4: Newtonian Mechanics

COMP270: MATHEMATICS FOR 3D WORLDS & SIMULATIONS

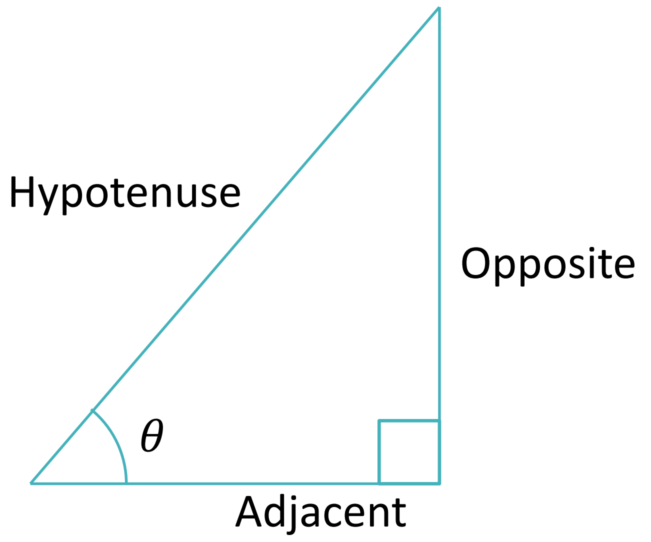
Trigonometric identities

Silly Old Harry Caught A Herring Trawling Off America

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



- Therefore

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Other trigonometric functions

- Secant: $\sec\theta = \dfrac{1}{\cos\theta}$
- Cosecant: $\csc\theta = \dfrac{1}{\sin\theta}$
- Cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

Named after Pythagoras of Samos (c570 – c495 BC), Greek philosopher

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1$$

Divide through by $\cos^2\theta$:

$$1 + \tan^2\theta = \sec^2\theta$$

Divide through by $\sin^2\theta$:

$$1 + \cot^2\theta = \csc^2\theta$$

Sum, difference, double angle formulae

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

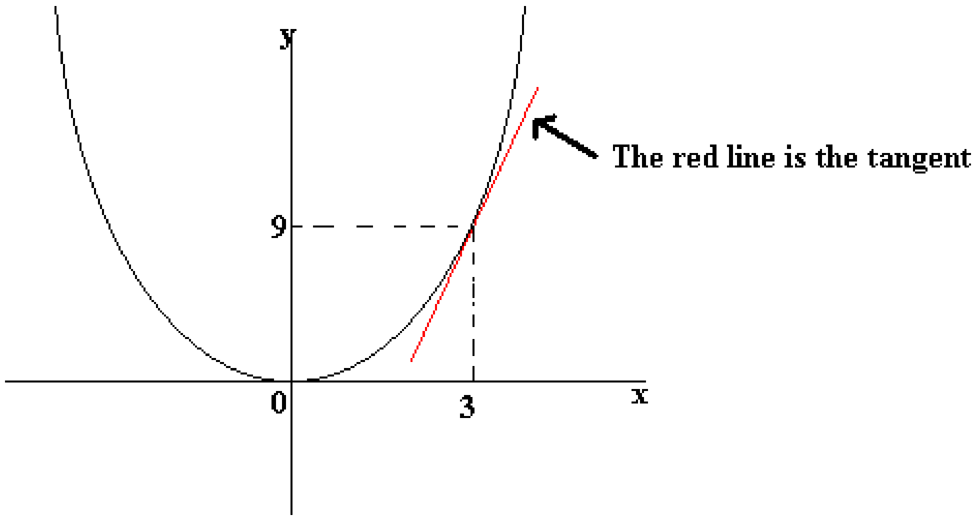
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Calculus

Rates of change

- Consider a quantity that changes over time
- $$\text{Rate of change} = \frac{\text{Change in quantity}}{\text{Change in time}}$$
- Same as the gradient of a graph:

$$\text{radient} \quad = \frac{\text{Change in y}}{\text{Change in x}}$$



Derivatives

- The **derivative** of a quantity x with respect to time t is the rate of change of x with respect to t
- Denoted $\frac{\mathrm{d}x}{\mathrm{d}t}$
- The mathematical process of finding $\frac{\mathrm{d}x}{\mathrm{d}t}$ given x is called **differentiation**

