COMP110: Principles of Computing

10: Algorithm Strategies

Worksheets

- Worksheet 6: due **today**
- Worksheet 7: due **next Monday**

FALMOUTH UNIVERSITY

Recursion

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- A **recursive** function is a function that **calls itself**

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int factorial(int n)
{
    if (n <= 1)
        return 1;
    else
        return n * factorial(n-1);
}
```

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- Recursive functions need a **base case** where they stop recursing, otherwise they will go **forever**
- (Or rather, until a **stack overflow**)

Thinking recursively

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Thinking recursively

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- If I already had a function to solve smaller instances of the problem, I could use it to write my function
- I can solve the smallest possible problem
- Therefore I can write a recursive function

The call stack

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- Recall: nested function calls are handled using a **stack**

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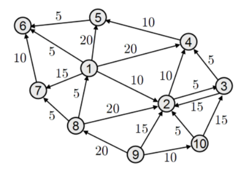
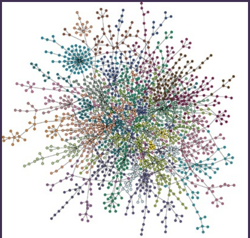
- Recursive functions are no different
- This means if a recursive function contains **local variables**, they are **independent** between instances of the function

Graphs and trees

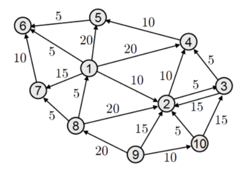
Graphs

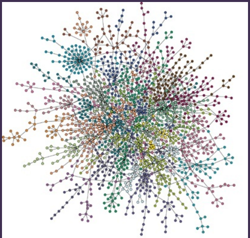
Graphs





Graphs





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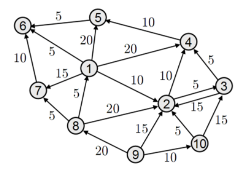
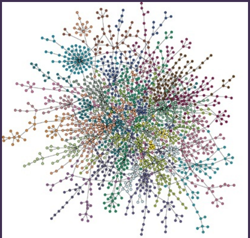
Graphs





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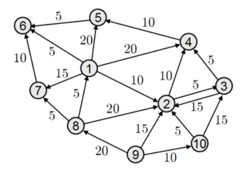
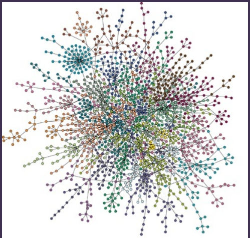
Graphs





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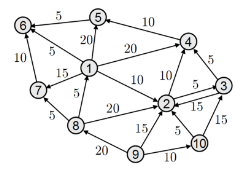

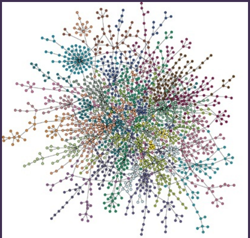

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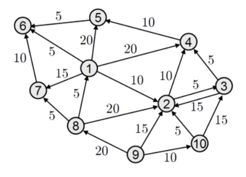

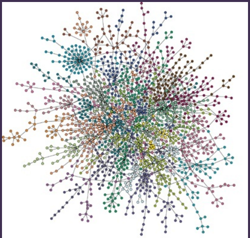

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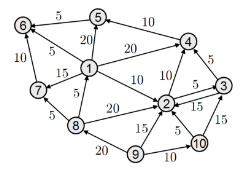
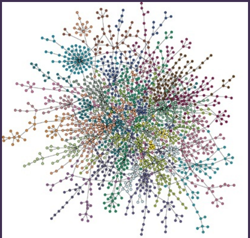
- **Directed** graph: edges are arrows

Graphs





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- Often used to model **networks** (e.g. social networks, transport networks, game levels, automata, ...)
- **Directed** graph: edges are arrows
- **Undirected** graph: edges are lines

Drawing graphs

Drawing graphs

- A graph does not necessarily specify the physical **positions** of its nodes

Drawing graphs

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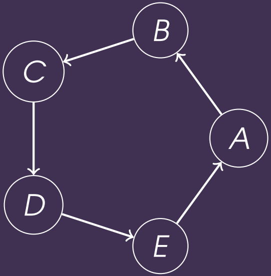
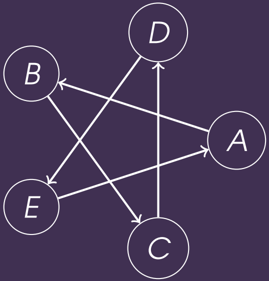
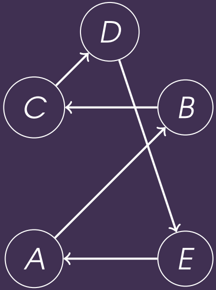
- E.g. these are technically the same graph:

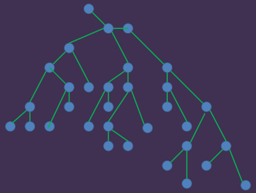


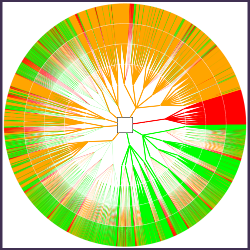




