# BE: Modeling and calculation of composite structures up to failure

Composite Materials

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16 Nov 2022

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# 1 Behavior of a single ply and change of basis

## 1.1 Question 1

*Plane stress stiffness matrix Cm in function of the elastic properties of the single ply E1 [Pa], E2 [Pa], v12 [ul], and G12 [Pa]*

```
function Cm=rigidCm(E1, E2, nu12, G12)

    nu21 = nu12*E2/E1;
    rat = 1-(nu12*nu21);
    Cm = [E1/rat nu12*E2/rat 0
        nu21*E1/rat E2/rat 0
        0 0 G12];
end
```

where Cm = Stiffness Matrix in Material Matrix (1,2,3)

## 1.2 Question 2

*Write a function in Matlab that gives the strain related to thermal effects epsilon th m in the basis (1, 2, 3) in function of coefficients of thermal expansion alpha1 and alpha2 the temperature difference delta T from the reference temperature.*

```

function tEps=thermEps(alpha1, alpha2, dT)
    alpha = [alpha1, alpha2, 0];
    tEps = alpha.*dT;
end
```

Where alpha 1 and alpha 2 the linear thermal expansion coefficients [1/C] in the material basis, and dT is the delta temperature.

## 1.3 Question 3

*Write a function in Matlab that gives the stiffness matrix Cp in the basis (x, y, z) in function of the matrix Cm defined in Question 1 and the angle theta.*

As previously defined, Cm is the stiffness matrix of the material in the material basis, however due to the stacking sequence, this stiffness will change in each ply if it is seen from a global basis (xyz). It will only remain constant if the orientation of the ply had an angle of 0.

Later for Cp, the angle theta (in radians) is used to construct the transformation matrix T needed to change from material to general basis.

```
function Cm=rigidCm(E1, E2, nu12, G12)

    nu21 = nu12*E2/E1;
    rat = 1-(nu12*nu21);
    Cm = [E1/rat nu12*E2/rat 0
        nu21*E1/rat E2/rat 0
        0 0 G12];
end

```

```
function Cp=rigidCp(Cm, theta)
    c = cos(theta);
    s = sin(theta);
    T = [c^2 s^2 c*s
        s^2 c^2 -c*s
        -2*c*s 2*c*s c^2-s^2];
    Cp = T'*Cm*T;
end
```

## 1.4 Question 4

*Write a function in Matlab that draws, in polar coordinates, the elastic modulus Ex in direction x in function of the angle theta.*
A first function was made to calculate the elastic modulus Ex [Pa], as a function of theta and the previously calculated Cm ( Cm is constant for the material). Later a second function was employed to plot the Ex for various angles, radians are needed so a deg2rad function was used.

```
function Ex = elasEx(Cm, theta)
    Cp = rigidCp(Cm,theta);
    Sp = inv(Cp);
    Ex = 1/Sp(1,1);
end

function plotEx(Cm)

    angle = 0:deg2rad(1):deg2rad(359);
    Ex = zeros(size(angle));
    for i=1:length(angle)
        Ex(i) = elasEx(Cm, angle(i));
    end
    polarplot(angle, Ex)
end
```

## 1.5 Question 5

*Draw Ex for the following materials:*
*Material 1:*

- *Material 1: E1= 100 Gpa, E2= 10 Gpa, v12= 0.3, G12= 5Gpa.*
- *Material 2: E1= 50 Gpa, E2= 50 Gpa, v12= 0.3, G12= 5Gpa.*
- *Material 3: E1=E2=E= 50 Gpa, v= 0.3, G= E/2(1+v).*

For the Material 1, the anisotropic behavior of the material is evident since the stiffness is much higher in one direction. Material 2 is slightly less anisotropic, since the stiffness is the same in direction 1 and 2. But on material 3 we can observe a circle since its behavior is completely isotropic.