Derivatives: example

- A car drives along a straight road at a constant speed
- In half an hour, it covers a distance of 20 miles
- Its speed (which we know is constant) is $\dfrac{20\text{ miles}}{0.5\text{ hours}} = 40\text{ miles per hour}$
- In other words…
  - "Distance travelled" is a quantity varying with time
  - We call the rate of change (the derivative) of this quantity "speed"
  - If x is distance travelled and t is time, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{20}{0.5} = 40$$

Integration

- Given $\frac{\mathrm{d}x}{\mathrm{d}t}$, find x
- x is the **integral** of $\frac{\mathrm{d}x}{\mathrm{d}t}$
- The process of finding this is called **integration** – the opposite of differentiation

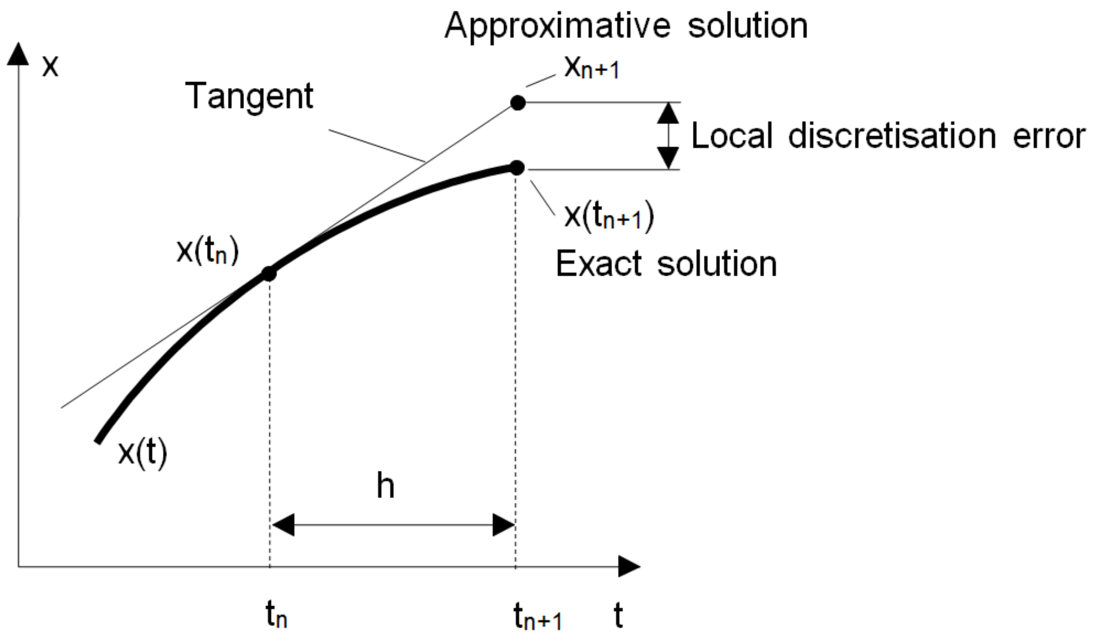
Leonhard Euler (1707-1783), Swiss mathematician

Pronounced “oiler”

Numerical integration – Euler’s method

- Given the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ at time t, we can estimate the value of x at time $t + h$ for small h:

$$x(t+h) \approx x(t) + h\frac{\mathrm{d}x}{\mathrm{d}t}$$



Calculus with vectors

- The rate of change of a vector is also a vector
- If $\mathbf{v} \in \mathbb{R}^n$ then $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \in \mathbb{R}^n$
- Differentiate component-wise: if $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} {}^{\mathrm{d}x}/_{\mathrm{d}t} \\ {}^{\mathrm{d}y}/_{\mathrm{d}t} \end{pmatrix}$$

