Week 4: Mechanics I
Part 4: Projectile motion

COMP270: Mathematics for 3D Worlds and Simulations

Objectives

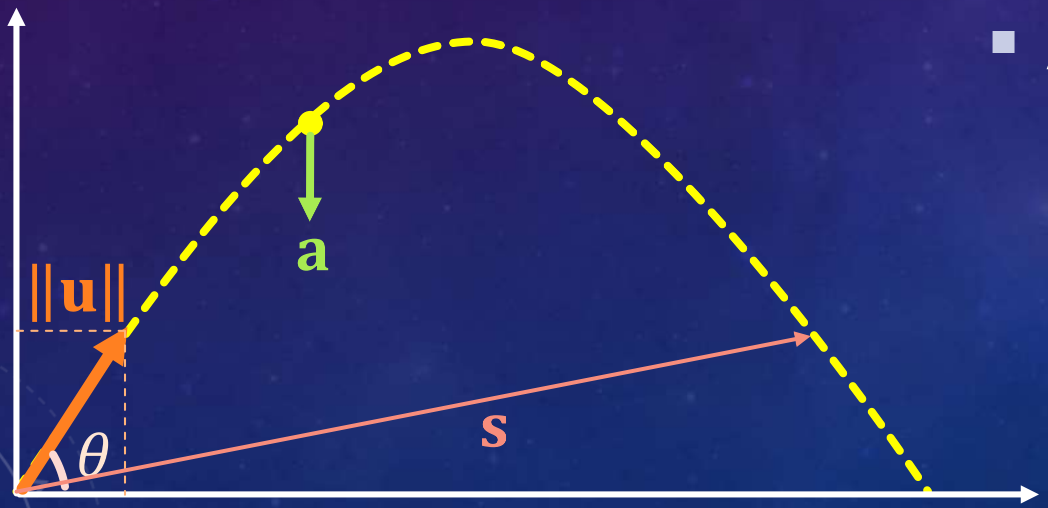
- **Derive** some general results from the equations of motion to discover how projectiles move
- **Apply** the results/equations to solve target-finding problems

Recap: Equations of motion ('suvat' equations)

1. $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ (no $\mathbf{s}$)
2. $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ (no $\mathbf{v}$)
3. $\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$ (no $\mathbf{a}$)
4. $\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{a} \cdot \mathbf{s}$ (no t)
5. $\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$ (no $\mathbf{u}$)

Projectiles

- Definition: a **projectile** is a body projected by external force and continuing in motion by its own inertia.
  - e.g. dropped, thrown, shot…



'cos' to 'close' the angle

- $\mathbf{u} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix} \quad (u = \|\mathbf{u}\|)$
- $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$
- At time t, the displacement is:

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$= \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ -g \end{pmatrix} t^2$$

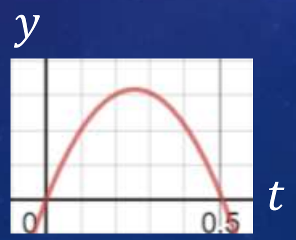
Projectile path

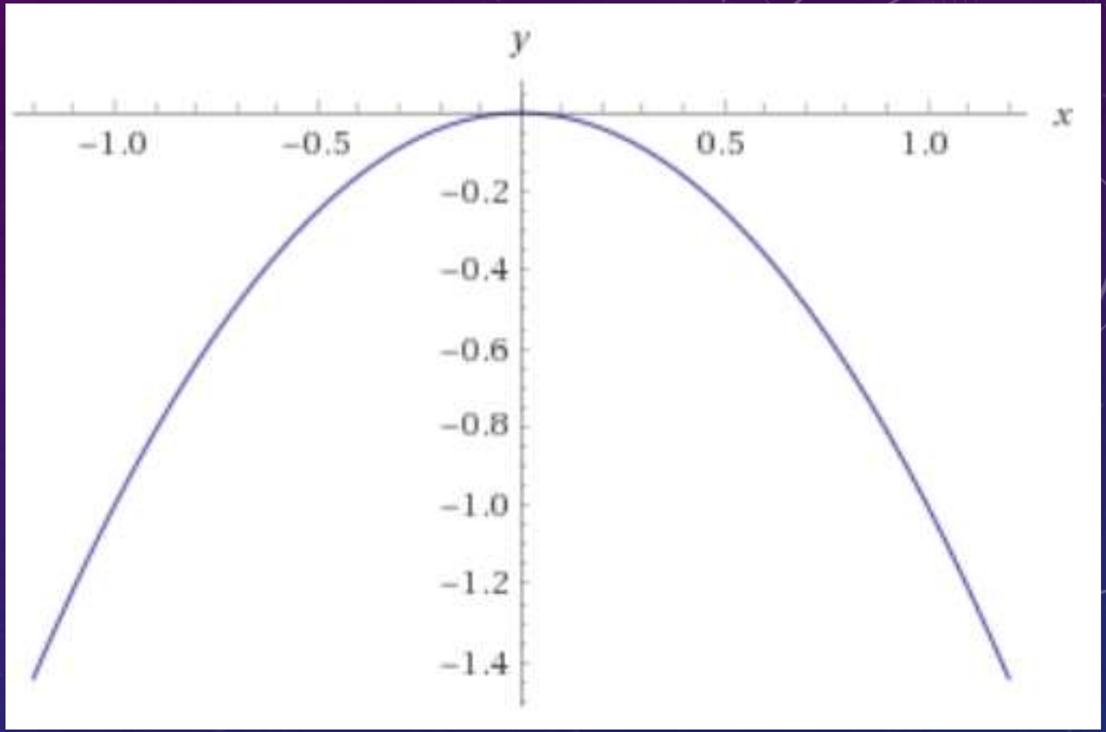
$$\mathbf{s} = \begin{pmatrix} u\cos\theta\, t \\ u\sin\theta\, t - \dfrac{gt^2}{2} \end{pmatrix}$$