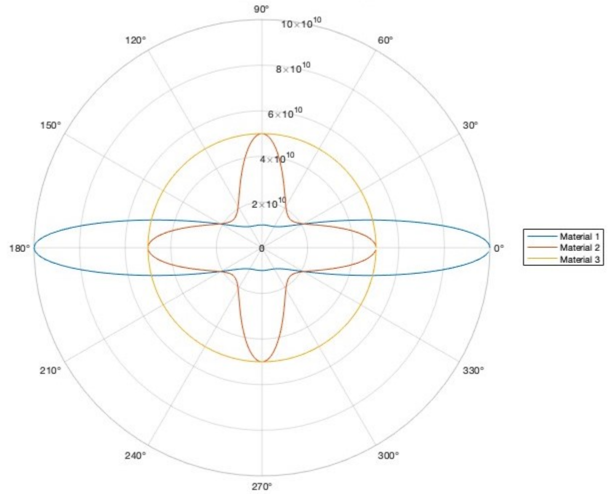



Figure 1: Elastic Modulus $E_x$ (GPa) as a function of stacking angle $\theta°$

# 2 Constitutive Behavior of a composite plate

## 2.1 Question 6

*Write a function in Matlab that gives the elastic behavior of the plate as a function of the properties of each ply and the information on the stacking sequence.*

To calculate this, the Cm is needed which is constant in the material, and the stacking sequence as an array that contains the orientation of the plies in angles, the last parameter t is the thickness of the plies on meters. In the z axis, 0 is defined at the center of the plaque. The units of each sub-matrix (A, B, and C will be latter mentioned)

```
function [S, A, B, D]=elasBehav(Cm, stack, t)
    height = t*[-length(stack)/2:length(stack)/2];
    nbLayers = length(stack);
    A = zeros(3);
    B = zeros(3);
    D = zeros(3);
    for i=1:nbLayers
        A = A + (rigidCp(Cm, deg2rad(stack(i)))*(height(i+1)-(height(i))));
        B = B + (rigidCp(Cm, deg2rad(stack(i)))*((height(i+1))^2-((height(i)))^2)/2);
        D = D + (rigidCp(Cm, deg2rad(stack(i)))*((height(i+1))^3-((height(i)))^3)/3);
    end
    S = [A B
        B D];
end
```

## 2.2 Question 7

*Write a function in Matlab that gives the thermal contribution into the plate behavior as a function of the properties of each ply and the information on the stacking sequence.*

Similar to the previous question, the stiffness matrix of the material Cm, the stacking sequence, and the thickness were needed. But also the thermal expansion coefficients alpha 1, alpha 2 [1/C], and the delta temperature in Celsius or Kelvin.

```
function sigUth=sigTherm(Cm, theta, alpha1, alpha2)
    alpha = [alpha1, alpha2, 0];
    c = cos(theta);
    s = sin(theta);
    T = [c^2 s^2 c*s
        s^2 c^2 -c*s
        -2*c*s 2*c*s c^2-s^2];
    sigUth = T'*Cm*alpha';
end

function [Nth, Mth]=thermBehav(Cm, stack, t, alpha1, alpha2, dT)
    height = t*[-length(stack)/2:length(stack)/2];
    nbLayers = length(stack);
    Nth = zeros(3,1);
    Mth = zeros(3,1);
    for i=1:nbLayers
        Nth = Nth + (sigTherm(Cm, deg2rad(stack(i)), alpha1, alpha2)*(height(i+1)-(height(i)))*dT);
        Mth = Mth + (sigTherm(Cm, deg2rad(stack(i)), alpha1, ...
            alpha2)*(height(i+1)^2-(height(i))^2)*dT/2);
    end
end
```

