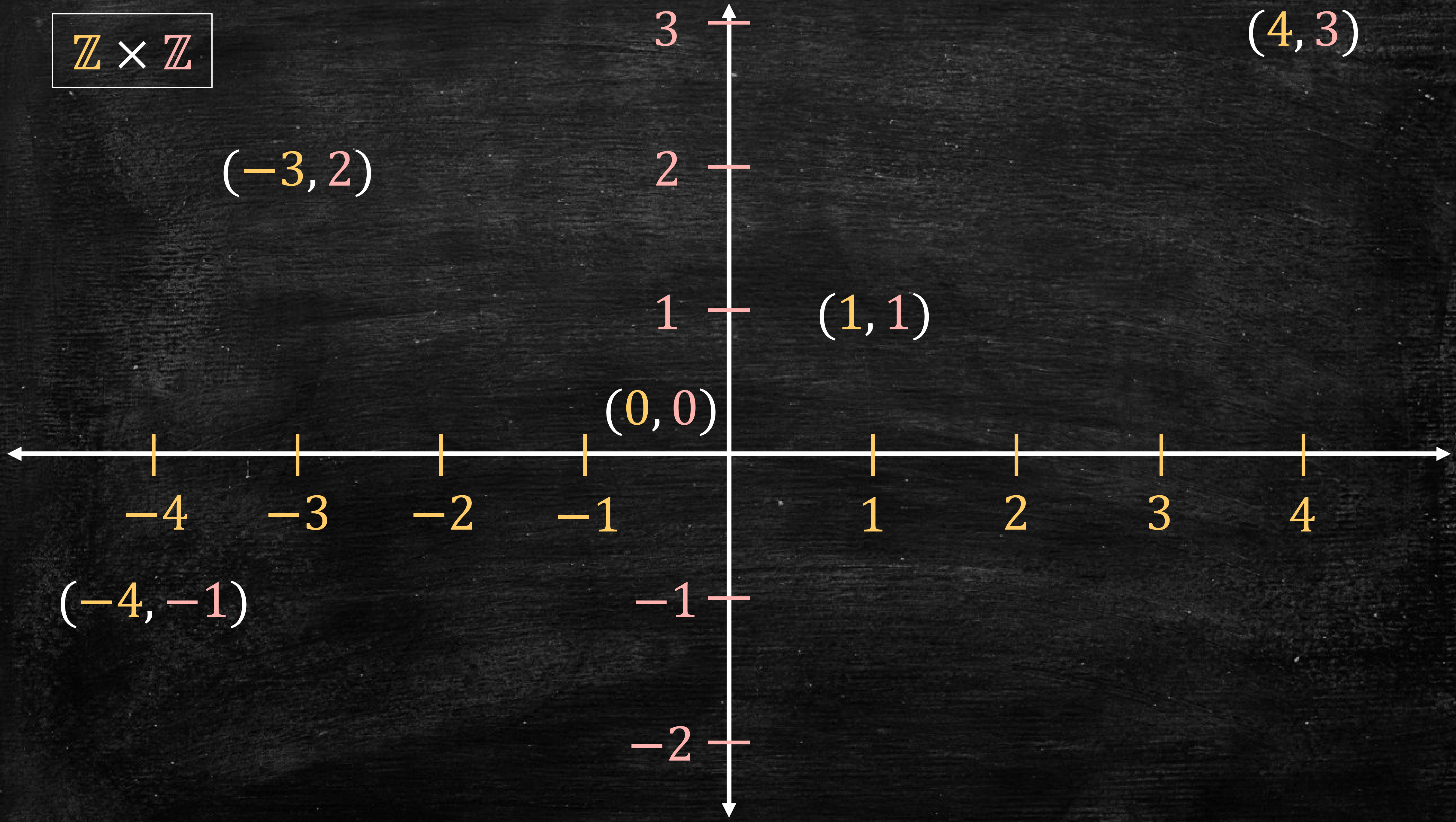
$$S \times T = \{(a, b) : a \in S, b \in T\}$$

		B		
		1	2	3
A	*x*	(*x*,1)	(*x*,2)	(*x*,3)
	y	(*y*,1)	(*y*,2)	(*y*,3)
	z	(*z*,1)	(*z*,2)	(*z*,3)

- Generalises to products of 3, 4, … sets

- Cartesian products of a set with itself give **Cartesian powers**:

$$S^2 = S \times S = \{(a, b) : a \in S, b \in S\}$$
$$S^3 = S \times S \times S = \{(a, b, c) : a \in S, b \in S, c \in S\}$$



$\mathbb{Z} \times \mathbb{Z}$
$(4, 3)$
$(-3, 2)$
$(1, 1)$
$(0, 0)$
$(-4, -1)$
3
2
1
−1
−2
−4
−3
−2
−1
1
2
3
4

Cartesian Coordinate Systems

- **Idea**: use the elements from a Cartesian product as **coordinates** to specify positions in a **space**.
  - $\mathbb{R}^1 = \mathbb{R}$ is 1-dimensional space, aka the space of **scalars**
  - $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is 2-dimensional space, aka the 2D **plane**
  - $\mathbb{R}^3$ is 3-dimensional space
  - $\mathbb{R}^n$ is n-dimensional space…