Structural assemblies: dealing with frictional contact problems

Practical work
Accounting for boundary conditions

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1 Structure of interest

Let's consider the plane truss shown in figure 1. All bars are assumed to have the same cross-section S and Young's modulus E. The exact solution is partially provided in part 7.

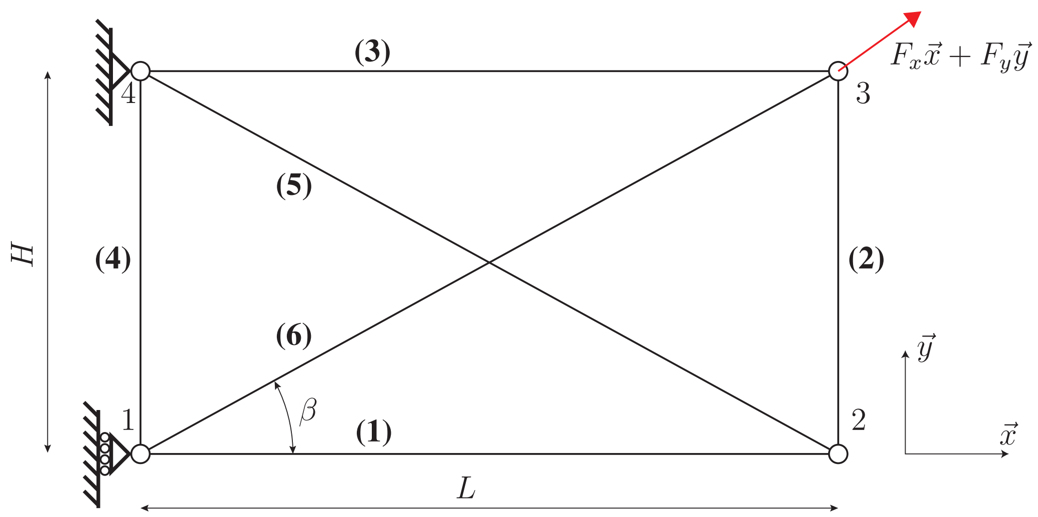

Figure 1: Structure of interest

Displacements of a node i along $\vec{x}$ and $\vec{y}$ axis are respectively denoted by $u_{i,x}$ and $u_{i,y}$.

2 Stiffness matrix

The element stiffness matrix for a linear bar element is:

$$[k_e] = \frac{ES}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where L_e is the element's length.

Longitudinal displacements U_i and U_j in the bar element's local reference frame are related to the displacements $u_{i,x}$, $u_{i,y}$, $u_{j,x}$ and $u_{j,y}$ by the following equations (see figure 2):

$$U_i = u_{i,x}\cos(\alpha) + u_{i,y}\sin(\alpha)$$
$$U_j = u_{j,x}\cos(\alpha) + u_{j,y}\sin(\alpha)$$

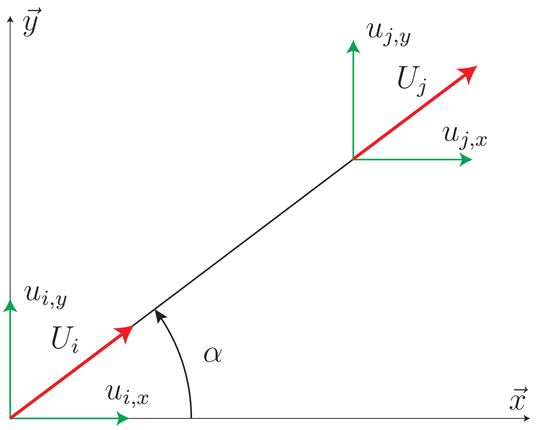

Figure 2: Displacements in the local reference frame of the bar element (in red) expressed in terms of the nodal displacements in the global frame (in green)

