COMP220: Graphics & Simulation

3: Mathematics for graphics

Learning outcomes

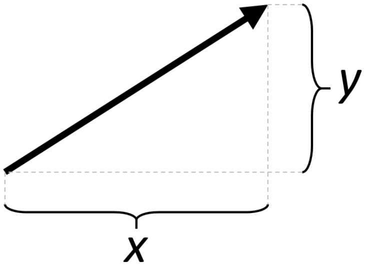
By the end of this session, you should be able to:

- **Explain** the role of vectors and matrices in computer graphics
- **Calculate** basic transformation matrices using the GLM library
- **Explain** the constituents of the model-view-projection matrix

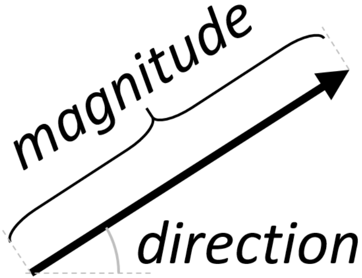
Vectors

Vectors

A vector has **components**



A vector also has **direction** and **magnitude** (or **length**)



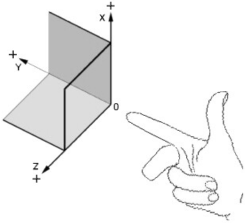
The **origin** is the point represented by the vector $(0, 0, \ldots)$

Radians

- We often measure angles in **radians**
- $\pi = 3.14159\ldots$
- π radians $=$ 180 degrees $=$ half a circle
- $\frac{\pi}{2}$ radians $=$ 90 degrees $=$ right angle
- Careful! Some things in OpenGL work in **degrees**, others in **radians** (just to confuse you...)

Right hand rule

OpenGL uses a **right-handed coordinate system**

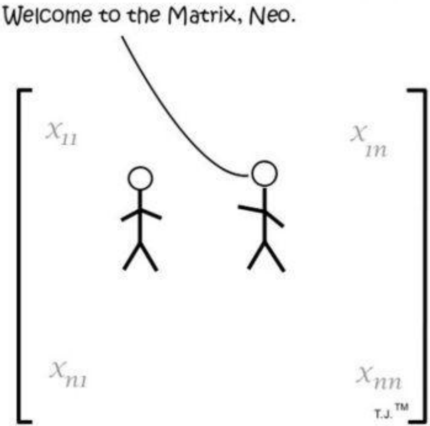


- The *x*-**axis** points towards the **right-hand side** of the screen
- The *y*-**axis** points towards the **top** of the screen
- The *z*-**axis** points **out** of the screen

Homogeneous coordinates

- In 3D graphics, it is useful to represent a **point in 3D space** as a **4-dimensional vector**
- The extra coordinate is called w
- Simple explanation: w should always equal 1 for points in 3D space; having w there makes certain calculations easier
  - (Actually, a point (x, y, z) can be represented as a vector $(x \times w, y \times w, z \times w, w)$ for any $w \neq 0$)
- In homogeneous coordinates, the origin is $(0, 0, 0, 1)$ not $(0, 0, 0, 0)$!

Matrices


Welcome to the Matrix, Neo.
x_{11}
x_{1n}
x_{n1}
x_{nn}
T.J.™

Matrices

- An $m \times n$ **matrix** is a rectangular array of numbers, having m rows and n columns

$$\begin{pmatrix} 3 & 0 & 2.4 \\ 1.7 & -6 & -4.5 \end{pmatrix} \qquad \leftarrow \text{A } 2 \times 3 \text{ matrix}$$

- Note: the plural of **matrix** is **matrices**
- In computer graphics we mostly work with **square** matrices (number of rows = number of columns)

Multiplying vectors and matrices

- Two $n \times n$ matrices can be **multiplied**, giving a new $n \times n$ matrix
- An $n \times n$ matrix and an n-vector can be **multiplied**, giving a new n-vector
- See `https://www.khanacademy.org/math/precalculus/precalc-matrices/multiplying-matrices-by-matrices/v/matrix-multiplication-intro`
- (But you don't really need to know how to calculate these manually...)

Commutativity

- Multiplication of numbers is **commutative**
  - $a \times b = b \times a$
  - e.g. $2 \times 3 = 3 \times 2$
- Multiplication of matrices is **not commutative**
  - In general, $A \times B \neq B \times A$
  - There may be some matrices where $A \times B = B \times A$, but they are the exception

Transformations

Transformations and matrices

- A **transformation** is a **mathematical function** that **changes points in space**
- E.g. shifts them, rotates them, scales them, ...
- Many useful transformations can be **represented** by matrices
- Multiplying these matrices together **combines** the transformations
- Multiplying a vector by the matrix **applies** the transformation

GLM

- We will use the **GLM** library to do matrix calculations for us
- `http://glm.g-truc.net/`
- GLM aims to mirror GLSL data types (`vec4`, `mat4` etc) in C++
- Lets us perform calculations with vectors and matrices in C++
- GLM types can be passed into shaders as uniforms, e.g.

```
// transformLocation points to a uniform of type ↩
    mat4
glm::mat4 transform = ...;
glUniformMatrix4fv(transformLocation, 1, GL_FALSE ↩
    , glm::value_ptr(transform));
```

Identity

The identity transformation does not change anything

```
// Default constructor for glm::mat4 creates an ↩
    identity matrix
glm::mat4 transform;
```

