SD Tools
VIBRATION SOFTWARE & CONSULTING

Arts et Métiers Institute of Technology

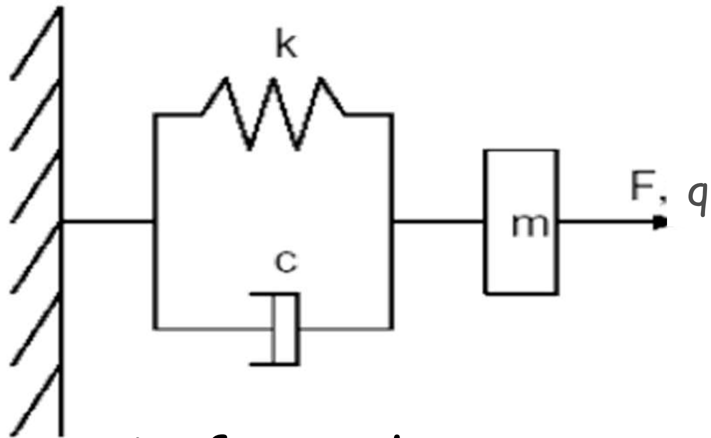
MS2SC http://savoir.ensam.eu/moodle/course/view.php?id=1874

PROVIR http://savoir.ensam.eu/moodle/course/view.php?id=490

Resonance (1 DOF oscillator), poly ch1

Dynamic equation :

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = F(t)$$



Harmonic excitation

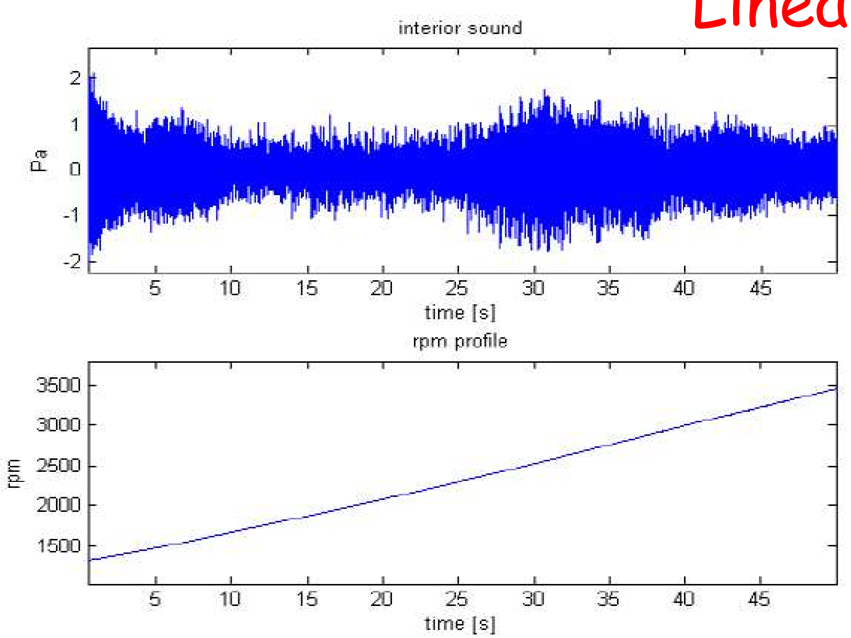
$$F(t) = \mathrm{Re}\left(F(\omega)e^{\imath\omega t}\right)$$

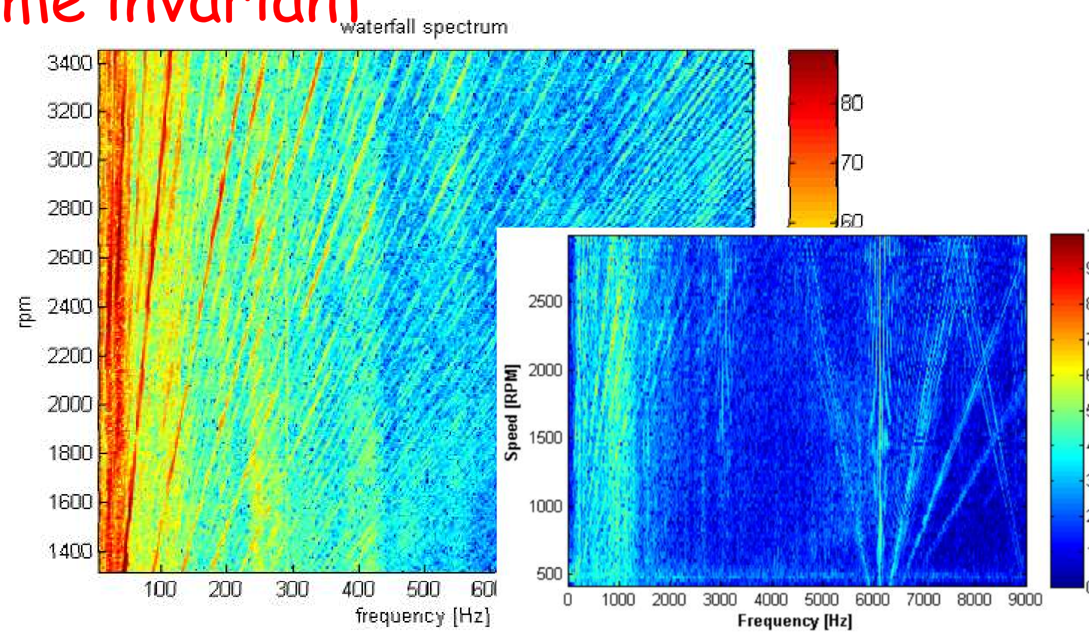
Hyp ?

Harmonic forced response

$$q(t) = \mathrm{Re}\left(q\left(\omega\right)e^{i\omega t}\right)$$

Linear time invariant





1 DOF time integration : state space (poly 3.1)

- Second order (meca : Abaqus, NASTRAN, ANSYS)

$$[M]\{\ddot{q}(t)\}+[C]\{\dot{q}(t)\}+[K]\{q\}=\{F(t)\}$$

- First order (ODE : Simulink, Simpack, …)

$$\left\{\begin{array}{c}\dot{q}\\ \ddot{q}\end{array}\right\}=\left[\begin{array}{cc}0 & I\\ -M^{-1}K & -M^{-1}C\end{array}\right]\left\{\begin{array}{c}q\\ \dot{q}\end{array}\right\}+\left[\begin{array}{c}0\\ M^{-1}b\end{array}\right]\{u(t)\}$$

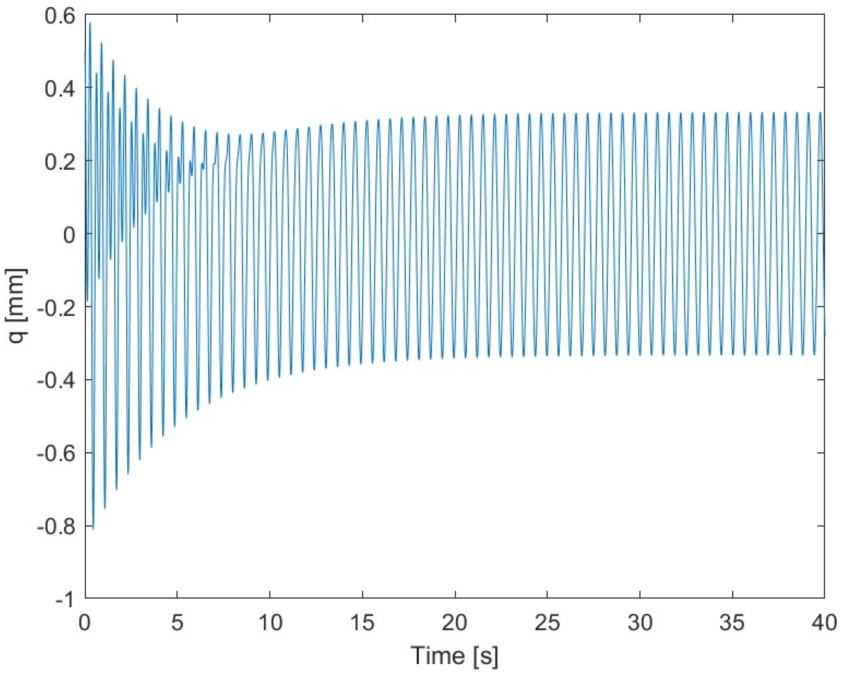
$$\{y(t)\}=\left[\begin{array}{cc}c & 0\end{array}\right]\left\{\begin{array}{c}q\\ \dot{q}\end{array}\right\}$$

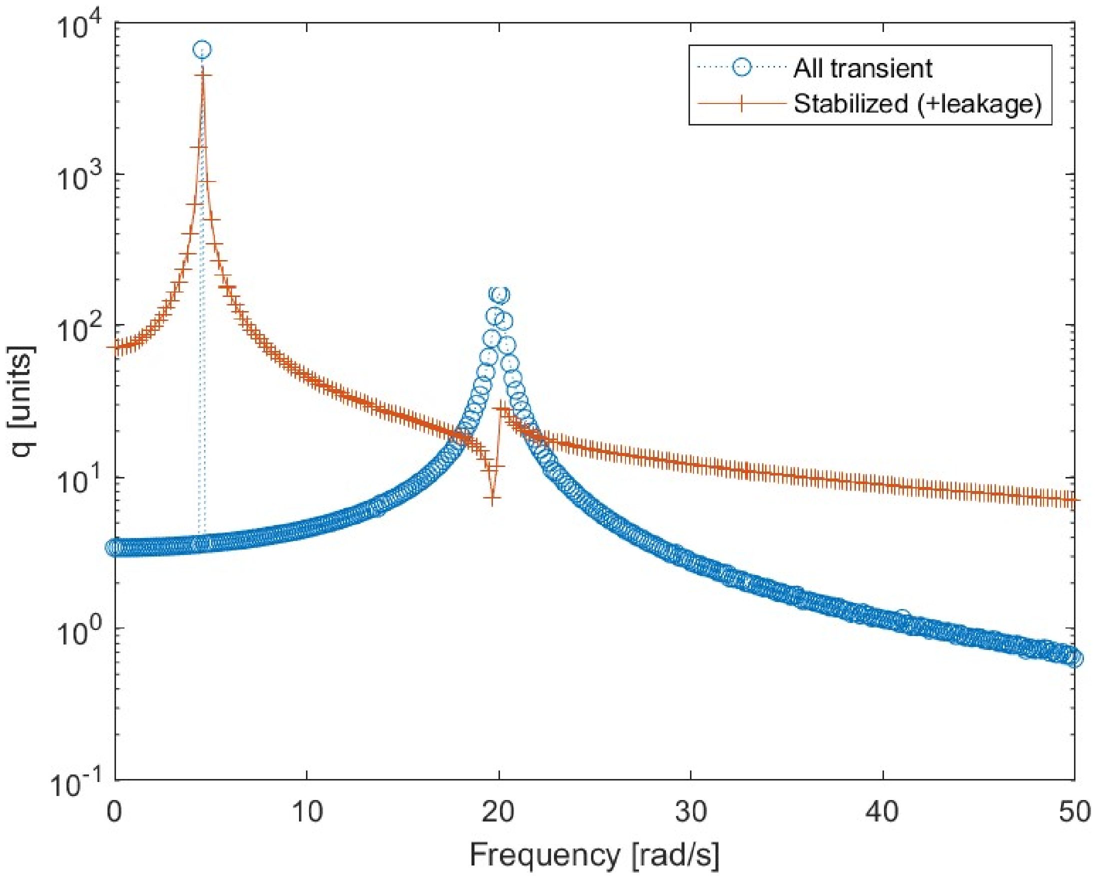
https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html
Matlab : ode45, …