- Horizontally: position changes linearly with t
- Vertically: position is a quadratic
- The shape of motion is a parabola

$$y = u\sin\theta\, t - \frac{gt^2}{2}$$

for $u = 5$, $\theta = 30°$



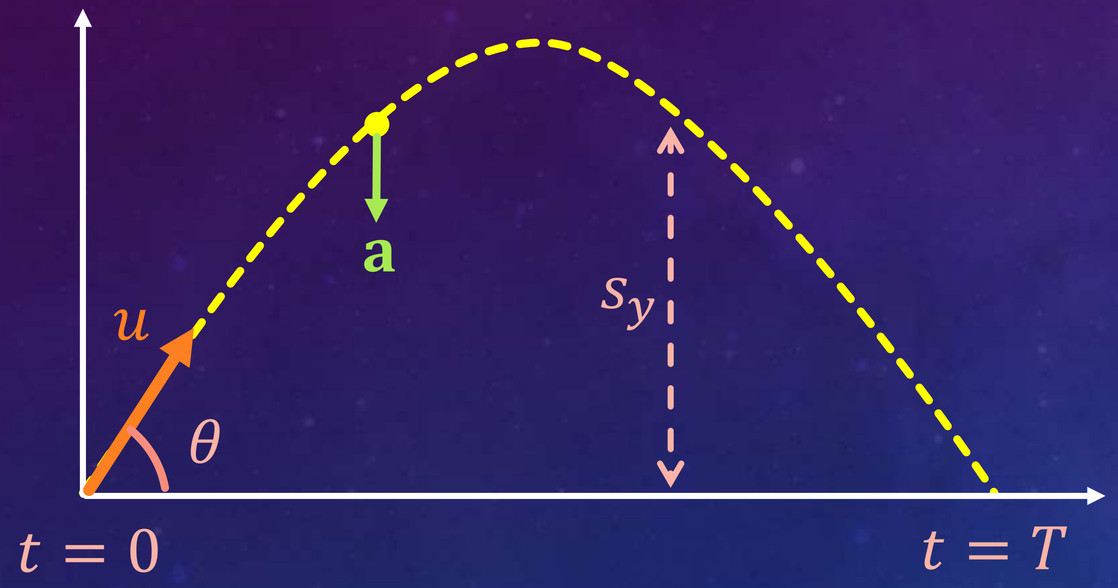


http://www.tom-e-white.com/2015/03/tennis-ball-parabola.html

Separating components

- It’s often useful to consider horizontal and vertical motion separately –
- We can do this because our basis vectors are orthogonal, i.e. the components are at right-angles and do not affect one another
- Further explanation here: https://www.physicsclassroom.com/class/vectors/Lesson-1/Independence-of-Perpendicular-Components-of-Motion

General results



- Time of flight:

At $t = T$, $s_y = 0$, so

use $s_y = u_y t + \frac{1}{2} a_y t^2$

$$0 = u \sin\theta \, t - \frac{1}{2} g t^2$$

$$0 = t\left(u \sin\theta - \frac{1}{2} g t\right)$$

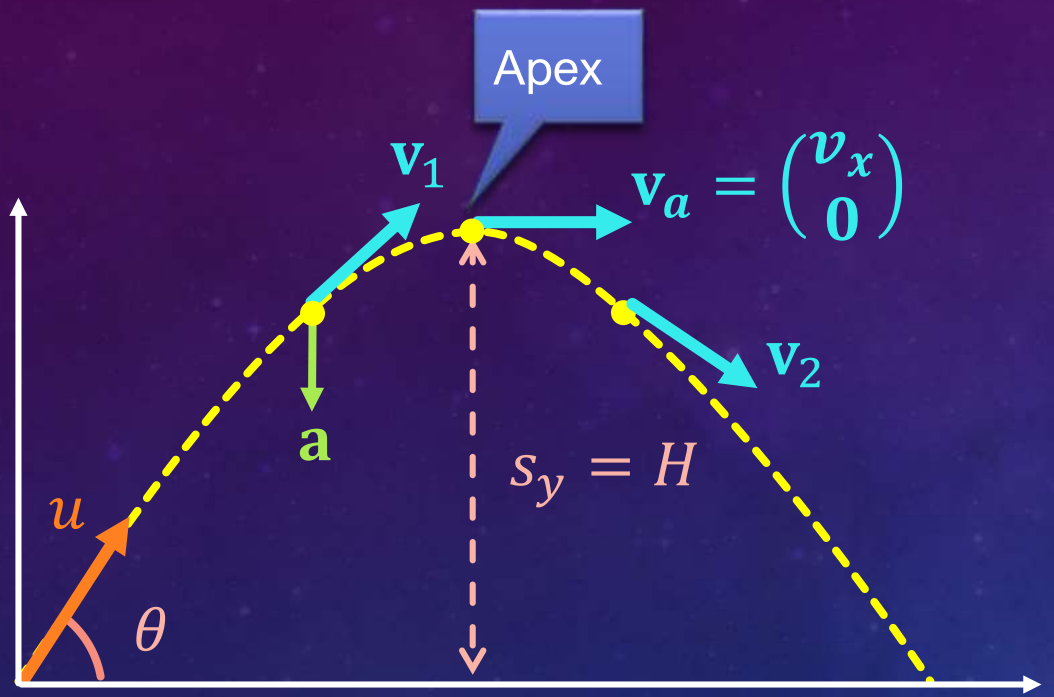
$t = 0$ at the origin, so

$$T = \frac{2u \sin\theta}{g}$$

Or the height difference between the start and end points

$a_y = -g$

General results



- Greatest height:

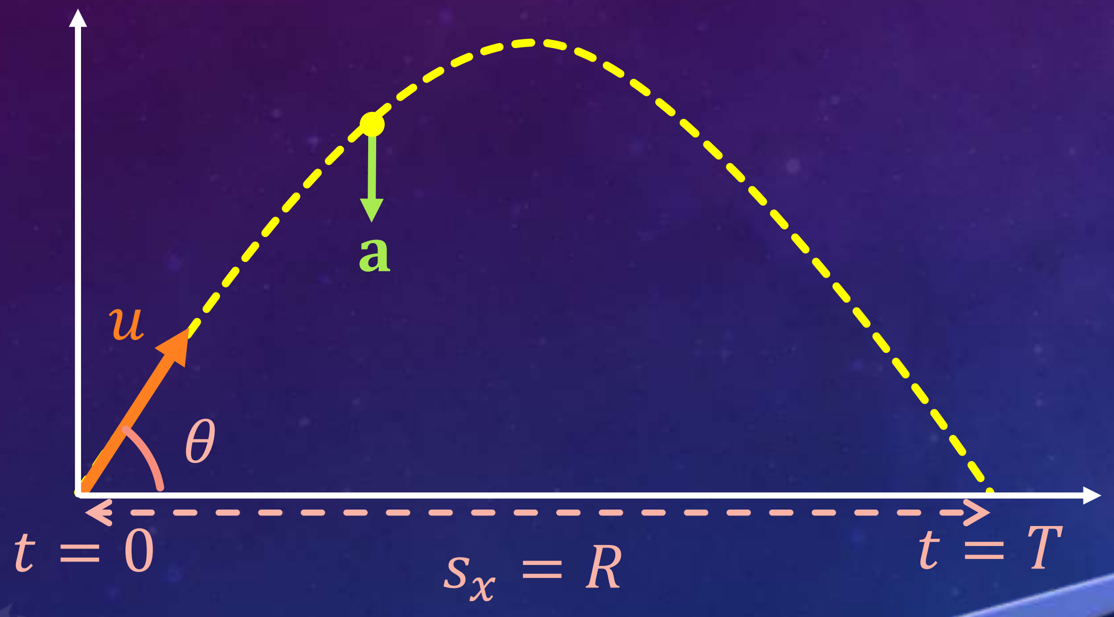
At $s_y = H$, $v_y = 0$, so

use ${v_y}^2 = {u_y}^2 + 2a_y s_y$

$$0 = (u \sin\theta)^2 - 2gH$$

$$H = \frac{u^2 \sin^2\theta}{2g}$$

General results



- Horizontal range:

At $s_x = R$, $t = T$, so

use $s_x = u_x t + \frac{1}{2} a_x t^2$

$a_x = 0$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g}$$

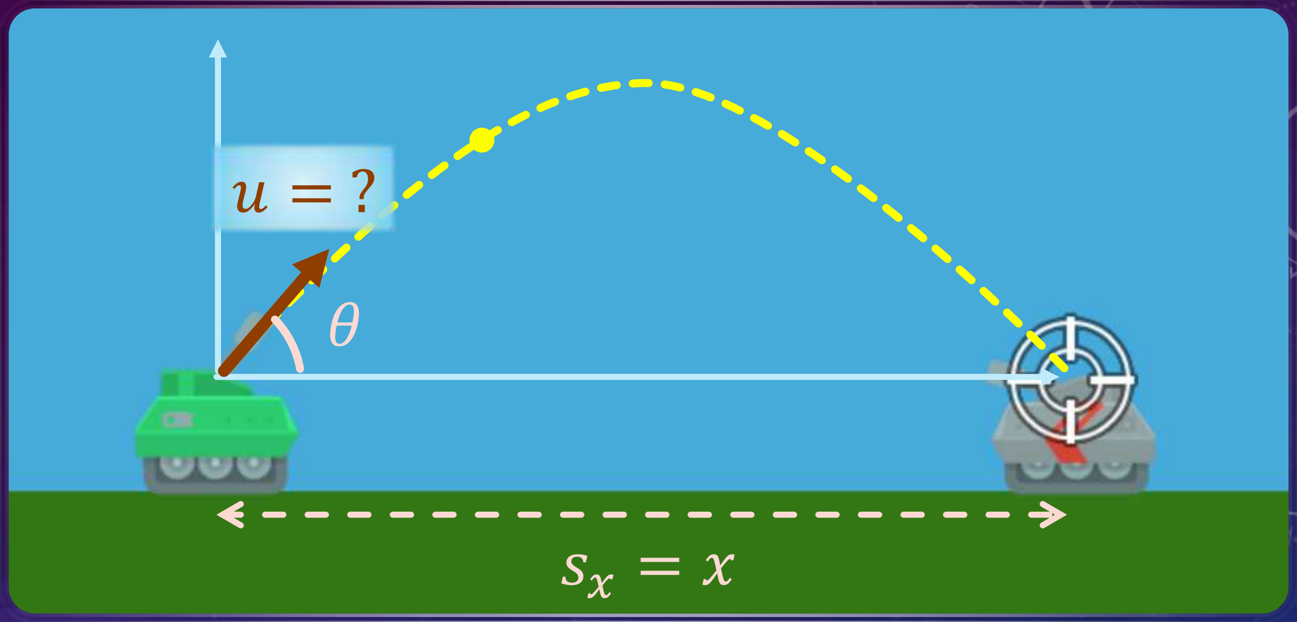
$$R = \frac{u^2 \sin 2\theta}{g}$$

Double angle identity:
$\sin 2\theta \equiv 2\sin\theta\cos\theta$
(more here).

Max. when $\sin 2\theta = 1$, i.e. $\theta = 45°$; $R = \frac{u^2}{g}$

Target practice

- The enemy tank is a distance of x units away, at the same elevation
- Given angle θ, what shot speed u is needed to hit the enemy tank?



$$x = \frac{u^2 \sin 2\theta}{g} \Rightarrow u = \sqrt{\frac{xg}{\sin 2\theta}}$$

- Equivalent to solving

$$\mathbf{s} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} u \cos\theta\, t \\ u \sin\theta\, t - \frac{1}{2} g t^2 \end{pmatrix}$$

When don't these equations work?

- For very small objects (quantum mechanics)
- For objects travelling close to the speed of light (theory of relativity)
- For objects that interact with other objects or forces that change their acceleration…

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