Basic mechanics

Basic quantities of mechanics

- **Position** describes an object's location in space
- **Velocity** is rate of change of position
- **Acceleration** is rate of change of velocity

Velocity and speed

- Velocity is a **vector** quantity – has a **magnitude** and a **direction**
- We call the magnitude of velocity the **speed**

Système international, or the International System of Units

Units

- In SI units:
- Position is usually measured in metres (m)
- Velocity is measured in metres per second ($\mathrm{m/s}$ or $\mathrm{ms^{-1}}$)
- Acceleration is measured in metres per second per second ($\mathrm{m/s^2}$ or $\mathrm{ms^{-2}}$)
- Other units are possible (e.g. pixels, miles, hours) but be consistent!

Force

Named after Isaac Newton (1642-1726/27), English mathematician

- SI unit: Newtons (N)
- Forces occur when objects interact
- E.g. gravity, air resistance, friction
- E.g. reaction force
- E.g. car engine, rocket engine, launched projectile, muscle, …

Newton’s Laws of Motion

1: An object remains at rest or moves at constant velocity unless acted upon by an external force

2: The sum of forces acting upon an object is equal to its mass multiplied by its acceleration ($\mathbf{F} = m\mathbf{a}$)

3: When one body exerts a force on another, the second body exerts an equal and opposite force on the first

Simulating Newtonian physics

- For each object, store its position and velocity
- On each time step:
- Apply numerical integration to the velocity to determine the new position
- Calculate the forces acting upon the object, and thus the acceleration from Newton's 2nd law
- Apply numerical integration to the aceleration to determine the new velocity

Gravity

- A force which pulls all objects with mass towards each other
- Tiny unless one or both objects has huge mass (e.g. a planet)
- Near the surface of a planet, gravity pulls objects downwards (towards the centre of the planet) with a force called weight
- $w = mg$, where w is weight, m is mass and g is the acceleration due to gravity
- Near Earth's surface, $g \approx 9.81\mathrm{ms}^{-2}$ -- same for all objects on Earth, but differs on other planets

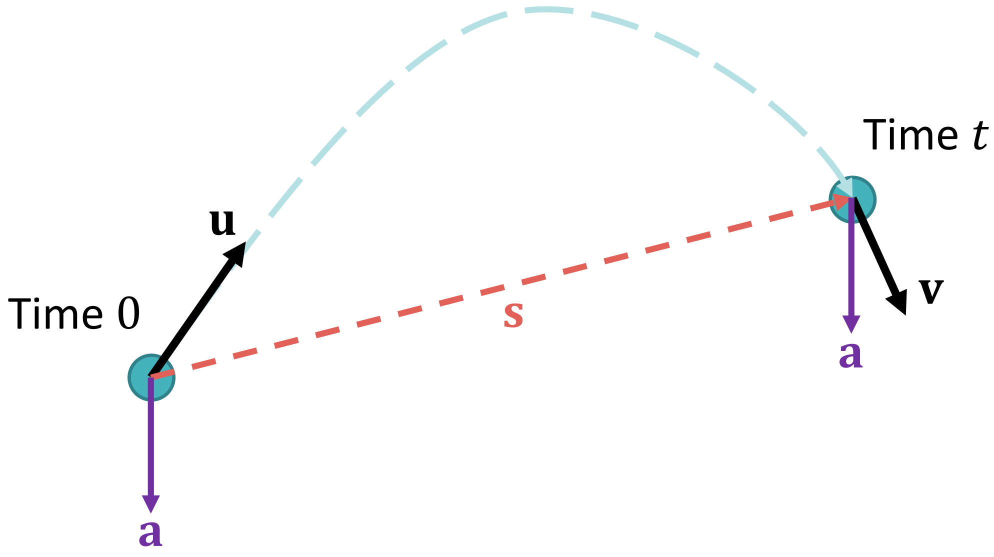
Gravity

- Gravity applies the same acceleration ($9.81\mathrm{ms}^{-2}$) to all objects on Earth, regardless of weight
- The only reason that some objects fall faster than others is air resistance – in a vacuum all objects fall at the same rate