To be discussed

- DOF / State / Initial condition
- Newmark, Runge Kutta, …

Forced harmonic/transient





1 DOF frequency domain / transfer

Dynamic equation

$$\mathrm{Re}\left(\left(-\omega^2 m + i\omega c + k\right) q\left(\omega\right) e^{i\omega t} - F\left(\omega\right) e^{i\omega t}\right) = 0$$

Transfer function

$$H\left(\omega\right) = \frac{q\left(\omega\right)}{F\left(\omega\right)} = \frac{1}{-\omega^2 m + i\omega c + k}$$

Fourier / Laplace transform

$$\mathcal{F}(y) = Y(\omega) = \int_{-\infty}^{+\infty} y(t) e^{-i\omega t} dt$$

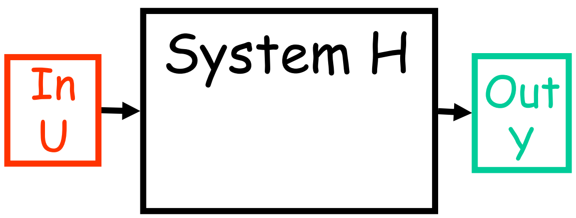
$$\mathcal{F}(\dot{y}) = i\omega \mathcal{F}(y)$$

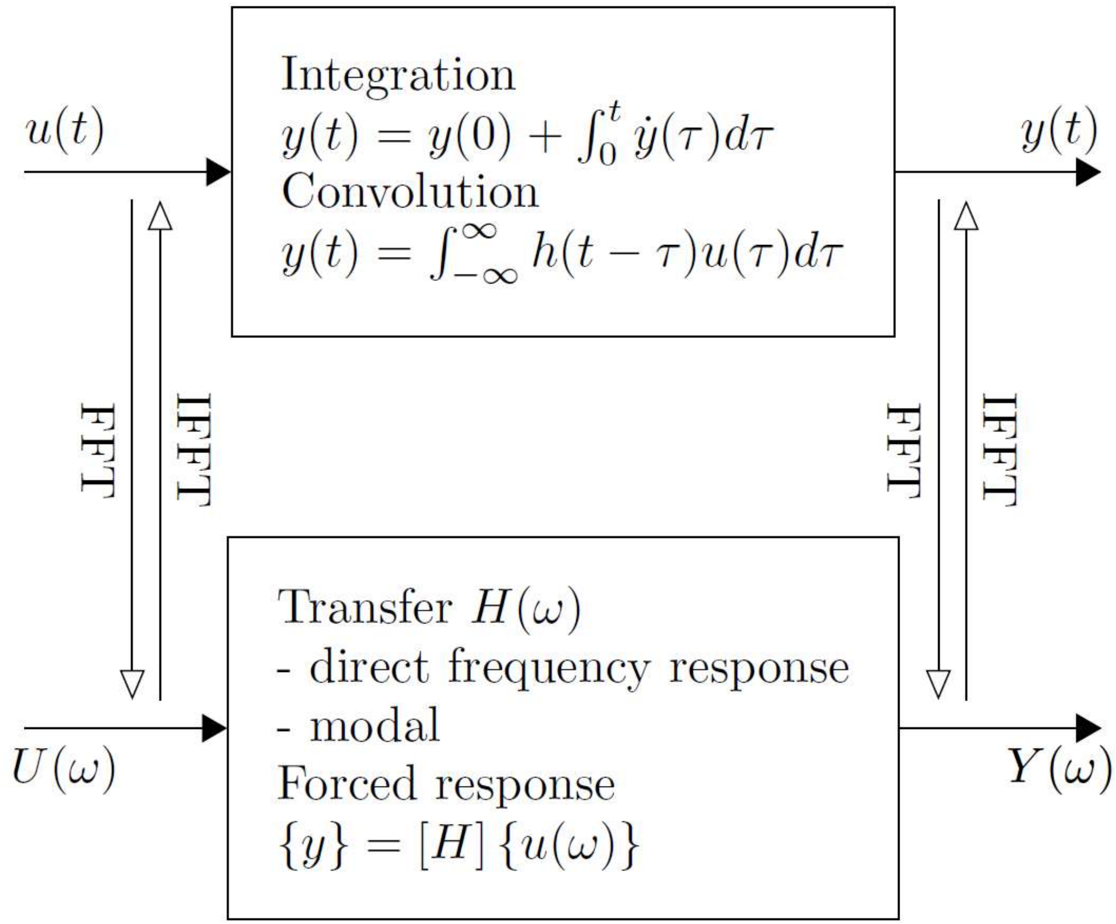
$$Y(s) = \int_{0}^{+\infty} y(t) e^{-st} dt$$

$$H(s) = \frac{q}{F} = \frac{1}{ms^2 + cs + k}$$

$s = i\omega$ (in France s often noted p)
Laplace/Fourier

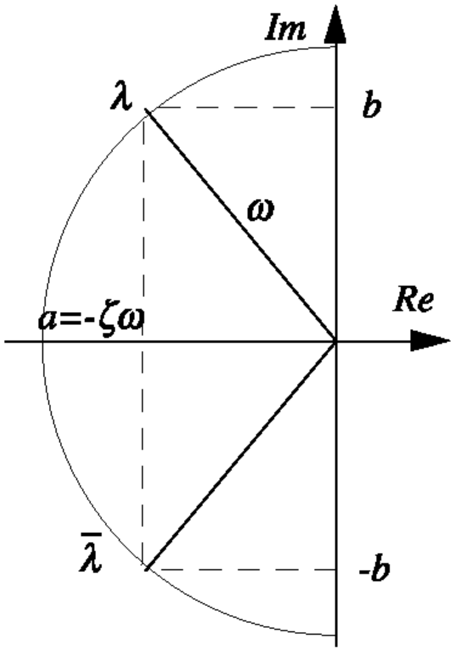
Frequency / time responses of systems





Transfer = assume linear time invariant

1 DOF (Bode plot)

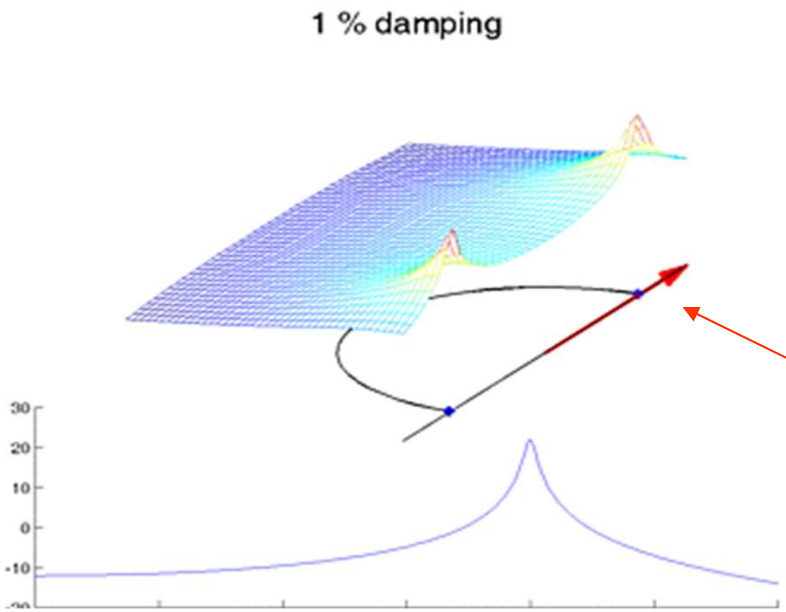


$$H(s) = \frac{1}{s^2 m + cs + k} = \frac{1}{m}\left(\frac{\beta}{s-\lambda} + \frac{\bar{\beta}}{s-\bar{\lambda}}\right)$$

Poles

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \quad , \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_n = \sqrt{k/m} = |\lambda| \quad , \qquad \zeta = \frac{c}{c_{crit}} = \frac{c}{2\sqrt{km}} = \frac{-Re(\lambda)}{|\lambda|}$$



1 DOF system (single mode for mechanical system) has

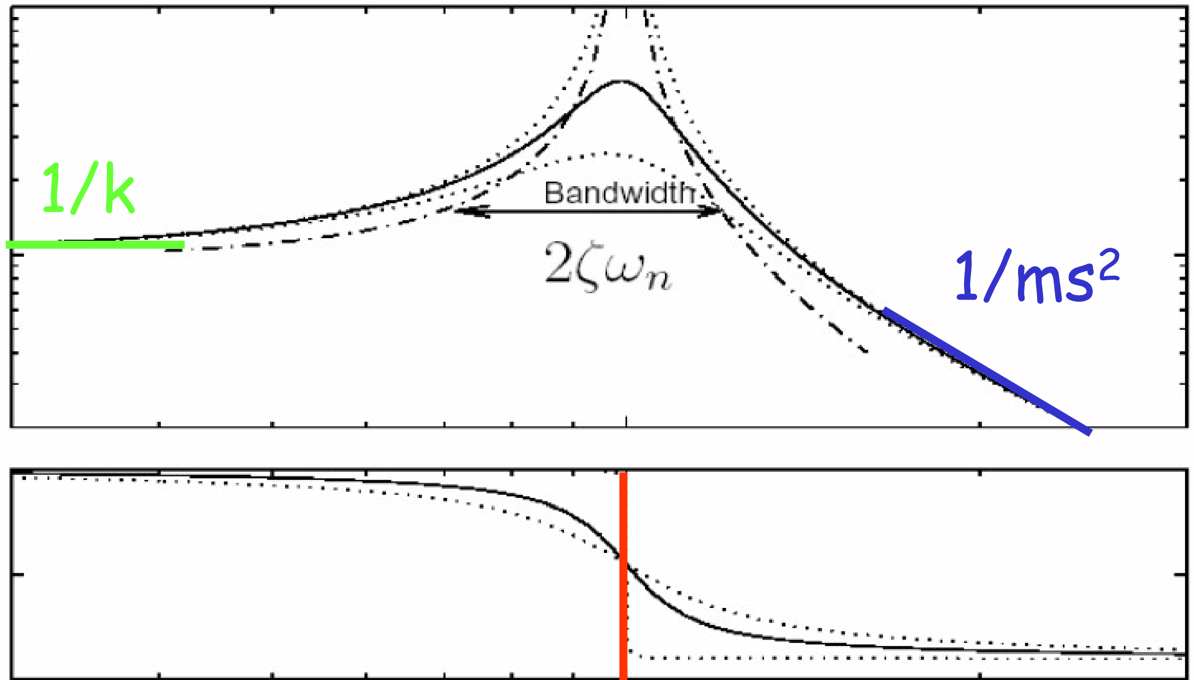
2 complex conjugate poles (linear system modes)