## 2.3 Question 8

*Write a function in Matlab that gives the local strains and stresses in the material basis at the given position of z as a function of the generalized strain state of the plate, the properties of each ply and the information on the stacking sequence.*

```
function [plyNum, Ep,curStack] = localStrains(nu,ki,z,t,stack)
    if (z>(length(stack)*t/2)) || (z<(-length(stack)*t/2))
        error('>:  The height asked is out of the max height of the stacked')
    end
    Ep = (nu+z*ki)';
    plyNum = (ceil(z/t))+(length(stack)/2)
    curStack = stack(plyNum)
end

function [plyNum, sigP,eP,sigM,eM] = localBeh(Cm,dT,t,nu,ki,z,stack,alpha1,alpha2)

    eP = zeros(3, length(z));
    sigM = zeros(3, length(z));
    sigP = zeros(3, length(z));
    eM = zeros(3, length(z));
    plyNum = zeros(size(z));


    for i=1:length(z)
        if (z(i)>(length(stack)*t/2)) || (z(i)<(-length(stack)*t/2))
            error('>:  The height asked is out of the max height of the stacked');
        end
        [PLYNUM, EP,curStack] = localStrains(nu,ki,z(i),t,stack)
        plyNum(i) = PLYNUM;
```

```
        eP(:,i) = EP;
        Cp = rigidCp(Cm,deg2rad(curStack))
        sigUth=sigTherm(Cm, deg2rad(curStack), alpha1, alpha2)
        SIGP = Cp*EP - sigUth*dT
        sigP(:,i) = SIGP;
        c = cos(deg2rad(curStack));
        s = sin(deg2rad(curStack));
        T = [c^2 s^2 c*s
            s^2 c^2 -c*s
            -2*c*s 2*c*s c^2-s^2];
        SIGM = T'\SIGP;
        sigM(:,i) = SIGM;
        EM = T*EP;
        eM(:,i) = EM;
    end
    eP
    eM
    sigM
end
```

A check for the asked height to be within the total height of the stack was also put.

## 2.4 Question 9

*Using the material 1 given in the previous question, calculate the matrices of the elastic behavior A, B and D for the following stacking sequences :*

- *Stacking Sequence 1 =[0/90]s*
- *Stacking Sequence 2 =[0/45/90/-45]s*
- *Stacking Sequence 3 =[0/90]*

*(Thickness of each ply is 0.125 mm)*

Based on function defined at question 6 and using the different stacking sequences as inputs on the array 'stack'.

```
    [S, A, B, D] = elasBehav(Cm, stack, thickness)
```

```
A =

   1.0e+07 *

    2.7750    0.1514    0.0000
    0.1514    2.7750    0.0000
    0.0000    0.0000    0.2500


B =

   1.0e-13 *

         0    0.1421         0
    0.1421    0.2842         0
         0         0         0


D =

    0.9329    0.0315    0.0000
    0.0315    0.2234    0.0000
    0.0000    0.0000    0.0521
```

```
A =

   1.0e+07 *

    4.4881    1.3645         0
    1.3645    4.4881    0.0000
         0    0.0000    1.5618


B =

   1.0e-12 *

         0   -0.0853         0
    0.0284   -0.1137         0
         0         0   -0.0568


D =

    5.8457    0.8053    0.5321
    0.8053    2.2982    0.5321
    0.5321    0.5321    0.9697
```

```
A =

   1.0e+07 *

    1.3875    0.0757    0.0000
    0.0757    1.3875    0.0000
    0.0000    0.0000    0.1250


B =

 -709.5106         0    0.0000
         0  709.5106    0.0000
    0.0000    0.0000         0


D =

    0.0723    0.0039    0.0000
    0.0039    0.0723    0.0000
    0.0000    0.0000    0.0065
```

Figure 2: Values of A* $(N/m)$, B* $(N/m^2)$ D $(N.m)$ for Stacking Sequence 1, 2, and 3 respectively

*In the Figure 2, from the membrane matrix, A, we expected to have an uncoupled shear and normal behavior for all the three stacking sequences.

Similarly, since the stacking sequence 1 and 2 are symmetric, we expected B, the coupling matrix to be equal to 0, which it is.

## 2.5 Question 10

*We now consider the stacking sequence 2, subjected to a uni-axial tensile test. One chooses to control the test in displacement in the direction x, and one imposes a constant strain $\mu x = 0.01$.*

- *Determine the associated generalized stresses.*

```
function genStress=genBehav(Cm, stack, t, nu, ki, alpha1, alpha2, dT)
    [S, A, B, D] = elasBehav(Cm, stack, t);
    [Nth, Mth] = thermBehav(Cm, stack, t, alpha1, alpha2, dT);
    genStress = S*[nu, ki]' - [Nth;Mth];

end
```

```
N =

   1.0e+05 *

    4.8030
    1.6794
   -0.0000

>> M

M =

   1.0e-15 *

         0
    0.2842
         0
```

Figure 3: N (N/m) and M (N) as generalized stresses for material 1 and stacking sequence 2

From the generalized stresses in Figure 3, we can see that the M is zero since $\chi$, the curvatures, is zero and $M = B * \mu + D * \chi$. We know that B is also 0 in this case, equaling M to zero