Trees

Trees





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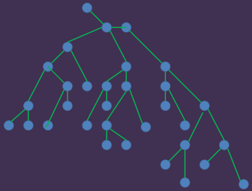
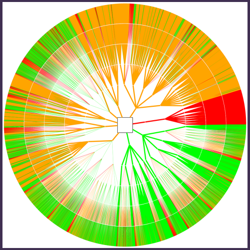
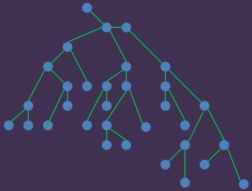
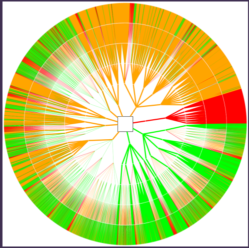




- A **tree** is a special type of directed graph where:

Trees





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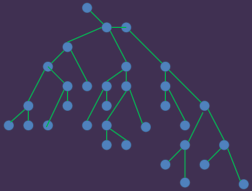
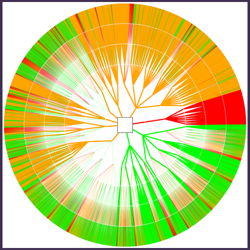
Trees





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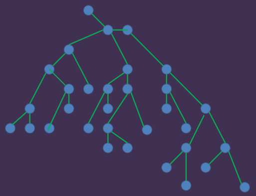

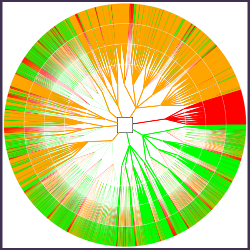

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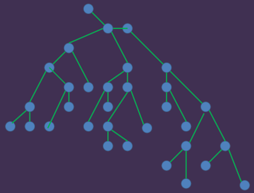

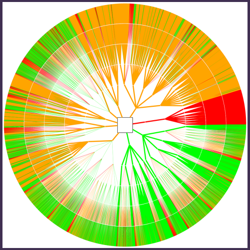

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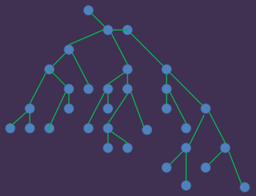

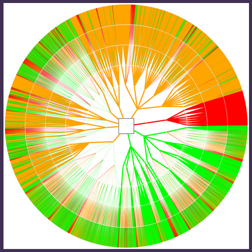

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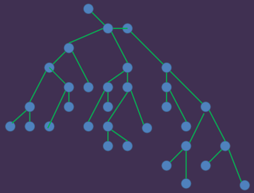

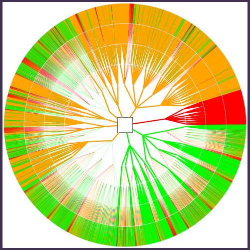

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- Used to model **hierarchies** (e.g. file systems, object inheritance, scene graphs, state-action trees, behaviour trees, ...)

Tree traversal

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  - Breadth first

Tree traversal

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procedure DepthFirstSearch

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Tree traversal

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    print n

Tree traversal

procedure DepthFirstSearch
  let *S* be a stack
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Tree traversal

procedure DepthFirstSearch
    let *S* be a stack
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    end while
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procedure BREADTHFIRSTSEARCH

Tree traversal

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    let Q be a queue
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    while Q is not empty do
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Tree traversal

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Tree traversal example

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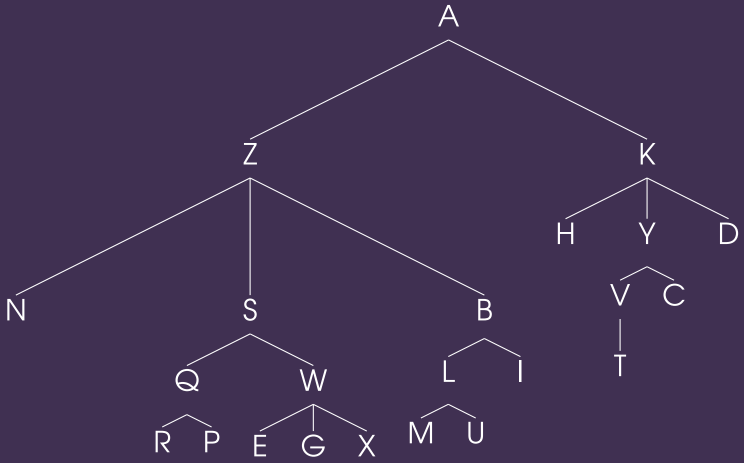

Recursive depth first search

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procedure DepthFirstSearch(n)

Recursive depth first search

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Recursive depth first search

procedure DEPTHFIRSTSEARCH(n)
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Recursive depth first search

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procedure DepthFirstSearch(n)
    print n
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```

Recursive depth first search

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procedure DepthFirstSearch(n)
    print n
    for each child c of n do
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    end for
end procedure
```