Damping ratio see also quality slide

$s = i\omega$

Laplace/Fourier

1 DOF :Bode plot



$$\boxed{\omega_n = \sqrt{k/m},}$$

1 DOF :Bode plot

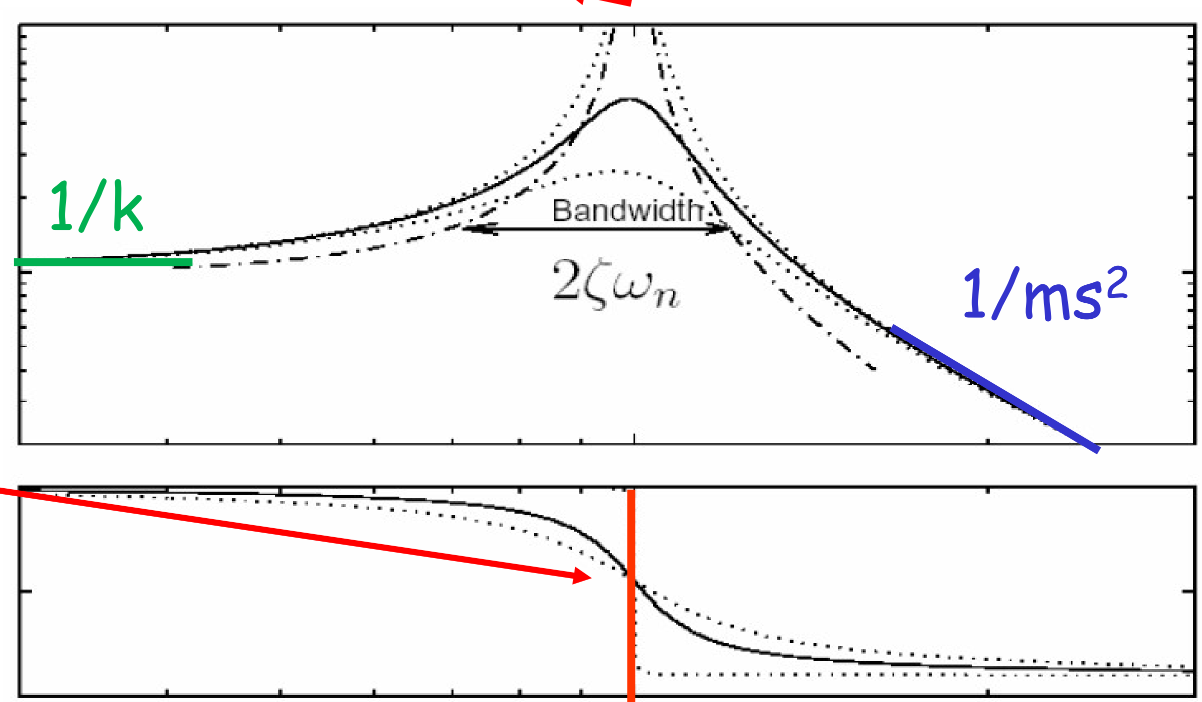
$$H(\omega) = \frac{1}{-\omega^2 m + i\omega c + k}$$

Asymptotes :
- Flexibility $1/k$
- Inertia (isolation) $1/ms^2$

Resonance
- Amplitude $\propto 1/\zeta$
- Phase resonance
- Bandwith $\propto \zeta$

Response at phase resonance

$H(w_n)=1/i2\zeta\omega_n^{\,2}$

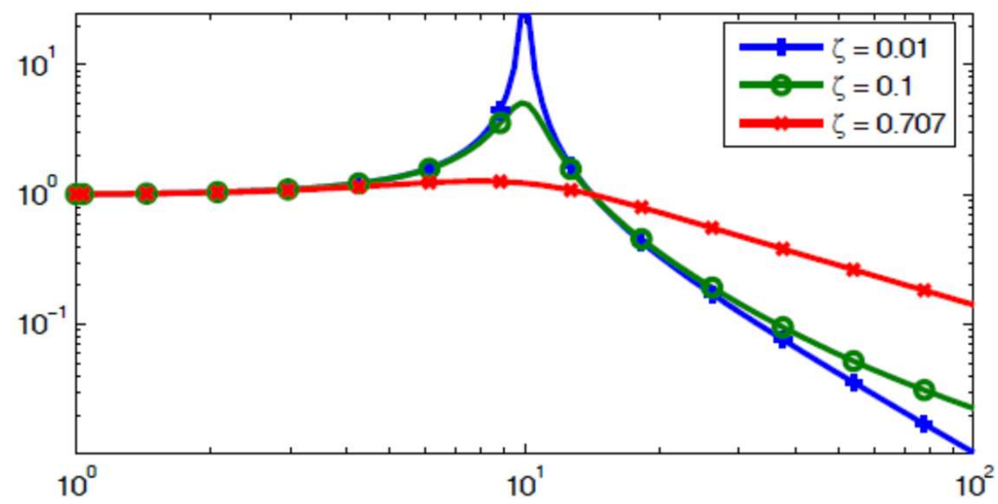


$$\omega_n = \sqrt{k/m},$$

Quality factor

- Other physics definition : $Q = 2\pi \frac{E_{max}}{E_{dis}} = \frac{1}{2\zeta_j}$
- Transmissibility

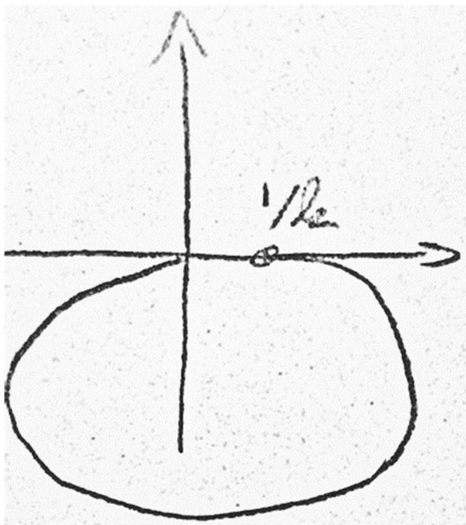
$$H = \frac{x_M}{x_B} = \left[\frac{s^2}{s^2 + 2\zeta_j \omega_j s + \omega_j^2}\right]_{\omega_j} = \frac{1}{2i\zeta_j} = Q$$

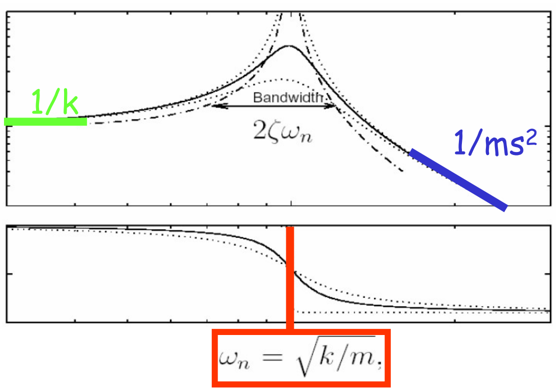


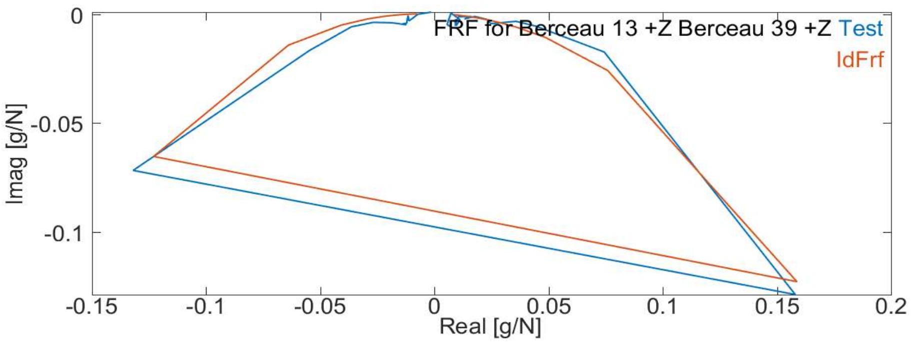
Nyquist & -3dB method

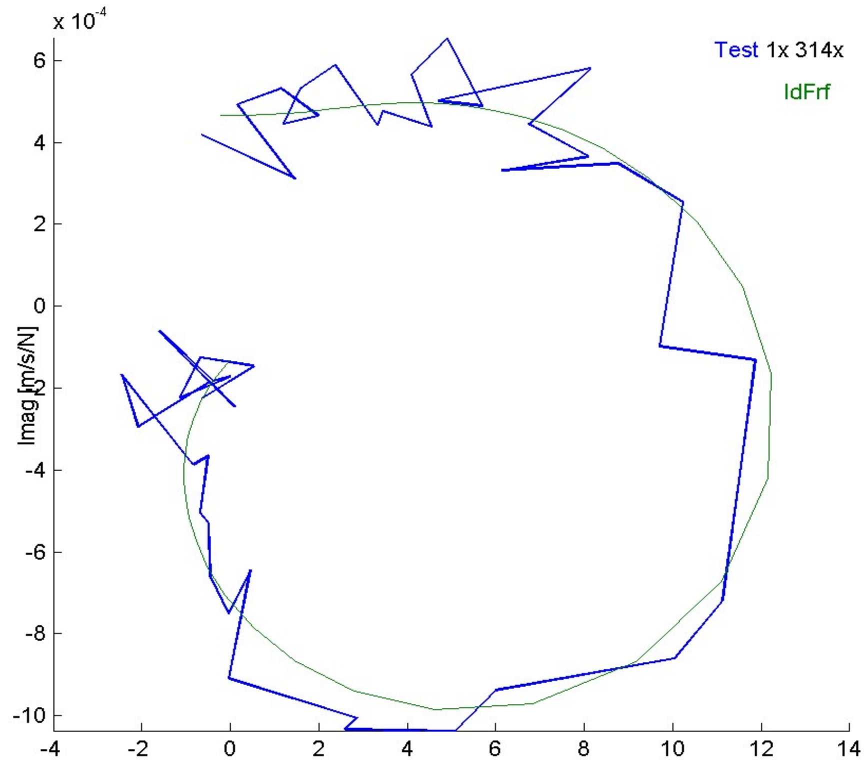
Failures : resolution, noise, multi-mode