- *Determine the local stresses in the material basis for each ply.*
  From function defined at question 8

```
[plyNum, sigP,eP,sigM,eM] = localBeh(Cm,dT,thickness,nu,ki,z,stack,alpha1,alpha2);
```

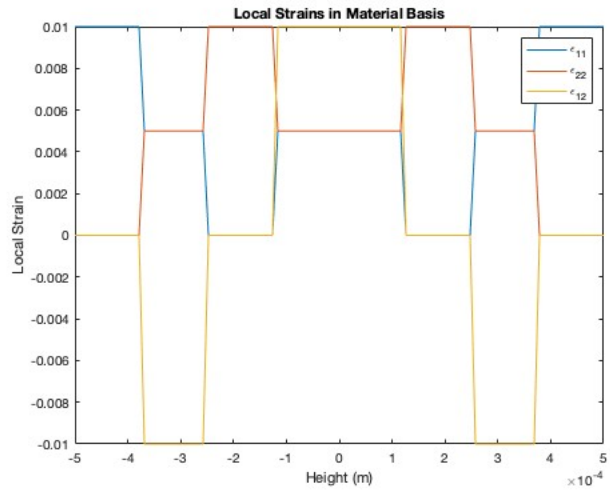



Figure 4: Local Strains(Unitless) against Height(m) in Material Basis for Material 1 and Stacking Sequence 2

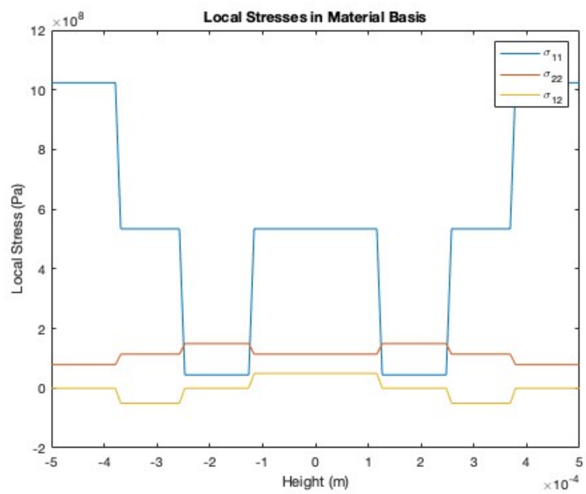



Figure 5: Local Stresses(Pa) against Height(m) in Material Basis for Material 1 and Stacking Sequence 2

- *Is the solution valid near a free edge? If not, explain how we can introduce a correction.*

The Love-Kirchoff approximation, the simplication of a 3D body into 2D seems accurate between the free edges but needs a correction at the edges.

At the free surfaces the condition $\sigma * n = 0$ must be fullfilled, where "n" is a outward pointing vector normal to the surface (in this case -/+ y for all plies). However this is not the case. For instance, at the ply with an angle of 0°, the direction 1 is completely aligned with the direction of the force, and the direction 2 is aligned with y. Based on the free surface condition of $\sigma * n = 0$, the $\sigma_{22}$ component should be equal to 0, but this is not true as seen on the figure 5. To compensate for this, a stress equal in magnitude but opposite in sign must be applied on this ply. Since we have three stacking angles of $0°, 45°$ and $90°$ and the negatives of them, this procedure should be repeated at each ply based on it's angle. The compensation stress is applied as a boundary condition.

Lastly, for checking the requirement of the delamination, we should see, whether the interfaces are in tension or compression caused by the rotation induced by the stress couples.

## 2.6 Question 11

*We consider again the stacking sequence 2, subjected this time to a thermal load delta T = 160C. Considering alpha 1 = 0 and alpha 2 = 30e-6 (1/C)*

- *Calculate the relaxed generalized strains nu and ki. Can this configuration be achieved in the real structure?*

```
nur =

   1.0e-03 *

   -0.5379
   -0.5379
    0.0000

>> kir

kir =

   1.0e-16 *

   -0.0629
   -0.2125
    0.1512
```

Figure 6: The relaxed stress $\mu$ (Unitless) and $\chi(m^{-1})$

Since, the temperature difference is $-160°$, in Figure 6, we can see that there is a relaxed strain that seems plausible. However, this does not satisfy the kinematic compatibility equations, which would require to accommodate the stress caused, into the actual strain and displacement fields. This means that if an additional loading is applied, the stresses and strains caused by this cooling process should be taken into account.