These equations can be written in matrix form:

$$\left\{\begin{array}{c} U_i \\ U_j \end{array}\right\} = \left[\begin{array}{cccc} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{array}\right] \left\{\begin{array}{c} u_{i,x} \\ u_{i,y} \\ u_{j,x} \\ u_{j,y} \end{array}\right\}$$

$$= [R] \left\{\begin{array}{c} u_{i,x} \\ u_{i,y} \\ u_{j,x} \\ u_{j,y} \end{array}\right\}$$

Using the local displacements, the strain energy can be written as:

$$E_d^e = \frac{1}{2}\{U_i \quad U_j\}[k_e]\left\{\begin{array}{c} U_i \\ U_j \end{array}\right\}$$

$$= \frac{1}{2}\{u_{i,x}\ u_{i,y}\ u_{j,x}\ u_{j,y}\}[R]^T[k_e][R]\left\{\begin{array}{c} u_{i,x} \\ u_{i,y} \\ u_{j,x} \\ u_{j,y} \end{array}\right\}$$

$$= \frac{1}{2}\{u_e\}^T[K_e]\{u_e\}$$

which makes it possible to introduce an elementary stiffness matrix $[K_e]$ involving the global degrees of freedom.
In our case, all forces are imposed on the nodes, which makes it easy to build the right-hand side $\{F_e\}$:

$$\{F_e\}^T = \{F_{i,x}\ F_{i,y}\ F_{j,x}\ F_{j,y}\}$$

After assembling the right-hand side and the global elementary stiffness matrices, we finally obtain the linear system to be solved:

$$[K]\{u\} = \{F\} \tag{1}$$

where vector $\{u\}$ is the unknown. It stores all the nodal displacements $u_{i,x}$ and $u_{i,y}$.

Question 1 • Following the procedure described above, assemble the stiffness matrix $[K]$ of the entire truss.

Question 2 • Build the nodal forces vector $\{F\}$ associated with the loading $\vec{F}$.

However, before solving the system, the boundary conditions have to be taken into account. For this purpose, we will implement different methods seen in class.

3 Method 1: rows and columns elimination

We need to take the following boundary conditions into account:

- Displacement at node 1 blocked along the $\vec{x}$ axis;
- Displacement at node 4 blocked along both $\vec{x}$ and $\vec{y}$ axis.

In order to do so, rows of the system (1) and columns of the matrix $[K]$ corresponding to these degrees of freedom have to be eliminated.

Question 3 • Build the 5×5 stiffness matrix $[\widetilde{K}]$.

Question 4 • Solve the linear system. Is the result consistent with the reference?

4 Method 2: penalty method

The above mentioned boundary conditions will now be taken into account by penalty method.

Question 5 • Build a matrix $[C]$ and a vector $\{B\}$ enabling to write the boundary conditions in the form $[C]\{U\} = \{B\}$.

Question 6 • After introducing the penalty coefficient k, modify the stiffness matrix $[K]$ accordingly.

Question 7 • Solve the linear system for a given parameter k and validate (or not) the obtained result.

Question 8 • Carry out a parametric study to visualize the influence of the penalty parameter k on the finite element solution's quality.

5 Method 3: with *one* Lagrange multiplier

In this part, the boundary conditions will be taken into account by using one Lagrange multiplier $\{\lambda\}$.

Question 9 • By using the assembled stiffness matrix $[K]$, build the $[A]$ matrix of the linear system to solve with this method.

Question 10 • By using the nodal forces vector $\{F\}$, build the system's right-hand side.

Question 11 • After solving the linear system, validate the obtained solution in displacement as well as the value obtained for the introduced Lagrange multiplier $\{\lambda\}$.

Question 12 • Repeat the previous procedure by modifying the boundary condition at node 1: $\vec{u}_1.\vec{n} = 0$, where $\vec{n} = \cos(\theta)\vec{x} + \sin(\theta)\vec{y}$.

Question 13 • Investigate the influence of the θ angle on the structure's response (you can observe the displacement at a given point for example). What can you say?