$$\zeta = \frac{\Delta\omega}{2\omega_{max}}$$

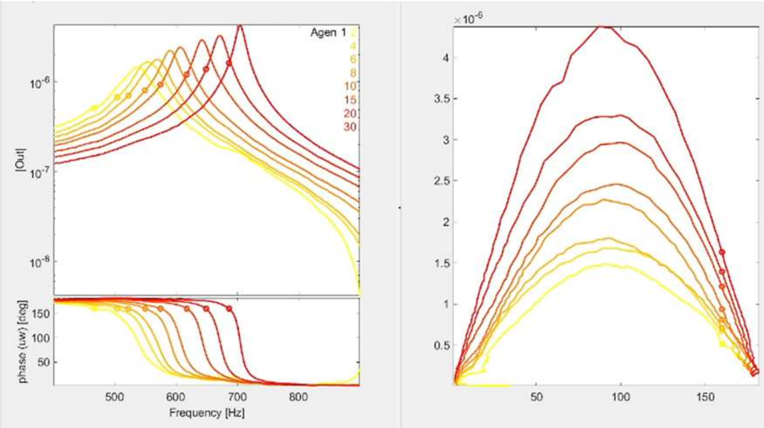




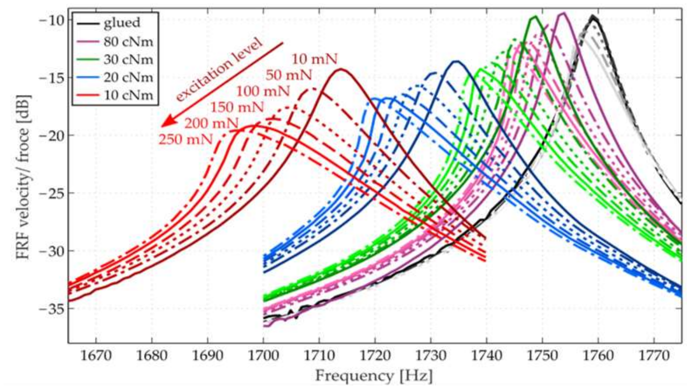


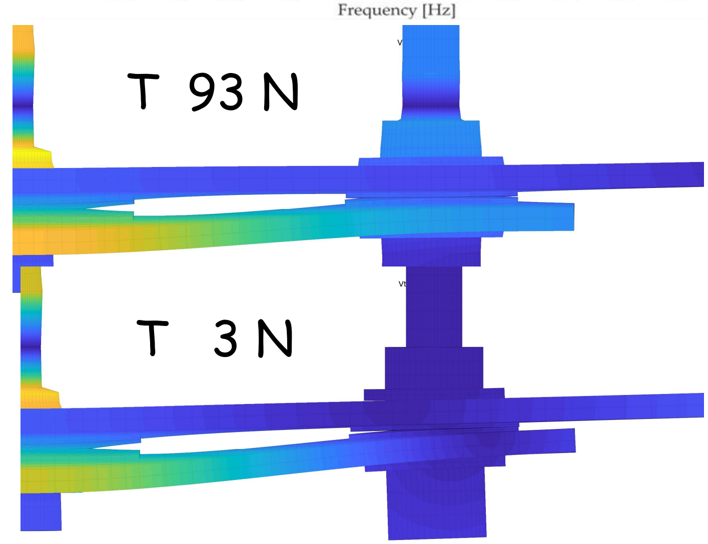


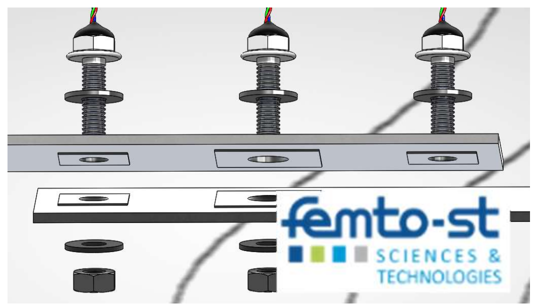
Resonance of NL / parametric system

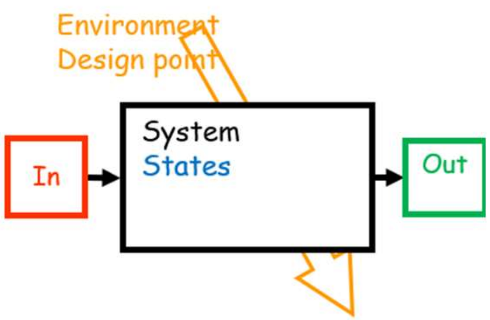


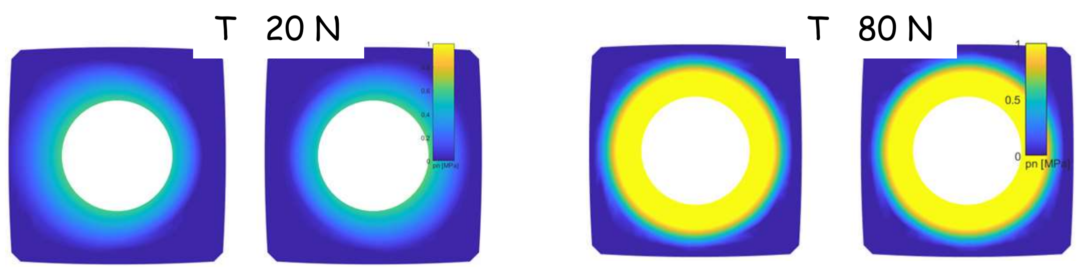
HITACHI
Inspire the Next







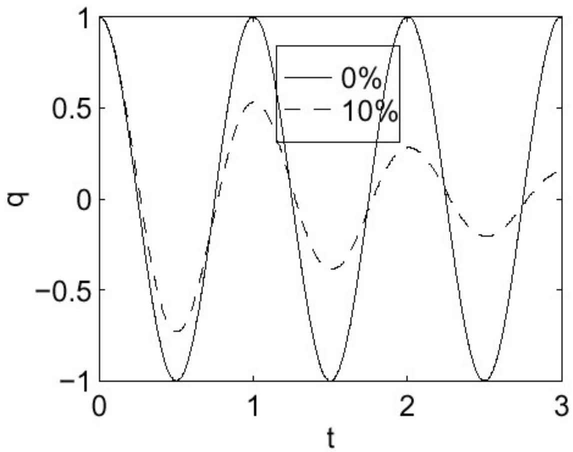




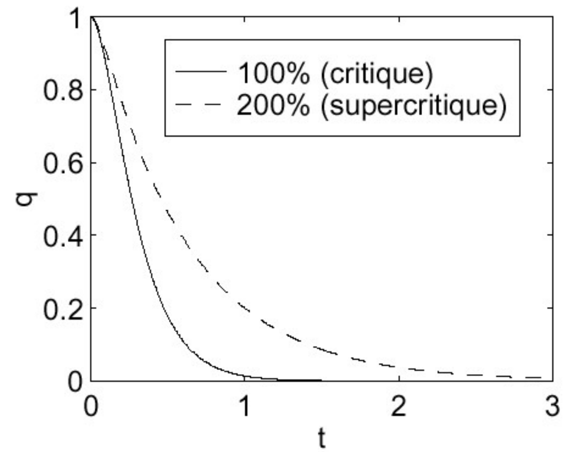
1 DOF : time response / poles

$$q(t) = Re\left(Ae^{\lambda_1 t} + Be^{\lambda_2 t}\right) = A\cos\left(\omega_j\sqrt{1-\zeta_j^2}\,t + \phi\right)e^{-\zeta_j\omega_j t}$$

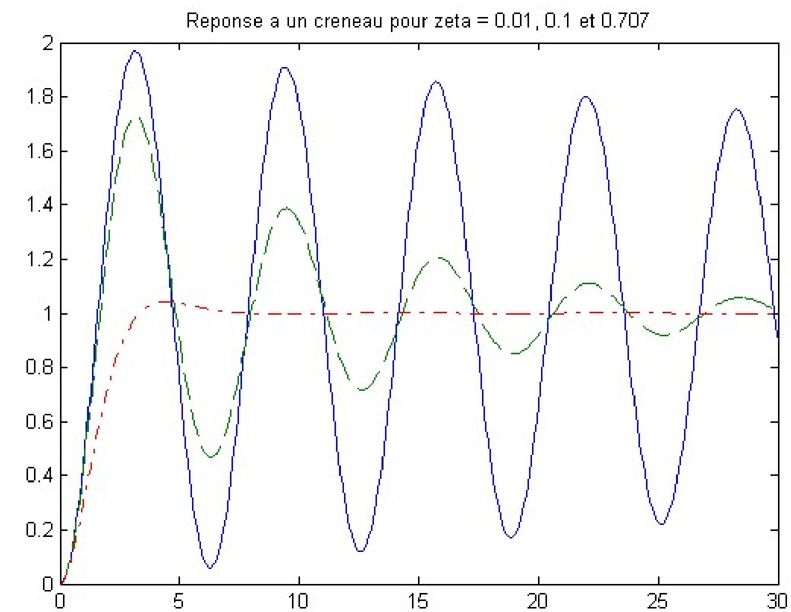
$$\mathcal{L}^{-1}\left(\frac{1}{s-\lambda_1}\right) = e^{\lambda_1 t}$$