- *Calculate the stresses due to the thermal loading in the material basis for each ply associated with the relaxed configuration.*

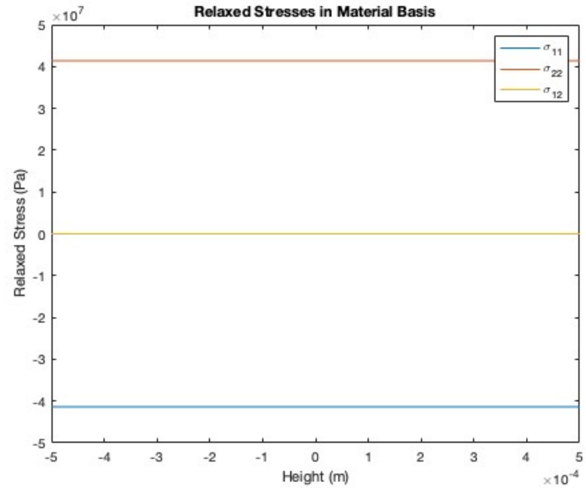



Figure 7: Relaxed Stresses $\sigma_{mr}$ (Pa) in Material Basis as a function of Height (m)

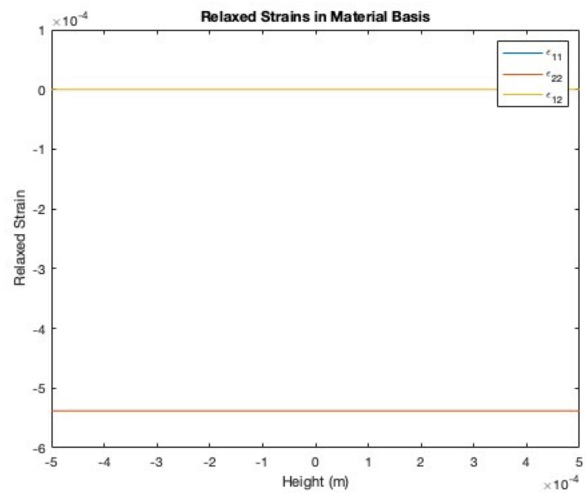
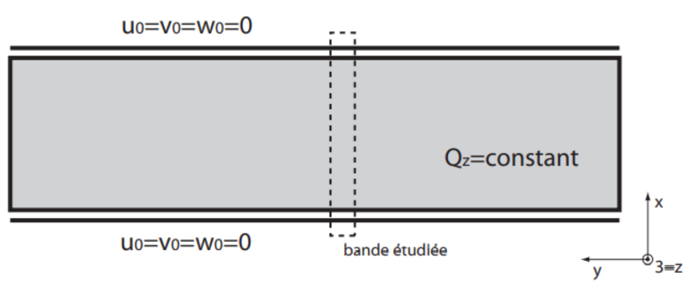



Figure 8: Relaxed Strains (Unitless) in Material Basis as a function of Height (m)

# 3 Analytical solution of a simplified plate problem




From the Figure 7 and 8, we can see there is a $-5.379e-03$ strain all over the stack height,

## 3.1 Question 12

*We consider a rectangular plate of dimensions 0.1 × 0.4 m (material 1, stacking sequence 2). It is supported (u0 = v0 = w0 = 0) on the long sides and free on the short sides, and subjected to a pressure distributed on the surface Qz = -5 · 104 N/m2 (see Figure 3). To determine the maximum deflection, we consider a band far enough from the free edges, which makes it possible to introduce the following simplifying assumptions:*

$$f(x, y, z) = f(x, z) \text{ and } \frac{\partial f}{\partial y} = 0$$

- *Considering the plate equations and the proposed simplification, determine the distribution of moments Mxx(x) along the band.*
  From plate equations the following formula can be extracted:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} * \begin{bmatrix} \mu \\ \chi \end{bmatrix}$$

  However, since the stacking sequence is symmetric, B = 0. And thus the expression can be further simplified to:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} * \begin{bmatrix} \mu \\ \chi \end{bmatrix} \text{ which yields } \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A * \mu \\ D * \chi \end{bmatrix}$$