6 Method 4: with *two* Lagrange multipliers

We are now using *two* Lagrange multipliers to take the boundary conditions into account. Compared to the standard Lagrange multipliers method, the advantage of this one is to lead to a matrix with no zero terms on the diagonal. Thus, we introduce two multipliers $\{\lambda_1\}$ and $\{\lambda_2\}$, as well as a positive real number a.

Question 14 • Build the matrix $[A]$ and the second member of the system to be solved. Keep the possibility to modify the factor a.

Question 15 • By using both multipliers $\{\lambda_1\}$ and $\{\lambda_2\}$, calculate the reaction forces. Does the result make sense?

Question 16 • Assess the influence of parameter a on the quality of the solution.

7 Exact solution

The reaction forces at nodes 1 and 2 can be determined by writing the global equilibrium of the truss, giving 3 equations:

$$\begin{aligned} R_{1,x} + R_{4,x} + F_x &= 0 \\ R_{4,y} + F_y &= 0 \\ H.R_{1,x} + L.F_y &= 0 \end{aligned}$$

We thus obtain:

$$\begin{aligned} R_{1,x} &= -F_y \frac{L}{H} \\ R_{4,x} &= -F_x + F_y \frac{L}{H} \\ R_{4,y} &= -F_y \end{aligned}$$

The equilibrium written at each of the 4 nodes gives 2 equations, but some of them are redundant with the previous ones:

- Node 1:
$$R_{1,x} + N_1 + N_6\cos(\beta) = 0 \quad N_4 + N_6\sin(\beta) = 0$$
- Node 2:
$$-N_1 - N_5\cos(\beta) = 0 \quad N_2 + N_5\sin(\beta) = 0$$
- Node 3:
$$F_x - N_3 - N_6\cos(\beta) = 0 \quad F_y - N_2 + N_6\sin(\beta) = 0$$
- Node 4:
$$R_{4,x} + N_3 + N_5\cos(\beta) = 0 \quad R_{4,y} - N_4 - N_5\sin(\beta) = 0$$

The problem is statically indeterminate. Therefore, the static unknowns can be expressed in terms of the given parameters and one reaction force, for example, N_6:

$$\begin{aligned}
N_1 &= F_y\frac{1}{\tan(\beta)} - N_6\cos(\beta)\\
N_2 &= F_y - N_6\sin(\beta)\\
N_3 &= F_x - N_6\cos(\beta)\\
N_4 &= -N_6\sin(\beta)\\
N_5 &= -F_y\frac{1}{\sin(\beta)} + N_6
\end{aligned}$$

The total strain energy is given by:

$$E_d = \frac{1}{2ES}\left[N_1^2 L + N_2^2 H + N_3^2 L + N_4^2 H + N_5^2\frac{L}{\cos(\beta)} + N_6^2\frac{L}{\cos(\beta)}\right]$$

The complementary energy is equal to the strain energy minus the imposed displacements' work. Since the imposed displacements are null in our case:

$$E_c = E_d$$

By using the principle of minimum complementary energy, we can write:

$$\frac{dE_c}{dN_6} = 0$$

which provides the missing equation.

$$\begin{aligned}
\frac{dE_c}{dN_6} &= \frac{dE_d}{dN_6}\\
&= \frac{1}{ES}\Big[(N_6\cos(\beta) - F_y\frac{1}{\tan(\beta)})L\cos(\beta) - (F_y - N_6\sin(\beta))H\sin(\beta)\\
&- (F_x - N_6\cos(\beta))L\cos(\beta) + N_6 H\sin^2(\beta)\\
&+ (-F_y\frac{1}{\sin(\beta)} + N_6)\frac{L}{\cos(\beta)} + N_6\frac{L}{\cos(\beta)}\Big] = 0
\end{aligned}$$

The expression of the normal effort N_6 can then be deduced:

$$2N_6\left[L\cos^2(\beta)+H\sin^2(\beta)+\frac{L}{\cos(\beta)}\right]=F_x(L\cos(\beta))+F_y\left[L\frac{\cos^2(\beta)}{\sin(\beta)}+H\sin(\beta)+\frac{L}{\cos(\beta)\sin(\beta)}\right]$$