Initial condition

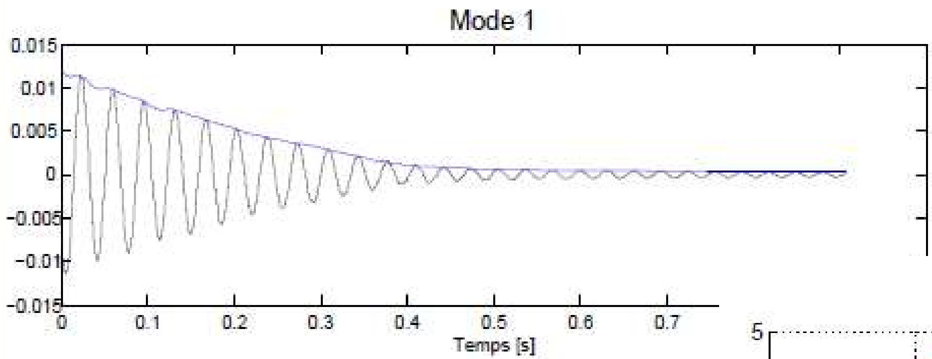


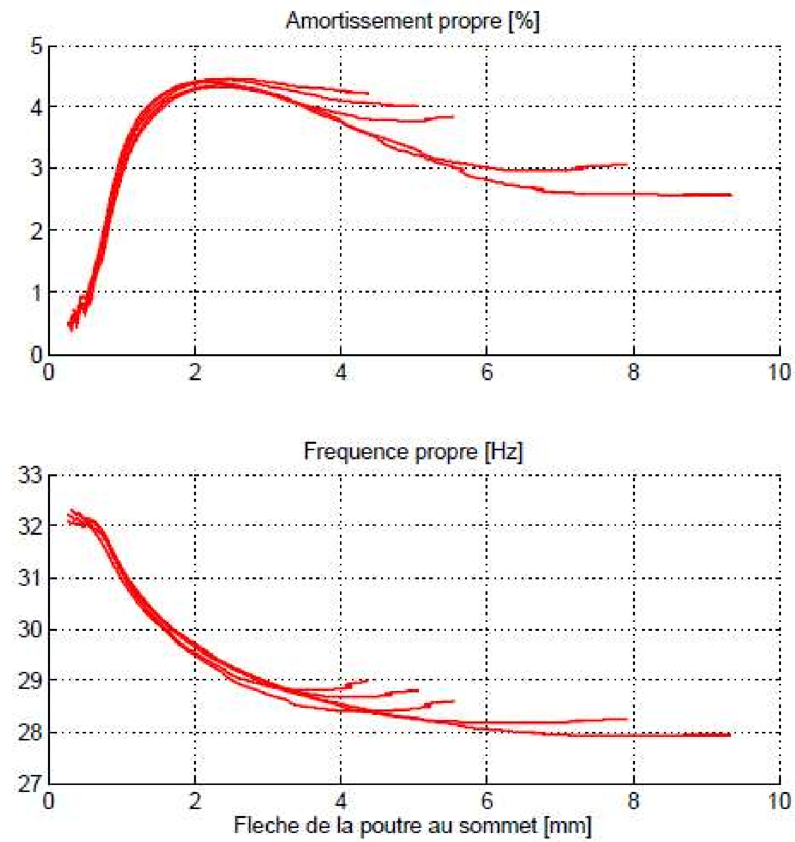
Step input



Work in Lab1

Logarithmic decrement, evolutions

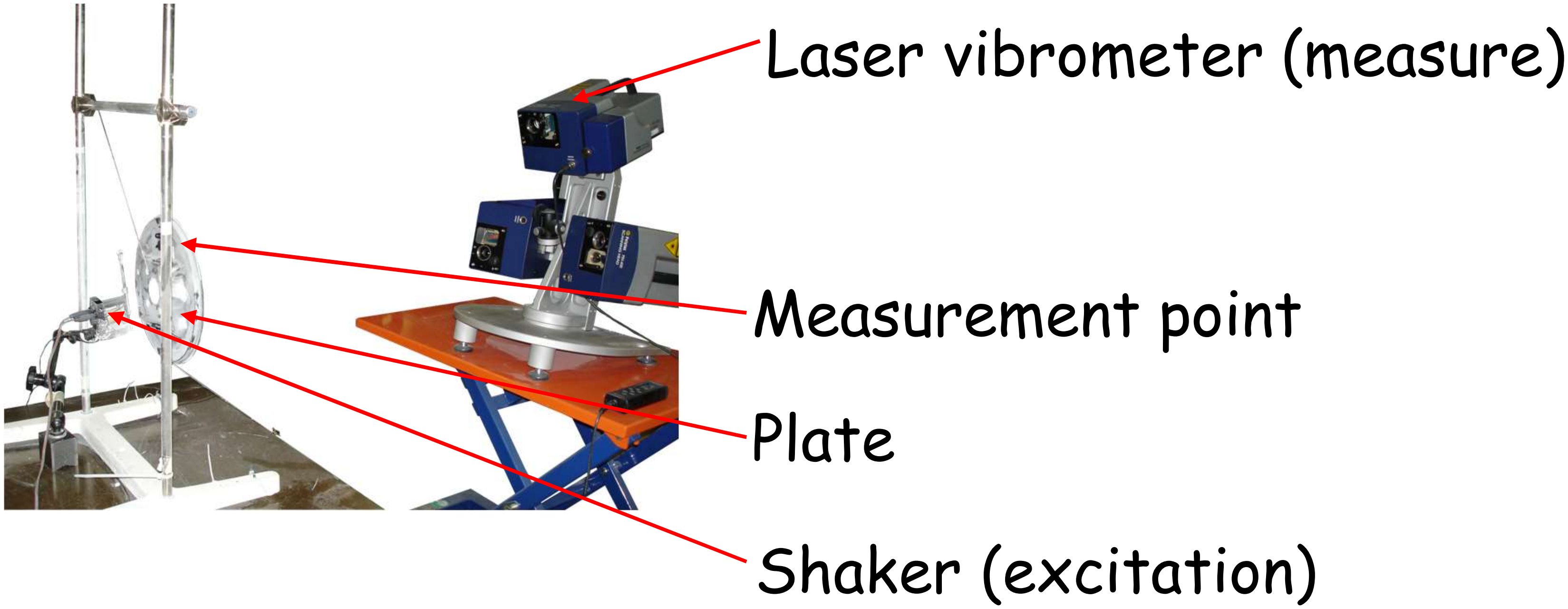


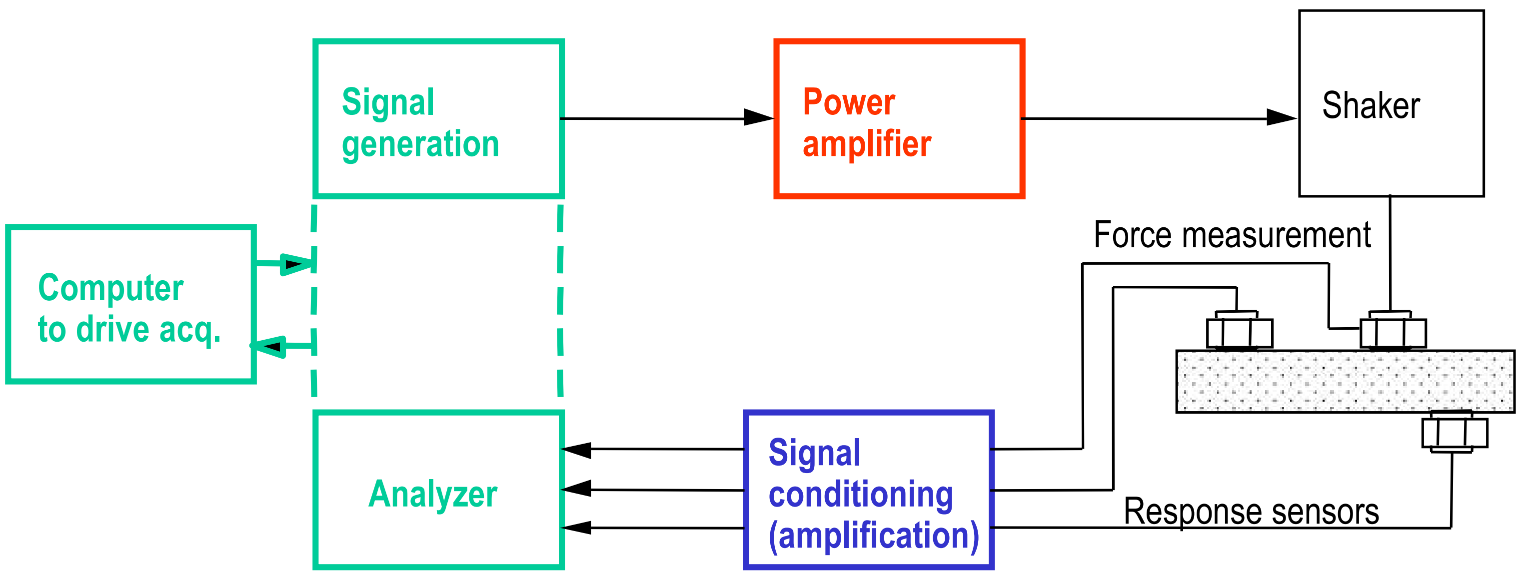


- For non-linear systems ω & ζ are amplitude dependent
- Hilbert transform, Kalman filt., analytic signal
- Sample application :
  NES (non-linear energy sink)

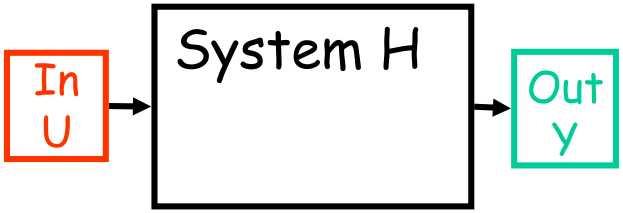
M. Feldman, *Hilbert Transform Applications in Mechanical Vibration.* John Wiley & Sons, Ltd, 2011.

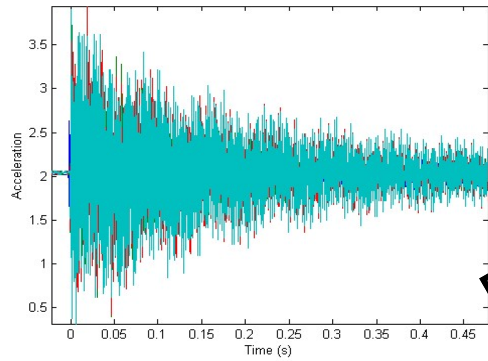
Experimental modal analysis : measurements