  Due to the assumptions made, $\chi$ has the following form:

$$\chi = \begin{bmatrix} -\frac{\partial^2 w_o}{\partial^2 x} \\ 0 \\ 0 \end{bmatrix}$$

  And we finally obtain:

$$\begin{bmatrix} Mxx \\ Myy \\ Mxy \end{bmatrix} = \begin{bmatrix} D11 * -\frac{\partial^2 w_o}{\partial^2 x} \\ D12 * -\frac{\partial^2 w_o}{\partial^2 x} \\ D13 * -\frac{\partial^2 w_o}{\partial^2 x} \end{bmatrix}$$

  With Mxx being : $Mxx = D11 * -\frac{\partial^2 w_o}{\partial^2 x}$

- *Determine the displacement w0(x) and deduce from it the relation between the imposed loading and the maximum deflection.*
  Equilibrium equation:

$$(\nabla * (\nabla * M)) + \nabla * C + Q_z = 0$$

  In our specific case we can simplify the equation to:

$$\frac{\partial^2 Mxx}{\partial^2 x} = -Q_z$$

  Which from the previous question can be rewritten as:

$$-D11 * \frac{\partial^4 w_o}{\partial^4 x} = -Q_z$$

Based on the physics of the problem, the following boundary conditions can be established: No displacement at the borders $w_o(0) = w_o(l) = 0$ (with "$l$" being the length of the short edge of the plate), and no moment at the edges $w_o''(0) = w_o''(l) = 0$. This leads to a solution in the form of a the following 4th degree polynomial :

$$w_o = \begin{bmatrix} -3.5639e+02 \\ 71.2773 \\ 0 \\ -0.3564 \\ 0 \end{bmatrix} * \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix}$$

That plotted looks like this:

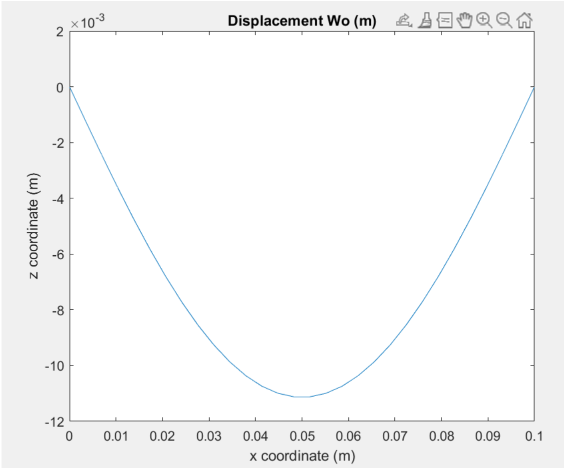



Figure 9: Displacement $w_\circ(x)$ (m) as a function of x coordinates (m)

With a min value of: -0.01114 m at x = 0.05 m.
The final expression of the the maximum value is the (at x = 0.05m):

$$w_{oMax} = \frac{5*Qz*L^4}{384*D11}$$

- *Compare the results obtained with the results of the finite element calculation (shell elements) given in Figure 4. Comment on and interpret any differences.*
  The results obtained almost match exactly those from figure 4, 11.14 mm (our solution) vs 11.11 mm (shell elements). The slight difference can be accounted to discretization errors on the finite element calculation.

- *Compare the results obtained with the results of the finite element calculations (3D elements) given in Figures 5 and 6. The two calculations differ in terms of the boundary conditions, specified in these figures. Comment on and interpret any differences.*

  The result from figure 5 don't match those previously obtained (5.44mm vs 11.1 mm). This is caused by the 0 moment boundary condition imposed at both edges to solve the differential equation. On the 3D model, a clamped condition is imposed at both elements (translation and rotation is restricted on the whole face of the 3D element) and thus moment obtains a considerable value.
  Figure 6's results match those obtained analytically. Because, contrary to the previous case only one of the edges is set to with a 0 displacement condition.

## 3.2 Question 13

*We consider a specimen at* $0°$*. From the html HTMLreport.html page in the Report _ deg directory, identify the Young's modulus E1 and the Poisson's ratio* $\nu_{12}$ *(give the formulas used in the notation presented in the course and in the notation used in the Abaqus report). Is the calculation performed representative of the tensile test? Why ?*

From the course, the constitutive behaviour of the material is given by the equation:

$$\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

From the ABAQUS tests, we can obtain the individual parameters of the constitutive matrix, given the strain and stress fields.

