Week 4: Mechanics I
Part 1: Calculus

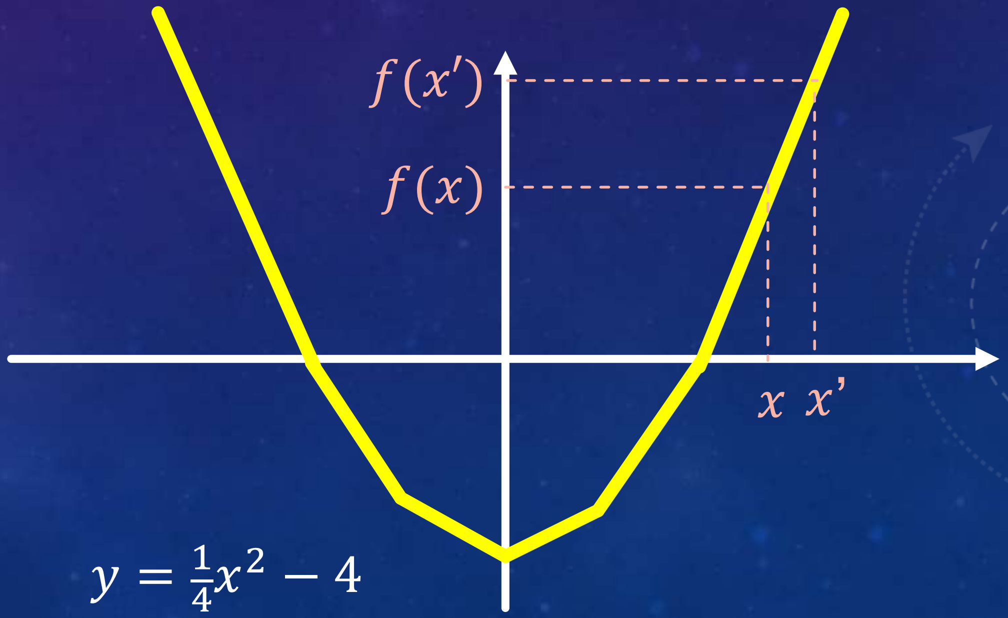
COMP270: Mathematics for 3D Worlds and Simulations

Objectives

- **Define** the derivative and the integral of a quantity
- **Understand** the relationship between changes in a quantity, time and their graphical representation
- **Estimate** values using numerical methods

Recap: functions and graphs

- Define a function $f : S \to T$ as $f(s) \in T$ for $s \in S$
- Represent the function $f : \mathbb{R} \to \mathbb{R}$ as a graph by plotting the points $(x, f(x))$ against 2D axes
- AKA $y = f(x)$

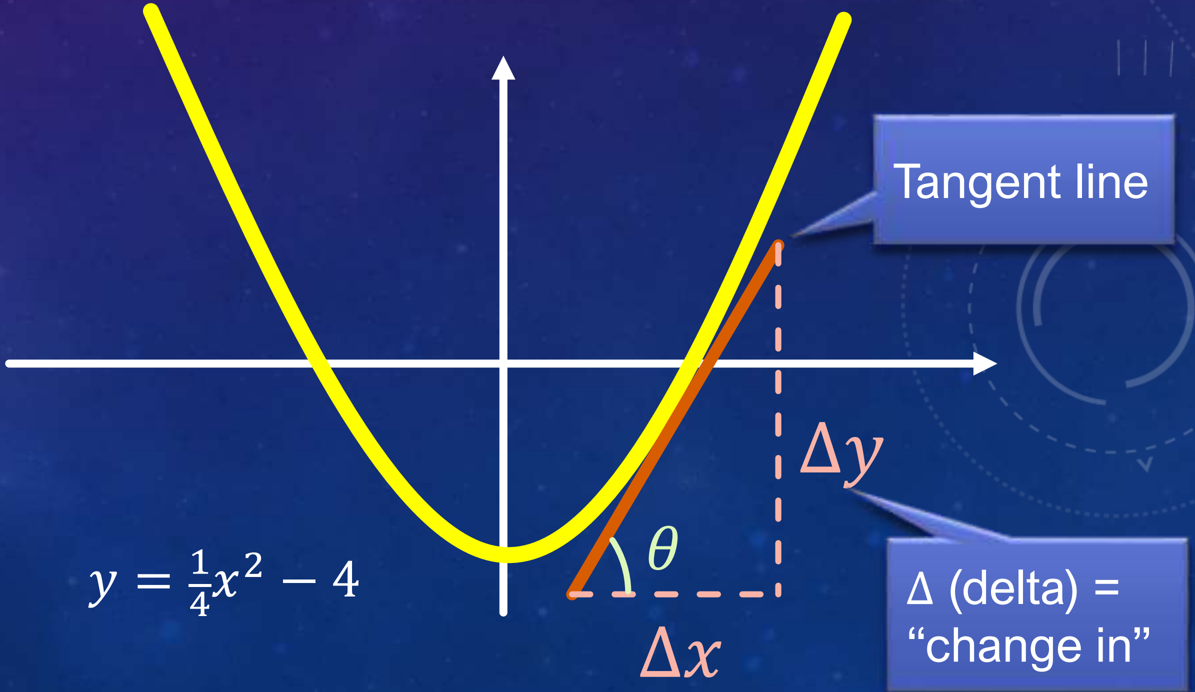


Graph properties: tangent and rate of change

- **Definition**: the **slope** (or gradient) is a quantity which gives the inclination of a curve or line with respect to another curve or line.
  - $m = \dfrac{\text{Change in } y}{\text{Change in } x} = \dfrac{\Delta y}{\Delta x} = \tan\theta$
- For a quantity that changes over time:

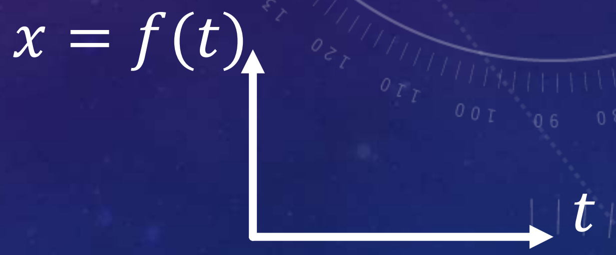
$$\text{Rate of change} = \frac{\text{Change in quantity}}{\text{Change in time}}$$

How much the quantity changes in a single unit of time



Derivatives

- **Definition**: the <u>**derivative**</u> of a quantity x with respect to time t is the rate of change of x with respect to t



- Denoted $\frac{\mathrm{d}x}{\mathrm{d}t}$ for $x = f(t)$

$\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = f(x)$

- The mathematical process of finding $\frac{\mathrm{d}x}{\mathrm{d}t}$ given x is called **differentiation**

Also written $f'(x)$ or $\dot{x}$

  - For polynomials:
    - Multiply coefficients by corresponding exponents
    - Lower each exponent by one degree
    - Remove constant (exponent = 0)

e.g. $x = 2t^3 + t^2 + 3t + 4$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 6t^2 + 2t + 3$

$t^0 = 1$

Derivatives: example

- A car drives along a straight road at a constant speed
- In half an hour, it covers a distance of 20 miles
- Its speed (which we know is constant) is $\dfrac{20\text{ miles}}{0.5\text{ hours}} = 40\text{mph}$
- In other words…
  - "Distance travelled" is a quantity varying with time
  - We call the rate of change (the derivative) of this quantity "speed"
  - If x is distance travelled and t is time, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{20}{0.5} = 40$$

Integrals

- **Definition**: the **integral** is the reverse of the derivative…
- Given $\frac{\mathrm{d}x}{\mathrm{d}t}$, x is the integral of $\frac{\mathrm{d}x}{\mathrm{d}t}$
- The process of finding this is called **integration** – the opposite of differentiation
  - For polynomials:
    - Increase each exponent by one degree
    - Divide coefficients by new exponents
    - Add a constant…

e.g. $\frac{\mathrm{d}x}{\mathrm{d}t} = 6t^2 + 2t + 3$

$\Rightarrow \; x = 2t^3 + t^2 + 3t + c$

Some constant value…

Numerical integration – Euler's method

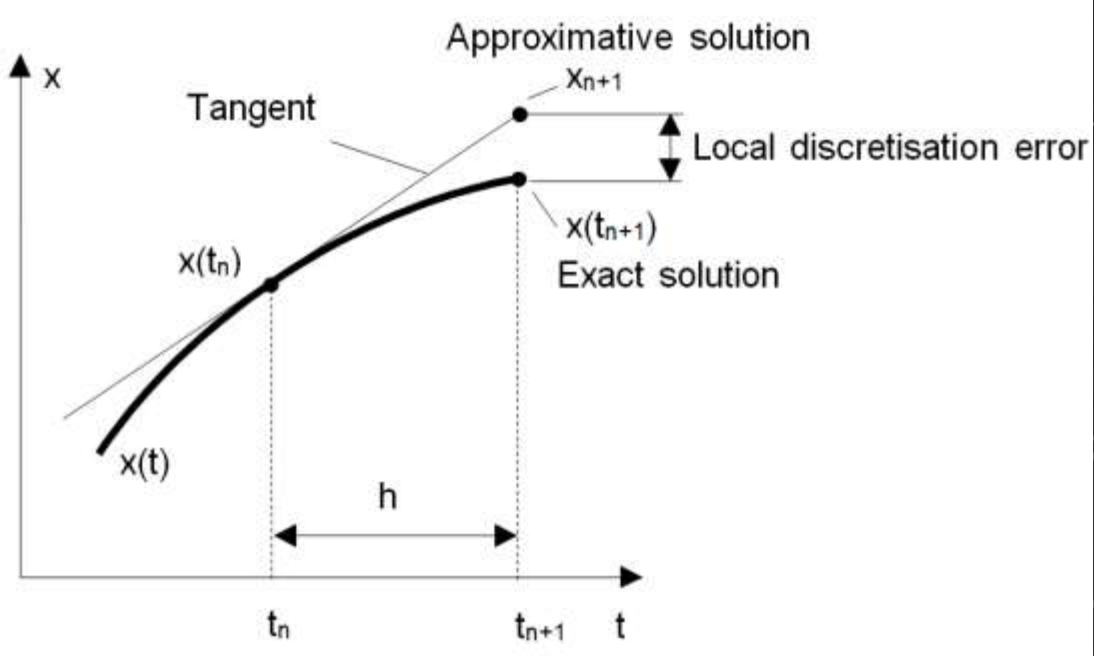
Leonhard Euler (1707-1783), Swiss mathematician

- Given the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ at time t, we can estimate the value of x at time $t + h$ for small h:

$$x(t+h) \approx x(t) + h\frac{\mathrm{d}x}{\mathrm{d}t}$$

How much x changes by if t changes by h

- Demonstration here



Calculus with vectors

- The rate of change of a vector is also a vector
- If $\mathbf{v} \in \mathbb{R}^n$ then $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \in \mathbb{R}^n$
- Differentiate component-wise: if $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} {}^{\mathrm{d}x}/_{\mathrm{d}t} \\ {}^{\mathrm{d}y}/_{\mathrm{d}t} \end{pmatrix}$$