Modal analysis : transfers



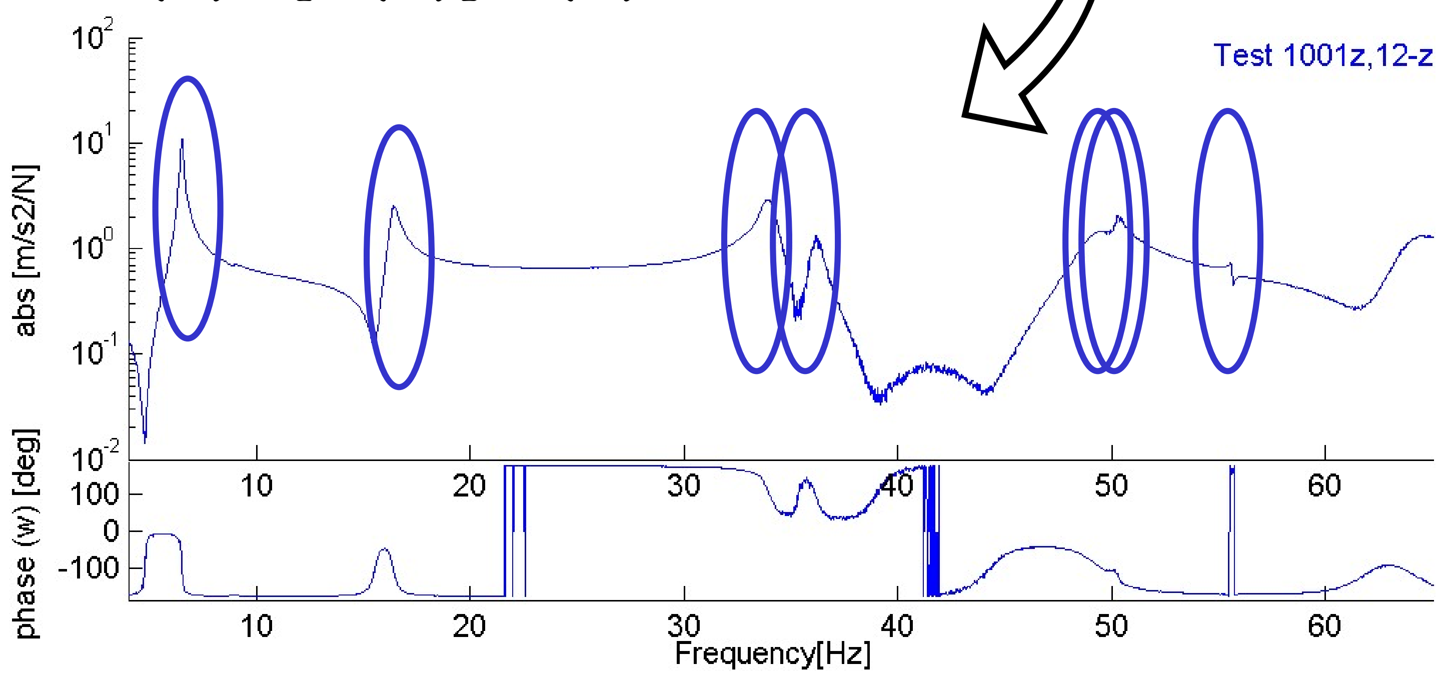


Transfers estimated from time response

ONE input

ONE output

$$\{Y(\omega)\}=[H(\omega)]\{U(\omega)\}$$

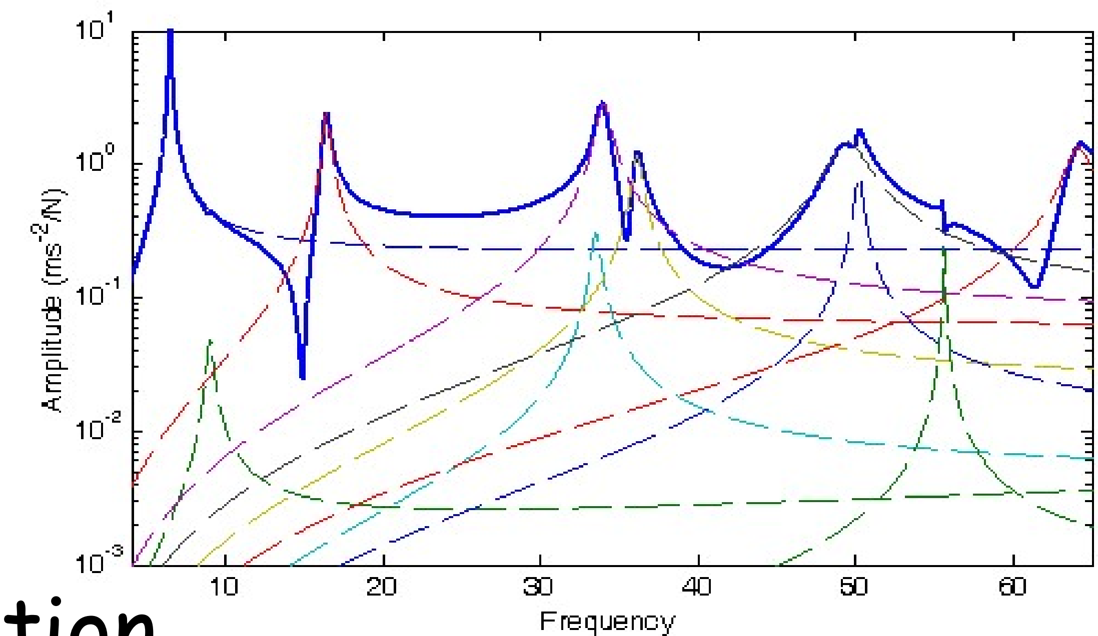


MANY resonances

Bode plot : visualization of transfer function

1 input, 1 output, many resonances

MDOF multiple degree of freedom
SISO single input single output



Spectral decomposition

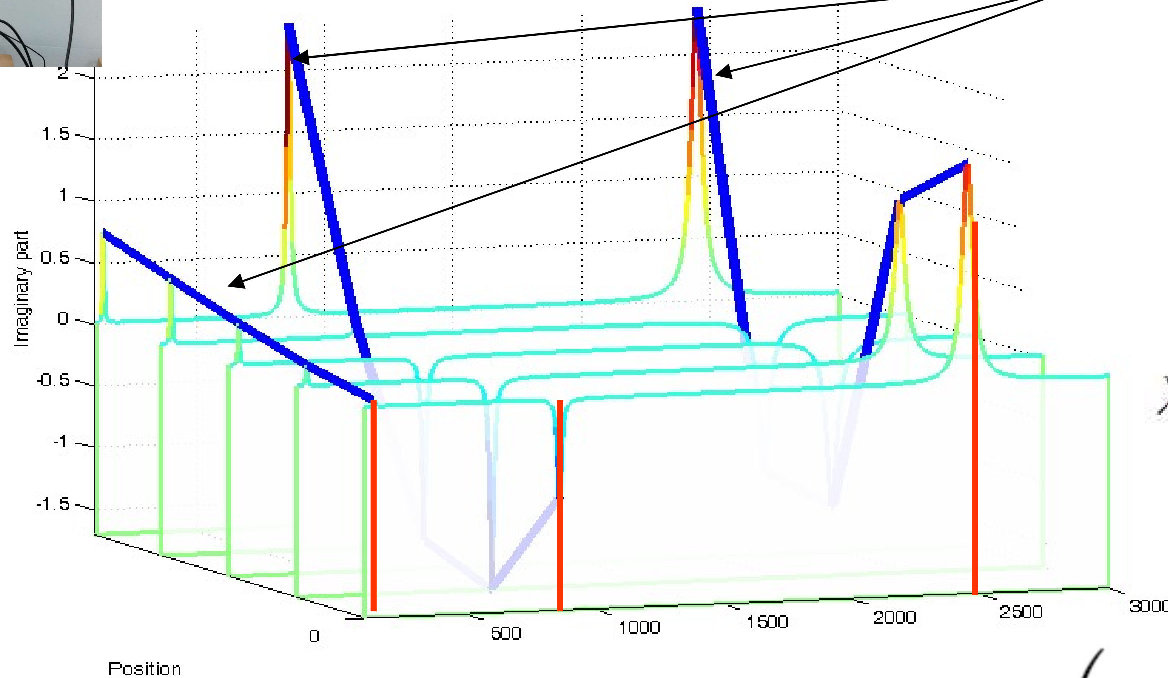
MDOF (multiple resonances)

SISO Tj is 1x1

$$[\alpha(s)] = \sum_{\boxed{j \in \mathtt{identified}}} \left(\frac{\boxed{[T_j]}}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \right)$$

Note : series truncated in practice
Constant approximation of high frequencies (D term in states space models)

MDOF MIMO system



The shapes

The poles

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$[\alpha(s)] = \sum_{\boxed{j\in\text{identified}}} \left(\frac{\boxed{[T_j]}}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} \right)$$

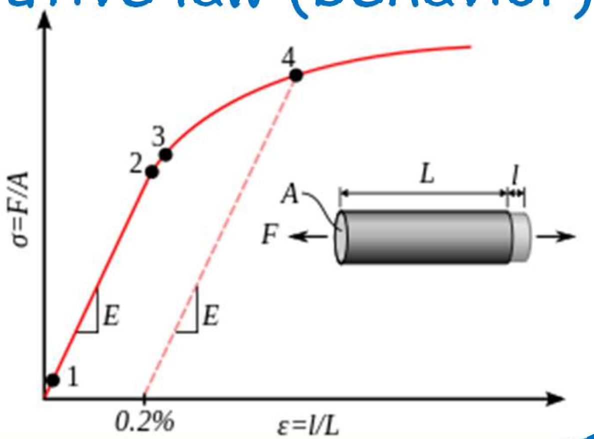
- Poles depend on the system (not the input/output)
- The shape is associated with the input/output locations

Lagrange equations / virtual power principle

Kinematics
- Displacement $u(x,t)$
- Strain $\epsilon(x,t)$

Statics/thermodynamics
- Load/Stress $\sigma(x,t)$
- Power : $\int_\Omega \sigma\dot{\epsilon}$

Constitutive law (behavior)



Equations of motion

$$[M]\{\ddot{q}(t)\}+[C]\{\dot{q}(t)\}+[K]\{q\}=\{F(t)\}$$

Modes : harmonic solution with no force

$$[Ms^2 + Cs + K]\{q(s)\} - \{F(s)\} = 0$$

$q(t) = Re\left(\{\psi_j\}e^{\lambda_j t}\right)$ complex mode (general definition)

$$Re\left([M\lambda_j^2 + C\lambda_j + K]\{\psi_j\}e^{\lambda_j t} - \{0\}\right) = 0$$

Eigenvalue problem

Linear time invariant

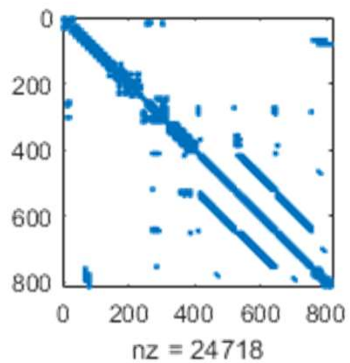
https://www.youtube.com/watch?v=zstmGnaaaCI

Normal modes of elastic structure

- Real mode : no damping / elastic / conservative

$$q(t) = Re\left(\{\phi_j\}e^{i\omega_j t}\right)$$

$$\left[K - \omega_j^2 M\right]_{N\times N}\{\phi_j\}_{N\times 1} = \{0\}$$



- M>0 & K≥0 ⇒ ϕ real
- There are N distinct modes for N DOF

- Full solver : scipy.linalg.eig (MATLAB eig, LAPACK Linear Algebra)
- Partial solvers exist, a few keywords
  - scipy.sparse.linalg.eigs (MATLAB eigs) : Arnoldi
  - FEM Solvers : Lanczos (Krylov+conjugate gradient), AMLS

Normal modes of elastic structure

- Orthogonality

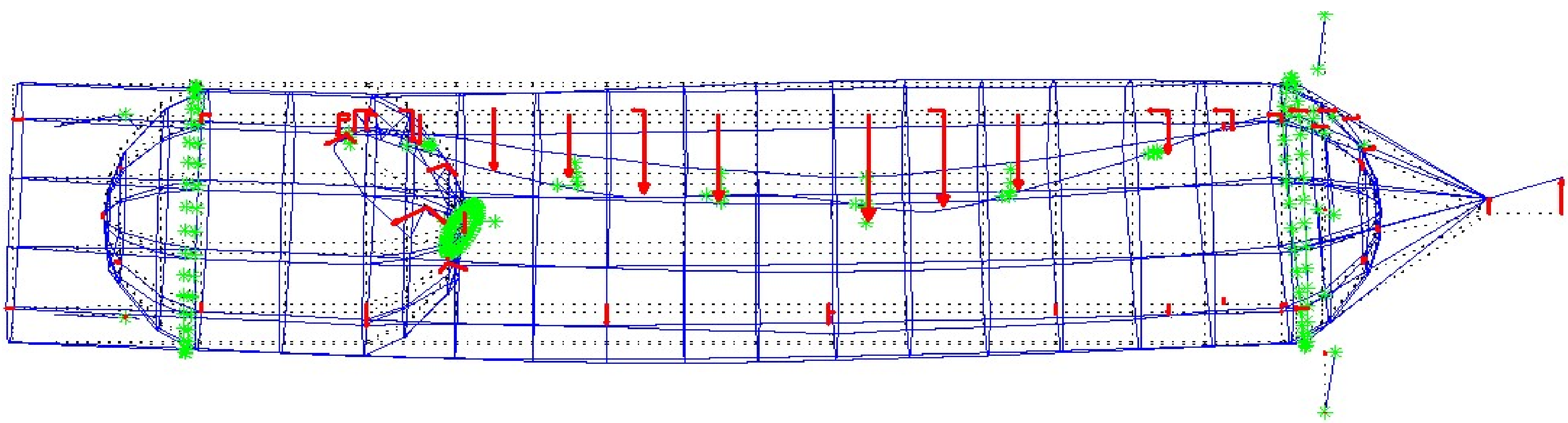
$$\{\phi_k\}^T[K - \omega_j^2 M]\{\phi_j\} = \{0\}$$
$$\{\phi_j\}^T[K - \omega_k^2 M]\{\phi_k\} = \{0\}$$

$$\boxed{\begin{aligned}\phi_j^T M \phi_k &= \mu_j \delta_{jk} \\ \phi_j^T K \phi_k &= \mu_j \omega_j^2 \delta_{jk}\end{aligned}}$$