**Poisson Coeff:**

*From Course:*

$\nu_{12} = -\frac{\epsilon_{22}}{\epsilon_{11}}$

*From ABAQUS:* $\nu_{12} = -\frac{E_{22}}{E_{11}} = 0.3$

**Youngs Modulus:**

*From Course:* $E_{11} = \frac{\sigma_{11}}{\epsilon_{11}}$

*From ABAQUS:* $E_{11} = \frac{S11}{E11} = 100GPa$

Yes, this test is a representative of a tensile test since the fibers are aligned to the loading, only longitudinal and transversal deformations are present, which can be obtained from the 1/4 plate simplification.

## 3.3 Question 14

*We consider a specimen at* $90°$*. From the html HTMLreport.html page in the Report _ deg directory, identify the Young's modulus E2 and the Poisson's ratio* $\nu_{12}$ *(give the formulas used in the notation presented in the course and in the notation used in the Abaqus report). Is the calculation performed representative of the tensile test? Why ?*

**Poisson Coeff:**

*From Course:* $\nu_{21} = -\frac{\epsilon_{11}}{\epsilon_{22}}$ and $\nu_{12} = -\nu_{21} * \frac{E_1}{E_2}$

*From ABAQUS:* $\nu_{12} = -\frac{E11}{E22} * \frac{E_1}{S22} * E22 = 0.3$

**Youngs Modulus:**

*From Course:* $E_2 = \frac{\sigma_{22}}{\epsilon_{22}}$

*From ABAQUS:* $E_2 = \frac{S22}{E22} = 10GPa$

Yes, this test is a representative of a tensile test. Similar to previous configuration, the material only experiences transversal and longitudinal deformations. Even if the load is not aligned to the fibers.

## 3.4 Question 15

*We consider a specimen at* $45°$*. From the html HTMLreport.html page in the Report _ deg directory, identify the Young's modulus G12 and the Poisson's ratio* $\nu_{12}$ *(give the formulas used in the notation presented in the course and in the notation used in the Abaqus report). Is the calculation performed representative of the tensile test? Why ?*

**Youngs Modulus:**

*From Course:* $G_{12} = \frac{\sigma_{12}}{2*\epsilon_{12}}$

*From ABAQUS:* $G_{12} = \frac{S12}{E12} = 5GPa$

Here, we didn't have to divide by 2, assuming that ABAQUS already takes the twice of the $\epsilon_{12}$, the shear modulus is also correct.

No, this test is not representative of the tensile test. The tensile test on the whole plate would create an angular deformation, but this can not be observed at the base of the simulation using the 1/4 plate simplification.

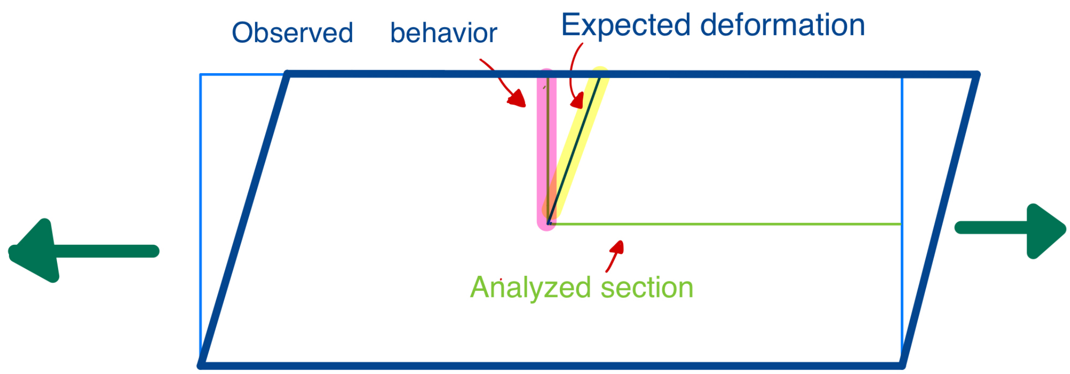



Figure 10: Representative image of angular behavior expected during the tensile test of the plate