Equations of motion

Setup

- Consider a particle under **constant acceleration**
- E.g. under gravity with no other forces acting
- At all times, the acceleration of the particle is $\mathbf{a}$
- At time 0, assume the particle is at the origin and has velocity $\mathbf{u}$
- At time t, let $\mathbf{s}$ be the particle's position and let $\mathbf{v}$ be its velocity



Equations of motion (SUVAT equations)

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

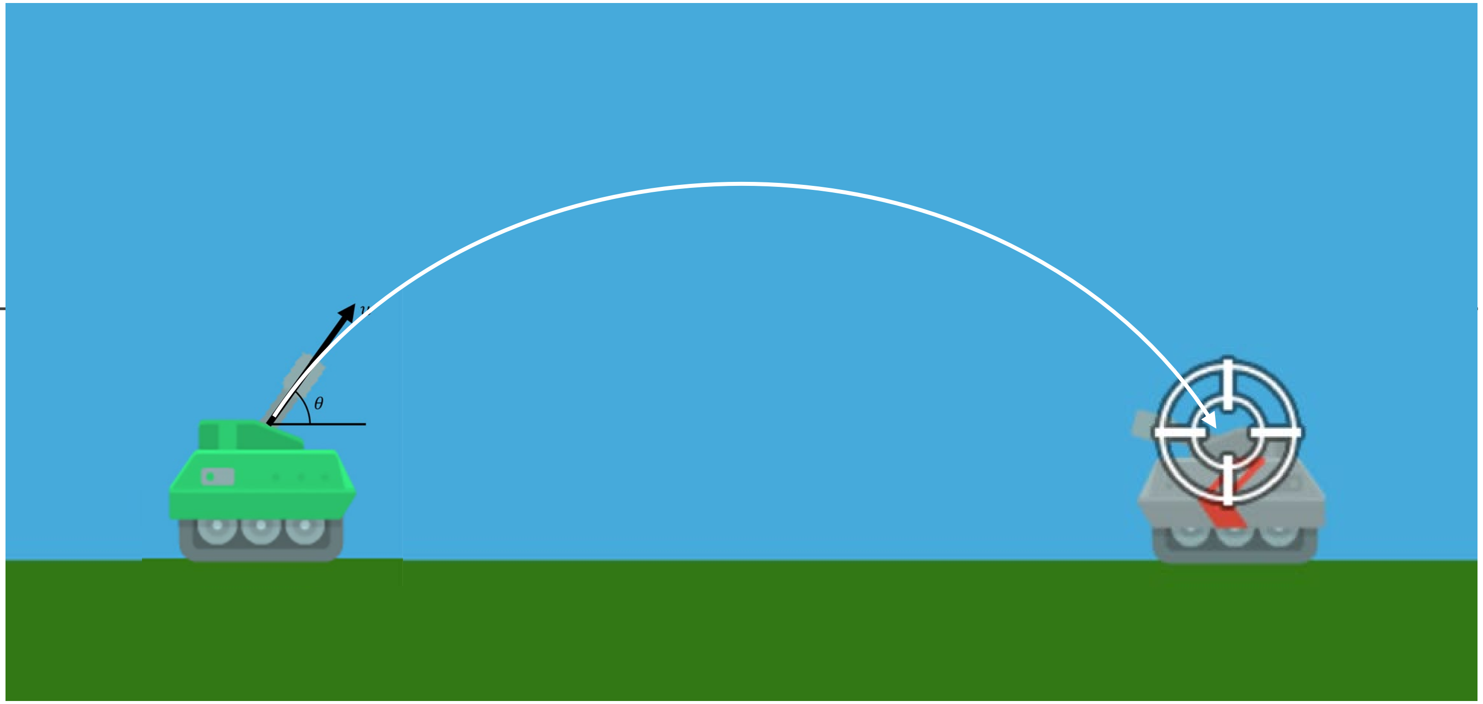
$$\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{a} \cdot \mathbf{s}$$

$$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

Example

- A particle is dropped and falls under gravity: $\mathbf{u} = 0$, $\mathbf{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$
- At time $t = 5$ seconds:
- $\mathbf{v} = \mathbf{u} + \mathbf{a}t = \begin{pmatrix} 0 \\ 5 \times -9.81 \end{pmatrix}$ -- the particle is falling downwards at 49.05 metres per second
- $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 = \frac{25}{2}\begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$ -- the particle has fallen down a distance of 122.625 metres