- Scaling conditions
  - Unit mass
  - Unit amplitude

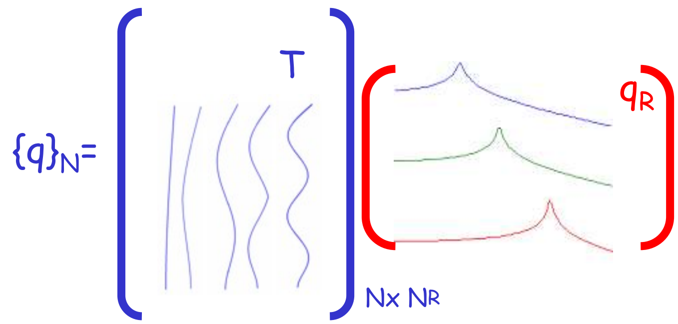
$$\boxed{\{\phi_j\}^T[M]\{\phi_j\} = 1}$$

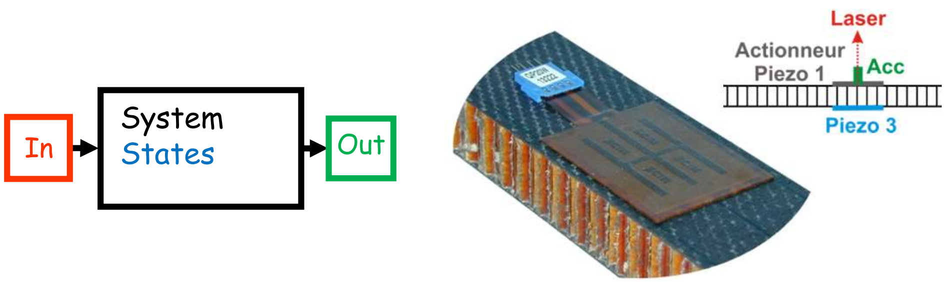
$$[c_s]\left\{\tilde{\phi}_j\right\} = 1 \quad \mu_j(c_s) = ([c_i]\{\phi_j\})^{-2}$$



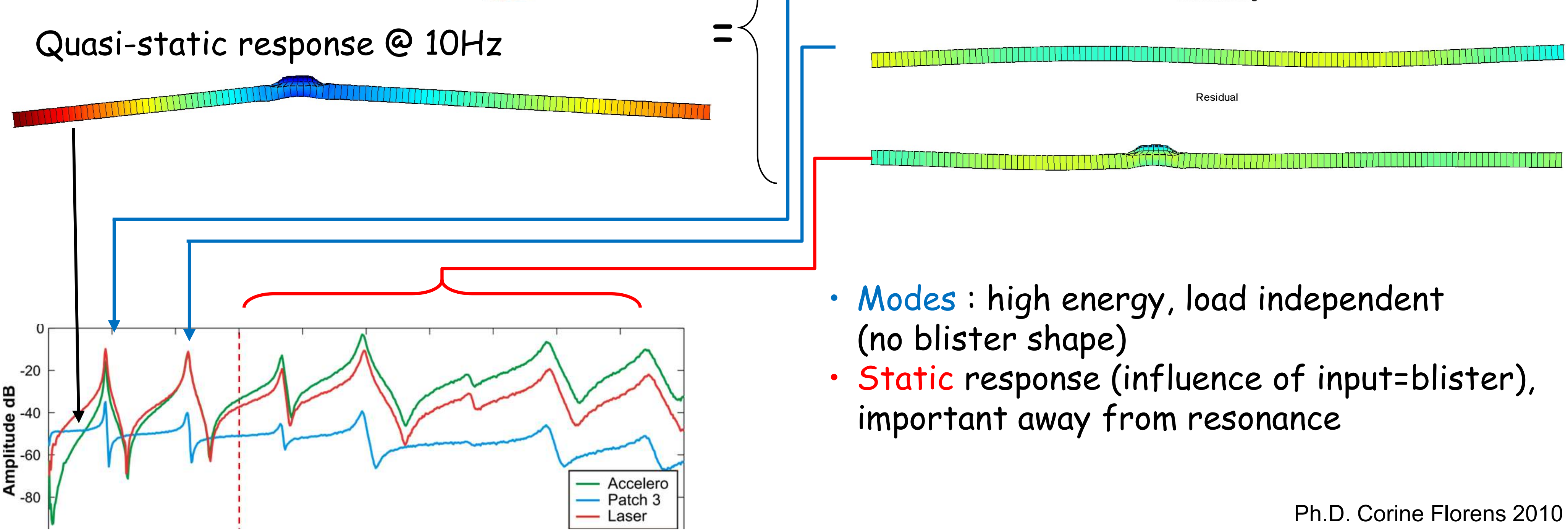
Kinematics / model reduction

- Displacement u(x,t)
  = shapes (x) x DOF (t)





Quasi-static response @ 10Hz



- Modes : high energy, load independent (no blister shape)
- Static response (influence of input=blister), important away from resonance

Ph.D. Corine Florens 2010

Modal (principal) coordinates

- Coordinate change (physical q, generalized q_R):

$$\{q(s)\} = [T]\{q_R(s)\} = [\phi_1 \ \dots \phi_{NM}]\{q_R(s)\}$$

- Inject in equations of motion (kinematics)

$$[Ms^2 + Cs + K]_{Nq \times Nq} \ [T]\{q_R(s)\}_{Nqr \times 1} = \{F(s)\}_{Nq \times 1}$$

- Over determined ($N_q \gg N_{qr}$) : compute "virtual work" $[T]^T$

$$[T]^T \left([Ms^2 + Cs + K] \ [T]\{q_R(s)\} - \{F(s)\}\right) = \{0\}$$

- For modes $[T] = [\phi_1 \ \dots \phi_{NM}]$

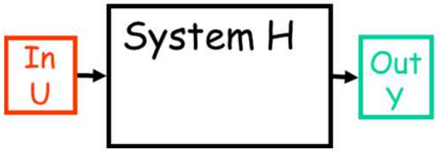
Reduced mass $M_R = T^T M T = [\phi_i^T M \phi_k] = \begin{bmatrix} \diagdown 1 \diagdown \end{bmatrix} = I$

Reduced stiffness $K_R = T^T K T = \begin{bmatrix} \diagdown \omega_j^2 \diagdown \end{bmatrix}$

Need : observation, loads, damping

Observation

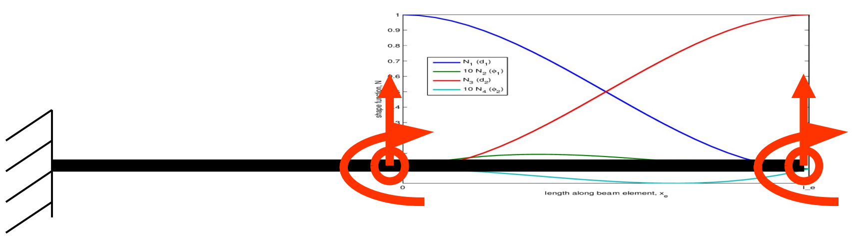
$$\left[Ms^2 + Cs + K\right]\{q\} = \{F(s)\}$$
$$\{y(s)\} = [c]\{q\}$$



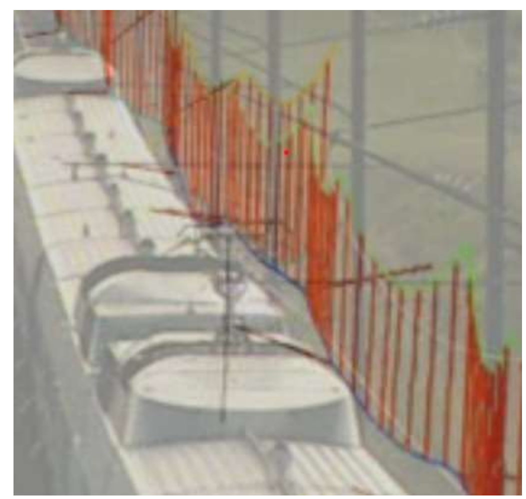
- {y} outputs are linearly related to DOFs {q} using an observation equation

$$\{y\} = [c]\{q(t)\}$$

- Simple case : extraction $\{w_2\} = [0\ \ 0\ \ 1\ 0]\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$

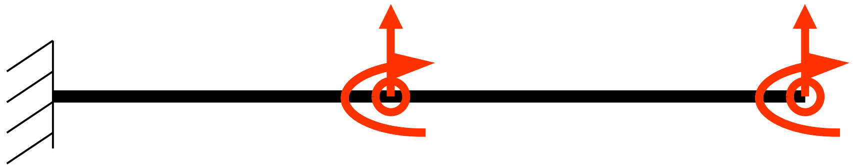


- More general : intermediate points, reactions, strains, stresses, …



Command

$$\left[Ms^2 + Cs + K\right]\{q\} = \{F(x,s)\} = [b(x)]\{u(s)\}$$



- Loads decomposed as spatially unit loads and inputs

$$\{F(t)\} = [b]\,\{u(t)\}$$

Abaqus : *CLOAD + *AMPLITUDE, ...

NASTRAN : FORCE-MOMENT + RLOAD

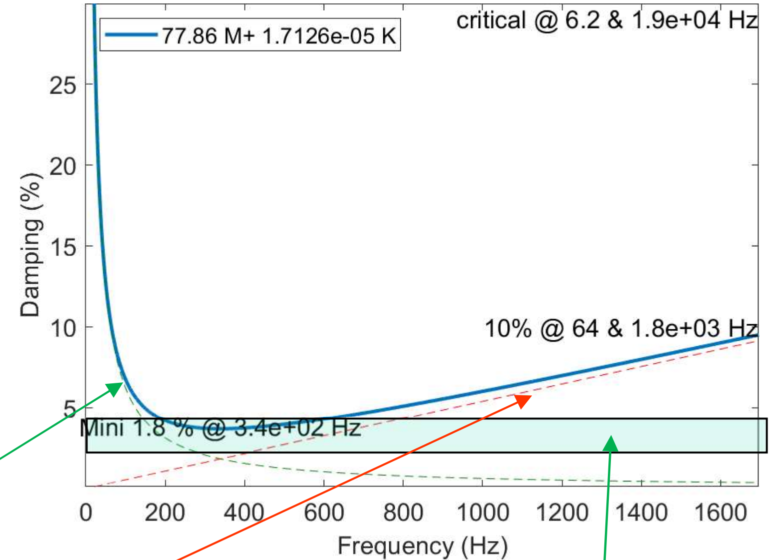
ANSYS, CODE Aster, ...