Translation

Translation shifts all points by the same vector offset

```
transform = glm::translate(transform, glm::vec3(0.3f, ↩
    0.5f, 0.0f));
```

Scaling

Scaling moves all points closer or further from the origin by the same factor

```
transform = glm::scale(transform, glm::vec3(1.2f, 0.5f ↩
    , 1.0f));
```

Rotation

- How do we represent a rotation in 3 dimensions?
- One way is by specifying the **axis** (as a vector) and the **angle** (in radians)
- Axis always runs through the origin

```
float angle = glm::pi<float>() * 0.5f;
glm::vec3 axis(0, 0, 1);
transform = glm::rotate(transform, angle, axis);
```

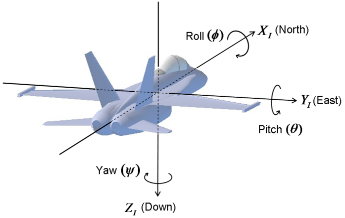
Combining transformations

```
transform = glm::translate(transform, glm::vec3(0.5f, ↩
    0.5f, 0.0f));
transform = glm::rotate(transform, angle, axis);
```

- Transformations **do not commute** in general — changing the order will change the result
- The order they are applied is the **reverse** of what you might think — i.e. the above rotates **then** translates

Euler angles

- Any orientation of an object in 3D space can be described by **three** rotations around:
  - The x-axis $(1, 0, 0)$
  - The y-axis $(0, 1, 0)$
  - The z-axis $(0, 0, 1)$
- These angles are sometimes called **roll**, **pitch** and **yaw**



Gimbal lock

https://youtu.be/rrUCBOlJdt4?t=1m55s

Model, View, Projection

Model, View, Projection

Drawing a 3D object on screen generally involves **three** transformations:

- **Model**: translate, rotate and scale the object into its place in the scene
- **View**: translate and rotate the scene to put the observer at the origin
- **Projection**: convert points in 3D space to points on the 2D screen

The **model-view-projection (MVP) matrix**:

$$M_{MVP} = M_{\text{projection}} \times M_{\text{view}} \times M_{\text{model}}$$

(remember, multiplication goes in reverse order)

The model matrix

Exactly what we've been doing so far today...

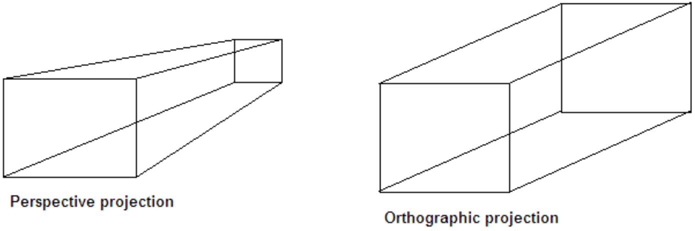
The view matrix

Need to translate and rotate the scene so that the "camera" is at $(0, 0, 0)$ and looking in the negative z direction

```
glm::mat4 view = glm::lookAt(
  glm::vec3(2, 0, 2),    // eye
  glm::vec3(0, 0, 0),    // centre
  glm::vec3 up(0, 1, 0)  // up
);
```

- `eye` is the position of the camera
- `centre` is a point for the camera to look at
- `up` is which direction is "up" for the camera (usually the positive y-axis)

Types of projection



- Generally use **perspective** for 3D graphics
- **Orthographic** is useful for 2D or pseudo-2D graphics (e.g. isometric perspective)

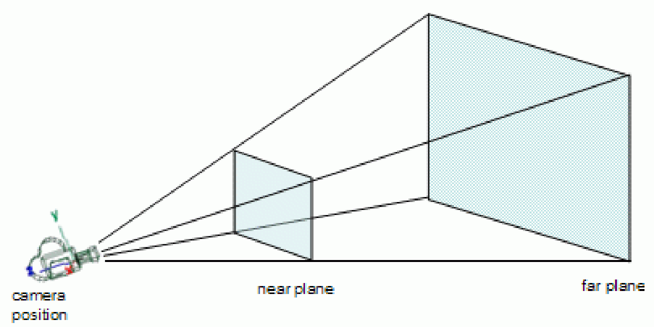
The projection matrix

```
glm::mat4 projection = glm::perspective(
    glm::radians(45.0f), // field of view
    4.0f / 3.0f,         // aspect ratio
    0.1f,                // near clip plane
    100.0f               // far clip plane
);
```

- **Field of view (FOV)**: how "wide" or "narrow" the view is
- **Aspect ratio**: should be `screenWidth / screenHeight`
- **Near and far clip planes**: fragments that fall outside this range of distances from the camera are not drawn

Also available: `glm::ortho` for orthographic projection

The view frustum



- Defined by the **near and far clipping planes** and the **edges of the screen**
- **Nothing outside** the view frustum is visible

Putting it together

```
glm::mat4 mvp = projection * view * modelTransform;
glUniformMatrix4fv(mvpLocation, 1, GL_FALSE, glm:: ↩
    value_ptr(mvp));
```

And in the vertex shader, simply multiply the vertex position (in homogeneous coordinates) by the MVP matrix:

```
uniform mat4 mvp;

void main()
{
  gl_Position = mvp * vec4(vertexPos, 1.0);
}
```