Projectile motion



- A tank fires a bullet with initial speed u at angle θ
- $\mathbf{u} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

Projectile motion

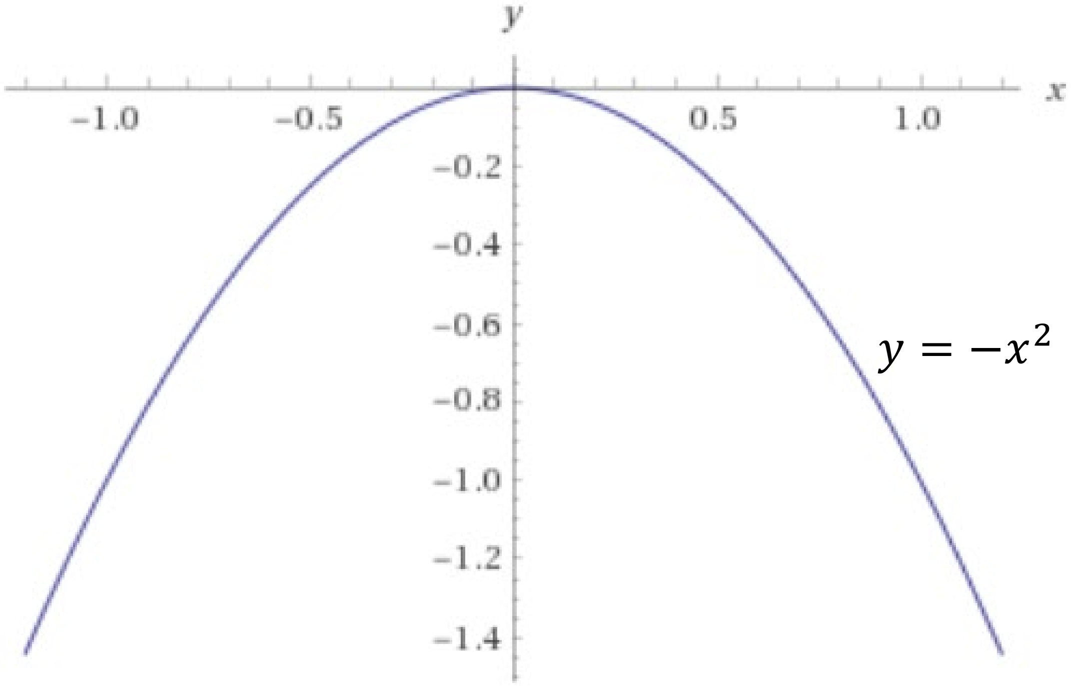
- At time t, the position of the bullet relative to the tank is

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$= \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ -g \end{pmatrix} t^2$$
$$= \begin{pmatrix} u\cos\theta\, t \\ u\sin\theta\, t - \dfrac{gt^2}{2} \end{pmatrix}$$