Modal damping

Rayleigh damping

- Physical domain $[C] = \alpha[M] + \beta[K]$
- Modal $\phi^T[C]\phi = \left[\, {}^{\backslash} 2\zeta_j\omega_{j_{\backslash}} \right] = \alpha[I] + \beta\left[\, {}^{\backslash}\omega_j^2{}_{\backslash} \right]$

$$\boxed{\zeta_j} = \boxed{\frac{\alpha}{2\omega_j}} + \boxed{\frac{\beta\omega_j}{2}}$$



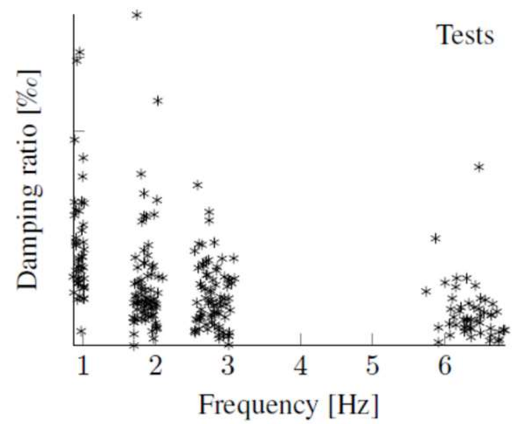
Mass

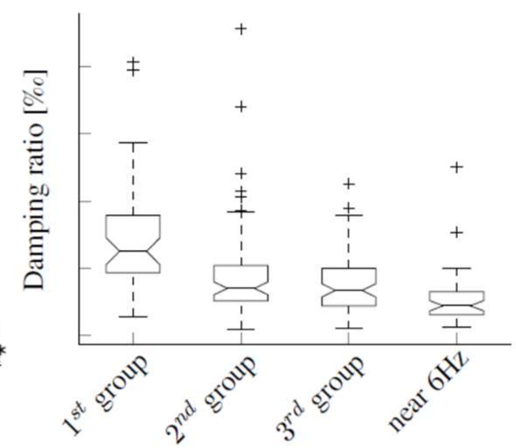
Stiffness
can be > 100%

Reality

Modal damping $\boxed{\zeta_j}$ derived from test

Physical domain $[C] = [M\phi]\left[\, {}^{\backslash} 2\zeta_j\omega_{j_{\backslash}} \right][\phi^T M]$



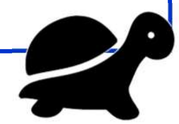


O. Vo Van, E. Balmes, et X. Lorang, « Damping characterization of a high speed train catenary », IAVSD, Graz, 2015, http://sam.ensam.eu/handle/10985/10918.

Physical / modal & spectral decomposition

- Physical

$$[Ms^2 + Cs + K]\,\{q(s)\}_{Nq} = [b]\{u(s)\}$$
$$\{y(s)\} = [c]\{q(s)\}$$



- Modal coordinate $\{q(s)\} = [T]\{q_R(s)\} = [\phi_1\ \ldots \phi_{NM}]\{q_R(s)\}$
- Modal equations (modal damping)

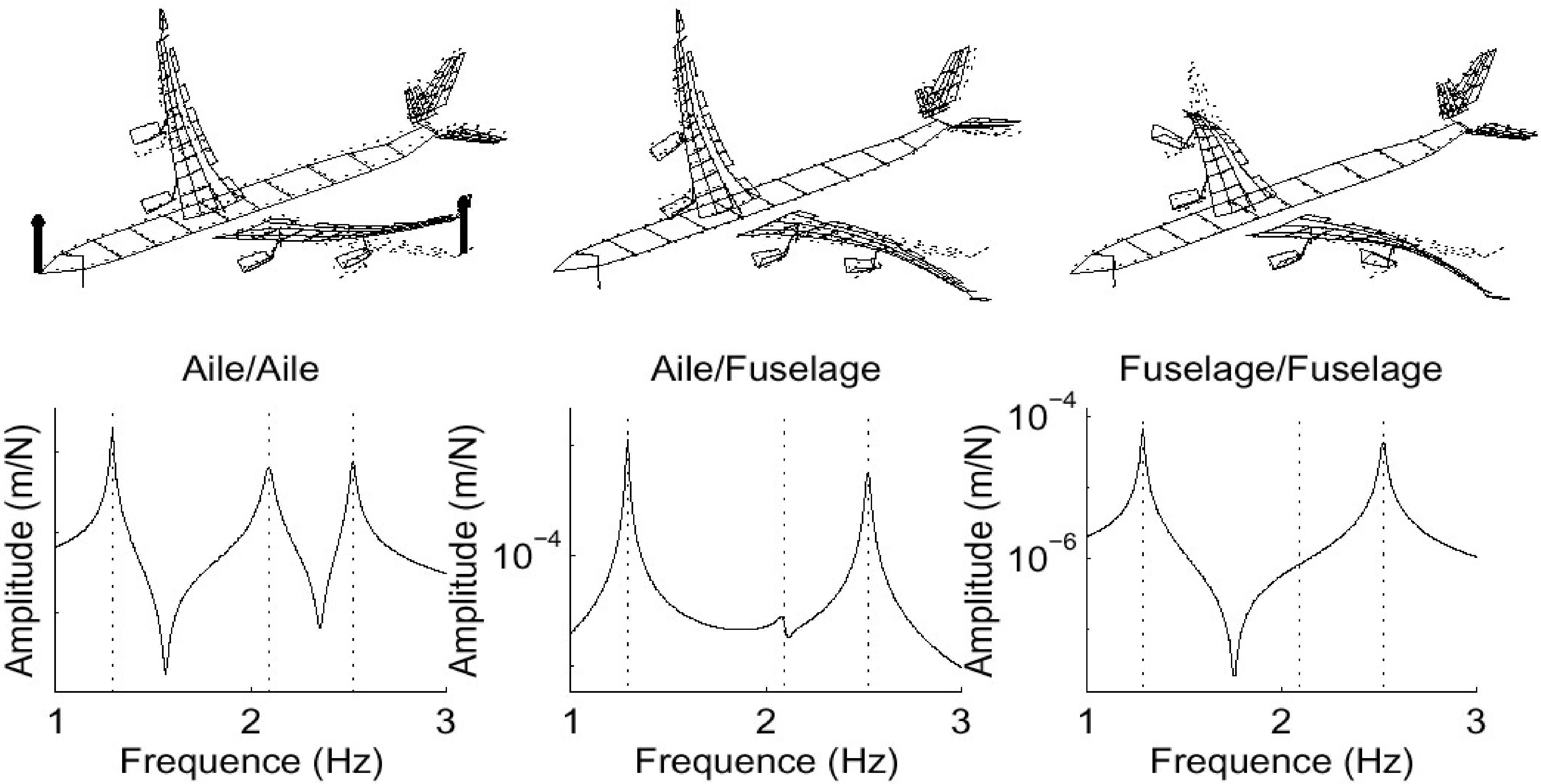
$$\left[Is^2 + \left[{}^{\backslash}2\zeta_j\omega_j{}_{\backslash}\right]s + \left[{}^{\backslash}\omega_j^2{}_{\backslash}\right]\right]\{q_R(s)\}_{Nqr} = [\phi_j^T b]\{u(s)\}$$
$$\{y(s)\} = [c\phi_j]\{q_R(s)\}$$

- Reduced matrices = diagonal
  Modal observability/commandability
- Spectral equations (inverse of diagonal matrix)

$$H(s) = [c][Ms^2 + Cs + K]^{-1}[b] = \sum_j \frac{\{c\phi_j\}\{\phi_j^T b\}}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}$$

Observability/controlability

$$H(s) = \sum_{j=1}^{N} \frac{\boxed{[c]\{\phi_j\}}\boxed{\{\phi_j\}^T[b]}}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} = \sum_{j=1}^{N} \frac{\boxed{[T_j]}}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}$$



Discretized/reduced equations of motion

FEM $\Leftrightarrow$ Reduction

	Finite elements Continuous $\rightarrow$ discrete full	Reduction Full $\rightarrow$ reduced
Support	Element: line, tria, tetra, …	FE mesh
Variable separ. Shape functions	$w(x,t) = N_i(x)q_i(t)$ $\epsilon(x,t) = B_i(x)q_i(t)$	$\{q(t)\} = \{T_i\}q_i(t)$ T_i simple FE solutions
Matrix comp. Weak form	$K_{ij} = \int_\Omega B_i^T \Lambda B_j = \sum_g B_i^T(g)\Lambda B_j w_g J_g$ numerical integration	$K_{ijR} = T_i^T K T_j$ FEM matrix projection
Assembly	Localization matrix	Boundary continuity, CMS
Validity	Fine mesh for solution gradients	Good basis for considered loading

[1] O. C. Zienkiewicz et R. L. Taylor, *The Finite Element Method*. MacGraw-Hill, 1989
[2] J. L. Batoz et G. Dhatt, *Modélisation des Structures par Éléments Finis*. Hermès, Paris, 1990
[3] K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall Inc., Englewood Cliffs, NJ, 1982

State-space form

- Principal coordinates are also used to build state-space models

$$\left\{ \begin{array}{c} \dot{p} \\ \ddot{p} \end{array} \right\} = \left[\begin{array}{cc} 0 & I \\ -\left[\diagdown \omega_j^2 \diagdown\right] & -\Gamma \end{array} \right] \left\{ \begin{array}{c} p \\ \dot{p} \end{array} \right\} + \left[\begin{array}{c} 0 \\ \phi_j^T b \end{array} \right] \{u(t)\}$$

$$\{y(t)\} = \left[\begin{array}{cc} c\phi_j & 0 \end{array} \right] \left\{ \begin{array}{c} p \\ \dot{p} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \dot{q} \\ \ddot{q} \end{array} \right\} = \left[\begin{array}{cc} 0 & I \\ -M^{-1}K & -M^{-1}C \end{array} \right] \left\{ \begin{array}{c} q \\ \dot{q} \end{array} \right\} + \left[\begin{array}{c} 0 \\ M^{-1}b \end{array} \right] \{u(t)\}$$

$$\{y(t)\} = \left[\begin{array}{cc} c & 0 \end{array} \right] \left\{ \begin{array}{c} q \\ \dot{q} \end{array} \right\}$$

Nearly modal real state-space

If non modal damping complex conjugate poles can be grouped as in elastic mode state-space

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} [0] & \left[\diagdown I_\diagdown\right] \\ -\left[\diagdown \omega_j^2 {}_\diagdown\right] & -\left[\diagdown 2\zeta_j\omega_j {}_\diagdown\right] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \{u(t)\}$$

$$\{y(t)\} = [C_1 \;\; C_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\begin{Bmatrix} B_{j1} \\ B_{j2} \end{Bmatrix} = 2\left[\mathrm{Re}\left(\theta_j^T B\right) \;\; \mathrm{Im}\left(\theta_j^T B\right)\right] \begin{bmatrix} 1 & 0 \\ \zeta_j\omega_j & -\omega_j\sqrt{1-\zeta_j^2} \end{bmatrix}$$

$$\begin{Bmatrix} C_{1j} \\ C_{2j} \end{Bmatrix} = \frac{1}{\omega_j\sqrt{1-\zeta_j^2}} \begin{bmatrix} \omega_j\sqrt{1-\zeta_j^2} & \zeta_j\omega_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Re}\left(C\theta_j\right) \\ \mathrm{Im}\left(C\theta_j\right) \end{bmatrix}$$

Collocated transfers

- Collocated $\Leftrightarrow \{u\}^T\{\dot{y}\} = power \Leftrightarrow [c] = [b]^T$
- Modal contributions positive real

$$H_c(s) = \sum_{j=1}^{NM} \frac{\left(c\phi_j\right)^2}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}$$

- Trivial ranking of mode contributions as fractions

$$cont_j = \frac{\left(c\phi_j\right)^2/\omega_j^2}{\sum\left(c\phi_j\right)^2/\omega_j^2} \in [0\ \ 1]$$