Recursive depth first search

```
procedure DepthFirstSearch(n)
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end procedure
```

- Compare to the pseudocode on the previous slide. Where is the stack?

Algorithm strategies

The knapsack problem

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- There is a set X of **items**

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- What subset $S \subseteq X$ maximises the total value, whilst not exceeding the maximum weight?

The knapsack problem

- There is a set X of **items**
- Each item x has a weight $\mathrm{weight}(x)$ and a value $\mathrm{value}(x)$
- There is a maximum weight W
- What subset $S \subseteq X$ maximises the total value, whilst not exceeding the maximum weight?
- In other words: find $S \subseteq X$ to maximise

$$\sum_{x \in S} \mathrm{value}(x)$$

subject to

$$\sum_{x \in S} \mathrm{weight}(x) \leq W$$

Algorithm strategies

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- Brute force

Algorithm strategies

- Brute force
- Greedy

Algorithm strategies

- Brute force
- Greedy
- Divide-and-conquer

Algorithm strategies

- Brute force
- Greedy
- Divide-and-conquer
- Dynamic programming

Brute force

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Brute force

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procedure KNAPSACK(X, W)

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Brute force

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Brute force

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    $S_{\text{best}} \leftarrow \{\}$
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        if $\text{weight}(S) \leq W$ and $\text{value}(S) > v_{\text{best}}$ **then**

Brute force

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      $S_{\text{best}} \leftarrow S$
      $v_{\text{best}} \leftarrow$ value(S)

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  $v_{\text{best}} \leftarrow 0$
  for every subset $S \subseteq X$ **do**
    if weight$(S) \leq W$ and value$(S) > v_{\text{best}}$ **then**
      $S_{\text{best}} \leftarrow S$
      $v_{\text{best}} \leftarrow$ value(S)
    end if
  end for
  return S_{best}

Brute force

- Try **every possible** solution and decide which is best

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- If X contains n elements, how many subsets of X are there?
- Therefore what is the time complexity of the brute force algorithm?
- If we add one element to X, what happens to the running time of the algorithm?

Greedy algorithm

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- At each stage of building a solution, take the **best** available option

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- In some problems an appropriately chosen greedy solution is **optimal**
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- **However** the greedy solution to the knapsack problem may not be optimal!

Divide and conquer

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- Example from last time: **binary search**
  - Problem: find an element in a list
  - Subproblem: find the element in a list of half the size

Divide and conquer for the knapsack problem

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- Consider an element $x \in X$ with $\mathrm{weight}(x) \leq W$

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- ... whichever has the greater value
- Base case: the solution to the knapsack problem on the empty set **is** the empty set

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- Thus the worst case time complexity is $O(2^n)$ — still exponential!
- However in the **average** case many of the calls have only a single recursive call, so this is still more efficient than brute force

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  - **Not** memorization!
- One of several techniques in the category of **dynamic programming**

Dynamic programming for the knapsack problem

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  if $k < 0$ **then**
    cache and return $\{\}$
  end if
  $S \leftarrow \textsc{Knapsack}(X, W, k-1)$
  if $\text{weight}(x_k) \leq W$ **then**
    $S' \leftarrow \textsc{Knapsack}(X, W - \text{weight}(x_k), k-1) \cup \{x_k\}$
    cache and return whichever of S, S' has the larger value
  else
    cache and return S
  end if
end procedure

Socrative `FALCOMPED`

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- What is the maximum possible number of entries in the table of intermediate results?

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- What is the maximum possible number of entries in the table of intermediate results?
- Therefore what is the time complexity of the dynamic programming algorithm?

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  - Good enough for small/simple problems
- Greedy
  - Efficient for certain problems, but doesn't always give optimal solutions
- Divide-and-conquer
  - Good if the problem can be broken down into simpler subproblems
- Dynamic programming
  - Makes divide-and-conquer more efficient if subproblems often reoccur