COMP110 EXERCISE SHEET III: RECURSION AND INDUCTION

Version 1.0
BSc Computing for Games
COMP110

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To complete this exercise sheet:

- **complete** the following problems using **pen and paper**; and
- **hand in** your solutions in the COMP110 workshop session in week 8.

1. Prove the following identities by induction. In all cases, $X_1, \ldots, X_n$ and Y are boolean variables.
   (a) $\neg(X_1 \vee X_2 \vee \ldots \vee X_n) = \neg X_1 \wedge \neg X_2 \wedge \ldots \wedge \neg X_n$
   (b) $\neg(X_1 \wedge X_2 \wedge \ldots \wedge X_n) = \neg X_1 \vee \neg X_2 \vee \ldots \vee \neg X_n$
   (c) $(Y \wedge X_1) \vee (Y \wedge X_2) \vee \ldots \vee (Y \wedge X_n) = Y \wedge (X_1 \vee X_2 \vee \ldots \vee X_n)$
   (d) $(Y \vee X_1) \wedge (Y \vee X_2) \wedge \ldots \wedge (Y \vee X_n) = Y \vee (X_1 \wedge X_2 \wedge \ldots \wedge X_n)$
2. Prove by induction that, for all positive integers n,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

3. The Fibonacci numbers are defined recursively as follows:

$$\begin{aligned} \mathrm{fib}(0) &= 1 \\ \mathrm{fib}(1) &= 1 \\ \mathrm{fib}(n) &= \mathrm{fib}(n-1) + \mathrm{fib}(n-2) \qquad \text{for } n \geq 2 \end{aligned}$$

   (a) Write down the first 10 Fibonacci numbers (i.e. $\mathrm{fib}(0), \ldots, \mathrm{fib}(9)$).
   (b) Consider the following Python function:

```
def fib(n):
    if n <= 1:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```

   Prove, by induction, that the total number of calls to `fib` required to compute `fib(n)` is $2 \times \mathrm{fib}(n) - 1$.