Leading to:

$$N_6=F_x\frac{1}{2}\frac{L\cos(\beta)}{L\cos^2(\beta)+H\sin^2(\beta)+\frac{L}{\cos(\beta)}}+F_y\frac{1}{2}\frac{L\frac{\cos^2(\beta)}{\sin(\beta)}+H\sin(\beta)+\frac{L}{\cos(\beta)\sin(\beta)}}{L\cos^2(\beta)+H\sin^2(\beta)+\frac{L}{\cos(\beta)}}$$

Displacement at node 3 can be found using Castigliano's theorem.

$$u_{3,x}=\frac{dE_d}{dF_x}$$

$$u_{3,y}=\frac{dE_d}{dF_y}$$

Each value represents the displacement in the same direction as the applied effort.

We finally obtain:

$$u_{3,y}=\frac{F_y\left(2H-2L+L\frac{1}{\sin^2(\frac{\beta}{2})\cos(\beta)}\right)-2F_x\frac{L}{\tan(\beta)}}{4ES}$$

The expression of $u_{3,x}$ being way more complicated, it won't be written explicitly here.

Following values can be used for a numerical application:

- $L=2$ m et $H=1$ m ;
- $S=10^{-4}$ m^2 ;
- $E=2.1\ 10^{11}$ N.m^{-2} ;
- $F_x=F_y=10^6$ N.

which leads to: $u_{3,y}=0.385$ m.

The final shape is ploted on figure 3.

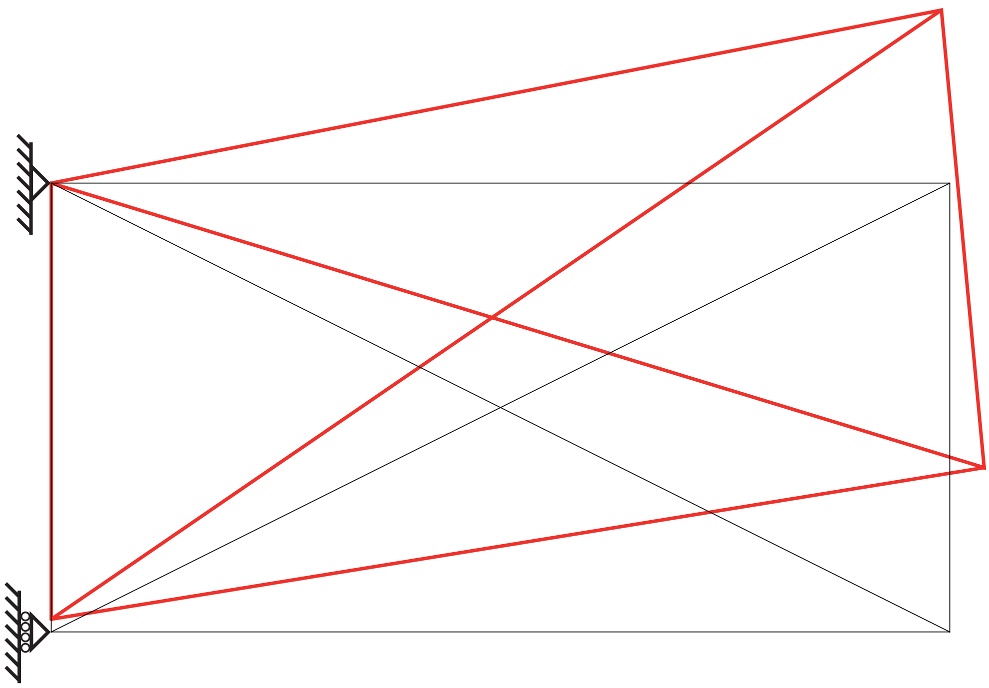

Figure 3: Final shape of the truss for a particular case: $L = 2$ m, $H = 1$ m, $S = 10^{-4}$ m^2, $E = 2.1\ 10^{11}$ N.m^{-2}, $F_x = F_y = 10^6$ N