Projectile motion

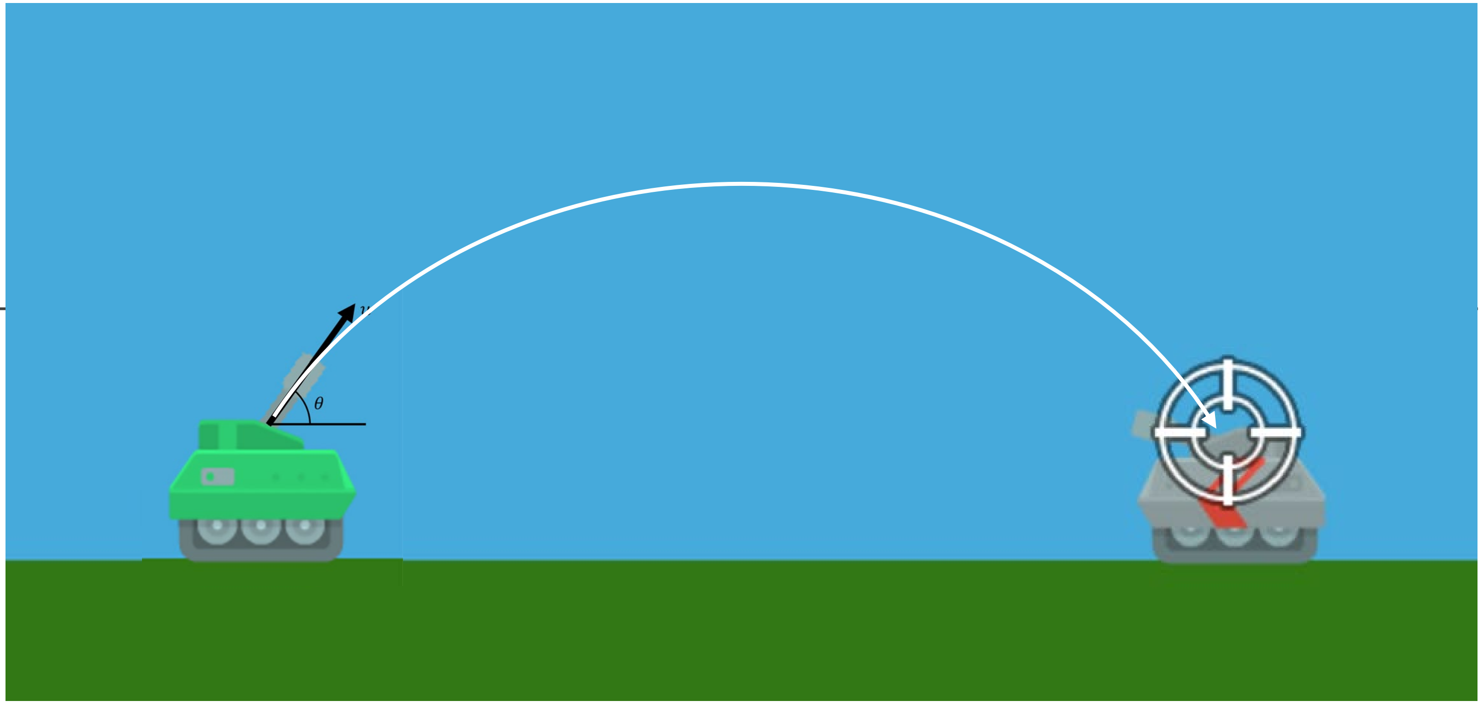
$$\mathbf{s} = \begin{pmatrix} u\cos\theta\, t \\ u\sin\theta\, t - \dfrac{gt^2}{2} \end{pmatrix}$$

- Horizontally: position changes **linearly** with t
- Vertically: position is a **quadratic**
- The shape of motion is a **parabola**



http://www.tom-e-white.com/2015/03/tennis-ball-parabola.html

Projectile motion



- The enemy tank is a distance of x units away, at the same elevation
- Given angle θ, what shot speed u is needed to hit the enemy tank?

Finding the shot speed

$$\mathbf{s} = \binom{x}{0} = \begin{pmatrix} u\cos\theta\, t \\ u\sin\theta\, t - \frac{1}{2}gt^2 \end{pmatrix}$$

- Considering x and y components gives simultaneous equations:

$$x = u\cos\theta\, t$$
$$y = u\sin\theta\, t - \frac{1}{2}gt^2$$

- Solving for u gives

$$u = \sqrt{\frac{xg}{\sin 2\theta}}$$

Worksheet B

Worksheets

- Worksheet A: due today!
- Worksheet B: now available on LearningSpace
- Due in 2 weeks (28th October)